MATH 122: Quiz Review Solutions

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Contents

1 Quiz 2 Review

1.1

$$\int x \sin{(x^2 + 5)} dx$$

Let $u = x^2 + 5$. Then $du = 2xdx$ and $dx = \frac{du}{2x}$

$$\int x \sin{(u)} \frac{du}{2x} = \int \frac{1}{2} \sin{(u)} du = \frac{-1}{2} \cos{(u)} + C$$

Answer: $-\frac{\cos{(x^2+5)}}{2} + C$

1.2

$$\int \sin^4 x \cos x dx$$

Let $u = \sin{(x)}$. Then $du = \cos{(x)} dx$ and $dx = \frac{du}{\cos x}$

$$\int u^4 \cos{(x)} \frac{du}{cos(x)} = \int u^4 du = \frac{1}{5} u^5 + C$$

Answer: $\frac{1}{5}(\sin{(x)}^5 + C$

1.3

$$\int \frac{1}{x \ln x} dx$$

Let $u = \ln x$. Then $du = \frac{dx}{x}$ and $dx = xdu$

$$\int \frac{1}{xu} xdu = \int \frac{1}{u} du = ln|u| + C$$

Answer: $\ln |\ln |x|| + C$

1.4

$$\int x^2 \ln x dx$$

Let $u = \ln x$, and $dv = x^2$. Then $du = \frac{1}{x} dx$, and $v = \frac{1}{3} x^3$.

$$= \ln x (\frac{1}{3} x^3) - \frac{1}{3} \int x^2 dx = \ln x (\frac{1}{3} x^3) - \frac{1}{3} (\frac{1}{3} x^3) + C$$

Answer: $\frac{1}{3}(\ln x(x^3) - \frac{1}{3} x^3) + C$

1.5

$$\int x^2 \cos x dx$$

Let $u = x^2$ and $dv = \cos x dx$. Then, $du = 2xdx$ and $v = \sin x$

$$= x^2 \sin x - \int \sin x 2x dx$$

Let $j = 2x$ and $dk = \sin x dx$. Then, $dj = 2dx$ and $k = -\cos x$.

$$= x^2 \sin(x) - ((2x)(-\cos(x)) + 2\int (\cos(x)dx))$$

$$= x^2 \sin(x) + ((2x)(-\cos(x)) + 2(\sin(x) + C))$$

$$= x^2 \sin(x) - (2x)(-\cos(x)) - 2(\sin(x)) + C$$

Answer: $x^2 \sin(x) + 2x\cos(x) - 2\sin(x) + C$

1.6

$$\int x^2 e^{2x} dx$$

Let $u = x^2$ and $dv = e^{2x}dx$. Then, $du = 2xdx$ and $v = \frac{1}{2}e^{2x}$

$$= x\frac{1}{2}(x^2)(e^{2x}) - \int xe^{2x}dx$$

Let $j = x$ and $dk = e^{2x}dx$. Then, $dj = dx$ and $k = \frac{1}{2}e^{2x}$.

$$= \frac{1}{2}(x^2)(e^{2x}) - (\frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

$$) = \frac{1}{2}(x^2)(e^{2x}) - (\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C)$$

Answer: $\frac{1}{2}(x^2)(e^{2x}) - \frac{1}{2}xe^{2x} + \frac{1}{4}e^{2x} + C$

1.7

$$\int xe^{x^2} dx$$

Let $u = x^2$. Then $du = 2xdx$ and $dx = \frac{du}{2x}$

$$= \int xe^u \frac{du}{2x} = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C$$

Answer: $\frac{1}{2}e^{x^2} + C$

1.8

$$\int xe^{2x}dx$$

Let $u = x$ and $dv = e^{2x}dx$. Then, $du = dx$ and $v = \frac{1}{2}e^{2x}$

$$= \frac{1}{2}xe^{2x} - \frac{1}{2}\int e^{2x}dx$$

Answer: $\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

1.9

$$\int \cos^3 2xdx$$

Let $u = \sin(2x)$. Then $du = 2\cos(2x)dx$ and $dx = \frac{du}{2\cos(2x)}$

$$= \int \cos^3(2x)\frac{du}{2\cos(2x)} = \frac{1}{2}\int \cos^2(2x)du$$

$$= \frac{1}{2}\int 1 - \sin^2(2x)du = \frac{1}{2}\int 1 - u^2du = \frac{1}{2}(u - \frac{1}{3}u^3) + C$$

Answer: $\frac{1}{2}(\sin(2x) - \frac{1}{3}(\sin^3(2x))) + C$

1.10

$$\int \cos^2 3x \sin^2 3xdx$$

Using the identities $\cos^2(a) = \frac{1+\cos 2a}{2}$ and $\sin^2(a) = \frac{1-\cos 2a}{2}$, we have:

$$\cos^2(3x)\sin^2(3x) = \left(\frac{1-\cos(6x)}{2}\right)\left(\frac{1+\cos(6x)}{2}\right)$$

$$= \frac{1}{4}\left(1 - \cos^2 6x\right) = \frac{1}{4}\sin^2 6x = \frac{1}{8}\left(1 - \cos 12x\right)$$

Then,

$$\int \frac{1}{8}\left(1 - \cos(12x)dx\right) = \frac{1}{8}(x - \frac{1}{12}\sin(12x)) + C$$

Answer:$\frac{1}{8}x - \frac{1}{96}\sin(12x) + C$

1.11

$$\int \sin^5 x \cos^2 x dx$$

Let $u = \cos x$. Then $du = -\sin x dx$ and $dx = \frac{du}{-\sin x}$

$$\int \sin^5 x u^2 \frac{du}{-\sin x} = -\int \sin^4 x u^2 du$$

$$= -\int (1 - \cos^2 x)^2 u^2 du = -\int (1-u)^2 u^2 du$$

$$= -\int (1 - 2u + u^4) u^2 du = -\int u^2 - 2u^4 + u^6 du$$

$$= -(\frac{1}{3}u^3 - 2\frac{1}{5}u^5 + \frac{1}{7}u^7) + C$$

Answer: $-\frac{1}{3}\cos^3 x + \frac{2}{5}\cos^5 x - \frac{1}{7}\cos^7 x + C$

1.12

$$\int \sec^3 x \tan^3 x dx$$

Let $u = \sec x$. Then $du = \sec x \tan x dx$ and $dx = \frac{du}{\sec x \tan x}$

$$\int \sec^3 x \tan^3 x \frac{du}{\sec x \tan x} = \int sec^2 x \tan^2 x du$$

$$= \int u^2 \tan^2 x du = \int u^2 (\sec^2 x - 1) du$$

$$= \int u^2 (u^2 - 1) du = \int u^4 - u^2 du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$$

Answer: $\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

1.13

$$\int \sec^4 x \tan^2 x dx$$

Let $u = \tan x$. Then $du = \sec^2 x dx$ and $dx = \frac{du}{\sec^2 x}$

$$\int \sec^4 x \tan^2 x \frac{du}{\sec^2 x} = \int \sec^2 x u^2 du = \int (1 + \tan^2 x) u^2 du$$

$$= \int (1 + u^2) u^2 du = \int u^2 + u^4 du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

Answer: $\frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$

1.14

$$\int \sqrt{\tan x} \sec^4 x dx$$

Let $u = \tan x$. Then $du = \sec^2 x dx$ and $dx = \frac{du}{\sec^2 x}$

$$\int \sqrt{u} \sec^4 x \frac{du}{\sec^2 x} = \int \sqrt{u} (\sec^2 x) du$$

$$= \int \sqrt{u}(1 + \tan^2 x) du = \int \sqrt{u}(1 + u^2) du$$

$$= \int u^{1/2} + u^{5/2} du = \frac{2}{3} u^{3/2} + \frac{2}{7} u^{7/2} + C$$

Answer: $\frac{2}{3} \tan^{3/2} x + \frac{2}{7} \tan^{7/2} x + C$

1.15

$$\int \sqrt{\sec x} \tan^3 x dx$$

Let $u = \sec x$. Then $du = \sec x \tan x dx$ and $dx = \frac{du}{\sec x \tan x}$

$$\int \sqrt{u} \tan^3 x \frac{du}{\sec x \tan x} = \int \sqrt{u} \tan^2 x \frac{du}{u}$$

$$= \int u^{-1/2} (\sec^2 x -) = \int u^{-1/2} (u^2 - 1)$$

$$= \int u^{3/2} - u^{1/2} du = \frac{2}{5} u^{5/2} - 2u^{1/2} + C$$

Answer: $\frac{2}{5} \sec^{5/2} x - 2 \sec^{1/2} x + C$

2 Quiz 3 Review

2.1

$$\int \frac{x^3}{\sqrt{1-x^2}}dx$$

Let $x = \sin\theta$. Then $dx = \cos\theta d\theta$

$$\int \frac{\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta d\theta = \int \sin^3\theta d\theta$$

Let $u = \cos\theta$. Then $du = -\sin\theta d\theta$ and $d\theta = \frac{du}{-\sin\theta}$

$$-\int \sin^2\theta du = -\int 1-\cos^2\theta du = -\int 1-u^2 du$$

$$= -(u - \frac{u^3}{3}) + C = -(\cos\theta - \frac{\cos^3\theta}{3}) + C$$

Answer: $-\sqrt{1-x^2} + \frac{1}{3}(\sqrt{1-x^3})^3 + C$

2.2

$$\int \frac{1}{\sqrt{x^2+4}}dx$$

Let $x = 2\tan\theta$. Then $dx = 2\sec^2\theta d\theta$.

$$\int \frac{1}{\sqrt{(2\tan\theta)^2+4}}2\sec^2\theta d\theta = \int \frac{1}{2\sec\theta}2sec^2\theta d\theta$$

$$= \int \sec\theta d\theta = \ln|\sec\theta + \tan\theta| + C$$

$$\ln|\frac{\sqrt{x^2+4}}{4} + \frac{x}{4}| + C = \ln|\sqrt{x^2+4} + x| - \ln 4 + C$$

Answer: $\ln|\sqrt{x^2+4} + x| + C$

2.3

$$\int \frac{1}{(16-x^2)^{3/2}}dx$$

Let $x = 4\sin\theta$. Then $dx = 4\cos\theta d\theta$

$$\int \frac{1}{\sqrt{16-(4\sin\theta)^2}^3} 4\cos\theta d\theta = \int \frac{1}{(4\cos\theta)^3} 4\cos\theta d\theta = \int \frac{1}{(4\cos\theta)^2} d\theta$$

$$\frac{1}{16}\int \sec^2\theta d\theta = \frac{1}{16}\tan\theta + C$$

Answer: $\frac{1}{16}\frac{x}{\sqrt{16-x^2}} + C$

2.4

$$\int \frac{1}{\sqrt{4x^2+9}} dx$$

Let $x = \frac{3}{2}\tan\theta$. Then $dx = \frac{3}{2}\sec^2\theta d\theta$

$$\int \frac{1}{\sqrt{(3\tan\theta)^2+9}} \frac{3}{2}\sec^2\theta d\theta = \frac{1}{2}\int \frac{1}{\sec\theta}\sec^2\theta d\theta$$

$$= \frac{1}{2}\int \sec\theta d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

Answer: $\frac{1}{2}\ln\sqrt{4x^2+9} + 2x| + C$

2.5

$$\int \tanh x dx$$

$$\int \frac{\sinh x}{\cosh x} dx$$

Let $u = \cosh x$. Then $du = \sinh x dx$ and $dx = \frac{du}{\sinh x}$

$$\int \frac{\sinh x}{u}\frac{du}{\sinh x} = \int \frac{1}{u} du = \ln|u| + C$$

Answer: $\ln|\cosh x| + C$

2.6

$$\int \frac{\cosh x}{3\sinh x + 4} dx$$

Let $u = 3\sinh x + 4$. Then $du = 3\cosh x dx$ and $dx = \frac{du}{3\cosh x}$

$$\int \frac{\cosh x}{u}\frac{du}{3\cosh x} = \frac{1}{3}\int \frac{1}{u} du = \frac{1}{3}\ln|u| + C$$

