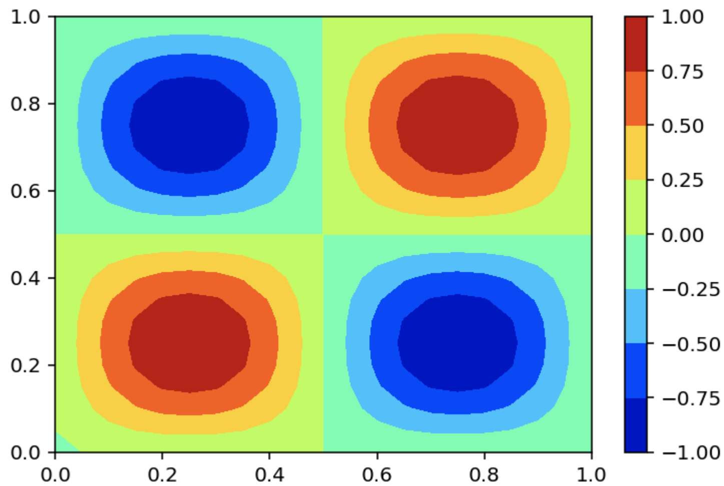
Homework 1, Tony Saad's uCFD Course

From University of Utah CHEN 6355



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1 Question 1: The Energy Equation

What is a thermodynamic relation between temperature and enthalpy?

The purpose of this question is to develop an equation for temperature from the Navier-Stokes energy equation:

$$\frac{\partial \rho h}{\partial t} = \frac{D\rho}{Dt} - \nabla \cdot \mathbf{u}\rho h \ - \tau_{ij}\frac{\partial u_i}{\partial x_j} - \nabla \cdot \mathbf{q}$$

Using the definition of the specific heat capacity at a constant temperature:

$$\frac{dh}{dT} = c_p$$

we can create a discretized expression for temperature in terms of enthalpy.

$$\int_{h_i}^{h_f} dh = \int_{T_i}^{Tf} c_p dT$$

$$T_f = T_i + \frac{1}{c_p}(h_f - h_i)$$

Write out the $\tau_{ij}\frac{\partial u_i}{\partial x_j}$ term. This term is the viscous dissipation. Note that repeated indices indicate a summation over all indices.

Expanding the term as indicated above yields:

$$\tau_{11}\frac{\partial u_1}{\partial x_1} + \tau_{12}\frac{\partial u_1}{\partial x_2} + \tau_{21}\frac{\partial u_2}{\partial x_1} + \tau_{22}\frac{\partial u_2}{\partial x_2}$$

Converting this expression to drop the indices for the velocity and spatial dimensions:

$$\tau_{11}\frac{\partial u}{\partial x} + \tau_{12}\frac{\partial u}{\partial y} + \tau_{21}\frac{\partial v}{\partial x} + \tau_{22}\frac{\partial v}{\partial y}$$

Now, to expand on the definition of the stress tensor, that is:

$$\tau_{ij} = -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2}{3}\mu\delta_{ij}\nabla \cdot \mathbf{u}$$

The diagonal terms contain the extra contribution from the dirac delta function, and are expanded as:

$$\tau_{11} = -\frac{4}{3}\mu\frac{\partial u}{\partial x} + \frac{2}{3}\mu\frac{\partial v}{\partial y}$$

$$\tau_{22} = -\frac{4}{3}\mu\frac{\partial v}{\partial y} + \frac{2}{3}\mu\frac{\partial u}{\partial x}$$

The off-diagonal terms contain only the contribution from the first term:

$$\tau_{12} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{21} = -\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

which are the same due to the symmetry of the stress tensor. Completing the multiplications for each term and simplifying results in:

$$-\frac{4}{3}\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2\right] - \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$

Assume Fourier's Law for heat transfer, i.e. $\boldsymbol{q} = -k\nabla T$

Inserting Fourier's law, and the definition of the specific heat capacity:

$$\rho c_p \frac{\partial T}{\partial t} = \frac{\partial p}{\partial t} + p\nabla\cdot\mathbf{u} - \rho c_p T\nabla\cdot\mathbf{u} + \frac{4}{3}\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 - \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial v}{\partial y}\right)^2\right] + \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + k\nabla^2 T$$

A further simplification can be made if the fluid is assumed to be an ideal gas. In that case, the substitution $p = \rho RT$ can be made, getting another expression for the time derivative of the temperature.

2 Question 2: Vorticity-streamfunction Navier-Stokes

If a function ψ *exists such that* $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial y}$, *show that continuity is satisfied by* ψ.

Substituting the definition of the streamfunction for each velocity into the continuity equation:

$$\frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\psi}{\partial x} = 0$$

$$\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial x\partial y} = 0$$

Derive an equation from the NS momentum equations that are a function only of vorticity and the streamfunction.

By differentiating (5) with respect to y and (6) with respect to x, we obtain:

$$\frac{\partial^2 u}{\partial y\partial t} + \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + u\frac{\partial^2 u}{\partial x\partial y}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 v}{\partial y^2}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^3 u}{\partial x^2\partial y} + \frac{\partial^3 u}{\partial y^3}\right)$$

$$\frac{\partial^2 v}{\partial x\partial t} + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x\partial y}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} + \frac{\mu}{\rho}\left(\frac{\partial^3 v}{\partial x^3} + \frac{\partial^3 v}{\partial x\partial y^2}\right)$$

Subtracting the latter equation from the former:

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\mu}{\rho}\left(\frac{\partial^2}{\partial x^2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] + \frac{\partial^2}{\partial y^2}\left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right]\right)$$

Using the definition of vorticity, $\omega = \nabla \times \mathbf{u} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\mathbf{j}$, the above equation becomes:

$$\frac{\partial \omega}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y} = \frac{\mu}{\rho}\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right)$$

where we have taken advantage of the fact that the fourth term on the left-hand side contains the divergence of the velocity, which is zero for the case of a constant density. Because the vorticity is equal to a Poisson equation for the streamfunction:

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}$$

you would be able to find an initial streamfunction given an initial condition for the vorticity. An iteration strategy could be the following:

- Find streamfunction from initial condition for vorticity using the Poisson equation
- Find the next timestep for vorticity using the NS equation
- Update the streamfunction using the new condition for vorticity
- Repeat