THE CLASSICAL TOPOLOGY AND CONE LATTICES

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ABSTRACT

In his 2005 dissertation, Antoine Vella[6] introduced a new topology on graphs and hypergraphs known as the ”classical” topology. In his 2015 thesis, Brian Frazier[7] characterized the prime spectrum for certain graphs' open set lattice under the classical topology and demonstrated how the graph may be recaptured from the prime spectrum. In this paper, we further explore the classical topology on simple, social graphs by characterizing the classical topology for simple, social graphs order theoretically, discussing graph posets and their connection with the classical topology, and characterizing the lattices which the classical topology yields, namely cone lattices.

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CHAPTER 1

INTRODUCTION

In a 2005 dissertation, Antoine Vella[6] described the classical topology on graphs and identified a number of properties. In his 2015 thesis, Brian Frazier[7] characterized the prime spectrum for certain graph's open set lattice under the classical topology and demonstrated how the graph may be recaptured from the prime spectrum.

The goal of this paper will be to characterize the lattices the classical topology yields, namely cone lattices. This is accomplished by studying the meet-prime and join-prime elements of the topology, by identifying graph posets within the classical topology, and concludes with defining cone lattices and relating them to the topologies generated from simple, social graphs by the classical topology.

This paper makes free use of standard concepts and notation from the realms of order theory, point-set topology, and graph theory. Readers desiring clarification are encouraged to consult Diestel [3] for more details on graph theory, Munkres [5] for more details about topology, and Birkhoff [1] or Davey and Priestley [2] for more details on order theory.

In the first section, we define terminology required in order to read the later sections. Key definitions from this section include Boolean lattices, atoms and co-atoms, and order isomorphism. These definitions are important in the understanding of the structures presented in the later chapters, as atomic, Boolean sublattices are key features of cone lattices and order isomorphism is an essential tool in proving theorems.

In the second section, we define the classical topology and proceed to show key order theoretic features of the classical topology. Some important features therein include the characterization of completely meet-prime elements of the classical topology and showing that completely join-prime elements of the classical topology are the join-prime elements of the classical topology and the completely meet-prime elements of the classical topology are the meet-prime elements of the classical topology.

In the third section, we describe graph posets and show how they relate to the classical topology. Namely, the meet-prime elements of the classical topology formed by a simple, social graph $\mathcal{G}$ form a graph poset which is associated with a graph which is graph-isomorphic to $\mathcal{G}$. Another result in section 3 is that the classical topology generated by a simple, social graph $\mathcal{G}$ is the set of all lowersets generated by the graph poset associated with $\mathcal{G}$.

In the fourth section, we define cone lattices and show that they are the lattices formed by the classical topology. In particular, we show that if given a cone lattice $\mathcal{L}$ whose meet-prime elements form a graph poset, then there is a simple, social graph $\mathcal{G}$ whose classical topology is order isomorphic to $\mathcal{L}$ and whose meet-prime elements of its classical topology are order isomorphic to the meet-prime elements of $\mathcal{L}$. This theorem concludes the results portion of the paper.

In the final section, a review of what has been said and suggestions for further research can be found.

1.1 Basic Definitions

Let G be a nonempty set, and let $E \subseteq (G \times G)$ satisfy the following properties:

- $xy \in E$ where $xy = \{(x,y),(y,x)\}$.
- $xx \notin E$ for any $x \in G$.

The pair $\mathcal{G} = (G, E)$ is called a *simple graph*. The elements of G are called the *vertices* of $\mathcal{G}$, and the elements of E are called the *edges* of $\mathcal{G}$. Two vertices x and y are said to be *adjacent* provided $x = y$ or $(x, y) \in E$. (In visual representations of a graph, vertices are denoted by points, and the adjacency edge between x and y is represented as a line segment connecting the point for x and the point for y.)

Note that every member of E must be the unique adjacency edge for two distinct vertices x and y. We will say that this edge is *incident* to the vertices x and y. It is worth noting that the ordered pairs (x, x) can be interpreted as *loops* connecting a vertex to itself. Simple graphs do not contain loops.

Definition 1.1. *Let $\mathcal{G} = (G, E)$ be a simple graph. We say that $\mathcal{G}$ is social provided every vertex is adjacent to an edge.*

Throughout this paper, we will restrict our attention to graphs that are simple and social. We let `SimSoc` denote the set of all simple, social graphs.

If X is any set, we will let $\mathtt{Su}[X]$ denote the powerset of X, and we will let $\mathtt{Fin}[X]$ denote the finite subsets of X.

Definition 1.2. *Let X be a set. We say the set $\tau \subseteq \mathtt{Su}(X)$ is a topology provided:*

1. $\emptyset, X \in \tau$
2. *For all $X, Y \in \tau$, $X \cap Y \in \tau$.*
3. *For all $\mathcal{F} \subseteq \tau$, $\cup\mathcal{F} \in \tau$.*

Definition 1.3. *A <u>partially ordered set</u>, or poset, is a system $\mathcal{P} = (P, \leq)$ consisting of a set P and a binary relation $\leq$ as a subset of $P \times P$ satisfying the following conditions:*

1. *For all $x \in P$, we have $x \leq x$ (<u>reflexivity</u>)*
2. *If $x \leq y$ and $y \leq x$, then $x = y$ (<u>antisymmetry</u>)*
3. *If $x \leq y$ and $y \leq z$, then $x \leq z$ (<u>transitivity</u>)*

Definition 1.4. *Let $\mathcal{P} = (\mathcal{P}, \leq)$ be any poset. The <u>order dual</u> of $\mathcal{P}$ is defined to be the system $\mathcal{P}^{op} = (\mathcal{P}, \leq_{op})$ where $x \leq_{op} y \iff y \leq x$. We usually denote the order dual of a poset P by simply writing P^{op}.*

Definition 1.5. *Let P be a poset and let $X \subseteq P$. We say that X is <u>bounded below</u> (or has a <u>lower bound</u>) in P provided*

$$\bigcap\{\downarrow x : x \in X\} \neq \emptyset$$

We say that X is upper-bounded in P provided it is lower-bounded in P^{op}. We let $m(X)$ and $j(X)$ denote the set of all lower-bounds and upper-bounds, respectively, for X.

Definition 1.6. *Let P be a poset. We say P has a least element provided P has exactly one minimal element. We say that P has a greatest element provided P^{op} has a least element. We use $\perp$ and $\top$ to denote the least and greatest elements, respectively, of P (when they exist).*

A poset which has a least element is said to be lower-bounded. A poset which has a greatest element is said to be upper-bounded. A bounded poset has both a least and a greatest element.

Definition 1.7. *Let P be a poset and let $X \subseteq P$. We say that X has an infimum (or greatest lower-bound) in P provided $m(X)$ has a greatest element. This element is known as the meet of X in P and is denoted by $\bigwedge X$. Likewise, we say that X has a supremum (or least upper bound) in P provided $j(X)$ has a least element. This element is known as the join of X in P and is denoted by $\bigvee X$.*

When $X = \{x_1, ..., x_n\}$ has a meet in a poset P, we often denote it by

$$\bigwedge X = x_1 \wedge ... \wedge x_n$$

and likewise denote the join of X in P by

$$\bigvee X = x_1 \vee ... \vee x_n$$

Definition 1.8. *A poset J is called a join semilattice provided every pair of elements in J has a join in J. We say that P is a meet semilattice provided P^{op} is a join semilattice.*

Definition 1.9. *A poset $\mathcal{L}$ is said to be a* <u>*lattice*</u> *provided it is both a join and a meet semilattice.*

Definition 1.10. *A lattice $\mathcal{L}$ is said to be complete provided for all $X \subseteq \mathcal{L}$, $\bigvee X \in \mathcal{L}$.*

Definition 1.11. *Let $\mathcal{L}$ be a lattice. We say that $\mathcal{L}$ is* <u>*distributive*</u> *provided joins distribute over meets and vice-versa. That is, for all $x, y, z \in \mathcal{L}$, we have*

- $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
- $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

Let P be a poset and let $a, b \in P$. We will let $[a, b] = (\uparrow a) \cap (\downarrow b)$. This subset of P is called an <u>interval</u> in P; and, of course, is nonempty if and only if $a \leq b$.

Definition 1.12. *Let $\mathcal{L}$ be a lattice and let $[a, b] \subseteq \mathcal{L}$. An element $x \in [a, b]$ has a* <u>*relative complement*</u> *in $[a, b]$ provided there exist $y \in [a, b]$ such that $x \wedge y = a$ and $x \vee y = b$. We say that $[a, b]$ is* <u>*relatively complemented*</u> *provided every element in $[a, b]$ has a relative complement in $[a, b]$. A lattice in which every interval is relatively complemented is called a* <u>*relatively complemented*</u> *lattice.*

If $\mathcal{L}$ is a bounded lattice, then relatively complemented elements in $[\perp, \top] = \mathcal{L}$ are said to be <u>complemented</u>. A complemented, distributive lattice is called a <u>Boolean</u> lattice in honor of George Boole, a prominent nineteenth century mathematician. (Notice that Boolean lattices are necessarily bounded.) Motivated by this classical definition, relatively complemented, distributive lattices are called <u>generalized</u> Boolean lattices. A generalized Boolean lattice is a Boolean lattice if and only if it is bounded.

Definition 1.13. *Let P be a lower-bounded poset. We say that an element $a \in P$ is an* <u>*atom*</u> *of P provided $\perp \prec a$. Likewise, an element c of an upper-bounded poset P is a* <u>*co-atom*</u> *of P provided c is an atom of P^{op}. We say that a lower-bounded poset P is* <u>*atomic*</u> *provided $\downarrow x$ contains an atom for all $\perp < x \in P$.*

We let the sets $\mathtt{Co}(\mathcal{L})$ and $\mathtt{At}(\mathcal{L})$ denote the set of co-atoms and atoms respectively of a poset $\mathcal{L}$.

Definition 1.14. *Let $\mathcal{L}$ and $\mathcal{Q}$ be lattices. A function $f : \mathcal{L} \longrightarrow \mathcal{Q}$ is said to be an* <u>*order isomorphism*</u> *provided the following:*

1. *f is bijective.*

2. *For all $x, y \in \mathcal{L}$, if $x \leq y$, then $f(x) \leq f(y)$.*

3. *For all $x, y \in \mathcal{L}$, if $f(x) \leq f(y)$, then $x \leq y$.*

If there exists an order isomorphism $f : \mathcal{L} \longrightarrow \mathcal{Q}$, then we say that $\mathcal{L}$ and $\mathcal{Q}$ are order isomorphic.

CHAPTER 2

CLASSICAL TOPOLOGY CHARACTERIZATION

2.1 Classical Topology

Let $\mathcal{G} = (G, E)$ be a simple, social graph. For any vertex x, let $E(x)$ denote the set of all edges incident to x. We will call this set the *edge neighborhood* for x. We will let $B(x) = E(x) \cup \{x\}$ represent the *edge-ball* of the vertex x.

Definition 2.15. *Let $\mathcal{G} = (G, E)$ be a simple, social graph. A subset X of $G \cup E$ is graph-open provided one of the following conditions is met.*

- *We have $X \subseteq E$.*
- *If $x \in X \cap G$, then $E(x) \subseteq X$.*

It is easy to see that the collection $\Omega(\mathcal{G})$ of graph-open sets forms a topology on $G \cup E$. This topology is called the *graph* or *classical* topology on $\mathcal{G}$. As such, the collection $\Omega(\mathcal{G})$ forms a complete, distributive lattice under subset inclusion. The join of any family from $\Omega(\mathcal{G})$ is simply the union of that family, while the meet of any family from $\Omega(\mathcal{G})$ will be the topological interior of the intersection of the family.

It is worth noting that the family

$$\mathtt{B}(\mathcal{G}) = \{\{e\} : e \in E\} \cup \{B(x) : x \in G\}$$

forms a basis for the graph-topology.