Answer: $\frac{1}{3}\ln|3\sinh x + 4|$

2.7

$$\int \operatorname{arctanh} x dx$$

Let $u = \operatorname{arctanh} x$ and $dv = dx$. Then $du = \frac{1}{1-x^2} dx$ and $v = x$.

$$\operatorname{arctanh} x * x - \int \frac{1}{1-x^2} dx = \operatorname{arctanh} x * x - \frac{1}{2} \ln |u| + C$$

Answer: $\operatorname{arctanh} x * x - \frac{1}{2} \ln |1 - x^2| + C$

2.8

$$\int \frac{2-x}{x^2+x} dx$$

$$\frac{A}{x} + \frac{B}{x+1} = \frac{2-x}{x(x+1)}$$
$$A(x+1) + Bx = 2 - x$$
$$A = 2$$
$$B = -3$$

$$\int \frac{2}{x} - \frac{3}{x+1} dx$$

Answer: $2 \ln |x| - 3 \ln |x+1|$

2.9

$$\int \frac{3x+11}{x^2+5x+6} dx$$

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2} dx = \int \frac{3x+11}{(x+3)(x+2)} dx$$

$$\frac{A}{x+3} + \frac{B}{x+2} = \frac{3x+11}{(x+3)(x+2)}$$
$$A(x+2) + B(x+3) = 3x + 11$$
$$A = -2$$
$$B = 5$$

$$\int \frac{-2}{x+3} + \frac{5}{x+2} dx$$

Answer: $-2 \ln |x+3| + 5 \ln |x+2| + C$

2.10

$$\int \frac{x^3+6x^2+3x+6}{x^3+2x^2}dx$$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx = \int \frac{x^3+2x^2}{x^3+2x^2} + \frac{4x^2+3x+6}{x^3+2x^2}dx$$

$$= x + \int \frac{4x^2+3x+6}{x^3+2x^2}dx$$

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2} = \frac{4x^2+3x+6}{x^3+2x^2}$$

$$A(x)(x+2) + B(x+2) + C(x^2) = 4x^2+3x+6$$

$$A = 0$$

$$B = 3$$

$$C = 4$$

$$x + \int \frac{3}{x^2} + \frac{4}{x+2}dx$$

Answer: $x - \frac{3}{x} + 4\ln|x+2| + C$

2.11

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx$$

$$\int \frac{2x^2+5x-1}{x^3+x^2-2x}dx = \int \frac{2x^2+5x-1}{x(x+2)(x-1)}$$

$$\frac{A}{x} + \frac{B}{x+2}\frac{C}{x-1} = \frac{2x^2+5x-1}{x(x+2)(x-1)}$$

$$A(x+2)(x-1) + B(x)(x-1) + C(x)(x+2) = 2x^2+5x-1$$

$$A = \frac{1}{2}$$

$$B = \frac{-1}{2}$$

$$C = 2$$

$$\int \frac{1/2}{x} - \frac{1/2}{x+2} + \frac{2}{x-1}dx$$

Answer: $\frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + 2\ln|x-1| + C$

2.12

$$\int \frac{3x+6}{x^3+2x^2-3x}dx$$

$$\int \frac{3x+6}{x^3+2x^2-3x}dx = \int \frac{3x+6}{x(x+3)(x-1)}dx$$

$$\frac{A}{x} + \frac{B}{x+3} + \frac{C}{x-1} = \frac{3x+6}{x(x+3)(x-1)}$$

$$A(x+3)(x-1) + Bx(x-1) + Cx(x+3 = 3x+6$$

$$A = -2$$
$$B = \frac{-1}{4}$$
$$C = \frac{9}{4}$$

$$\int \frac{-2}{x} - \frac{1/4}{x+3} + \frac{9/4}{x-1}dx$$

Answer: $-2\ln|x| - \frac{1}{4}\ln|x+3| + \frac{9}{}]\ln|x-1| + C$

2.13

$$\int \frac{6x^2-10x+2}{x^3-3x^2+2x}dx$$

$$\int \frac{6x^2-10x+2}{x^3-3x^2+2x}dx = \int \frac{6x^2-10x+2}{x(x-2)(x-1)}$$

$$\frac{A}{x} + \frac{B}{x-2} + \frac{C}{x-1} = \frac{6x^2-10x+2}{x(x-2)(x-1)}$$

$$A(x-2)(x-1) + Bx(x-1) + Cx(x-2) = 6x^2 - 10x + 2$$

$$A = 1$$
$$B = 3$$
$$C = 2$$

$$\int \frac{1}{x} + \frac{3}{x-2} + \frac{2}{x-1}dx$$

Answer: $\ln|x| + 3\ln|x-2| + 2\ln|x-1| + C$

2.14

$$\int \frac{x^2-3}{x^3+x}dx$$

$$\int \frac{x^2-3}{x^3+x}dx = \int \frac{x^2-3}{x(x^2+1)}dx$$

$$\frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{x^2-3}{x(x^2+1)}$$

$$A(x^2+1) + (Bx+C)x = x^2-3$$

$$A = -3$$
$$B = 4$$
$$C = 0$$

$$\int \frac{-3}{x} + \frac{4x}{x^2+1}dx$$

Answer: $-3\ln|x| + 2\ln|x^2+1| + C$

2.15

$$\int \frac{3x^2-6x+9}{(x^2+9)(x-3)}dx$$

$$\frac{Ax+B}{x^2+9} + \frac{C}{x-3} = \frac{6x^2-10x+2}{x^3-3x^2+2x}$$

$$(Ax+B)(x-3) + C(x^2+9) = x^3-3x^2+2x$$

$$A = 2$$
$$B = 0$$
$$C = 1$$

$$\int \frac{2x}{x^2+9} + \frac{1}{x-3}$$

Answer: $\ln|x^2+9| + \ln|x-3|$

3 Quiz 4 Review

3.1 Let $p(x) = \frac{x^2}{9}$ for $0 \leq x \leq 3$.

3.1.1 Is $p(x)$ a valid probability density function?

To verify this, we need to make sure that the result of the function integrated from 0 to 3 is 1.

$$\int_0^3 \frac{x^2}{9} dx = \frac{1}{9}\frac{1}{3}x^3\Big|_0^3 = \frac{1}{9}\frac{1}{3}27 = 1$$

Answer: yes.

3.1.2 Find $P(1 \leq X \leq 2)$.

Answer: $\frac{7}{27}$

3.1.3 Find the mean μ.

The formula for mean of a PDF is $\mu = \int f(x)xdx$.

$$\int_0^3 \frac{x^2}{9} xdx = \frac{1}{9}\frac{1}{4}x^4\Big|_0^3 = \frac{1}{9}\frac{1}{4}81$$

Answer: $\frac{9}{4}$

3.2 Consider a random variable X with probability density function $p(x) = k\sqrt{x+10}$ on the interval $-6 \leq x \leq 6$.

3.2.1 For what value(s) of k is $p(x)$ a valid probability density function?

To find the value of K, we need to set the probability density function equal to 1.

$$\int_{-6}^{6} K\sqrt{x+10}dx = 1$$

Let $u = x + 10$. Then $du = dx$.

$$\int K\sqrt{u}du = K(\frac{2}{3}u^{3/2}) = K(\frac{2}{3}(x+10)^{3/2})\Big|_{-6}^{6} = \frac{2}{3}K(16^{3/2} - 4^{3/2})$$

$$= \frac{2}{3}K(64-8) = \frac{2}{3}K(56) = 1$$

Answer: $K = \frac{3}{112}$

3.2.2 Compute $P(X \leq 0)$.

$$\int_{-6}^{0} \frac{3}{112}\sqrt{x+10}dx = \frac{3}{112}\frac{2}{3}(x+10)^{3/2}\Big|_{-6}^{0} = \frac{1}{56}(10^{3/2}-8)$$

Answer: 0.421835

3.2.3 Find the mean μ.

$$\mu = \int_{-6}^{6} \frac{3}{112}x\sqrt{x+10}dx$$

Let $u = x+10$. Then $du = dx$ and $x = u - 10$.

$$\frac{3}{112}\int \sqrt{u}(u-10)du = \frac{3}{112}\int u^{3/2} - 10u^{1/2}du = \frac{3}{112}(\frac{2}{5}u^{5/2} - 10\frac{2}{3}u^{3/2})$$

$$= \frac{3}{112}(\frac{2}{5}(x+10)^{5/2} - \frac{20}{3}(x+10)^{3/2})\Big|_{-6}^{6}$$

Answer: $\frac{22}{35}$

3.3 For any positive value β the function $p(x) = (\beta+1)(\beta+2)x^{\beta}(1-x)$ is a probability density function on the interval $0 \leq x \leq 1$. Find the mean in terms of β.

$$\begin{aligned}
1 &= \int_0^1 x(\beta+1)(\beta+2)x^{\beta}(1-x)dx \\
\frac{1}{(\beta+1)(\beta+2)} &= \int_0^1 x^{\beta+1} - x^{\beta+2}dx \\
\frac{1}{(\beta+1)(\beta+2)} &= \frac{x^{\beta+2}}{\beta+2} - \frac{x^{\beta+3}}{\beta+3}\Big|_0^1 \\
\frac{1}{(\beta+1)(\beta+2)} &= \frac{1^{\beta+2}}{\beta+2} - \frac{1^{\beta+3}}{\beta+3} \\
\frac{1}{(\beta+1)(\beta+2)} &= \frac{1}{\beta+2} - \frac{1}{\beta+3} \\
\frac{1}{(\beta+1)(\beta+2)} &= \frac{(\beta+3)-(\beta+2)}{(\beta+2)(\beta+3)} \\
\frac{1}{(\beta+1)(\beta+2)} &= \frac{1}{(\beta+2)(\beta+3)} \\
\frac{1}{(\beta+1)} &= \frac{1}{(\beta+3)} \\
1 &= \frac{\beta+1}{\beta+3}
\end{aligned}$$

Answer: $\frac{\beta+1}{\beta+3}$

3.4 Find the arc length of the curve $y = \ln \cos x$ over the interval $[0, \frac{\pi}{4}]$.

$$f(x) = \ln \cos x$$
$$\frac{dy}{dx} = \frac{1}{\cos x - \sin x} = -\tan x$$
$$(\frac{dy}{dx})^2 = \tan^2 x$$

$$S = \int_0^{\pi/4} \sqrt{1 + \tan^2 x} dx = \int_0^{\pi/4} \sqrt{\sec^2 x} dx$$

$$= \int_0^{\pi/4} \sec x dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4}$$

Answer: $\ln|\sqrt{2} + 1|$

3.5 Find the arc length of the curve $y = \frac{1}{4}x^{3/2}$ over the interval $[0, 4]$.

$$f(x) = \frac{1}{4}x^{3/2}$$
$$\frac{dy}{dx} = \frac{3}{8}x^{1/2}$$
$$(\frac{dy}{dx})^2 = \frac{9}{64}x$$

$$\int_0^4 \sqrt{1 + \frac{9}{64}x} dx$$

Let $u = 1 + \frac{9}{64}x$. Then $du = \frac{9}{64}dx$ and $dx = \frac{64}{9}du$

$$S = \int \sqrt{u} du = \frac{64}{9} * \frac{2}{3} * u^{3/2} = \frac{128}{27}(1 + \frac{9}{64}x)^{3/2}\Big|_0^4$$