Lemma 2.16. *Let $\mathcal{G} = (G, E)$ be a simple, social graph. An element $U \in \Omega(\mathcal{G})$ is compact if and only if one of the following statements is true:*

1. *We have $U \in \mathtt{Fin}[E]$.*

2. There exists a finite $\mathcal{F} \subseteq B(\mathcal{G})$ so that $U = \cup\mathcal{F}$.

Proof. Let $\mathcal{G} = (V, E)$. Let $U \in \Omega(\mathcal{G})$. Suppose U is compact. Then, for all $\mathcal{F} \subseteq \Omega(\mathcal{G})$, if $U \subseteq \cup\mathcal{F}$, then there exists a finite $F \subseteq \mathcal{F}$ so that $U \subseteq \cup F$. Since $U \in \Omega(\mathcal{G})$, then either $U \subseteq E$ or, if $x \in U \cap G$, then $E(x) \subseteq U$.

Suppose $U \subseteq E$. Then $U \subseteq \bigcup_{e \in E}\{e\}$. Then, since U is compact, there exists a finite $\mathcal{E} \subseteq E$ so that $U \subseteq \cup\mathcal{E}$. Thus, $U \in \mathtt{Fin}[E]$.

Suppose $U \nsubseteq E$. Let $x \in U$. If $x \in \mathcal{G}$, then $B(x) \subseteq U$. Let $\beta = \{B(x) : x \in U \cap G\}$ and $\mathcal{E} = \{\{e\} : e \in U - \cup\beta\}$. Then, since U is compact, there exists a finite $\mathcal{F} \subseteq \beta \cup \mathcal{E}$ so that $U \subseteq \cup\mathcal{F}$. Note, $\mathcal{F} \subseteq B(\Omega(\mathcal{G}))$. Indeed, it must be that $\cup\mathcal{F} \subseteq U$ as well, since for all $x \in G \cap U$, $B(x) \subseteq U$ and for all $e \in U \cap E$, $\{e\} \subseteq U$. Thus, $U = \cup\mathcal{F}$. Hence, U satisfies (2). Thus, if $U \in \Omega(\mathcal{G})$ is compact, then U satisfies (1) or (2).

On the other hand, let $U \in \Omega(\mathcal{G})$. Suppose U satisfies (1). Let $\mathcal{F} \subseteq \Omega(\mathcal{G})$ such that $U \subseteq \cup\mathcal{F}$. Since U is finite, there must exist a finite $F \subseteq \mathcal{F}$ so that $U \subseteq \cup F$. Hence, U is compact in this case.

Suppose U satisfies (2). Let $\mathcal{F} \subseteq \Omega(\mathcal{G})$ such that $U \subseteq \cup\mathcal{F}$. There exists a finite $K \subseteq B(\Omega(\mathcal{G}))$ so that $U = \cup K$. Then it must be that there is a finite family $F \subseteq \mathcal{F}$ so that for all $X \in K$, there exists a $Y \in F$ so that $X \subseteq Y$. Thus, $U \subseteq \cup F$. Hence, U is compact in this case.

Therefore, by the above cases, if U satisfies one of (1) or (2), then U is compact. Hence, U is compact in $\Omega(\mathcal{G})$ if and only if U satisfies (1) or (2).

□

If $\mathcal{G}$ is any simple, social graph, the previous lemma tells us that $\Omega(\mathcal{G})$ is an *algebraic* lattice; that is, every member of $\Omega(\mathcal{G})$ is the union of a directed family of compact graph-open sets.

Definition 2.17. *Let $\mathcal{L} = (L, \leq)$ be any lattice. An element $a \in L$ is join-prime (or coprime) provided, whenever $X \in \mathtt{Fin}[L]$ is such that $a \leq \bigvee X$, then $a \leq x$ for some $x \in X$.*

Notice that the least element of a lattice (when it exists) cannot be join-prime. An element p of a lattice L is said to be meet-prime (or simply *prime*) provided p is join-prime in the order-dual of $\mathcal{L}$.

If $\mathcal{G}$ is any simple, social graph, then it is easy to see that every member of $\mathtt{B}(\mathcal{G})$ is join-prime. Since every member of $\Omega(\mathcal{G})$ is the union of a directed family of compact, join-prime graph-open sets, we know that $\Omega(\mathcal{G})$ is *bialgebraic*. That is, both $\Omega(\mathcal{G})$ and its order-dual are algebraic lattices.

Theorem 2.18. *Let $\mathcal{G} = (G, E)$ be a simple, social graph. A member $U \in \Omega(\mathcal{G})$ is join-prime if and only if $U \in \mathtt{B}(\mathcal{G})$.*

Proof. Let $\mathcal{G} = (G, E)$ be a simple, social graph. Let $U \in \Omega(\mathcal{G})$ so that U is join-prime. Then, whenever $F \in \mathtt{Fin}[\Omega(\mathcal{G})]$ is such that $U \subseteq \bigcup F$, then there exists an $X \in F$ so that $U \subseteq X$. Suppose, by way of contradiction, $U \notin B(\mathcal{G})$. Since $B(\mathcal{G})$ is a basis for $\Omega(\mathcal{G})$, there exists a $\mathcal{F} \subseteq B(\mathcal{G})$ so that $U = \cup\mathcal{F}$. Choose an $X \in \mathcal{F}$. Let $\mathfrak{F} = \{X, \cup(\mathcal{F} - \{X\})\}$. Then $U = \cup\mathcal{F} = \cup\mathfrak{F}$. Observe, $\mathfrak{F} \in \mathtt{Fin}[\Omega(\mathcal{G})]$. Hence, since $U = \cup\mathcal{F} = \cup\mathfrak{F}$, $\mathfrak{F} \in \mathtt{Fin}[\Omega(\mathcal{G})]$, and U is join-prime, either $U \subseteq X$ or $U \subseteq \cup(\mathcal{F} - \{X\})$, a contradiction to U being join-prime. Therefore, it must be that $U \in B(\mathcal{G})$.

On the other hand, suppose $U \in B(\mathcal{G})$. Let $F \in \mathtt{Fin}[\Omega(\mathcal{G})]$ so that $U \subseteq \cup F$. Suppose $U = \{e\}$. Then, since $U \subseteq \cup F$, there exists an $X \in F$ so that $U \subseteq X$. Suppose, then, that $U = B(v)$ for some $v \in G$. Since $U \subseteq \cup\mathcal{F}$, $v \in \cup F$. Then, there exists a $K \in F$ so that $v \in K$. Since $v \in K$, $B(v) \subseteq K$. Therefore, there exists a $K \in F$ so that $U \subseteq K$. Hence, by the above cases, if $U \in B(\mathcal{G})$, then U is join-prime.

Hence, $U \in \Omega(\mathcal{G})$ is join-prime if and only if $U \in B(\mathcal{G})$.

□

Let $\mathcal{L} = (L, \leq)$ be a complete lattice. It is easy to show that the following statements are equivalent for any $j \in L$:

1. The element j is compact and join-prime in $\mathcal{L}$.

2. Whenever $X \subseteq L$ is such that $j \leq \bigvee X$, then $j \leq x$ for some $x \in X$.

Compact, join-prime elements of a complete lattice are often called *completely* join-prime for this reason. We define completely meet-prime elements to be the completely join-prime elements of $\mathcal{L}^{op}$. For any complete lattice $\mathcal{L}$, we will let $\mathtt{CJP}(\mathcal{L})$ and $\mathtt{CMP}(\mathcal{L})$ denote its subposets of completely join-prime and completely meet-prime elements, respectively.

It is well-known that $\mathtt{CJP}(\mathcal{L})$ and $\mathtt{CMP}(\mathcal{L})$ are order-isomorphic for any complete lattice $\mathcal{L}$. The isomorphism is accomplished via the mappings $\phi : \mathtt{CMP}(\mathcal{L}) \longrightarrow \mathtt{CJP}(\mathcal{L})$ and $\gamma : \mathtt{CJP}(\mathcal{L}) \longrightarrow \mathtt{CMP}(\mathcal{L})$ defined by

$$\phi(j) = \bigwedge\{x \in L : x \not\leq j\} \qquad \gamma(m) = \bigvee\{y \in L : m \not\leq y\}$$

Theorem 2.19. *Let $\mathcal{G} = (G, E)$ be a simple, social graph. A member P of $\Omega(\mathcal{G})$ is completely meet-prime if and only if one of the following conditions holds.*

1. *The set P is missing exactly one vertex.*

2. *The set P is missing exactly two vertices and the edge incident to these vertices.*

Proof. Let $P \in \mathtt{CMP}(\Omega(\mathcal{G}))$. This is true if and only if $\phi(P) \in \mathtt{CJP}(\Omega(\mathcal{G}))$, which is true if and only if $\phi(P) = \{e\}$ for some $e = (v_0, v_1) \in E$ or $\phi(P) = B(v)$ for some $v \in G$. This is true if and only if

$$\begin{aligned} P &= \gamma(\phi(P)) \\ &= \gamma(\{e\}) \\ &= \bigvee\{X \in \Omega(\mathcal{G}) : \{e\} \not\subseteq X\} \\ &= G \cup E - \{e, v_0, v_1\} \end{aligned}$$

or

$$
\begin{aligned}
P &= \gamma(\phi(P)) \\
&= \gamma(B(v)) \\
&= \bigvee\{X \in \Omega(\mathcal{G}) : B(v) \nsubseteq X\} \\
&= G \cup E - \{v\},
\end{aligned}
$$

which is true if and only if P satisfies (1) or (2). Hence, $P \in \mathtt{CMP}(\Omega(\mathcal{G}))$ if and only if P satisfies (1) or (2). □

Lemma 2.20. *Let $\mathcal{G} = (G, E)$ be a simple, social graph containing at least three vertices. If $P \in \Omega(\mathcal{G})$ is completely meet-prime, then $P \notin \mathtt{Fin}[E]$.*

Proof. Let $P \in \Omega(\mathcal{G})$ be completely meet-prime. Then $\phi(P) \in \mathtt{CJP}(\Omega(\mathcal{G}))$. Then $\phi(P)$ is join prime. Then $\phi(P) \in B(\mathcal{G})$, so $\phi(P) = \{e\}$ for some $e = (v_0, v_1) \in E$ or $\phi(P) = B(v)$ for some $v \in G$.

If $\phi(P) = \{e\}$, then $P = \gamma(\phi(P)) = \gamma(\{e\}) = G \cup E - \{e, v_1, v_2\} \notin \mathtt{Fin}[E]$. Hence, $P \notin \mathtt{Fin}[E]$ in this case.

If $\phi(P) = B(v)$, then $P = \gamma(\phi(P)) = \gamma(B(v)) = G \cup E - \{v\} \notin \mathtt{Fin}[E]$. Hence, $P \notin \mathtt{Fin}[E]$ in this case.

Thus, by the above cases, if P is completely meet-prime in $\Omega(\mathcal{G})$, then $P \notin \mathtt{Fin}[E]$. □

It is well known that every prime ideal of a Boolean lattice is maximal. A result due to Frazier[7] is that in $\Omega(\mathcal{G})$, $\uparrow E$ and $\downarrow E$ are complete, atomic Boolean lattices.

Lemma 2.21. *Let $\mathcal{L}$ be a distributive lattice. Suppose $I = \downarrow x$ is principal. Then $\downarrow x$ is meet-prime in $\mathtt{Idl}(\mathcal{L})$ if and only if x is meet-prime in $\mathcal{L}$.*

Proof. Let $F \subseteq \mathcal{L}$ be finite so that $\bigwedge F \leq x$. Since $\downarrow x$ is meet-prime in $\mathtt{Idl}(\mathcal{L})$, $\downarrow x$ is prime in $\mathtt{Idl}(\mathcal{L})$. Since $\downarrow x$ is prime, F is finite, and $\bigwedge F \leq x$, there exists an $f \in F$

so that $f \in \downarrow x$. Thus, there exists an $f \in F$ so that $f \leq x$. Hence, x is meet-prime in $\mathcal{L}$.

On the other hand, let $\downarrow x$ be principal. Suppose $\downarrow x$ is not meet-prime in $\mathtt{Idl}(\mathcal{L})$. Then $\downarrow x$ is not a prime ideal in $\mathcal{L}$. Since $\downarrow x$ is not a prime ideal and $\downarrow x$ is proper, it must be that there exists $a \wedge b \in \downarrow x$ so that $a, b \in \mathcal{L} - (\downarrow x)$. Then $x \geq a \wedge b$ and $a > x$ and $b > x$. Therefore, x is not meet-prime in $\mathcal{L}$. Thus, by contraposition, if $x \in \mathcal{L}$ is meet-prime, then $\downarrow x$ is meet-prime in $\mathtt{Idl}(\mathcal{L})$.

Hence, $\downarrow x$ is meet-prime in $\mathtt{Idl}(\mathcal{L})$ if and only if x is meet-prime in $\mathcal{L}$.

$\square$

Lemma 2.22. *Let $\mathcal{G} = (G, E)$ be a simple, social graph, and suppose $U \in \Omega(\mathcal{G})$ contains E. If U is not maximal in $\Omega(\mathcal{G})$, then U is not meet-prime.*

Proof. Let $U \in \Omega(\mathcal{G})$ so that $E \subseteq U$ and suppose U is not maximal. Consider $\uparrow E$ in $\Omega(\mathcal{G})$. Then $U \in \uparrow E$. Note, U is not maximal. Then, since $\uparrow E$ is a complete, atomic Boolean lattice, $\downarrow U$ is not maximal in $\mathtt{Idl}(\uparrow E)$, so $\downarrow U$ is not prime in $\uparrow E$. Then $\downarrow U$ is not meet-prime in $\mathtt{Idl}(\uparrow E)$. Then, by Lemma 2.21, U is not meet-prime in $\uparrow E$. Then there exists a finite $F \subseteq \uparrow E$ so that $\cap F \subseteq U$ and for all $K \in F$, $U \subset K$. Then U is not meet-prime in $\Omega(\mathcal{G})$, since $F \subseteq \Omega(\mathcal{G})$. Hence, if U is not maximal in $\Omega(\mathcal{G})$, then U is not meet-prime in $\Omega(\mathcal{G})$.

$\square$

Lemma 2.23. *Let $\mathcal{G} = (G, E)$ be a simple, social graph. If $P \in \Omega(\mathcal{G})$ is meet-prime, then $P \cap E = E$, or $P \cap E$ is meet-prime in the sublattice $\mathtt{Su}[E]$.*

Proof. Let $\mathcal{G} = (G, E)$ be a simple, social graph. Let $P \in \Omega(\mathcal{G})$ be meet-prime. Suppose $E \subseteq P$. Then $P \cap E = E$. Suppose, then, that $E \nsubseteq P$. Let $\mathcal{F} \in \mathtt{Fin}[\mathtt{Su}[E]]$ so that $\bigcap \mathcal{F} \subseteq P \cap E$. Observe, for all $F \in \mathcal{F}$, $F \subseteq E$. Since P is meet-prime and $\bigcap \mathcal{F} \subseteq P$, there exists an $F \in \mathcal{F}$ so that $F \subseteq P$. Therefore $F \subseteq P \cap E$. Hence, $P \cap E$ is meet-prime. Thus, since $\mathcal{F} \in \mathtt{Fin}[\mathtt{Su}[E]]$ was arbitrary, if $P \in \Omega(\mathcal{G})$ is meet-prime and $E \nsubseteq P$, then $P \cap E$ is meet-prime in $\mathtt{Su}[E]$.

Thus, by the above, if $P \in \Omega(\mathcal{G})$ is meet-prime, then $P \cap E = E$ or $P \cap E$ is meet-prime in the sublattice $\mathtt{Su}[E]$.

□

Theorem 2.24. *Let $\mathcal{G} = (G, E)$ be a simple, social graph. If $P \in \Omega(\mathcal{G})$ is meet-prime but not maximal, then $P = \gamma(\{e\})$ for some $e \in E$. In particular, P is completely meet-prime.*

Proof. Let $\mathcal{G} = (G, E)$ be a simple, social graph. Let $P \in \Omega(\mathcal{G})$ be meet-prime but not maximal. Then P does not contain E by Lemma 2.21. Thus, there exists an $e \in E$ so that $e \notin P$. Observe, $e = (v_0, v_1)$. It must be that $v_0, v_1 \notin P$, otherwise $e \in P$. Since P does not contain E, $P \cap E$ is meet-prime in the sublattice $\mathtt{Su}[E]$ by Lemma 2.22. It must be that $P \cap E$ is maximal in $\mathtt{Su}[E]$, otherwise $P \cap E = A \cap B$ for some $A, B \in \mathtt{Su}[E]$ such that $P \cap E \subsetneq A, B$, contradicting $P \cap E$ being meet-prime. Thus, $P \cap E \prec E$, so $P \cap E$ is missing exactly one edge, e. Thus, P is missing exactly one edge, e.

Suppose P is missing more vertices than v_0 and v_1. Let $F = \{v \in G - P : v_0 \neq v \neq v_1\}$. Then $F \neq \emptyset$. Let $K = (G \cup E) - \{e, v_0, v_1\}$ and $\mathcal{F} = (G \cup E) - (\{e\} \cup F)$. Then $F = K \cap \mathcal{F}$ and $F \subsetneq K$ and $F \subsetneq \mathcal{F}$, a contradiction to P being meet-prime. Thus, P is missing exactly the vertices v_0, v_1. Hence, $P = G \cup E - \{e, v_0, v_1\}$.

Let $Q = \{Y \in \Omega(\mathcal{G}) : \{e\} \nsubseteq Y\}$. Observe, $\bigvee Q = P$. Thus, $\gamma(\{e\}) = \bigvee Q = P$, so $\gamma(\{e\}) = P$. Hence, since $\{e\}$ is completely join-prime and $\gamma : \mathtt{CJP}(\Omega(\mathcal{G})) \longrightarrow \mathtt{CMP}(\Omega(\mathcal{G}))$, $\gamma(\{e\}) = P$ is completely meet-prime. Thus, if $P \in \Omega(\mathcal{G})$ is meet-prime but not maximal, then $P = \gamma(\{e\})$ for some $e \in E$ and P is completely meet-prime.

□

Theorem 2.25. *Let $\mathcal{G}$ be a simple, social graph. Let $U \in \Omega(\mathcal{G})$. The following statements are true:*

1. *U is completely join-prime if and only if U is join-prime in $\Omega(\mathcal{G})$.*

2. *U is completely meet-prime if and only if U is meet-prime in $\Omega(\mathcal{G})$.*

Proof. Clearly, if $U \in \Omega(\mathcal{G})$ is completely join-prime, then U is join-prime. Suppose, then, that U is join-prime. Then $U \in B(\mathcal{G})$. Then $U \in \texttt{Fin}[E]$ or there exists a finite $\mathcal{F} \subseteq B(\mathcal{G})$ so that $U = \cup\mathcal{F}$. Then U is compact by 2.2. Then, since U is join-prime and compact, U is completely join-prime. Thus, if U is join-prime in $\Omega(\mathcal{G})$, then U is completely join-prime in $\Omega(\mathcal{G})$. Hence, U is completely join-prime in $\Omega(\mathcal{G})$ if and only if U is join-prime in $\Omega(\mathcal{G})$.

Clearly, if $U \in \Omega(\mathcal{G})$ is completely meet-prime, then U is meet-prime. Suppose, then, that U is meet-prime in $\Omega(\mathcal{G})$. Then $U \cap E = E$ or $U \cap E$ is meet-prime in $\texttt{Su}[E]$ by Theorem 2.23.

Suppose $U \cap E = E$. Then, since U is meet-prime, U is maximal in $\Omega(\mathcal{G})$ by Lemma 2.22. Then U is missing exactly one vertex. Then U is completely meet-prime by 2.5. Thus, if U is meet-prime, then U is completely meet-prime in this case.

Suppose $U \cap E$ is meet-prime in $\texttt{Su}[E]$. Then U is completely meet-prime by 2.10, since U is not maximal in $\Omega(\mathcal{G})$. Hence, if U is meet-prime, then U is completely meet-prime in this case. Thus, if U is meet-prime, then U is completely meet-prime. Hence, $U \in \Omega(\mathcal{G})$ is completely meet-prime if and only if U is meet-prime in $\Omega(\mathcal{G})$.

□

CHAPTER 3

GRAPH POSETS

3.1 Graph Posets

Definition 3.26. *A poset P is called a graph poset provided the following conditions are met.*

1. *There exist disjoint, nonempty antichains G and E such that $P = G \cup E$.*
2. *Every member of E is covered by exactly two members of G.*
3. *If $e, f \in E$ are distinct, then at least one cover for e is not a cover for f.*

Let $\mathcal{G} = (G, E)$ be a simple, social graph with vertex set G and edge set E. Define a binary relation $\leq \subseteq (V \cup E) \times (V \cup E)$ as follows:

- The pair $(\alpha, \beta) \in \leq$ if and only if one of the following statements is true:
    1. We have $\alpha = \beta$.
    2. We have $\beta \in V$, and $\alpha \in E(\beta)$.

It is easy to see that $(G \cup E, \leq)$ is a graph poset. Indeed, distinct elements of G and distinct elements of E must be incomparable by Assumption 1; hence, G and E are disjoint antichains. Furthermore, since an edge in a graph must be incident to exactly two vertices, it follows that every member of E is covered by exactly two members of G — namely the two vertices that are adjacent via the edge e. Since the graph $\mathcal{G}$ is simple, it contains no loops. Therefore, it is not possible for two vertices u and v to be adjacent via distinct edges e and f. Therefore, if e and f are distinct edges, at least one of the covers for e is not a cover for f.

We will let $\mathcal{P}_G$ represent the graph poset associated with a simple, social graph $\mathcal{G}$.

On the other hand, suppose that $P = (G \cup E, \leq)$ is a graph poset. The set $\mathcal{G}_P = (G, E)$ can be made into a simple, social graph in a straightforward way: Two members x and y of G are adjacent if and only if $x = y$ or they cover the same member of E.

We will let $\mathcal{G}_P$ represent the graph associated with a graph poset P.

Definition 3.27. *Let $\mathcal{P} = (G \cup E, \leq)$ be a graph poset. We say that $\mathcal{P}$ is social provided the principal lowerset $\downarrow x = \{x\}$ if and only if $x \in E$.*

Definition 3.28. *Let $\mathcal{G} = (G, E)$ and $\mathcal{H} = (H, F)$ be graphs. A function*

$$f : \mathcal{G} \longrightarrow \mathcal{H}$$

is said to be a graph isomorphism provided the following are true:

1. *f is a bijection between G and H.*

2. *$v, u \in G$ are adjacent if and only if $f(v), f(u) \in H$ are adjacent.*

3. *If (1), then $f((v, u)) = (f(u), f(v))$.*

Lemma 3.29. *Let $\mathcal{G} = (G, E)$ and $\mathcal{H} = (H, F)$ be graphs. Then $\mathcal{G}$ is graph isomorphic to $\mathcal{H}$ if and only if $\mathcal{P}_\mathcal{G}$ is order isomorphic $\mathcal{P}_\mathcal{H}$.*

Proof. Suppose $\mathcal{G}$ is graph isomorphic to $\mathcal{H}$. Then there exists a function $f : \mathcal{G} \longrightarrow \mathcal{H}$ so that $v, u \in G$ are adjacent if and only if $f(v), f(u) \in H$ are adjacent. Define $g : \mathcal{P}_\mathcal{G} \longrightarrow \mathcal{P}_\mathcal{H}$ by $g(x) = f(x)$. Then g is a bijection.

Let $x, y \in \mathcal{P}_\mathcal{G}$ so that $x \leq y$. If $x = y$, then $g(x) = g(y)$. Suppose, then, that $x < y$. Then $x \in E$ and $y \in G$ and there exists a $k \in G$ so that $x = (y, k)$. Then $f(y)$ is adjacent to $f(k)$ in $\mathcal{P}_\mathcal{H}$ nd $(f(y), f(k)) = f((y, k)) = f(x)$. Hence, $f(x) \in F$ and $f(y) \in H$. Thus, $g(x) \leq g(y)$. Hence, g is an order isomorphism.