Answer: $\frac{122}{27}$

3.6 Find the arc length of the curve $y = \frac{1}{3}x^3 + \frac{1}{4x}$ over the interval $[1, 3]$.

$$f(x) = \frac{1}{3}x^3 + \frac{1}{4}x$$
$$\frac{dy}{dx} = x^2 - \frac{1}{4}x^{-2}$$
$$(\frac{dy}{dx})^2 = (x^2 - \frac{1}{4}x^{-2})^2 = x^4 - \frac{1}{4} - \frac{1}{4} + \frac{1}{16}x^4 = x^4 - \frac{1}{2} + \frac{1}{16}x^4$$

$$S = \int_1^3 \sqrt{1 + x^4 - \frac{1}{2} + \frac{1}{16}x^4}dx = \int_1^3 \sqrt{\frac{1}{2} + x^4 + \frac{1}{16}x^4}dx$$

$$= \int_1^3 \sqrt{(x^2 + \frac{1}{4x^2})^2}dx$$
$$= \int_1^3 x^2 + \frac{1}{4x^2}dx = \frac{x^3}{3} - \frac{1}{4x}\Big|_1^3$$

Answer: $\frac{53}{6}$

3.7 Find the surface area generated by rotating $y = 7x$ from $x = 0$ and $x = 1$ about the x-axis.

$$f(x) = 7x$$
$$\frac{dy}{dx} = 7$$
$$(\frac{dy}{dx})^2 = 49$$

$$\int_0^2 2\pi(7x)\sqrt{1 + 49}dx = 14\pi\sqrt{5}\int_0^1 xdx = 14\pi\sqrt{5}(\frac{x^2}{2})\Big|_0^1$$

Answer: $35\sqrt{2}\pi$

3.8 Find the surface area generated by rotating $y = \sqrt{x}$ from $x = 0$ and $x = 2$ about the x-axis.

$$f(x) = \sqrt{x}$$
$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$
$$(\frac{dy}{dx})^2 = \frac{1}{4x}$$

$$\int_0^2 2\pi\sqrt{x}\sqrt{1+\frac{1}{4x}}dx = 2\pi\int_0^2 \sqrt{x+\frac{1}{4}}dx$$

Let $u = x + \frac{1}{4}$. Then $du = dx$.

$$2\pi\int\sqrt{u}du = 2\pi\frac{2}{3}u^{3/2} = \frac{4\pi}{3}(x+\frac{1}{4})^{3/2}\Big|_0^2$$

Answer: $\frac{13\pi}{3}$

3.9 Find the surface area generated by rotating $y = \sqrt{x} - \frac{1}{3}x^{3/2}$ from $x = 1$ and $x = 3$ about the x-axis.

$$f(x) = \sqrt{x} - \frac{1}{3}x^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2}$$

$$(\frac{dy}{dx})^2 = (\frac{1}{2}x^{-1/2} - \frac{1}{2}x^{1/2})^2 = \frac{1}{4x} - \frac{1}{2} + \frac{x}{4}$$

$$\int_1^3 2\pi(\sqrt{x} - \frac{1}{3}x^{3/2})\sqrt{\frac{1}{4x} - \frac{1}{2} + \frac{x}{4} + 1}dx$$

$$= \int_1^3 2\pi(\sqrt{x} - \frac{1}{3}x^{3/2})\sqrt{\frac{1}{4x} + \frac{1}{2} + \frac{x}{4}}dx$$

$$= \int_1^3 2\pi(\sqrt{x} - \frac{1}{3}x^{3/2})\sqrt{(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2})^2}dx$$

$$= \int_1^3 2\pi(\sqrt{x} - \frac{1}{3}x^{3/2})(\frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{2})dx$$

$$= 2\pi\int_1^3 \frac{1}{2} + \frac{x}{2} - \frac{x}{6} - \frac{x^2}{6}dx = 2\pi(\frac{x}{2} + \frac{x^2}{4} - \frac{x^2}{12} - \frac{x^3}{18})\Big|_1^3$$

Answer: $\frac{16\pi}{9}$

3.10 Find the surface area generated by rotating $y = x^3$ from $x = 0$ and $x = 1$ about the x-axis.

$$f(x) = x^3$$
$$\frac{dy}{dx} = 3x^2$$
$$(\frac{dy}{dx})^2 = 9x^4$$

$$\int 2\pi x^3 \sqrt{1 + 9x^4}$$

Let $u = 1 + 9x^4$. Then $du = 36x^3 dx$ and $dx = \frac{du}{36x^3}$

$$\frac{2\pi}{36} \int \sqrt{u} du = \frac{2\pi}{36}(\frac{2}{3}u^{3/2} = \frac{2\pi}{36}(\frac{2}{3}(1 + 9x^4)^{3/2}\Big|_0^1$$

Answer: $\frac{\pi}{27}(10^{3/2} - 1)$

3.11 Find the surface area generated by rotating $y = \sqrt{4 - x^2}$ from $x = -1$ and $x = 1$ about the x-axis.

$$f(x) = \sqrt{4 - x^2}$$
$$\frac{dy}{dx} = \frac{1}{2}(4 - x^2)^{-1/2}(-2x) = -x(4 - x^2)^{-1/2}$$
$$(\frac{dy}{dx})^2 = x^2(4 - x^2)^{-1}$$

$$2\pi \int_{-1}^{1} \sqrt{4 - x^2}\sqrt{1 + x^2(4 - x^2)^{-1}} dx = 2\pi \int_{-1}^{1} \sqrt{4 - x^2}\sqrt{\frac{(4 - x^2) + x^2}{4 - x^2}} dx$$

$$2\pi \int_{-1}^{1} \sqrt{4 - x^2}\frac{\sqrt{4}}{\sqrt{4 - x^2}} dx = 2\pi \int_{-1}^{1} 2dx = 2\pi(2x)\Big|_{-1}^{1}$$

Answer: 8π

4 Quiz 5 Review

4.1 The vertical wall on the end of a swimming pool is 20 ft wide and 8 ft high. If water in the swimming pool is filled to a height of 7 ft, find the force exerted on the wall by the water ($\rho g = 62.4\text{lb/ft}^3$).

The fluid force equation is:

$$\text{FF} = \int_a^b \rho g h(x) w(x) dx$$

We gather the following information from Figure 1:

$$a = 0 \quad b = 7 \quad h(x) = x \quad w(x) = 20$$

$$\text{FF} = \int_0^7 \rho g(x)(20) dx = 10x^2 \Big|_0^7 = (62.4)(490) = 3,057,616$$

Answer: 3,057,616

4.2 The vertical gate of a damn has the shape of a trapezoid, 12 ft at the top, 8 ft at the bottom and 4 ft high. What is the force on the gate when the surface of the water is 2 ft above the top of the gate?

We first split the trapezoid into it's corresponding rectangle and triangle components as shown in Figure 2.

We gather the following information:

$$a = 0 \quad b = 4 \quad h_1(x) = 2 + x \quad w_1(x) = 8 \quad h_2(x) = 6 - x \quad w_2(x) = x$$

$$\text{FF} = \rho g \Bigg[\int_0^4 (2 + x) 8 dx + \int_0^4 (6 - x) x dx \Bigg]$$

$$= \rho g \Bigg[8 \bigg(2x + \frac{x^2}{2} \bigg) \Big|_0^4 + \bigg(3x^2 - \frac{x^3}{3} \bigg) \Big|_0^4 \Bigg]$$

$$= \rho g \Bigg[8 \bigg(8 + 8 \bigg) + \bigg(48 - \frac{64}{3} \bigg) \Bigg] = \rho g \bigg(\frac{464}{3} \bigg) = 9,651.2$$

Answer: 9,651.2

4.3 The viewing port of a submarine is a circle of radius 1 ft. If the center of the viewing port is 100 ft below the surface, find the force exerted by the water on it.

We gather the following information from Figure 3:

$$a = -1 \quad b = 1 \quad h(x) = 100 + x \quad w(x) = 2\sqrt{1 - x^2}$$

$$\text{FF} = \int_{-1}^{1} \rho g(100 + x)2\sqrt{1 - x^2}dx = 2\rho g \int_{-1}^{1} (100 + x)\sqrt{1 - x^2}dx$$

$$= 2\rho g\left[\int_{-1}^{1} 100\sqrt{1 - x^2}dx + \int_{-1}^{1} x\sqrt{1 - x^2}dx\right]$$

For the first integral, we can look at the area directly and compute the area using the formula $A = \pi r^2$ for a circle. For the second integral, let $u = 1 - x^2$, $du = -2dx$, and $dx = \frac{du}{-2x}$. After substitution, we have:

$$= 2\rho g\left[100\frac{\pi}{2} + \int x\sqrt{u}\frac{du}{-2x}\right] = 2\rho g\left[50\pi - \frac{1}{2}\int \sqrt{u}du\right]$$

$$= 2\rho g\left[50\pi - \frac{1}{2}\frac{2}{3}u^{3/2}\right] = 2\rho g\left[50\pi - \frac{1}{2}\left(\frac{2}{3}(1 - x^2)^{3/2}\right)\Bigg|_{-1}^{1}\right]$$

$$= 2\rho g(50\pi - 0) = 100\rho g\pi = 6,240\pi = 19,604$$

Answer: 19,604

4.4 Find the force on a vertical flat plate in the form of a semicircle 5 meters in radius that is submerged in water ($\rho g = 9810\text{N/m}^3$).

We gather the following information from Figure 4:

$$a = 0 \quad b = 3 \quad h(x) = x \quad w(x) = 2\sqrt{25 - x^2}$$

$$\text{FF} = \int_{0}^{3} \rho g(x)(2\sqrt{25 - x^2})dx$$

Let $u = 25 - x^2, du - -2xdx$, and $dx = \frac{du}{-2x}$.

$$= 2\rho g \int x\sqrt{u}\frac{du}{-2x} = -\rho g \int \sqrt{u}du = -\rho g\frac{2}{3}u^{3/2}$$

$$= -\rho g\frac{2}{3}(25 - x^2)^{3/2}\Bigg|_{0}^{3} = 817,500$$

Answer: 817,500

Find the center of mass for:

4.5 The region bounded by $f(x) = x^2$ and the x-axis for $[0, 2]$.

$$m_y = \int x f(x) dx = \int_0^2 x(x^2) dx = \left.\frac{x^4}{4}\right|_0^2 = 4$$

$$m_x = \int \frac{f(x)^2}{2} dx = \int_0^2 \frac{x^4}{2} dx = \left.\frac{x^5}{10}\right|_0^2 = \frac{16}{5}$$

$$m = \int f(x) dx = \int_0^2 x^2 dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{8}{3}$$

$$\bar{y} = \frac{m_x}{m} = \frac{16/5}{8/3} = \frac{6}{5}$$

$$\bar{x} = \frac{m_y}{m} = \frac{4}{8/3} = \frac{3}{2}$$

Answer: $\left(\frac{3}{2}, \frac{6}{5}\right)$

4.6 The region bounded by $f(x) = \sqrt{x}$ and the x-axis for $[0, 4]$.

$$m_y = \int x f(x) dx = \int_0^4 x\sqrt{x} dx = \int_0^4 x^{3/2} dx$$

$$= \left.\frac{2}{5} x^{5/2}\right|_0^4 = \frac{2}{5}(32) = \frac{64}{5}$$

$$m_x = \int \frac{f(x)^2}{2} dx = \int_0^4 \frac{x}{2} dx = \left.\frac{x^2}{4}\right|_0^4 = 4$$

$$m = \int f(x) dx = \int_0^4 \sqrt{x} dx = \left.\frac{2}{3} x^{3/2}\right|_0^4 = \frac{2}{3}(8) = \frac{16}{3}$$

$$\bar{x} = \frac{m_y}{m} = \frac{64/5}{16/3} = \frac{12}{5}$$

$$\bar{y} = \frac{4}{16/3} = \frac{3}{4}$$

Answer: $\left(\frac{12}{5}, \frac{3}{4}\right)$

4.7 The region bounded by $f(x) = 2x - x^2$ and the x-axis.

See Figure 5.