Let $a, b \in \mathcal{P}_\mathcal{H}$ be so that $a \leq b$. Then there exists $x, y \in \mathcal{P}_\mathcal{G}$ so that $g(x) = a$ and $g(y) = b$. Then $f(x) \leq f(y)$. If $f(x) = f(y)$, then $x = y$ since f is an injection.

Suppose, then, that $f(x) < f(y)$. Then $f(x) \in F$ and $f(y) \in H$. Then there exists a $k \in G$ so that $f(k) \in H$ and $f(x) = (f(y), f(k))$. Then $(y, k) \in E$ since f is a graph isomorphism.

Let $e = (y, k)$. Then $e < y$ and $e < k$. Then $f(e) < f(y)$ and $f(e) < f(k)$. Then $f(e) = (f(y), f(k))$. Then $f(e) = f(x)$. Then $x = e = (y, k)$, since f is injective. Then $x \leq y$. Thus, if $g(x) \leq g(y)$, then $x \leq y$. Hence, since g is a bijection, g is an order homomorphism, and if $g(x) \leq g(y)$, then $x \leq y$, g is an order isomorphism. Thus, if $\mathcal{G}$ and $\mathcal{H}$ are graph isomorphic, then $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{P}_{\mathcal{H}}$ are order isomorphic.

On the other hand, suppose $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{P}_{\mathcal{H}}$ are order isomorphic. Then there exists an order isomorphism $g : \mathcal{P}_{\mathcal{G}} \longrightarrow \mathcal{P}_{\mathcal{H}}$. Define $f : \mathcal{G} \longrightarrow \mathcal{H}$ by $f(x) = g(x)$. Then f is a bijection, since g is a bijection.

Let $x, y \in G$ so that $e = (x, y) \in E$. Then $e < x$ and $e < y$ in $\mathcal{P}_{\mathcal{G}}$. Then $g(e) < g(x)$ and $g(e) < g(y)$ since g is an order isomorphism. Thus, $g(e) = (g(x), g(y))$. Hence, $f(e) = (f(x), f(y))$. Thus, if $(x, y) \in E$, then $(f(x), f(y)) \in F$.

Let $a, b \in H$ so that $(a, b) \in F$. Then, since g is a bijection, there exists $x, y \in G$ and $e \in E$ so that $g(x) = a$, $g(y) = b$, and $g(e) = (a, b)$. Then $g(e) < g(x)$ and $g(e) < g(y)$. Then, since g is an order isomorphism, $e < x$ and $e < y$. Then, $e = (x, y)$. Then $(x, y) \in E$. Hence, if $(f(x), f(y)) \in F$, then $(x, y) \in E$.

Therefore, $(x, y) \in E$ if and only if $(f(x), f(y)) \in F$. Furthermore, we have shown that $f(e) = (f(x), f(y))$. Hence, f is a graph isomorphism. Therefore, if $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\mathcal{P}_{\mathcal{H}}$, then $\mathcal{G}$ is graph isomorphic to $\mathcal{H}$.

Therefore, $\mathcal{G}$ and $\mathcal{H}$ are graph isomorphic if and only if $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{P}_{\mathcal{H}}$ are order isomorphic. $\square$

Theorem 3.30. *Let $\mathcal{G}$ be a simple, social graph. Then $\mathtt{MP}(\Omega(\mathcal{G}))$ is a graph poset that is order isomorphic to $\mathcal{P}_{\mathcal{G}}$.*

Proof. Observe, $\mathtt{MP}(\Omega(\mathcal{G})) = \mathtt{CMP}(\Omega(\mathcal{G}))$ by Theorem 2.25. Then,

$$\mathtt{MP}(\Omega(\mathcal{G})) = \mathtt{Max}(\mathtt{MP}(\Omega(\mathcal{G}))) \cup \mathtt{Min}(\mathtt{MP}(\Omega(\mathcal{G}))).$$

Note, $\texttt{Max}(\texttt{MP}(\Omega(\mathcal{G})))$ and $\texttt{Min}(\texttt{MP}(\Omega(\mathcal{G})))$ are disjoint antichains. Thus, $\texttt{MP}(\Omega(\mathcal{G}))$ is the union of two disjoint antichains.

Let $X \in \texttt{Min}(\texttt{MP}(\Omega(\mathcal{G})))$. Then X is missing two vertices u, v and the edge e incident to it by Theorem 2.19. For all $Y \in \texttt{Max}(\texttt{MP}(\Omega(\mathcal{G})))$, Y is missing exactly one vertex. Thus, there are exactly two members of $\texttt{Max}(\texttt{MP}(\Omega(\mathcal{G})))$ covering X, namely the two members $A, B \in \texttt{Max}(\texttt{MP}(\Omega(\mathcal{G})))$ so that $A = G \cup E - \{u\}$ and $B = G \cup E - \{v\}$. Hence, for any member $e \in \texttt{Min}(\texttt{MP}(\Omega(\mathcal{G})))$, there are exactly two members of $\texttt{Max}(\texttt{MP}(\Omega(\mathcal{G})))$ covering e.

Let $P, Q \in \texttt{Min}(\texttt{MP}(\Omega(\mathcal{G})))$ so that $P \neq Q$. Then, since exactly two vertices and the edge incident to those two vertices are missing from P and exactly two vertices and the edge incident to those two vertices are missing from Q, and since $P \neq Q$, at least one of the vertices missing from P must be different from one of the vertices missing from Q. Thus, since P and Q are covered by exactly two members of $\texttt{Max}(\texttt{MP}(\Omega(\mathcal{G})))$ each and those covering members correspond to the missing vertices of P and Q, it must be that at least one of the covers of P is different from one of the covers of Q. Thus, for all $e, f \in \texttt{Min}(\texttt{MP}(\Omega(\mathcal{G})))$, at least one cover of e is not a cover for f.

Hence, by the above, $\texttt{MP}(\Omega(\mathcal{G}))$ is a graph poset.

Observe, $\mathcal{P}_{\mathcal{G}} = G \cup E$ as in Definition 3.26. Define $f : \mathcal{P}_{\mathcal{G}} \longrightarrow \texttt{MP}(\Omega(\mathcal{G}))$ by

$$f(x) := \begin{cases} (G \cup E) - \{x\} & , \text{if } x \in G \\ \\ (G \cup E) - \{x, a, b\} & , \text{if } x = (a, b) \in E \end{cases}$$

Let $Y \in \texttt{MP}(\Omega(\mathcal{G}))$. Then $Y = G \cup E - \{x\}$ or $Y = G \cup E\{x, a, b\}$ for some $x \in G$ or for some $x = (a, b) \in E$ respectively. Thus, in either case there exists an $x \in \mathcal{P}_{\mathcal{G}}$ so that $f(x) = y$. Hence, f is surjective.

Let $x, y \in \mathcal{P}_{\mathcal{G}}$ so that $f(x) = f(y)$. Then $G \cup E - \{x\} = f(x) = f(y) = G \cup E - \{y\}$ where $x, y \in G$ or $G \cup E - \{x, a_x, b_x\} = f(x) = f(y) = G \cup E - \{y, a_y, b_y\}$ where $x = (a_x, b_x) \in E$ and $y = (a_y, b_y) \in E$. Thus, $\{x\} = \{y\}$ or $\{x, a_x, b_x\} = \{y, a_y, b_y\}$. Hence, in either case, $x = y$. Therefore, if $x, y \in \mathcal{P}_{\mathcal{G}}$ so that $f(x) = f(y)$, then $x = y$.

Thus, f is injective. Hence, since f is surjective and injective, f is bijective.

Let $x, y \in \mathcal{P}_{\mathcal{G}}$ so that $x \leq y$. Then $x = y$ or $x < y$. If $x = y$, then $f(x) = f(y)$. Thus, if $x \leq y$, then $f(x) \subseteq f(y)$ in this case. Suppose, then, that $x < y$. Then $x \in E$ and $y \in G$. Then $f(x) = G \cup E - \{x, y, z\} \subseteq G \cup E - \{y\} = f(y)$. Thus, if $x \leq y$, then $f(x) \subseteq f(y)$ in this case. Hence, by the above cases, if $x \leq y$, then $f(x) \subseteq f(y)$.

Let $x, y \in \mathcal{P}_{\mathcal{G}}$ so that $f(x) \subseteq f(y)$. If $f(x) = f(y)$, then $x = y$ since f is bijective. Thus, in this case, if $f(x) \subseteq f(y)$, then $x \leq y$. Suppose, then, that $f(x) \subset f(y)$. Then it must be that $f(x) = G \cup E - \{x, y, z\} \subset G \cup E - \{y\} = f(y)$ where $x = (y, z) \in E$. Hence, $x \leq y$ in $\mathcal{P}_{\mathcal{G}}$. Thus, if $f(x) \subseteq f(y)$, then $x \leq y$ in this case. Hence, by the above cases, for all $x, y \in \mathcal{P}_{\mathcal{G}}$, if $f(x) \subseteq f(y)$, then $x \leq y$.

Hence, since f is a bijection, for all $x, y \in \mathcal{P}_{\mathcal{G}}$ if $x \leq y$, then $f(x) \subseteq f(y)$, and for all $x, y \in \mathcal{P}_{\mathcal{G}}$, if $f(x) \subseteq f(y)$, then $x \leq y$, f is an order isomorphism. Therefore, $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\mathcal{P}_{\mathcal{G}}$.

Hence, if $\mathcal{G}$ is a simple, social graph, then $\mathtt{MP}(\Omega(\mathcal{G}))$ is a graph poset that is order isomorphic to $\mathcal{P}_{\mathcal{G}}$. $\square$

Let $\mathcal{P} = (P, \leq)$ be any poset and let $X \subseteq P$. We say that X is a *lowerset* of P provided $a \in X$, $y \in P$, and $y \leq a$ together imply that $y \in X$. We will let $\mathtt{Low}(\mathcal{P})$ represent the poset of lowerset of $\mathcal{P}$, partially ordered by subset inclusion. Note that $\mathtt{Low}(\mathcal{P})$ is a complete, distributive lattice. The join and meet of any family from $\mathtt{Low}(\mathcal{P})$ is the union and intersection, respectively, of that family.

Theorem 3.31. *Let $\mathcal{G} = (G, E)$ be a simple, social graph, and let $\mathcal{P}_G$ represent its graph poset. For $X \subseteq G \cup E$, the following statements are equivalent.*

1. *We have $X \in \Omega(\mathcal{G})$.*

2. *We have $X \in \mathtt{Low}(\mathcal{P}_G)$.*

Proof. Let $\mathcal{G} = (G, E)$ be a simple, social graph and let $\mathcal{P}_G$ represent its graph poset. Let $X \subseteq G \cup E$. Suppose $X \in \Omega(\mathcal{G})$. Observe, for all $v \in X \cap G$, $E(v) \subseteq X$. Written another way, $B(v) \subseteq X$ for all $v \in X \cap G$. Thus, for all $v \in X \subseteq \mathcal{P}_G$ so that $v \in G$, $\downarrow v \subseteq X$. Also, for all $e \in X \subseteq \mathcal{P}_G$ so that $e \in E$, $\downarrow e \subseteq X$, since $E \subseteq \mathcal{P}_G$ is an antichain. Hence, for all $x \in X$, $\downarrow x \subseteq X$. Thus, $X \in \texttt{Low}(\mathcal{P}_G)$. Therefore, if $X \in \Omega(\mathcal{G})$, then $X \in \texttt{Low}(\mathcal{P}_G)$.

On the other hand, suppose $X \in \texttt{Low}(\mathcal{P}_G)$. Then, for all $x \in X$, $\downarrow x \subseteq X$. In particular, if $v \in X \cap G$, then $\downarrow v \subseteq X$. Note, $\downarrow v = B(v)$. Thus, for all $v \in X \cap G$, $B(v) \subseteq X$. Let $F = \bigcup_{v \in X \cap G} B(v)$. Then, $X = (F) \cup (\bigcup_{e \in X - F} \{e\})$. Thus, $X \in \Omega(\mathcal{G})$. Hence, if $X \in \texttt{Low}(\mathcal{P}_G)$, then $X \in \Omega(\mathcal{G})$. Therefore, $X \in \Omega(\mathcal{G})$ if and only if $X \in \texttt{Low}(\mathcal{P}_G)$.

□

Let $\mathcal{P} = (P, \leq)$ be a poset, and let $D \subseteq P$. We say that D is *directed* provided every finite subset of D has an upper bound in D. Note that directed sets cannot be empty.

Lemma 3.32. *Let $\mathcal{G} = (G, E)$ be a simple, social graph, and let $\mathcal{P}_G$ represent its graph poset. A nonempty $D \subseteq G \cup E$ is directed if and only if one of the following conditions is met:*

1. *The set D is a singleton.*

2. *We have $D \cap G = \{x\}$ and $D \cap E \subseteq E(x)$.*

Proof. Let $\mathcal{G} = (G, E)$ be a simple, social graph and let $\mathcal{P}_G$ represent its graph poset. Let $D \subseteq G \cup E$ be directed. Then every finite subset of D has an upper bound in D. It is clear that D could be a singleton, since all singletons are directed. Suppose, then, that D is not a singleton.

Suppose $D \cap G \neq \{x\}$. Then D contains more than one vertex or $D \cap G = \emptyset$. Suppose D contains more than one vertex. Consider $F \subseteq D \cap G$ where F is finite. Then F has no upper bound in D since G is an antichain, a contradiction to D being directed. Thus, it must be that $D \cap G = \emptyset$. Then $D \subseteq E$. Again, for any finite subset of D there will be no upper bound in D since E is an antichain, contradicting D being directed. Hence, it cannot be that $D \cap G \neq \{x\}$. Thus, $D \cap G = \{x\}$.

Note, $D \cap E \neq \emptyset$, since D is not a singleton. Suppose there exists an $e \in D \cap E$ so that $e \notin E(x)$. Then $\{e, x\} \subseteq D$ has no upper bound in D, a contradiction to D being directed. Therefore, for all $e \in D \cap E$, $e \in E(x)$. Thus, $D \cap E \subseteq E(x)$. Thus, in this case, $D \cap G = \{x\}$ and $D \cap E \subseteq E(x)$. Therefore, if $D \subseteq G \cup E$ and D is directed, then D satisfies (1) or (2).

On the other hand, suppose $D \subseteq G \cup E$ is a singleton. Then $D = \{x\}$. Then every finite subset of D has an upper bound in D, x. Thus, D is directed in this case.

Suppose $D \subseteq G \cup E$ satisfies (2). Then $D \cap G = \{x\}$. Let $F \subseteq D$ be finite. For all $f \in F$, $f \leq x$. Thus, F has an upper bound in D, x. Hence, since F was an arbitrary finite subset of D, D is directed in this case. Hence, by the previous two cases, if $D \subseteq G \cup E$ satisfies (1) or (2), then D is directed.

Hence, $D \subseteq G \cup E$ is directed if and only if (1) or (2).

$\square$

CHAPTER 4

CONE LATTICES

4.1 Cone Lattices

An element m of a complete lattice $\mathcal{L}$ is *completely meet-irreducible* provided, whenever $X \subseteq L$ is such that $m = \bigwedge X$, then $m = x$ for some $x \in X$. It is easy to see that every completely meet-prime member of $\mathcal{L}$ is also completely meet-irreducible. It it well-known that the converse is true precisely when $\mathcal{L}$ is biaglebraic and distributive.

If $\mathcal{L}$ is an algebraic lattice, it is well-known that every member of $\mathcal{L}$ is the meet of a family of completely meet-irreducible elements. (This famous fact due to G. Birkhoff is often called *Birkhoff's subdirect product theorem.*) Combining these facts with the following Lemma gives us the following Theorem.

Lemma 4.33. *Let $\mathcal{L}$ be a bialgebraic, distributive lattice in which* $\mathtt{MP}(\mathcal{L})$ *is a graph poset. Then* $\mathtt{MP}(\mathcal{L}) = \mathtt{CMP}(\mathcal{L})$.

Proof. Let $\mathcal{L}$ be a $\mathcal{L}$ be a bialgebraic, distributive lattice in which $\mathtt{MP}(\mathcal{L})$ is a graph poset. Let $x \in \mathtt{MP}(\mathcal{L})$. Let $F \subseteq \mathcal{L}$ so that $x \geq \bigwedge F$. If F is finite, then, since x is meet prime, there exists an $f \in F$ so that $x \geq f$. Suppose, then, that F is not finite.

Since $\mathcal{L}$ is bialgebraic, $x = \bigwedge \mathcal{F}$, where $\mathcal{F}$ is a family of completely meet prime elements. Hence, $\bigwedge \mathcal{F} = x \geq \bigwedge F$. Therefore, for all $y \in \mathcal{F}$, $y \geq \bigwedge F$.

Suppose x is a vertex in $\mathtt{MP}(\mathcal{L})$. Then x is maximal in $\mathtt{MP}(\mathcal{L})$. Then for all $y \in \mathcal{F}$, since each y is completely meet prime and so is meet prime, $y = x$. Then, since each y is completely meet prime, x is completely meet prime.

Suppose x is not a vertex in $\mathtt{MP}(\mathcal{L})$. Then x is not maximal in $\mathtt{MP}(\mathcal{L})$. Then, since $x \in \mathtt{MP}(\mathcal{L})$, x is covered by exactly two vertices $a, b \in \mathtt{MP}(\mathcal{L})$. Note, since $\mathcal{L}$ is bialgebraic and distributive, $x = \bigwedge F$ for some $F \subseteq \mathtt{CMI}(\mathcal{L}) = \mathtt{CMP}(\mathcal{L})$. However, since x is covered by exactly two members a, b of $\mathtt{Max}(\mathtt{CMP}(\mathcal{L}))$ and $x \neq \bigwedge\{a, b\}$, we know

$F \cap \mathtt{Min}(\mathtt{CMP}(\mathcal{L})) \neq \emptyset$. Thus, since $\mathtt{Min}(\mathtt{CMP}(\mathcal{L}))$ is an antichain, $x \in \mathtt{Min}(\mathtt{CMP}(\mathcal{L}))$. Hence, if $x \in \mathtt{MP}(\mathcal{L})$, then $x \in \mathtt{CMP}(L)$. Therefore $\mathtt{MP}(\mathcal{L}) \subseteq \mathtt{CMP}(\mathcal{L})$. Thus, since $\mathtt{CMP}(\mathcal{L}) \subseteq \mathtt{MP}(\mathcal{L})$ and $\mathtt{MP}(\mathcal{L}) \subseteq \mathtt{CMP}(\mathcal{L})$, $\mathtt{MP}(\mathcal{L}) = \mathtt{CMP}(\mathcal{L})$. □

Theorem 4.34. *Suppose that $\mathcal{L}$ is a biaglebraic, distributive lattice. The following statements are equivalent.*

1. *The poset $\mathtt{MP}(\mathcal{L})$ is a graph poset.*

2. *The poset $\mathtt{CJP}(\mathcal{L})$ is a graph poset that is order isomorphic to $\mathtt{MP}(\mathcal{L})$.*

Proof. Suppose (1). Then $\mathtt{MP}(\mathcal{L}) = \mathtt{CMP}(\mathcal{L})$ by Lemma 3.32. Note, since $\mathcal{L}$ is complete, $\mathtt{CMP}(\mathcal{L})$ is order-isomorphic to $\mathtt{CJP}(\mathcal{L})$. Thus, $\mathtt{MP}(\mathcal{L})$ is order-isomorphic to $\mathtt{CJP}(\mathcal{L})$. Therefore there exists an order-isomorphism $f : \mathtt{MP}(\mathcal{L}) \longrightarrow \mathtt{CJP}(\mathcal{L})$. Since $\mathtt{MP}(\mathcal{L})$ is a graph poset and f is an order-isomorphism and therefore preserves all order structure of $\mathtt{MP}(\mathcal{L})$ in $\mathtt{CJP}(\mathcal{L})$, $\mathtt{CJP}(\mathcal{L})$ is a graph poset. Hence, (1) $\implies$ (2).

Suppose (2). Then there exists an order-isomorphism $f : \mathtt{CJP}(\mathcal{L}) \longrightarrow \mathtt{MP}(\mathcal{L})$ Since $\mathtt{CJP}(\mathcal{L})$ is a graph poset and f is an order-isomorphism and therefore preserves all order structure of $\mathtt{CJP}(\mathcal{L})$ in $\mathtt{MP}(\mathcal{L})$, $\mathtt{MP}(\mathcal{L})$ is a graph poset. Therefore (2) $\implies$ (1).

Hence, since (1) $\implies$ (2) and (2) $\implies$ (1), (1) $\iff$ (2).

□

Recall that a complete lattice $\mathcal{L} = (L, \leq)$ is *join-continuous* provided for all $X \subseteq L$ and $a \in L$, we have

$$a \wedge \bigvee X = \bigvee \{a \wedge x : x \in X\}$$

We say that $\mathcal{L}$ is meet-continuous provided the order-dual of $\mathcal{L}$ is join-continuous. Note that join or meet continuous lattices are automatically distributive. A join-continuous lattice is not, however, automatically meet-continuous. This will be true for bialgebraic, distributive lattices.

Lemma 4.35. *Let $L = [\bot, \top]$ be a complete, meet-continuous, co-atomic lattice. Let $\mathtt{Co}(\mathcal{L})$ denote the set of co-atoms of L. Let $Z \subseteq \mathtt{Co}(\mathcal{L})$ so that $\mathtt{Co}(\mathcal{L}) \neq Z \neq \emptyset$ and let $K = \mathtt{Co}(\mathcal{L}) - Z$. Then $\bigwedge Z \vee \bigwedge K = \top$.*

Proof. Suppose, by way of contradiction, that $\bigwedge Z \vee \bigwedge K < \top$. Suppose $\bigwedge Z \vee \bigwedge K \in \mathtt{Co}(\mathcal{L})$. Then $\bigwedge Z \vee \bigwedge K \in Z$ or $\bigwedge Z \vee \bigwedge K \in K$. Suppose, without loss of generality, that $\bigwedge Z \vee \bigwedge K \in Z$. Then $\bigwedge Z \vee \bigwedge K = z$ for some $z \in Z$. Then $z \geq \bigwedge K$. Then $z = z \vee \bigwedge K = \bigwedge \{z \vee k : k \in K\} = \top$, since $\mathcal{L}$ is meet continuous. This is a contradiction, since p is a co-atom, so $p < \top$. Therefore it cannot be that $\bigwedge Z \vee \bigwedge K \in \mathtt{Co}(\mathcal{L})$.

Suppose, then, that $\bigwedge Z \vee \bigwedge K \in \mathcal{L} - (\mathtt{Co}(\mathcal{L}) \cup \{\top\})$. Then, since $\mathcal{L}$ is co-atomic, there exists a co-atom $p \in \uparrow (\bigwedge Z \vee \bigwedge K)$. Then $p = p \vee (\bigwedge Z \vee \bigwedge K) = (p \vee \bigwedge Z) \vee \bigwedge K = \bigwedge \{p \vee z : z \in Z\} \vee \bigwedge K = \top \vee \bigwedge K = \top$, since $\mathcal{L}$ is meet-continuous. This is a contradiction, since p is a co-atom, so $p < \top$. Therefore, it cannot be that $\bigwedge Z \vee \bigwedge K \in \mathcal{L} - (\mathtt{Co}(\mathcal{L}) \cup \{\top\})$.

Hence, since $\bigwedge Z \vee \bigwedge K \not< \top$, it must be that $\bigwedge Z \vee \bigwedge K = \top$.

$\square$

Lemma 4.36. *Suppose that $\mathcal{L}$ is a co-atomic, complete, meet-continuous lattice. If, for all $x \in L$ there exists a set Z of co-atoms such that $x = \bigwedge Z$, then $\mathcal{L}$ is an atomic Boolean lattice.*

Proof. Let $\mathcal{L}$ be a co-atomic, complete, meet-continuous lattice. Suppose for all $x \in \mathcal{L}$, there exists a Z of co-atoms such that $x = \bigwedge Z$. Observe, $\bigwedge \emptyset = \top$. It must be that $\bot = \bigwedge \mathtt{Co}(\mathcal{L})$, since if there were a $Z \subsetneq \mathtt{Co}(\mathcal{L})$ so that $\bigwedge Z = \bot$, then $\bigwedge \mathtt{Co}(\mathcal{L}) < \bigwedge Z = \bot$, a contradiction. Therefore $\mathcal{L} = [\bot, \top] = [\bigwedge \mathtt{Co}(\mathcal{L}), \bigwedge \emptyset]$. Since $\mathcal{L}$ is meet-continuous, $\mathcal{L}$ is distributive.