$$m_y = \int xf(x)dx = \int_0^2 x(2x - x^2)dx = \int_0^2 2x^2 - x^3 dx = \frac{2}{3}x^3 - \frac{1}{4}x^4\bigg|_0^2$$

$$= \frac{2}{3}(8) - \frac{1}{4}(16) = \frac{16}{3} - 4 = \frac{4}{3}$$

$$m_x = \int \frac{f(x)^2}{2}dx = \int_0^2 \frac{(2x - x^2)^2}{2}dx = \frac{1}{2}\int_0^2 4x^2 - 4x^3 + x^4 dx$$

$$= \frac{1}{2}\left[\frac{4}{3}x^3 - x^4 + \frac{1}{3}x^3\right]\bigg|_0^2 = \frac{1}{2}\left[\frac{4}{3}(8) - 16 + \frac{1}{3}(32)\right] = \frac{8}{15}$$

$$m = \int f(x)dx = \int_0^2 2x - x^2 dx = x^2 - \frac{1}{3}x^3\bigg|_0^2 = 4 - \frac{1}{3}(8) = \frac{4}{3}$$

$$\bar{y} = \frac{m_x}{m} = \frac{8/15}{4/3} = \frac{2}{5}$$

$$\bar{x} = \frac{m_y}{m} = \frac{4/3}{4/3} = 1$$

Answer: $\left(1, \frac{2}{5}\right)$

4.8 The region bounded by $f(x) = x^2 - 3$ and $g(x) = -x^2 + 2x + 1$.

See Figure 6.

$$x^2 - 3 = -x^2 + 2x + 1 \implies 2x^2 - 2x - 4 = 0$$

$$\implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0 \implies x = -1, 2$$

Let $f(x) = x^2 - 3$ and $g(x) = -x^2 + 2x + 1$.

$$m_y = \int x(g(x) - f(x))dx = \int_{-1}^2 x(-x^2 + 2x + 1 - (x^2 - 3))dx$$

$$= \int_{-1}^2 x(-2x^2 + 2x + 4)dx = \int_{-1}^2 -2x^3 + 2x^2 + 4x dx$$

$$= \frac{-2}{4}x^4 + \frac{2}{3}x^3 + 2x^2\Big|_{-1}^{2} = \Big(-8 + \frac{16}{3} + 8\Big) - \Big(-\frac{1}{2} - \frac{2}{3} + 2\Big)$$

$$= \frac{16}{3} - \frac{5}{6} = \frac{9}{2}$$

$$m_x = \int \frac{[g(x)^2 - f(x)^2]}{2}dx = \frac{1}{2}\int_{-1}^{2}(-x^2 + 2x + 1)^2 - (x^2 - 3)^2 dx$$

$$= \frac{1}{2}\int_{-1}^{2}(x^4 - 4x^3 + 2x^2 + 4x + 1) - (x^4 - 6x^2 + 9)dx$$

$$= \frac{1}{2}\int_{-1}^{2} -4x^3 + 8x^2 + 4x - 8dx = \frac{1}{2}\Big[-x^4 + \frac{8}{3}x^3 + 2x^2 - 8x\Big]\Big|_{-1}^{2}$$

$$= \frac{1}{2}\Big[\Big(-16 + \frac{64}{3} + 8 - 16\Big) - \Big(-1 - \frac{8}{3} + 2 + 8\Big)\Big]$$

$$= \frac{1}{2}\Big[-\frac{8}{3} - \frac{19}{3}\Big] = -\frac{9}{2}$$

$$m = \int g(x) - f(x)dx = \int_{-1}^{2} -x^2 + 2x + 1 - (x^2 - 3)dx$$

$$= \int_{-1}^{2} -2x^2 + 2x + 4dx = -\frac{2}{3}x^3 + x^2 + 4x\Big|_{-1}^{2}$$

$$= \Big[-\frac{2}{3}(8) + 4 + 8\Big] - \Big[\frac{2}{3} + 1 - 4\Big] = -\frac{16}{3} + 12 - \frac{2}{3} + 3 = 9$$

$$\bar{y} = \frac{m_x}{m} = \frac{-9/2}{9} = -\frac{1}{2}$$

$$\bar{x} = \frac{m_y}{m} = \frac{9/2}{9} = \frac{1}{2}$$

Answer: $\left(\frac{1}{2}, -\frac{1}{2}\right)$

4.9 The region bounded by $\frac{x}{a} + \frac{y}{b} = 1$, the x-axis, and the y-axis.

See Figure 7. We can rearrange to solve for y.

$$y = \frac{-xb}{a} + b = f(x)$$

$$m_y = \int xf(x)dx = \int_0^a \left(\frac{-xb}{a} + b\right)dx = b\int_0^a \left(-\frac{x^2}{a} + x\right)dx$$

$$= b\left[-\frac{x^3}{3a} + \frac{x^2}{2}\right]\Bigg|_0^a = b\left[-\frac{a^3}{3a} + \frac{a^2}{2}\right]\Bigg|_0^a = b\left[-\frac{a^2}{3} + \frac{a^2}{2}\right]$$

$$= b\left[\frac{a^2}{6}\right] = \frac{a^2b}{6}$$

$$m_x = \int \frac{f(x)^2}{2}dx = \frac{1}{2}\int_0^a \left(\frac{-xb}{a} + b\right)^2 dx = \frac{1}{2}\int_0^a \left(\frac{x^2b^2}{a^2} - 2\frac{xb^2}{a} + b^2\right)dx$$

$$= \frac{1}{2}\left[\frac{x^3b^2}{3a^2} - \frac{x^2b^2}{a} + b^2x\right]\Bigg|_0^a = \frac{1}{2}\left[\frac{a^3b^2}{3a^2} - \frac{a^2b^2}{a} + b^2a\right]$$

$$= \frac{1}{2}\left[\frac{ab^2}{3} - ab^2 + ab^2\right] = \frac{ab^2}{6}$$

$$m = \int f(x)dx = \int_0^a \left(\frac{-xb}{a} + b\right)dx = \left[\frac{-x^2b}{2a} + bx\right]\Bigg|_0^a$$

$$= \frac{-a^2b}{2a} + ab = -\frac{ab}{2} + ab = \frac{ab}{2}$$

$$\bar{x} = \frac{m_y}{m} = \frac{a^2b/6}{ab/2} = \frac{a}{3}$$

$$\bar{y} = \frac{m_x}{m} = \frac{ab^2/6}{ab/2} = \frac{b}{3}$$

Answer: $\left(\frac{a}{3}, \frac{b}{3}\right)$

4.10 Verify that $y = \frac{x^4}{16}$ is a solution of the differential equation

$$\frac{dy}{dx} = xy^{1/2}$$

$$\int y^{-1/2} dy = \int x dx$$

$$2y^{1/2} + C = \frac{x^2}{2} + C$$

$$y^{1/2} = \frac{x^2}{4} + C$$

$$y = \frac{x^4}{16} + C$$

4.11 Verify that $y = x^2+2x+2+Ce^x$ is a solution of the differential equation

$$y' - y + x^2 = 0$$

First find y'.

$$y' = 2x + 2 + Ce^x$$

Plug in.

$$(2x + 2 + Ce^x) - (x^2 + 2x + 2 + Ce^x) + x^2$$

$$= (-x^2 + x^2) + (2x - 2x) + (2 - 2) + (Ce^x - Ce^x) = 0$$

Find the general solution of:

4.12

$$\frac{dy}{dx} = (x + 1)^2$$

$$\int dy = \int (x + 1)^2 dx$$

$$y = \frac{1}{3}(x + 1)^3 + C$$

Answer: $y = \frac{1}{3}(x + 1)^3 + C$

4.13

$$y^2 y' = 3x^2$$

$$\int y^2 dy = \int 3x^2 dx$$

$$\frac{y^3}{3} = x^3 + C$$

$$y^3 = 3x^3 + C$$

$$y = \sqrt[3]{3x^3 + C}$$

Answer: $y = \sqrt[3]{3x^3 + C}$

4.14

$$y' = x^3 y^2 + y^2$$

$$\frac{dy}{dx} = y^2(x^3 + 1)$$

$$\int \frac{1}{y^2} dy = \int x^3 + 1 dx$$

$$-y^{-1} = \frac{x^4}{4} + x + C$$

$$y^{-1} = -\frac{x^4}{4} - x + C = -\frac{-x^4 - 4x + C}{4}$$

$$y = -\frac{4}{x^4 + 4x + C}$$

Answer: $y = -\frac{4}{x^4+4x+C}$

4.15

$$y' = 5 - 2y$$

$$\frac{dy}{dx} = 5 - 2y$$

$$\int \frac{1}{5-2y} dy = \int dx$$

$$-\frac{1}{2} \ln |5 - 2y| = x + C$$

$$\ln |5 - 2y| = -2x + C$$

$$5 - 2y = e^{-2x+C}$$

$$-2y = Ce^{-2x} - 5$$

$$y = Ce^{-2x} + \frac{5}{2}$$

Answer: $y = Ce^{-2x} + \frac{5}{2}$

4.16

$$\frac{dy}{dx} = \frac{x}{y^2}$$

$$\int y^2 dy = \int x dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} + C$$

$$y^3 = \frac{3x^2}{2} + C$$

$$y = \sqrt[3]{\frac{3x^2}{2} + C}$$

Answer: $y = \sqrt[3]{\frac{3x^2}{2} + C}$

4.17

$$\frac{dy}{dx} = \frac{7}{y}$$

$$\int y dy = 7 \int dx$$

$$\frac{y^2}{2} = 7x + C$$

$$y^2 = 14x + C$$

$$y = \pm\sqrt{14x + C}$$

Answer: $y = \pm\sqrt{14x + C}$

4.18

$$x(y-1)y' = y$$

$$(y-1)\frac{dy}{dx}\frac{1}{y} = \frac{1}{x}$$

$$\int 1 - \frac{1}{y} dy = \int \frac{1}{x} dx$$

$$y - \ln y = \ln x + C$$

Answer: $y - \ln y = \ln x + C$

4.19 Solve $\frac{dy}{dx} = 1 + y \quad y(0) = 5.$

$$\int \frac{1}{1+y} dy = \int dx$$

$$\ln|1+y| = x + C$$

$$1 + y = e^{x+C}$$

$$1 + y = Ce^x$$

$$y = Ce^x - 1$$

$$5 = Ce^0 - 1$$

$$6 = C$$

$$y = 6e^x - 1$$

Answer: $y = 6e^x - 1$

4.20 Solve $\frac{dy}{dx} = y^{2/3} \quad y(0) = 8.$

$$\int y^{-2/3} dy = \int dx$$

$$3y^{1/3} = x + C$$

$$y^{1/3} = \frac{x}{3} + C$$

$$y = \left(\frac{x}{3} + C\right)^3$$

$$8 = \left(\frac{0}{3} + C\right)^3 = C^3$$

$$2 = C$$

$$y = \left(\frac{x}{3} + 2\right)^3$$

Answer: $y = \left(\frac{x}{3} + 2\right)^3$

4.21 Solve $\frac{dy}{dx} = y \quad y(0) = 3$.

$$\int \frac{1}{y} dy = \int dx$$

$$\ln y = x + C$$

$$y = Ce^x$$

$$3 = Ce^0$$

$$3 = C$$

$$y = 3e^x$$

Answer: $y = 3e^x$

5 Quiz 6 Review

5.1 A pie is taken out of the oven at a temperature of 200F and put in a room with a temperature of 70F. The temperature of the pie is 160F after 15 min.