Let $x \in \mathcal{L}$. Then there exists a $Z \subseteq \mathtt{Co}(\mathcal{L})$ so that $x = \bigwedge Z$. Let $K = \mathtt{Co}(\mathcal{L}) - Z$

and $y = \bigwedge K$. Then:

$$
\begin{aligned}
x \wedge y &= \bigwedge Z \wedge \bigwedge K \\
&= \bigwedge (Z \cup (\mathtt{Co}(\mathcal{L}) - Z)) \\
&= \bigwedge \mathtt{Co}(\mathcal{L}) \\
&= \bot.
\end{aligned}
$$

Also,

$$
\begin{aligned}
x \vee y &= (\bigwedge Z) \vee (\bigwedge K) \\
&= \top \text{ by Lemma 4.34}
\end{aligned}
$$

Hence y is the complement of x. Therefore, since $x \in \mathcal{L}$ was arbitrary, $\mathcal{L}$ is relatively complemented. Therefore, since $\mathcal{L}$ is bounded, $\mathcal{L}$ is distributive, and $\mathcal{L}$ is relatively complemented, $\mathcal{L}$ is Boolean. Since $\mathcal{L}$ is Boolean and co-atomic, $\mathcal{L}$ is atomic. Hence, $\mathcal{L}$ is an atomic Boolean lattice, as desired.

□

Lemma 4.37. *Suppose that $\mathcal{L} = (L, \leq)$ is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34. If $\mathtt{MP}(\mathcal{L}) = G \cup E$ as specified in Definition 3.26, then the following statements are true.*

1. *The elements of G are coatoms of $\mathcal{L}$.*

2. *We have $\bigwedge G \not\leq e$ for any $e \in E$; and if G contains at least three elements, $\bigwedge G$ is incomparable to e.*

3. *If $\bigwedge G \leq y$, then there exist $Y \subseteq G$ such that $y = \bigwedge X$.*

4. *If G contains at least three members, then $e \vee \bigwedge G$ is the smallest member of $\uparrow \bigwedge G$ that exceeds e.*

5. *If $e, f \in E$ so that $e \neq f$, then $e \vee f \geq \bigwedge G$.*

Proof. Let $x \in G$. Then $x \in \mathtt{MP}(\mathcal{L})$. Suppose $x \not\prec \top$. Then there exists a $y \in \mathcal{L}$ so that $x < y < \top$. Then, since $\mathcal{L}$ is bialgebraic and distributive, $y = \bigwedge K$ for some $K \subseteq \mathtt{MP}(\mathcal{L})$. Then there exists a $k \in K$ so that $x \leq k$. This is a contradiction, since x is maximal in $\mathtt{MP}(\mathcal{L})$ and G is an antichain. Thus, $x \prec \top$, so $x \in \mathtt{Co}(\mathcal{L})$. Thus, elements of G are coatoms of $\mathcal{L}$.

Suppose there exists an $e \in E$ so that $\bigwedge G \leq e$. Then there exists a $g \in G$ so that $e = g$, since $e \in \mathtt{MP}(\mathcal{L}) = \mathtt{CMP}(\mathcal{L})$. This is a contradiction, since E and G are disjoint. Hence, for all $e \in E$, $\bigwedge G \not\leq e$.

Let $e \in E$. Suppose $|G| \geq 3$. Suppose $e \leq \bigwedge G$. Then $e \leq g$ for all $g \in G$. This is a contradiction since e is covered by exactly two members of G and $|G| \geq 3$. Hence, $e || \bigwedge G$ for all $e \in E$. Thus, if $|G| \geq 3$, then $\bigwedge G || e$.

Let $y \in \mathcal{L}$ so that $\bigwedge G \leq y$. Then, since $\mathcal{L}$ is bialgebraic and distributive, $y = \bigwedge K$ for some $K \subseteq \mathtt{MP}(\mathcal{L})$. Thus, $\bigwedge G \leq \bigwedge K$. It must be that $K \subseteq G$, since if $K \cap E \neq \emptyset$, then there exists an $e \in E$ so that $\bigwedge G \leq e$, a contradiction. Hence, there exists a $K \subseteq G$ so that $y = \bigwedge K$. Therefore, if $y \in \mathcal{L}$ so that $\bigwedge G \leq y$, then there exists a $Y \subseteq G$ such that $y = \bigwedge Y$.

Suppose $|G| \geq 3$. Let $e \in E$. Suppose, by way of contradiction, that there exists a $y \in \uparrow \bigwedge G$ so that $e < y < e \vee \bigwedge G$. Then $\bigwedge G \leq y$ and $e \leq y$. Then $y = e \vee \bigwedge G$, a contradiction, since $y < e \vee \bigwedge G$. Therefore, for all $y \in \uparrow \bigwedge G$, $e < e \vee \bigwedge G \leq y$. Hence, if $|G| \geq 3$, then $e \vee \bigwedge G$ is the smallest member of $\uparrow \bigwedge G$ that exceeds e.

Let $e, f \in E$ so that $e \neq f$. Suppose, by way of contradiction, that $e \vee f < \bigwedge G$. Then $e, f < \bigwedge G$. Then $e, f < v$ for all $v \in G$. Since $\mathtt{MP}(\mathcal{L})$ is a graph poset and so every edge is covered by exactly two vertices and $e, f < v$ for all $v \in G$, it must be that $G = \{a, b\}$. Thus $e = (a, b)$ and $f = (a, b)$. Then $e = f$, a contradiction. Hence, $e \vee f \geq \bigwedge G$. Thus, if $e, f \in E$ so that $e \neq f$, then $e \vee f \geq \bigwedge G$. $\square$

Lemma 4.38. *Suppose that $\mathcal{L}$ is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34. If $\mathtt{MP}(\mathcal{L}) = G \cup E$ as specified in Definition 3.26* *and* $B_d = \{\bigwedge G \wedge \bigwedge Z : Z \subseteq E\}$*, then* B_d *is coatomic and* $\mathtt{Co}(B_d) = \{e \wedge \bigwedge G : e \in E\}$*.*

Proof. Let $e \in E$. Observe, $e \wedge \bigwedge G < \bigwedge G$. Suppose, by way of contradiction, that there exists an $f \in E$ so that $\bigwedge G \wedge e \leq \bigwedge G \wedge f$. Then

$$\begin{aligned}\bigwedge G \wedge f &= (\bigwedge G \wedge e) \vee (\bigwedge G \wedge f)\\ &= \bigwedge G \wedge (e \vee f)\\ &= \bigwedge G \text{ by Lemma 4.37(5).}\end{aligned}$$

Then $f \geq \bigwedge G$, a contradiction since $\bigwedge G \not\leq x$ for all $x \in E$. Hence, for all $e, f \in E$, $e \wedge \bigwedge G || f \wedge \bigwedge G$.

Suppose there exists a nonempty $Z \subseteq E$ so that $e \wedge \bigwedge G < \bigwedge Z \wedge \bigwedge G$. Then

$$\begin{aligned}\bigwedge Z \wedge \bigwedge G &= (\bigwedge Z \wedge \bigwedge G) \vee (e \wedge \bigwedge G)\\ &= \bigwedge G \wedge (e \vee \bigwedge Z).\end{aligned}$$

Observe, by meet continuity, $e \vee \bigwedge Z = \bigwedge\{e \vee z : z \in Z\} \geq \bigwedge G$ by Lemma 4.37(5). Then $\bigwedge Z \wedge \bigwedge G = \bigwedge G \wedge (e \vee \bigwedge Z) = \bigwedge G$, a contradiction. Therefore, for all nonempty $Z \subseteq E$, $\bigwedge Z \wedge \bigwedge G \leq e \wedge \bigwedge G$. Also, it must be that for all $e \in E$, $e \wedge \bigwedge G \prec \bigwedge G$. Hence, B_d is coatomic and $\mathtt{Co}(B_d) = \{e \wedge \bigwedge G : e \in E\}$.

□

Theorem 4.39. *Suppose that* $\mathcal{L}$ *is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34. If* $\mathtt{MP}(\mathcal{L}) = G \cup E$ *as specified in Definition 3.26, then the subposets*

$$B_u = \{\bigwedge Z : Z \subseteq G\} \quad \textit{and} \quad B_d = \{\bigwedge G \wedge \bigwedge Z : Z \subseteq E\}$$

are complete, atomic Boolean lattices.

Proof. Since $\mathcal{L}$ is complete, $\bigwedge B_u$ and $\bigvee B_u$ exist, where $\bigwedge B_u = \bigwedge \emptyset$ and $\bigwedge B_u = \bigwedge G$. Thus, B_u is bounded, $\bigvee B_u = \top_u$, and $\bigwedge B_u = \bot_u$.

Let $x \in [\bot_u, \top_u]$. If $x = \top_u = \bigwedge \emptyset$ or $x = \bot_u = \bigwedge G$, then there exists a $Z \subseteq G$ so that $x = \bigwedge Z$. Suppose, then, that $x \neq \top_u$ and $x \neq \bot_u$. Since $\mathcal{L}$ is bialgebraic and distributive, $x = \bigwedge K$, where $K \subseteq \mathtt{MP}(\mathcal{L})$.

Suppose there exists a $k \in K$ so that $k \notin G$. Then $k \in E$. Then $k \geq \bigwedge G$, so there exists a $v \in G$ so that $k \geq v$, since k is completely meet prime. Then, since $E \cap G = \emptyset$, this is a contradiction, since $k = v$ since $v \in G$ are maximal in $\mathtt{MP}(\mathcal{L})$, but $k \in E$. Hence, it must be that $K \subseteq G$. Therefore, since $x \in [\bot_u, \top_u]$ was arbitrary, for all $x \in [\bot_u, \top_u]$, there exists a $Z \subseteq G$ so that $x = \bigwedge G$. Therefore $[\bot_u, \top_u] \subseteq B_u$.

Let $Z \subseteq G$. Then $\bigwedge G \in B_u$ and $\bot_u = \bigwedge G \leq \bigwedge Z \leq \bigwedge \emptyset = \top_u$. Hence, $\bigwedge Z \in [\bot_u, \top_u]$. Therefore, since $Z \subseteq G$ was arbitrary, for all $Z \subseteq G$, $\bigwedge Z \in [\bot_u, \top_u]$. Therefore $B_u \subseteq [\bot_u, \top_u]$.

Hence, since $[\bot_u, \top_u] \subseteq B_u$ and $B_u \subseteq [\bot_u, \top_u]$, $B_u = [\bot_u, \top_u]$.

Let $x \in B_u$ so that $x < \top_u$. Then $x = \bigwedge Z$ for some $Z \subseteq G$. Then there exists a $k \in G$ so that $k \in Z$. Then $\bigwedge\{k\} = k \in \uparrow x$. Observe, $\top_u = \top_{\mathcal{L}}$, so by Lemma 4.37, since $k \in \mathtt{Co}(\mathcal{L})$, $k \prec \top_{\mathcal{L}} = \top_u$. Therefore k is a coatom of B_u. Hence, since x was arbitrary, for all $x \in B_u$, $\uparrow x$ contains a coatom. Therefore B_u is coatomic. Note, for all $x \in G$, $\bigwedge\{x\}$ will be a coatom in G.

Therefore, since B_u is a coatomic, complete, meet-continuous lattice such that for all $x \in B_u$, there exists a set $Z \subseteq G$ of coatoms of B_u such that $x = \bigwedge Z$, by Lemma 4.36, B_u is an atomic Boolean lattice. Hence, B_u is a complete, atomic Boolean lattice.