5.1.1 What is the temperature of the pie after 30 min?

We gather the following information:

$$t(0) = 200 \quad T_0 = 70 \quad t(15) = 160$$

Now to find $y(30)$

$$y = T_0 + Ce^{kt}$$

$$y = 70 + Ce^{kt}$$

$$200 = 70 + Ce^{0}$$

$$130 = C$$

$$y = 70 + 130e^{kt}$$

$$160 = 70 + 130e^{15k}$$

$$\frac{\ln(9/13)}{15} = k$$

$$y = 70 + 130e^{\frac{\ln 9/13}{15}t}$$

$$y(30) = 70 + 130e^{\frac{\ln 9/13}{15}30}$$

$$y = 132.3$$

Answer: 132.3

5.1.2 When will the pie be 120F?

Now to find $y(?) = 120$

$$120 = 70 + 130e^{\frac{\ln 9/13}{15}t}$$

$$\frac{5}{13} = e^{\frac{\ln 9/13}{15}t}$$

$$\ln 5/13 = \frac{\ln 9/13}{15}t$$

$$15\left(\frac{\ln \frac{5}{13}}{\ln \frac{9}{13}}\right) = t = 38.9$$

Answer: $t = 38.9$

5.2 Henry hides his bagel in a refrigerator (temperature 32F). After 10 minutes, the bagel's temperature is 80F and after 20 minutes it is 44F. What is the temperature of the bagel when it was first put in the refrigerator?

We gather the following information:

$$T_0 = 32 \quad t(10) = 80 \quad t(20) = 44$$

Now to find $t(0)$:

$$y = T_0 + Ce^{kt}$$

$$y = 32 + Ce^{kt}$$

$$80 = 32 + Ce^{10k} \implies 48 = Ce^{10k}$$

$$44 = 32 + Ce^{20k} \implies 12 = Ce^{20k}$$

$$\frac{48 = Ce^{10k}}{12 = Ce^{20k}} \implies 4 = e^{-10k} \implies \ln 4 = -10k \implies -0.1386 = k$$

$$y = 32 + Ce^{-0.1386t}$$

$$80 = 32 + Ce^{-0.1386t(10)}$$

$$192 = C$$

$$y = 32 + 192e^{-0.1386t}$$

$$y(0) = 32 + 192e^{0}$$

$$y = 224$$

Answer: $y = 224$ at $t = 0$

5.3 Sketch the slope field for $\frac{dy}{dx} = x(6 - y)$ and draw the solution that goes through $(0, 0)$.

See Figure X.

5.4 Sketch the slope field for $\frac{dy}{dx} = xy$ and draw the solution that goes through $(0, 1)$.

See Figure X. Use Euler's method:

5.5

$$\frac{dy}{dx} = y \quad y(0) = 1 \text{ find } y(1) \text{ with } h = 0.1$$

See Table 1.

Answer: $y(1) = 2.593$

x	y	$\frac{dy}{dx}$	$h\frac{dy}{dx}$	$h\frac{dy}{dx} + y$
0	1	1	0.1	1.1
0.1	1.1	1.1	0.11	1.21
0.2	1.21	1.21	0.121	1.331
0.3	1.331	1.331	0.1331	1.4641
0.4	1.4641	1.4641	0.14641	1.61051
0.5	1.61051	1.61051	0.161051	1.77156
0.6	1.77156	1.77156	0.177156	1.9487
0.7	1.9487	1.9487	0.19487	2.1435
0.8	2.1435	2.1435	0.21435	2.3579
0.9	2.3579	2.3579	0.23579	2.593
1	2.593			

Table 1: Table for 5.5

5.6

$$\frac{dy}{dx} = 2y - 1 \quad y(0) = 1 \text{ find } y(1) \text{ with } h = 0.1$$

See Table 2.

Answer: $y(1) = 3.595$

x	y	$\frac{dy}{dx}$	$h\frac{dy}{dx}$	$h\frac{dy}{dx} + y$
0	1	1	0.1	1.1
0.1	1.1	1.2	0.12	1.22
0.2	1.22	1.44	0.144	1.364
0.3	1.364	1.728	0.1728	1.5368
0.4	1.5368	2.0736	0.20736	1.74468
0.5	1.74416	2.48832	0.248832	1.9929
0.6	1.9929	2.985	0.2985	2.291
0.7	2.291	3.583	0.3583	2.649
0.8	2.6499	4.299	0.4299	3.079
0.9	3.0798	5.159	0.5159	3.595
1	3.595			

Table 2: Table for 5.6

5.7

$$\text{Solve } y' = 1.5y\left(1 - \frac{y}{4}\right) \quad y(0) = 1$$

$$B = \frac{y_0}{y_0 - A} = \frac{1}{1-4} = -\frac{1}{3}$$

$$y = \frac{4}{1 + 3e^{-1.5t}}$$

Answer: $y = \frac{4}{1+3e^{-1.5t}}$

5.8 In one of the dorms there are 1000 students. After fall break, 20 students return with the flu and 5 days later, 35 students have the flu. If the number of students with the flu follows the logistic model, how many students will have the flu after 2 weeks (14 days)?

We gather the following information:

$$A = 1000 \quad y(0) = 20 \quad y(5) = 35$$

Now to find $y(14)$:

$$B = \frac{y_0}{y_0 - A} = \frac{20}{20 - 1000} = -\frac{1}{49}$$

$$y = \frac{1000}{1 = 49e^{kt}}$$

$$35 = \frac{1000}{1 + 49e^{5k}}$$

$$35 + 1715e^{5k} = 1000$$

$$\frac{\ln 193/343}{5} = k$$

$$y = \frac{1000}{1 + 49e^{\frac{\ln 193/343}{5}t}}$$

$$y(14) = \frac{1000}{1 + 49e^{\frac{\ln 193/343}{5}(14)}}$$

$$y = 92.649 = 93$$

Answer: 93

5.9 A fish farm is stocked with 100 fish. Suppose that the fish population satisfies the logistic equation and that the carrying capacity of the pond is 2000. If after 1 year the population has increased to 250,

5.9.1 find an equation for the number of fish after t years.

We gather the following information:

$$y(0) = 100 \quad y(1) = 250 \quad A = 2000$$

Now to find $y(t)$:

$$B = \frac{y_0}{y_0 - A} = \frac{100}{100 - 2000} = -\frac{1}{19}$$

$$y = \frac{2000}{1 + 19e^{kt}}$$

$$250 = \frac{2000}{1 + 19e^k}$$

$$250 + 4750e^k = 2000$$

$$e^k = \frac{7}{19}$$

$$k = \ln 7/19$$

$$y = \frac{2000}{1 + 18e^{\ln(7/19)t}} = \frac{2000}{1 + 19e^{-0.9985t}}$$

Answer: $y = \frac{2000}{1+18e^{\ln(7/19)t}} = \frac{2000}{1+19e^{-0.9985t}}$

5.9.2 How long will it take for the fish population to reach 1000?

Now to find $y(t) = 1000$:

$$1000 = \frac{2000}{1 + e^{\ln(7/19)t}}$$

$$1000 + 19000e^{\ln(7/19)t} = 2000$$

$$e^{\ln(7/19)t} = \frac{1}{19}$$

$$\ln(7/19)t = \ln(1/19)$$

$$t = 2.948$$

Answer: $t = 2.948$

6 Quiz 7 Review

Determine if the following sequences converge or diverge. If it converges, find the limit.

6.1

$$\left\{\frac{2n-1}{3n^2+1}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty}\frac{2n-1}{3n^2+1}=0$$

Answer: 0

6.2

$$\left\{\frac{-9+(-1)^n}{n!}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty}\frac{-9+(-1)^n}{n!}=0$$

Answer: 0

6.3

$$\left\{\left(\frac{n+1}{n}\right)^n\right\}_{n=1}^{\infty}$$

$$y=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$$

$$\ln y=\lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n=\lim_{n\to\infty}n\ln\left(1+\frac{1}{n}\right)=\infty\times 0$$

$$\ln y=\lim_{n\to\infty}\frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}}=\frac{0}{0}$$

Use L'Hopital's Rule:

$$\ln y=\lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)\left(-\frac{1}{n^2}\right)\left(\frac{1}{\frac{-1}{n^2}}\right)=1\implies y=e$$

Answer: e

6.4

$$\left\{\frac{2n-\sqrt{n}}{n}\right\}_{n=1}^{\infty}$$

$$\lim_{n\to\infty}\frac{2n-\sqrt{n}}{n}=\lim_{n\to\infty}\left(\frac{2n}{n}-\frac{\sqrt{n}}{n}\right)=2$$

Answer: 2

Determine if the following sequences are increasing or decreasing.

6.5

$$\left\{7-\frac{1}{n^2}\right\}_{n=1}^{\infty}$$

Answer: Increasing.

6.6

$$\left\{\frac{2^n}{n!}\right\}_{n=4}^{\infty}$$

Answer: Decreasing.

6.7

$$\{ne^{-n}\}_{n=1}^{\infty}$$

Answer: Decreasing.

6.8

$$\{12\sin(3n)\}_{n=1}^{\infty}$$

Answer: Neither increasing nor decreasing.

6.9 Use the fact that $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ to find $\sum_{n=3}^{\infty}\frac{1}{n^2}$.

$$\sum_{n=3}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}-a_1-a_2=\frac{\pi^2}{6}-1-\frac{1}{4}=\frac{\pi^2}{6}-\frac{5}{4}$$

Answer: $\frac{\pi^2}{6}-\frac{5}{4}$

Determine if the following series converge or diverge, and if it converges, find the sum:

6.10

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \implies a = 1 \quad r = \frac{1}{3}$$

$$S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}$$

Answer: $\frac{3}{2}$

6.11

$$\sum_{n=1}^{\infty} 2(-0.9)^n$$

$$a = 2 \quad r = -0.9$$

$$\frac{2}{1 - (-0.9)} - a_0 = \frac{2}{1.9} - 2 = -\frac{18}{19}$$

Answer: $-\frac{18}{19}$

6.12

$$\sum_{n=1}^{\infty} \frac{n-6}{n}$$

$$\sum_{n=1}^{\infty} \frac{n-6}{n} = \infty \implies \text{Diverges}$$

Answer: Diverges

6.13

$$\sum_{n=1}^{\infty} \frac{1}{1+e^{-n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{1+e^{-n}} = 1 \implies \text{Diverges}$$

Answer: Diverges

6.14

$$\sum_{n=0}^{\infty} 0.5\left(-\frac{4}{3}\right)^n$$

$$a = 0.5 \quad r = -\frac{4}{3} \implies \text{Diverges}$$

Answer: Diverges

6.15

$$\sum_{n=0}^{\infty} \left(\frac{\sqrt{5}}{1+\sqrt{5}}\right)^n$$

$$a = 1 \quad r = \frac{\sqrt{t}}{1+\sqrt{t}}$$

$$\frac{1}{1-\frac{\sqrt{5}}{1+\sqrt{5}}} = \frac{1}{\frac{1+\sqrt{5}}{1+\sqrt{5}} - \frac{\sqrt{5}}{1+\sqrt{5}}} = \frac{1}{\frac{1}{1+\sqrt{5}}} = 1+\sqrt{5}$$

Answer: $1+\sqrt{5}$

6.16

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$= \left(1-\frac{1}{3}\right) + \left(\frac{1}{2}-\frac{1}{4}\right) + \left(\frac{1}{3}-\frac{1}{5}\right) + \cdots = \frac{3}{2}$$

Answer: $\frac{3}{2}$

6.17

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(n+3)(n+1)} = \sum_{n=1}^{\infty} \frac{A}{n+3} + \frac{B}{n+1}$$

$$A(n+1) + B(n+3) = 1 \implies B = \frac{1}{2} \quad A = \frac{-1}{2}$$

$$= \sum_{n=1}^{\infty} -\frac{1}{2(n+3)} + \frac{1}{2(n+1)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3}$$

$$= \frac{1}{2} \left[\left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + ... \right]$$

$$= \frac{1}{2} \left[\frac{1}{2} + \frac{1}{3} \right] = \frac{1}{2} \frac{5}{6} = \frac{5}{12}$$

Answer: $\frac{5}{12}$

6.18

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$= \sum_{n=1}^{\infty} \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn \implies A = \frac{1}{2} \quad B = \frac{-1}{2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2n} - \frac{1}{2(n+2)}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$$

$$= \frac{1}{2} \left[\left(\frac{1}{1} - \frac{1}{3} \right) + \left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \dots \right]$$

$$= \frac{1}{2} \left[1 + \frac{1}{2} \right] = \frac{3}{4}$$

Answer: $\frac{3}{4}$

7 Quiz 8 Review

Determine if the following series converge or diverge.

7.1

$$\sum_{n=2}^{\infty} \frac{(\ln n)^2}{n}$$

Let's use the integral test.

$$f(x) = \frac{(\ln x)^2}{x}$$

$$\int_2^{\infty} \frac{(\ln x)^2}{x} dx$$

Let $u = \ln x, du = \frac{1}{x}dx$, and $dx = xdu$.

$$\int \frac{u^2}{x} xdu = \int u^2 du = \frac{1}{3}u^3 = \frac{1}{3}(\ln x)^3 \Big|_2^{\infty} = \infty \implies \text{Diverges}$$

Answer: Diverges

7.2

$$\sum_{n=2}^{\infty} \frac{1}{n^2} \cos\left(\frac{1}{n}\right)$$

Let's use the integral test.

$$f(x) = \frac{1}{x^2} \cos\left(\frac{1}{x}\right)$$

$$\int_1^{\infty} \frac{1}{x^2} \cos\left(\frac{1}{x}\right) dx$$

Let $u = \frac{1}{x}, du = -\frac{1}{x^2}dx$, and $dx = -x^2 du$.

$$\int -\cos u du = -\sin u = -\sin \frac{1}{x} \Big|_1^{\infty}$$

$$= -\sin 0 - \Big(-\sin 1 \Big) = sin1 \implies \text{Converges}$$

Answer: Converges

7.3

$$\sum_{n=1}^{\infty} \frac{1}{n^{-2}}$$

Let's use the divergence test.

$$\lim_{n\to\infty} n^2 = \infty^2 \implies \text{Diverges}$$

Answer: Diverges

7.4

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{n} + \sqrt[3]{n}}$$

Let's use the limit comparison test with $\sum \frac{1}{n}$

$$\lim_{n\to\infty} \frac{\frac{1}{2\sqrt{n}+\sqrt[3]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{2\sqrt{n} + \sqrt[3]{n}} > 0$$

$$\frac{1}{n} \text{ diverges because it is the harmonic series, so } \frac{1}{2\sqrt{n} + \sqrt[3]{n}} \text{ diverges}$$

Answer: diverges

7.5

$$\sum_{n=1}^{\infty} \frac{1 + \cos n}{n^2}$$

Let's use the direct comparison test with $\sum \frac{2}{n^2}$

$$\frac{1 + \cos n}{n^2} < \frac{2}{n^2}$$

$$\frac{2}{n^2} \text{ converges by the p-test, so } \frac{1 + \cos n}{n^2} \text{ converges}$$

Answer: converges

7.6

$$\sum_{n=1}^{\infty} \frac{1}{\ln(\ln n)}$$

Let's use the direct comparison test with $\sum \frac{1}{\ln n} and \sum \frac{1}{n}$

$$\frac{1}{\ln(\ln n)} > \frac{1}{\ln n} > \frac{1}{n}$$

$$\frac{1}{n} \text{ diverges because it is the harmonic series, so } \frac{1}{\ln(\ln n)} \text{ diverges}$$

Answer: diverges

7.7

$$\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$$

Let's use the direct comparison test with $\sum \frac{1}{2^n}$

$$\frac{\sin^2 n}{2^n} < \frac{1}{2^n}$$

$$\frac{1}{2^n} \text{ converges by the geometric series, so } \frac{\sin^2 n}{2^n} \text{ converges}$$

Answer: converges

7.8

$$\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$$

Let's use the root test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{3n+1}\right)^n} = \lim_{n\to\infty} \frac{n}{3n+1} = \frac{1}{3} < 1 \implies \text{converges}$$

Answer: converges

7.9

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{(n+1)!}{10^{n+1}} \frac{10^n}{n!} = \lim_{n\to\infty} \frac{n+1}{10} > 1 \implies \text{diverges}$$

Answer: diverges

7.10

$$\sum_{n=1}^{\infty} \frac{n^{\sqrt{2}}}{2^n}$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{2^{n+1}} \frac{2^n}{n^{\sqrt{2}}} = \lim_{n\to\infty} \frac{(n+1)^{\sqrt{2}}}{2n^{\sqrt{2}}} = \frac{1}{2} < 1 \implies \text{converges}$$

Answer: converges

7.11

$$\sum_{n=1}^{\infty} \frac{1}{1+\ln n}$$

Let's use the direct comparison test with $\sum \frac{1}{n}$

$$\frac{1}{1+\ln n} > \frac{1}{n}$$

$$\frac{1}{n} \text{ diverges because it is the harmon series, so } \frac{1}{1+\ln n} \text{ diverges}$$

Answer: diverges

7.12

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}\ln n}$$

Let's use the direct comparison test with $\sum \frac{1}{n}$

$$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n}$$

$$\frac{1}{n} \text{ diverges because it is the harmon series, so } \frac{1}{\sqrt{n}\ln n} \text{ diverges}$$

Answer: diverges

7.13

$$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

Let's use the root test

$$\lim_{n\to\infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0 < 1 \implies \text{converges}$$

Answer: converges

7.14

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)^n$$

Let's use the root test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{1}{n} - \frac{1}{n^2}\right)^n} = \lim_{n\to\infty} \frac{1}{n} - \frac{1}{n^3} = 0 < 1 \implies \text{converges}$$

Answer: converges

7.15

$$\sum_{n=1}^{\infty} \frac{4 + |\cos n|}{n^3}$$

Let's use the direct comparison test with $\sum \frac{5}{n^3}$

$$\frac{4 + |\cos n|}{n^3} < \frac{5}{n^3}$$

$$\frac{5}{n^3} \text{ converges by the p-test, so } \frac{4 + |\cos n|}{n^3} \text{ converges}$$

Answer: converges

7.16

$$\sum_{n=1}^{\infty} \frac{(n+4)!}{4!n!4^n}$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{(n+5)!}{4!(n+1)!4^{n+1}} \frac{4!n!4^n}{(n+4)!} = \lim_{n\to\infty} \frac{n+5}{4(n+1)}$$

$$= \lim_{n\to\infty} \frac{n+5}{4n+4} = \frac{1}{4} < 1 \implies \text{converges}$$

Answer: converges

8 Quiz 9 Review

Determine if the following series converge absolutely, converge conditionally, or diverge:

8.1

$$\sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n+1}}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n \frac{1}{\sqrt{n+1}} = 0 \implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{1}{\sqrt{n+1}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$$

Let's use the integral test

$$\int_1^{\infty}\frac{1}{\sqrt{x+1}}dx$$

Let $u = x+1$ and $du = dx$

$$\int u^{-\frac{1}{2}}du = 2\sqrt{u} = 2\sqrt{x+1}\Big|_1^{\infty} = \infty$$

Let's use the conditional convergence test

$$\left|(-1)^{n+1}\frac{1}{\sqrt{n+2}}\right| < \left|(-1)^n \frac{1}{\sqrt{n+1}}\right|$$

This statement is true, so the sum converges conditionally.

Answer: converges conditionally

8.2

$$\sum_{n=1}^{\infty}(-1)^n \frac{n!}{2^n}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n \frac{n!}{2^n} \neq 0 \implies \text{diverges}$$

Answer: diverges

8.3

$$\sum_{n=1}^{\infty}(-1)^n\frac{\arctan n}{n^2+1}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n\frac{\arctan n}{n^2+1}=0 \implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{\arctan n}{n^2+1}\right|=\sum_{n=1}^{\infty}\frac{\arctan n}{n^2+1}$$

Let's use the direct comparison test with $\frac{2}{n^2}$

$$\frac{1}{n^2}>\frac{\arctan n}{n^2+1}$$

$$\frac{1}{n^2} \text{ converges because of p-test, so } \frac{\arctan n}{n^2+1} \text{ converges}$$

Answer: converges absolutely

8.4

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{\ln n}{\ln n^2}\right)^n$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n\left(\frac{\ln n}{\ln n^2}\right)^n \implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty}\left|(-1)^n\left(\frac{\ln n}{\ln n^2}\right)^n\right|=\sum_{n=1}^{\infty}\left(\frac{\ln n}{\ln n^2}\right)^n$$

Let's use the root test

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{\ln n}{\ln n^2}\right)^n}=\lim_{n\to\infty}\frac{\ln n}{2\ln n}=\frac{1}{2}<1 \implies \text{converges}$$

Answer: converges absolutely

8.5

$$\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+1}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n\frac{n}{n+1}=1\neq 0\implies \text{diverges}$$

Answer: diverges

8.6

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n\frac{\sin n}{n^2}=0\implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{\sin n}{n^2}\right|=\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$$

Let's use the direct comparison test with $\frac{1}{n^2}$

$$\frac{1}{n^2}<\frac{\sin n}{n^2}$$

$$\frac{1}{n^2}\text{ converges because of the p-test, so }\frac{\sin n}{n^2}\text{ converges}$$

Answer: converges absolutely

8.7

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n\frac{1}{\sqrt{n+1}+\sqrt{n}}=0\implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$$

Let's use the limit comparison test with $\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}\frac{\sqrt{n}}{1} = \frac{1}{2} > 0 \implies \text{diverges}$$

Let's use the conditional convergence test

$$\left|\frac{1}{\sqrt{n+2}+\sqrt{n+1}}\right| < \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right|$$

This inequality holds true, so $(-1)^n\frac{1}{\sqrt{n+1}+\sqrt{n}}$ converges conditionally.

Answer: converges conditionally

8.8

$$\sum_{n=1}^{\infty}(-1)^n\frac{(2n)!}{2^n n! n}$$

Let's use the divergence test

$$\lim_{n\to\infty}(-1)^n\frac{(2n)!}{2^n n! n} \neq 0 \implies \text{diverges}$$

Answer: diverges

8.9

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^3}{e^n}$$

Let's use the divergence test

$$\lim_{n\to\infty}\frac{(-1)^{n+1}n^3}{e^n} = 0 \implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}n^3}{e^n}\right| = \sum_{n=1}^{\infty}\frac{n^3}{e^n}$$

Let's use the ratio test

$$\lim_{n\to\infty}\frac{(n+1)^3}{e^{n+1}}\frac{e^n}{n^3} = \lim_{n\to\infty}\frac{(n+1)^3}{en^3} = \frac{1}{e} > 1 \implies \text{converges}$$

Answer: converges absolutely

8.10

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ln n}{n}$$

Let's use the divergence test

$$\lim_{n\to\infty} \frac{(-1)^n \ln n}{n} = 0 \implies \text{does not diverge}$$

Let's use the absolute convergence test

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \ln n}{n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{n}$$

Let's use the limit comparison test with $\frac{1}{n}$

$$\lim_{n\to\infty} \frac{\ln n}{n} \frac{n}{1} = \lim_{n\to\infty} \ln n = \infty > 0$$

$$\frac{1}{n} \text{ diverges because it is the harmonic series}$$

Let's use the conditional convergence test

$$\left| \frac{(-1)^{n=1} \ln n + 1}{n+1} \right| < \left| \frac{(-1)^n \ln n}{n} \right|$$

This inequality is true, so $\frac{(-1)^n \ln n}{n}$ converges conditionally.