Since $\mathcal{L}$ is complete, $\bigwedge B_d$ and $\bigvee B_d$ exist, where $\bigwedge B_d = \bigwedge G \wedge \bigwedge E$ and $\bigvee B_d = \bigwedge G \wedge \bigwedge \emptyset = \bigwedge G$. Hence, B_d is bounded where $\bot_d = \bigwedge G \wedge \bigwedge E$ and $\top_d = \bigwedge G$.

Let $x \in [\bot_d, \top_d]$. If $x = \bot_d$ or $x = \top_d$, then there exists a $Z \subseteq E$ so that $x = \bigwedge G \wedge \bigwedge Z$. Suppose, then, that $x \neq \bot_d$ and $x \neq \top_d$. Then $x < \bigwedge G$. Since L is bialgebraic and distributive, $x = \bigwedge K$ where $K \subseteq \mathtt{MP}(\mathcal{L})$. Then $\bigwedge K < \bigwedge G$, so since

$K \subseteq \mathtt{MP}(\mathcal{L})$, it must be that $K \cap E \neq \emptyset$. Let $K_E = K \cap E$ and $K_G = K \cap G$. Then

$$\begin{aligned} x &= \bigwedge K \\ &= \bigwedge K \wedge \bigwedge G \\ &= \bigwedge K_E \wedge \bigwedge K_G \wedge \bigwedge G \\ &= \bigwedge K_E \wedge \bigwedge G. \end{aligned}$$

Hence, there exists a $Z \subseteq E$ so that $x = \bigwedge G \wedge \bigwedge Z$. Therefore $[\perp_d, \top_d] \subseteq B_d$. Further, note that $\bigwedge G \wedge \bigwedge K_E = \bigwedge \{\bigwedge G \wedge e : e \in K_E\}$. Observe, $\{\bigwedge G \wedge e : e \in K_E\} \subseteq \mathtt{Co}(B_d)$ by Lemma 4.38. Hence, there exists a $Z \subseteq \mathtt{Co}(B_d)$ so that $x = \bigwedge Z$.

Let $Z \subseteq E$. Then $\perp_d = \bigwedge G \wedge \bigwedge E \leq \bigwedge G \wedge \bigwedge Z \leq \bigwedge G = \top_d$. Therefore $\bigwedge G \wedge \bigwedge Z \in [\perp_d, \top_d]$. Hence, $B_d \subseteq [\perp_d, \top_d]$.

Thus, since $[\perp_d, \top_d] \subseteq B_d$ and $B_d \subseteq [\perp_d, \top_d]$, $B_d = [\perp_d, \top_d]$.

By Lemma 4.37, B_d is coatomic and $\mathtt{Co}(B_d) = \{e \wedge \bigwedge G : e \in E\}$. Therefore, since B_d is a coatomic, complete, meet-continuous lattice such that for all $x \in B_d$ there exists a set Z of coatoms of B_d so that $x = \bigwedge Z$, B_d is an atomic Boolean lattice by Lemma 4.35. Hence, B_d is a complete, atomic Boolean lattice.

Therefore, if $\mathcal{L}$ is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34 and if $\mathtt{MP}(\mathcal{L}) = G \cup E$ as specified in Definition 3.26, then $B_u = \{\bigwedge Z : Z \subseteq G\}$ and $B_d = \{\bigwedge G \wedge \bigwedge Z : Z \subseteq E\}$ are complete, atomic Boolean lattices. $\square$

Definition 4.40. *Let $\mathcal{L} = (L, \leq)$ be a complete lattice. We say that $\mathcal{L}$ is a cone lattice provided there exists an element $\perp < \eta < \top$ such that*

1. *The posets $\uparrow \eta$ and $\downarrow \eta$ are complete atomic Boolean lattices.*

2. *If $x \in L - (\uparrow \eta \cup \downarrow \eta)$, then $\downarrow \eta \cap \downarrow x$ has a maximal member, and $\uparrow \eta \cap \uparrow x$ has a minimal member.*

We let $\texttt{Cone}(\mathcal{L}) = \downarrow \eta \cup \uparrow \eta$ and call this poset the *Boolean cone* of L. We will say that a cone lattice is *proper* provided $L - \texttt{Cone}(\mathcal{L})$ is nonempty. We let $\texttt{Cloud}(\mathcal{L}) = \mathcal{L} - \texttt{Cone}(\mathcal{L})$.

A social graph has no isolated vertices. Note that a simple, social graph $\mathcal{G}$ is social if and only if its corresponding graph poset $\mathcal{P}_G$ is social.

Corollary 4.41. *Suppose that $\mathcal{L}$ is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34. If $\texttt{MP}(\mathcal{L}) = G \cup E$ as specified in Definition 3.26 and $\texttt{MP}(\mathcal{L})$ is social and has at least three maximal members, then $\mathcal{L}$ is a proper cone lattice.*

Proof. Suppose that $\mathcal{L}$ is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34, $\texttt{MP}(\mathcal{L}) = G \cup E$ as specified in Definition 3.26, $\texttt{MP}(\mathcal{L})$ is social and has at least three maximal members.

By Theorem 4.33, $B_u = [\bot_u, \top_u]$ and $B_d = [\bot_d, \top_d]$ are complete, atomic Boolean lattices where $\bot_{\mathcal{L}} < \top_d = \bigwedge G = \bot_u < \top_{\mathcal{L}}$. Thus, $\uparrow \bigwedge G$ and $\downarrow \bigwedge G$ are complete, atomic Boolean lattices. Therefore $\mathcal{L}$ satisfies Definition 4.40(1).

If $\mathcal{L} - \texttt{Cone}(\mathcal{L}) = \emptyset$, then for all $e \in E$, $e \leq \bigwedge G$. This is a contradiction by Lemma 4.37(2) since $|G| \geq 3$. Hence, $\mathcal{L} - \texttt{Cone}(\mathcal{L}) \neq \emptyset$.

Let $x \in \texttt{Cloud}(\mathcal{L})$. Since $\mathcal{L}$ is complete, $k = \bigvee(\downarrow x \cap \downarrow \bigwedge G)$ exists. Then, for all $y \in \downarrow x \cap \downarrow \bigwedge G$, $y \leq k$. Note, for all $y \in \downarrow x \cap \downarrow \bigwedge G$, $y \leq x$ and $y \leq \bigwedge G$. Thus, since k is the least upper bound of $\downarrow x \cap \downarrow \bigwedge G$, it must be that for all $y \in \downarrow x \cap \downarrow \bigwedge G$, $y \leq k \leq x$ and $y \leq k \leq \bigwedge G$. Thus, $k \in \downarrow x$ and $k \in \downarrow \bigwedge G$. Hence, $k \in \downarrow x \cap \downarrow \bigwedge G$. Thus, $\downarrow x \cap \downarrow \bigwedge G$ has a maximal member for all $x \in \texttt{Cloud}(\mathcal{L})$.

Let $x \in \texttt{Cloud}(\mathcal{L})$. Since L is complete, $l = \bigwedge(\uparrow x \cap \uparrow \bigwedge G)$ exists. Then, for all $y \in \uparrow x \cap \uparrow \bigwedge G$, $y \leq l$. Note, for all $y \in \uparrow x \cap \uparrow \bigwedge G$, it must be that $x \leq y$ and $\bigwedge G \leq y$. Thus, since l is the greatest lower bound of $\uparrow x \cap \uparrow \bigwedge G$, it must be that for all $y \in \uparrow x \cap \uparrow \bigwedge G$, $x \leq l \leq y$ and $\bigwedge G \leq l \leq y$. Thus, $l \in \uparrow x$ and $l \in \uparrow \bigwedge G$. Hence, $l \in \uparrow x \cap \uparrow \bigwedge G$. Thus, $\uparrow x \cap \uparrow \bigwedge G$ has a minimal member for all $x \in \texttt{Cloud}(\mathcal{L})$.

Hence, if $x \in \texttt{Cloud}(\mathcal{L})$, then $\downarrow x \cap \downarrow \bigwedge G$ has a maximal member and $\uparrow x \cap \uparrow \bigwedge G$ has a minimal member. Therefore, $\mathcal{L}$ satisfies Definition 4.40(2). Therefore, since $\mathcal{L}$ satisfies Definition 4.40(1), Definition 4.40(2), and $\texttt{Cloud}(\mathcal{L}) \neq \emptyset$, $\mathcal{L}$ is a proper cone lattice. Hence, if $\mathcal{L}$ is a bialgebraic, distributive lattice that satisfies the equivalent conditions of Theorem 4.34, $\texttt{MP}(\mathcal{L}) = G \cup E$ as specified in Definition 3.26, and $\texttt{MP}(\mathcal{L})$ is social and has at least three maximal members, then $\mathcal{L}$ is a proper cone lattice.

□

Lemma 4.42. *Suppose $\mathcal{L}$ and $\mathcal{M}$ are algebraic lattices. If $\texttt{CMI}(\mathcal{L})$ is order-isomorphic to $\texttt{CMI}(\mathcal{M})$, then $\mathcal{L}$ is order isomorphic to $\mathcal{M}$.*

Proof. Suppose $\mathcal{L}$ and $\mathcal{M}$ are algebraic lattices. Suppose $\texttt{CMI}(\mathcal{L})$ is order-isomorphic to $\texttt{CMI}(\mathcal{M})$. Then there exists an order isomorphism $f : \texttt{CMI}(\mathcal{L}) \longrightarrow \texttt{CMI}(\mathcal{M})$. Note, since $\mathcal{L}$ is an algebraic lattice, by Birkhoff's Theorem, for all $x \in \mathcal{L}$, there exists an $X \subseteq \texttt{CMI}(\mathcal{L})$ so that $x = \bigwedge X$. Similarly, since $\mathcal{M}$ is an algebraic lattice, for all $y \in \mathcal{M}$, there exists a $Y \subseteq \texttt{CMI}(\mathcal{M})$ so that $y = \bigwedge Y$. Define $g : \texttt{CMI}(\mathcal{L}) \longrightarrow \texttt{CMI}(\mathcal{M})$ and $m : \texttt{CMI}(\mathcal{M}) \longrightarrow \texttt{CMI}(\mathcal{L})$ by

$$g(x) = g(\bigwedge X) = \bigwedge \{f(z) : z \in X\}$$

and

$$m(y) = m(\bigwedge Y) = \bigwedge \{f^{-1}(z) : z \in Y\}$$

where $X \subseteq \texttt{CMI}(\mathcal{L})$ and $x = \bigwedge X$ and $Y \subseteq \texttt{CMI}(\mathcal{M})$ and $y = \bigwedge Y$.

Let $x \in \mathcal{L}$. Then there exists an $X \subseteq \texttt{CMI}(\mathcal{L})$ so that $x = \bigwedge X$. Consider,

$$\begin{aligned}
m(g(x)) &= m(g(\bigwedge X)) \\
&= m(\bigwedge\{f(z) : z \in X\}) \\
&= m(\bigwedge f(X)) \\
&= \bigwedge\{f^{-1}(f(z)) : f(z) \in f(X)\} \\
&= \bigwedge\{z : z \in X\} \\
&= \bigwedge X \\
&= x.
\end{aligned}$$

Thus, since $x \in \mathcal{L}$ was arbitrary, for all $x \in \mathcal{L}$, $m(g(x)) = x$.

Let $y \in \mathcal{M}$. Then there exists a $Y \subseteq \texttt{CMI}(\mathcal{M})$ so that $y = \bigwedge Y$. Consider,

$$\begin{aligned}
g(m(y)) &= g(m(\bigwedge Y)) \\
&= g(\bigwedge\{f^{-1}(z) : z \in Y\}) \\
&= g(\bigwedge f^{-1}(Y)) \\
&= \bigwedge\{f(f^{-1}(z)) : f^{-1}(z) \in f^{-1}(Y)\} \\
&= \bigwedge\{z : z \in Y\} \\
&= \bigwedge Y \\
&= y.
\end{aligned}$$

Thus, since $y \in \mathcal{M}$ was arbitrary, for all $y \in \mathcal{M}$, $g(m(y)) = y$.

Hence, since for all $x \in \mathcal{L}$, $m(g(x)) = x$ and for all $y \in \mathcal{M}$, $g(m(y)) = y$, g is a bijection and $m = g^{-1}$.