Answer: converges conditionally

Find the radius and interval of convergence for:

8.11

$$\sum_{n=1}^{\infty} 3^n x^n$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{3^{n+1} x^{n+1}}{3^n x^n} = 3x < 1 \implies x < \frac{1}{3}$$

We obtain the following:

$$R = \frac{1}{3} \quad C = 0 \quad \text{interval:} \ -\frac{1}{3}, \frac{1}{3}$$

$$x = -\frac{1}{3} \implies \sum_{n=1}^{\infty} 3^n \left(-\frac{1}{3} \right)^n \implies \text{diverges}$$

$$x = \frac{1}{3} \implies \sum_{n=1}^{\infty} 3^n \left(\frac{1}{3} \right)^n \implies \text{diverges}$$

Answer: $\left(-\frac{1}{3}, \frac{1}{3} \right)$

8.12

$$\sum_{n=1}^{\infty} \frac{n!}{3^n}(x-2)^n$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{(n+1)!(x-2)^{n+1}}{3^{n+1}} \frac{3^n}{n!(x-2)^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)(x-2)}{3} \implies x = 2 \text{ converges}$$

We gather the following information:

$$R = 0 \quad C = 2$$

Answer: $R = 0$

8.13

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n+1}$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{(x+2)^{n+1}}{n+2} \frac{n+1}{(x+2)^n} = \lim_{n\to\infty} \frac{(x+2)(n+1}{(n+2)} = x+2 < 1$$

We gather the following information:

$$R = 1 \quad C = -2 \quad \text{interval: } -3, -1$$

$$x = -3 \quad \sum_{n=1}^{\infty} \frac{(-3+2)^n}{n+1} \implies \text{converges}$$

$$x = -1 \quad \sum_{n=1}^{\infty} \frac{(-1+2)^n}{n+1} \implies \text{diverges}$$

Answer: $\Big[-3, -1 \Big)$

8.14

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}(x-3)^n$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{(x-3)^{n+1}}{(n+1)^2+1}\frac{n^2+1}{(x-3)^n} = x-3 < 1$$

We gather the following information:

$$R=1 \quad C=3 \quad \text{interval:} \ -2,4$$

$$x=-2 \implies \sum_{n=1}^{\infty} \frac{(-2-3)^n}{n^2+1} \implies \text{converges}$$

$$x=4 \implies \sum_{n=1}^{\infty} \frac{(4-3)^n}{n^2+1} \implies \text{converges}$$

Answer: $\left[-2,4\right]$

8.15

$$\sum_{n=1}^{\infty} c9^n n!(x-1)^n$$

Let's use the ratio test

$$\lim_{n\to\infty} \frac{9^{n+1}(x-1)^{n+1}}{(n+1)!}\frac{n!}{9^n(x-1)^n} = 0$$

We gather the following information:

$$R=\infty$$

Answer: converges for all x

Find the n-th degree Taylor polynomial centered at $x=a$ for:

8.16

$$f(x) = \ln x \quad a = 1 \quad n = 4$$

$$f(x) = \ln x \quad f(1) = 0$$

$$f'(x) = \frac{1}{x} \quad f'(1) = 1$$

$$f''(x) = -x^{-2} \quad f''(1) = -1$$

$$f'''(x) = 2x^{-3} \quad f'''(1) = 2$$

$$f''''(x) = -6x^{-4} \quad f''''(1) = -6$$

Answer: $T_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

8.17

$$f(x) = \arctan x \quad a = 0 \quad n = 2$$

$$f(x) = \arctan(x) \quad f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \quad f'(0) = 1$$

$$f''(x) = -1(1+x^2)^{-2}(2x) \quad f''(0) = 0$$

Answer: $T_2(x) = x$

8.18

$$f(x) = \cos x \quad a = \frac{\pi}{4} \quad n = 3$$

$$f(x) = \cos x \quad f(0) = \frac{1}{\sqrt{2}}$$

$$f'(x) = -\sin x \quad f'(0) = \frac{-1}{\sqrt{2}}$$

$$f''(x) = -\cos x \quad f''(0) = \frac{-1}{\sqrt{2}}$$

$$f'''(x) = \sin x \quad f'''(0) = \frac{1}{\sqrt{2}}$$

Answer: $T_3(x) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) - \frac{1}{\sqrt{2}}\frac{1}{2}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{\sqrt{2}}\frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3$

8.19

$$f(x) = \frac{1}{\sqrt{1-x}} \quad a = 0 \quad n = 4$$

$$f(x) = \left(1-x\right)^{-1/2} \quad f(0) = 1$$

$$f'(x) = \frac{1}{2}\left(1-x\right)^{-3/2} \quad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{3}{4}\left(1-x\right)^{-5/2} \quad f''(0) = \frac{3}{4}$$

$$f'''(x) = \frac{15}{8}\left(1-x\right)^{-7/2} \quad f'''(0) = \frac{15}{8}$$

$$f''''(x) = \frac{105}{16}\left(1-x\right)^{-9/2} \quad f''''(0) = \frac{105}{16}$$

Answer: $T_4(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4$

9 Quiz 10 Review

Write out the first four non-zero terms for the Maclaurin series for:

9.1

$$f(x) = \sin 2x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5} - \frac{128x^7}{7!}$$

Answer: $\sin 2x = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5} - \frac{128x^7}{7!}$

9.2

$$f(x) = x \sin x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$x \sin x = x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right) = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!}$$

Answer: $x \sin x = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!}$

9.3

$$f(x) = xe^x$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$xe^x = x\left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}\right) = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \frac{x^5}{4!}$$

Answer: $xe^x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \frac{x^5}{4!}$

9.4

$$f(x) = e^x \sin x$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$e^x \sin x = x + x^2 + \frac{x^3}{2} + \frac{x^3}{4!} - \frac{x^3}{3!} - \frac{x^4}{3!} - \frac{x^5}{12} + \frac{x^5}{5!} + \dots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$$

Answer: $e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30}$

9.5

$$f(x) = \ln\left[\frac{1+x}{1-x}\right]$$

$$\ln\left[\frac{1+x}{1-x}\right] = \ln(1-x) - \ln(1+x)$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5$$

$$\int \frac{1}{1-x} = \ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5}$$

$$\int \frac{1}{1+x} = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}$$

$$\ln\left[\frac{1+x}{1-x}\right] = \ln(1-x) - \ln(1+x) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}$$

Answer: $\ln\left[\frac{1+x}{1-x}\right] = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7}$

9.6 For the parametric curve: $x = 4t + 6, y = 8t + 2$, eliminate the parameter and express in rectangular form.

$$t = \frac{x-6}{4} \quad t = \frac{y-2}{8}$$

$$\frac{x-6}{4} = \frac{y-2}{8} \implies 2(x-6) = y-2 \implies 2x - y - 10 = 0$$

Answer: $2x - y - 10 = 0$

9.7 For the parametric curve: $x = 2 + 2\cos t, y = 3 + 4\sin t$, eliminate the parameter and express in rectangular form.

$$\frac{x-2}{2} = \cos t \quad \frac{y-3}{4} = \sin t$$

$$\cos^2 t + \sin^2 t = 1 \implies \left(\frac{x-2}{2}\right)^2 + \left(\frac{y-3}{4}\right)^2 = 1$$

$$\implies \frac{(x-2)^2}{4} + \frac{(y-3)^2}{16} = 1$$

$$\implies 4(x-2)^2 + (y-3)^2 = 16$$

$$\implies 4x^2 + y^2 - 16x - 6y + 9 = 0$$

Answer: $4x^2 + y^2 - 16x - 6y + 9 = 0$

9.8 Find the equation of the tangent line to the curve $x = 2t^2 + 2t, y = 2t^2 + 4t$ at $t = 1$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t+4}{4t+2} = \frac{4+4}{4+2} = \frac{4}{3}$$

$$x = 2 + 2 = 4 \quad y = 2 + 4 = 6$$

$$\implies y - 6 = \frac{4}{3}(x-4) \implies y = \frac{4}{3}x + \frac{2}{3}$$

Answer: $y = \frac{4}{3}x + \frac{2}{3}$

9.9 Find the equation of the tangent line to the curve $x = \sec t, y = \tan t$ at $t = \frac{\pi}{4}$.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{sec^2 t}{\sec t \tan t} = \frac{\sec t}{\tan t} = \frac{\sin t}{\cos^2 t}$$

$$\frac{\frac{\sqrt{2}}{2}}{\left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{2}$$

$$x = \sec\left(\frac{\pi}{4}\right) = \sqrt{2} \quad y = 1$$

$$y - 1 = \sqrt{2}(x - \sqrt{2}) \quad y = \sqrt{2}x - 1$$

Answer: $y = \sqrt{2}x - 1$

9.10 Find $\frac{d^2y}{dx^2}$ if $x = \sin t, y = 2\cos t$.

$$\frac{dy^2}{d^2x} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$$

$$dy = -\sin t \quad dx = \cos t \implies \frac{dy}{dx} = -\tan t$$

$$\implies \frac{dy^2}{d^2x} = \frac{-\sec^2 t}{\cos t} = -\sec^3 t$$

Answer: $\sec^3 t$

9.11 Find the arc length of $x = 4\sin 2t, y = 4\cos 2t$, over the interval $0 \leq t \leq \frac{\pi}{2}$.

$$s = \int_a^b \sqrt{dx^2 + dy^2} dt$$

$$x = 4\sin 2t \quad dx = 8\cos 2t$$

$$y = 4\cos 2t \quad dy = -8\sin 2t$$

$$s = \int_0^{\pi/2} \sqrt{(8\cos 2t)^2 + (-8\sin 2t)^2} dt = \int_0^{\pi/2} \sqrt{64(\cos^2 2t + \sin^2 2t} dt$$

$$= \int_0^{\pi/2} 64 dt = \int_0^{\pi/2} 8 dt = 8t \Big|_0^{\pi/2} = 4\pi$$

Answer: 4π

9.12 Find the arc length of $x = t^2, y = t^3$, from $(1,1)$ to $(4, 8$.

$$x = t^2 \quad dx = 2t$$

$$y = t^3 \quad dy = 3t^2$$

$$(1,1) \implies 1 = t^2 \quad 1 = t^3 \implies t = 1$$

$$(4,8) \implies 4 = t^2 \quad 8 = t^3 \implies t = 2$$

$$s = \int_1^2 \sqrt{(2t)^2 + (3t^2)^2} dt = \int_1^2 \sqrt{4t^2 + 9t^4} dt = \int_1^2 \sqrt{t^2(4 + 9t^2)} dt$$

$$= \int_1^2 t\sqrt{4 + 9t^2} dt = \frac{1}{18}\frac{2}{3}(4 + 9t^2)^{3/2}\Big|_1^2 = \frac{1}{27}[(40)^{3/2} - (13)^{3/2}]$$

$$= \frac{1}{27}[40\sqrt{40} - 13\sqrt{13}] = \frac{1}{27}[80\sqrt{10} - 13\sqrt{13}]$$

Answer: $\frac{1}{27}[80\sqrt{10} - 13\sqrt{13}]$

9.13 Find the arc length of $x = a(t - \sin t), y = a(1 - \cos t)$, for $0 \leq t \leq 2\pi$.

$$x = a(t - \sin t) = at - a\sin t \quad dx = a - a\cos t$$

$$y = a(1 - \cos t) = a - a\cos t \quad dy = a\sin t$$

$$s = \int_0^{2\pi} \sqrt{(a - a\cos t)^2 + (a\sin t)^2} dt$$

$$= \int_0^{2\pi} \sqrt{a^2 - 2a^2 \cos t + a^2 \cos^2 t + a^2 \sin^2 t} dt$$

$$= \int_0^{2\pi} a\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} dt$$

$$= \int_0^{2\pi} a\sqrt{1 - 2\cos t + 1} dt = \int_0^{2\pi} 2 - 2\cos t dt = \int_0^{2\pi} a\sqrt{2}\sqrt{1 - \cos t} dt$$

Remember the identity

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \implies \sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2} = 2\sin^2\frac{\theta}{2} = 1 - \cos\theta$$

$$= \int_0^{2\pi} a\sqrt{2}\sqrt{2\sin^2\frac{t}{2}} dt = \int_0^{2\pi} 2a\sqrt{\sin^2\frac{t}{2}} dt = 2a\int_0^{2\pi} \sin\frac{t}{2} dt$$

$$= 2a\bigg(-2\cos\frac{t}{2}\bigg)\bigg|_0^{2\pi} = 4a(-\cos\pi - (\cos 0)) = 4a(1 + 1) = 8a$$

Answer: 8a

10 Quiz 11 Review

10.1 A vector $\vec{v}$ has initial point $(-4, -1)$ and terminal point $(-2, -5)$. Find $\vec{v}$.

$$\vec{v} =< -4 - (-2), -1 - (-5) >=< -2, 4 >$$

Answer: $< -2, 4 >$

10.2 A vector $\vec{v}$ has initial point $(1, 8)$ and terminal point $(3, -7)$. Find its magnitude.

$$\vec{v} =< 1 - 3, 8 - (-7) >=< -2, 15 >$$

$$||\vec{v}|| = \sqrt{(-2)^2 + (15)^2} = \sqrt{229}$$

Answer: $\sqrt{229}$

10.3 Given $\vec{w} = \hat{i}$ and $\vec{u} = 4\hat{i} - 2\hat{j}$, find $\vec{v} = 3\vec{w} + 5\vec{u}$.

$$\vec{v} = 3\hat{i} + 5(4\hat{i} - 2\hat{j}) = 23\hat{i} - 10\hat{j}$$

Answer: $23\hat{i} - 10\hat{j}$

10.4 Determine which vector is parallel to the vector $\vec{v} =< 2, -3, -1 >$

$$< 4, 6, -2 > \quad < \frac{-2}{3}, 1, \frac{1}{3} > \quad < 1, \frac{-3}{2}, \frac{1}{2} >$$

$$< 6, -9, 3 > \quad \text{None of these}$$

Answer: $< \frac{-2}{3}, 1, \frac{1}{3} >$

10.5 Find the center and radius of the sphere given by:

$$x^2 + y^2 + z^2 + 2x - 2y + 6z + 7 = 0$$

$$x^2 + 2x + 1 + y^2 - 2y + 1 + z^2 + 6z + 9 + 7 = 1 - 1 + 9$$

$$(x + 1)^2 + (y - 1)^2 + (z + 3)^2 = -7 + 9 = 2$$

Answer: center: $(-1, 1, -3)$, radius: $\sqrt{2}$

10.6 Find the center and radius of the sphere given by:

$$x^2 + y^2 + z^2 - 4x + 8y - 10z + 20 = 0$$

$$x^2 - 4x + 4 + y^2 + 8y + 16 + z^2 - 10z + 25 = -20 + 4 + 16 + 25$$

$$(x-2)^2 + (y+4)^2 + (z-5)^2 = 25$$

Answer: center $(2, -4, 5)$, radius: 5

10.7 Is the origin inside or outside the sphere

$$x^2 - 2x + y^2 + 4y + z^2 - 6z = 2$$

$$x^2 - 2x + 1 + y^2 + 4y + 4 + z^2 - 6z + 9 = 2 + 1 + 4 + 9$$

$$(x-1)^2 + (y+2)^2 + (z-3)^2 = 16 \implies \text{center} : (1, -2, 3), \text{radius} : 4$$

$$\sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{14} \implies \text{distance away from origin}$$

$$\sqrt{14} < 4 = \sqrt{16} \implies \text{inside}$$

Answer: inside

10.8 Find the parametric equations of the line through the points $(2, 0, 3)$ and $(4, 3, 3)$.

$$< 4-2, 3-0, 3-3 >=< 2, 3, 0 >$$

$$x = 2 + 2t \quad y = 0 + 3t \quad z = 3$$

Answer: $x = 2 + 2t \quad y = 0 + 3t \quad z = 3$

10.9 Find the parametric equations of the line through the points $(-3, 2, 0)$ and $(4, 3, 3)$.

$$< 4-(-3), 3-2, 3-0 >=< 7, 1, 3 >$$

$$x = 4 + 7t \quad y = 3 + t \quad 3 + 3t$$

Answer: $x = 4 + 7t \quad y = 3 + t \quad 3 + 3t$

10.10 Find numbers x and y such that the point $(x, y, 1)$ lies on the line passing through the points $(2, 5, 7)$ and $(0, 3, 2)$.

$$< 2-0, 5-3, 7-2 > = < 2, 2, 5 >$$

$$x = 2+2t \quad y = 5+2t \quad z = 7+5t$$

$$1 = 7+5t \implies t = \frac{-6}{5}$$

$$x = 2+2\left(\frac{-6}{5}\right) = \frac{-2}{5}$$

$$y = 5+2\left(\frac{-6}{5}\right) = \frac{13}{5}$$

Answer: $x = \frac{-2}{5}, y = \frac{13}{5}$

10.11 Find the point where the line through $(3, 2, 4)$ with direction vector $\vec{v} = < 7, 5, -4 >$ intersects the xy-place.

$$x = 3+7t \quad y = 2+5t \quad z = 4-4t$$

$$0 = 4-4t \implies t = 1$$

$$x = 3+7(1) = 10$$

$$y = 2+t(1) = 7$$

Answer: $(10, 7, 0)$

10.12 Determine whether the lines intersect, are parallel, or skew:

$$L_1 : x = 4t+2 \quad y = 3 \quad z = -t+1$$

$$L_2 : x = 2s+2 \quad y = 2s+3 \quad z = s+1$$

$$v_1 = < 4, 0, -1 > \quad v_2 = < 2, 2, 1 > \implies \text{not parallel}$$

$$4t+2 = 2s+2 \implies 4(-t+1) = 4(s+1)$$

$$\implies -t = s \implies t = s = 0$$

$$L_1 : < 2, 3, 1 > \quad L_2 : < 2, 3, 1 >$$

Answer: Parallel

10.13 Determine whether the lines intersect, are parallel, or skew:

$$L_1 : x = 1 - 4t \quad y = 2 + 3t \quad z = 4 - 2t$$

$$L_2 : x = 2 - s \quad y = 1 + s \quad z = 2 + 6s$$

$$v_1 =< -4, 3, -2 > \quad v_2 =< -1, 1, 6 > \Longrightarrow \text{ not parallel}$$

$$x = 1 - 4t = 2 - s \implies s = 1 + 4t$$

$$y = 2 + 3t = 1 + (1 + 4t) \implies t = 0$$

$$x = 1 + 4(0) = 1$$

$$z_1 = 4 - 2(0) = 4 \quad z_2 = 2 + 6(1) = 8$$

Answer: skew

10.14 For vectors $\vec{a} =< 1, 2, -1 >$ and $\vec{b} =< 3, 5, -1 >$ find:

10.14.1 Find $2\vec{a} - 5\vec{b}$

$$= 2 < 1, 2, -1 > -5 < 3, 5, -1 >=< -13, -21, 3 >$$

Answer: $< -13, -21, 3 >$

10.14.2 $\vec{a} \cdot \vec{b}$

$$= (1)(3) + (2)(5) + (-1)(-1) = 14$$

Answer: 14

10.14.3 The $\cos\theta$ where θ is the angel between $\vec{a}$ and $\vec{b}$.

$$= \frac{14}{\sqrt{6}\sqrt{35}}$$

Answer: $\frac{14}{\sqrt{6}\sqrt{35}}$

10.14.4 A unit vector in the direction of $\vec{a}$.

$$= \frac{< 1, 2, -1 >}{\sqrt{6}}$$

Answer: $\frac{<1,2,-1>}{\sqrt{6}}$

10.14.5 A vector of length 3 in the direction of $\vec{a}$.

$$= 3\vec{e}_{\vec{a}} = \frac{3 < 1, 2, -1 >}{\sqrt{6}}$$

Answer: $\frac{3<1,2,-1>}{\sqrt{6}}$

10.15 For vectors $\vec{a} =< 1, 1, -2 >$ and $\vec{b} =< 1, 2, -1 >$ find:

10.15.1 Find $4\vec{a} - 3\vec{b}$

$$= 4 < 1, 1, -2 > -3 < 1, 2, -1 >=< 1, -2, -5 >$$

Answer: $< 1, -2, -5 >$

10.15.2 $\vec{a} \cdot \vec{b}$

$$= (1)(1) + (1)(2) + (-2)(-1) = 5$$

Answer: 5

10.15.3 The $\cos\theta$ where θ is the angel between $\vec{a}$ and $\vec{b}$.

a

$$= \frac{5}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

Answer: $\frac{5}{6}$

10.15.4 A unit vector in the direction of $\vec{a}$.

$$= \frac{< 1, 1, -2 >}{\sqrt{6}}$$

Answer: $\frac{<1,1,-2>}{\sqrt{6}}$

10.15.5 A vector of length 4 in the direction of $\vec{a}$.

$$= 4\vec{e}_{\vec{a}} = \frac{4 < 1, 1, -2 >}{\sqrt{6}}$$

Answer: $\frac{4<1,1,-2>}{\sqrt{6}}$

10.16 Find the value of x so that $\vec{c} =< 2, x, -3 >$ and $\vec{d} =< -1, 3, -2 >$ are perpendicular.

$$0 = \vec{c} \cdot \vec{d} =< 2, x, -3 > \cdot < -1, 3, -2 >$$

$$0 = (2)(-1) + (x)(3) + (-3)(-2) = -2 + 3x + 6$$

$$-4 = 3x \implies \frac{-4}{3} = x$$

Answer: $x = \frac{-4}{3}$

10.17 Let $\vec{a} =< 3, -1, 2 >$ and $\vec{b} =< -2, -3, 2 >$. Calculate $proj_{\vec{b}}\vec{a}$

$$= \frac{\vec{a} \cdot \vec{b}}{||\vec{b}||} \frac{\vec{b}}{||\vec{b}||} = \frac{-6 + 3 + 4}{\sqrt{17}} \frac{< -2, -3, 2 >}{\sqrt{17}} = \frac{< -2, -3, 2 >}{17}$$

Answer: $\frac{<-2,-3,2>}{17}$

10.18 Let $\vec{a} =< 3, -1, 2 >$ and $\vec{b} =< -2, -3, 2 >$. Calculate $proj_{\vec{a}}\vec{b}$

$$= \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||} \frac{\vec{a}}{||\vec{a}||} = \frac{-6 + 3 + 4}{\sqrt{14}} \frac{< 3, -1, 2 >}{\sqrt{14}} = \frac{< 3, -1, 2 >}{14}$$

Answer: $\frac{<3,-1,2>}{14}$