Let $x, y \in \mathcal{L}$ so that $x \leq y$. Note, $x = \bigwedge X$ and $y = \bigwedge Y$ for some $X, Y \subseteq \texttt{CMI}(\mathcal{L})$. Then $\bigwedge X \leq \bigwedge Y$. Then $\uparrow y \subseteq \uparrow x$. Then $Y \subseteq X$. Then $f(Y) \subseteq f(X)$, since f is a bijection. Then $\bigwedge f(X) \leq \bigwedge f(Y)$. Then $g(x) \leq g(y)$. Hence, for all $x, y \in \mathcal{L}$, if $x \leq y$, then $g(x) \leq g(y)$. That is, g is an order homomorphism.

Let $p, q \in \mathcal{M}$ so that $p \leq q$. Then there exists $x, y \in \mathcal{L}$ so that $g(x) = p$ and $g(y) = q$. Then $g(x) \leq g(y)$. Observe, there exists $P, Q \subseteq \texttt{CMI}(\mathcal{M})$ so that $\bigwedge P = p$ and $\bigwedge Q = q$. Then $\bigwedge P \leq \bigwedge Q$. Then $\uparrow q \subseteq \uparrow p$. Then $Q \subseteq P$. Then $f^{-1}(Q) \subseteq f^{-1}(P)$ since f is a bijection. Then $\bigwedge f^{-1}(P) \leq \bigwedge f^{-1}(Q)$. Observe,

$$\begin{aligned} g(\bigwedge f^{-1}(P)) &= \bigwedge\{f(f^{-1}(k)) : k \in P\} \\ &= \bigwedge\{k : k \in P\} \\ &= \bigwedge P \\ &= p \\ &= g(x). \end{aligned}$$

Hence, since g is injective, $x = \bigwedge f^{-1}(P)$. Similarly, $y = \bigwedge f^{-1}(Q)$. Thus, $x \leq y$. Hence, for all $p, q \in \mathcal{M}$, if $g(x) = p \leq q = g(y)$, then $x \leq y$.

Thus, since g is bijective, g is an order homomorphism, and for all $p, q \in \mathcal{M}$, if $g(x) = p \leq q = g(y)$, then $x \leq y$, g is an order isomorphism. Therefore $\mathcal{L}$ is order isomorphic to $\mathcal{M}$.

Hence, if $\mathcal{L}$ and $\mathcal{M}$ are algebraic lattices so that $\texttt{CMI}(\mathcal{L})$ is order isomorphic to $\texttt{CMI}(\mathcal{M})$, then $\mathcal{L}$ is order isomorphic to $\mathcal{M}$. □

Theorem 4.43. *If $\mathcal{L}$ is a bialgebraic cone lattice such that $\texttt{MP}(\mathcal{L})$ is a graph poset, then there exists a simple, social graph $\mathcal{G}$ so that $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\texttt{MP}(\mathcal{L})$ and $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be a bialgebraic cone lattice such that $\texttt{MP}(\mathcal{L})$ is a graph poset. Since $\texttt{MP}(\mathcal{L})$ is a graph poset, there exists a simple, social graph $\mathcal{G}$ so that $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\mathtt{MP}(\mathcal{L})$. Observe, $\mathtt{MP}(\Omega(\mathcal{G}))$ is order isomorphic to $\mathcal{P}_{\mathcal{G}}$ by Theorem 4.38, and so is order isomorphic to $\mathtt{MP}(\mathcal{L})$.

Note, since $\mathcal{L}$ and $\Omega(\mathcal{G})$ are bialgebraic, $\mathtt{CMP}(\mathcal{L}) = \mathtt{CMI}(\mathcal{L})$ and $\mathtt{CMP}(\Omega(\mathcal{G})) = \mathtt{CMI}(\Omega(\mathcal{G}))$. In particular, since $\mathtt{MP}(\mathcal{L}) = \mathtt{CMP}(\mathcal{L})$ and $\mathtt{MP}(\Omega(\mathcal{G})) = \mathtt{CMP}(\Omega(\mathcal{G}))$, $\mathtt{MP}(\mathcal{L}) = \mathtt{CMI}(\mathcal{L})$ and $\mathtt{MP}(\Omega(\mathcal{G})) = \mathtt{CMI}(\Omega(\mathcal{G}))$. Thus, $\mathtt{MP}(\mathcal{L})$ is order isomorphic to $\mathtt{MP}(\Omega(\mathcal{G}))$.

Then, since $\Omega(\mathcal{G})$ and $\mathcal{L}$ are algebraic lattices such that $\mathtt{MP}(\mathcal{L})$ and $\mathtt{MP}(\Omega(\mathcal{G}))$ are order isomorphic, $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$. Hence, if $\mathcal{L}$ is a cone lattice such that $\mathtt{MP}(\mathcal{L})$ is a graph poset, then there exists a simple, social graph $\mathcal{G}$ so that $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$.

□

Theorem 4.44. *Let $\mathcal{G}$ be a simple, social graph and $\Omega(\mathcal{G})$ be its associated classical topology. Then there is a simple, social graph $\mathcal{G}_{\Omega(\mathcal{G})}$ associated with $\Omega(\mathcal{G})$ so that $\mathcal{G}_{\Omega(\mathcal{G})}$ is graph isomorphic to $\mathcal{G}$.*

Proof. Let $\mathcal{G}$ be a simple, social graph and $\Omega(\mathcal{G})$ be its associated cone lattice. Then $\mathtt{MP}(\Omega(\mathcal{G}))$ is a graph poset. Observe, $\mathcal{P}_{\mathcal{G}}$ is the graph poset associated with $\mathcal{G}$. Observe, $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\mathtt{MP}(\Omega(\mathcal{G}))$ by Theorem 3.27. Since $\mathtt{MP}(\Omega(\mathcal{G}))$ is a graph poset, there exists a graph $\mathcal{G}_{\Omega(\mathcal{G})}$ so that $\mathcal{P}_{\mathcal{G}_{\Omega(\mathcal{G})}}$ is order isomorphic to $\mathtt{MP}(\Omega(\mathcal{G}))$. Then $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\mathcal{P}_{\mathcal{G}_{\Omega(\mathcal{G})}}$. Therefore, since $\mathcal{P}_{\mathcal{G}}$ and $\mathcal{P}_{\mathcal{G}_{\Omega(\mathcal{G})}}$ are graph posets which are order isomorphic, $\mathcal{G}_{\Omega(\mathcal{G})}$ is graph isomorphic to $\mathcal{G}$.

Hence, if G is a simple, social graph and $\Omega(\mathcal{G})$ is its associated cone lattice, then there is a simple, social graph $\mathcal{G}_{\Omega(\mathcal{G})}$ associated with $\Omega(\mathcal{G})$ so that $\mathcal{G}_{\Omega(\mathcal{G})}$ is graph isomorphic to $\mathcal{G}$.

□

Theorem 4.45. *Let $\mathcal{L}$ be a bialgebraic cone lattice such that $\mathtt{MP}(\mathcal{L})$ is a graph poset. Then there is a simple, social graph $\mathcal{G}$ so that $\Omega(\mathcal{G}_{\Omega(\mathcal{G})})$ is order isomorphic to $\mathcal{L}$.*

Proof. Let $\mathcal{L}$ be a cone lattice. Then $\mathtt{MP}(\mathcal{L})$ is a graph poset. Then there is a simple, social graph $\mathcal{G}$ so that $\mathcal{P}_{\mathcal{G}}$ is order isomorphic to $\mathtt{MP}(\mathcal{L})$ and $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$ by Theorem 4.43. Then there is a simple, social graph $\mathcal{G}_{\Omega(\mathcal{G})}$ associated with $\Omega(\mathcal{G})$ so that $\mathcal{G}_{\Omega(\mathcal{G})}$ is graph isomorphic to $\mathcal{G}$ by Theorem 4.44. Since $\mathcal{G}$ and $\mathcal{G}_{\Omega(\mathcal{G})}$ are graph isomorphic, it must be that $\Omega(\mathcal{G})$ and $\Omega(\mathcal{G}_{\Omega(\mathcal{G})})$ are order isomorphic. Therefore, since $\mathcal{L}$ is order isomorphic to $\Omega(\mathcal{G})$ and $\Omega(\mathcal{G})$ is order isomorphic to $\Omega(\mathcal{G}_{\Omega(\mathcal{G})})$, it must be that $\mathcal{L}$ is order isomorphic to $\Omega(\mathcal{G}_{\Omega(\mathcal{G})})$.

Hence, if $\mathcal{L}$ is a bialgebraic cone lattice such that $\mathtt{MP}(\mathcal{L})$ is a graph poset, then there is a simple, social graph $\mathcal{G}$ so that $\Omega(\mathcal{G}_{\Omega(\mathcal{G})})$ is order isomorphic to $\mathcal{L}$.

$\square$

CHAPTER 5

CONCLUSION

5.1 Conclusion

In this thesis, several things have been shown. In section 2, we identified important order theoretic properties of the classical topology. In particular, we showed that the meet-prime elements are the completely meet-prime elements in $\Omega(\mathcal{G})$ and the join-prime elements are the completely join-prime elements in $\Omega(\mathcal{G})$.

In section 3, we defined graph posets and showed that for every simple, social graph $\mathcal{G}$, $\mathtt{MP}(\Omega(\mathcal{G}))$ is a graph poset whose associated graph is graph isomorphic to $\mathcal{G}$. We also showed that given a simple, social graph $\mathcal{G}$, $\mathtt{Low}(\mathcal{P}_{\mathcal{G}}) = \Omega(\mathcal{G})$.

In section 4, we defined cone lattices and showed that if we are given a cone lattice $\mathcal{L}$ whose meet-prime elements form a graph poset, then there is a simple, social graph $\mathcal{G}$ so that $\Omega(\mathcal{G})$ is order isomorphic to $\mathcal{L}$ and so that $\mathtt{MP}(\mathcal{L})$ is order isomorphic to $\mathtt{MP}(\Omega(\mathcal{G}))$.

5.2 Further Research

There are many topics for further research related to this thesis. In particular,

1. Generalize cone lattices to hypergraphs.
2. Show how graph homomorphisms between graphs $\mathcal{G}$ and $\mathcal{H}$ can be described in the context of the cone lattices $\Omega(\mathcal{G})$ and $\Omega(\mathcal{H})$.
3. Show that if $\mathcal{L}$ is a cone lattice so that $\mathtt{MP}(\mathcal{L})$ is a graph poset, then $\mathcal{L}$ is a bialgebraic, distributive lattice.
4. If $\mathcal{L}$ is a cone lattice, what happens if $\mathtt{MP}(\mathcal{L})$ is not a graph poset? What other kinds of posets can $\mathtt{MP}(\mathcal{L})$ be?

5. Given a simple, social graph $\mathcal{G}$ with n vertices,

$$|\Omega(\mathcal{G})| \leq |\Omega(K_n)| = 2^n + 2^{\frac{n(n-1)}{2}} - 1 + f(n),$$

   where $f(n) = |\texttt{Cloud}(\Omega(\mathcal{G}))|$. What is $f(n)$ for all n?

6. If $\mathcal{G}$ is a weighted graph, what is a good way to, if at all, to translate the notion of weights to $\Omega(\mathcal{G})$?

7. Can we view $\Omega(\mathcal{G})$ from the context of spectral graph theory?

8. The structure of a cone lattice $\mathcal{L}$ can be described by taking two atomic Boolean lattices B_1 and B_2 and identifying the bottom element of B_2 with the top element of B_1, called $\texttt{Cone}(\mathcal{L})$, with $\texttt{Cloud}(\mathcal{L}) = \mathcal{L} - \texttt{Cone}(\mathcal{L})$. If we consider a lattice $\mathcal{L}$ for which we can consider a sequence of atomic Boolean lattices $\{B_n\}_{k=1}^{\infty}$ so that we identify $\top_k = \bot_{k+1}$ for all $1 \leq k \leq n-1$ and call this $\texttt{Cone}(\mathcal{L})$ and take $\texttt{Cloud}(\mathcal{L}) = \mathcal{L} - \texttt{Cone}(\mathcal{L})$, what are the structural properties?

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