An Introduction to Pi (π)

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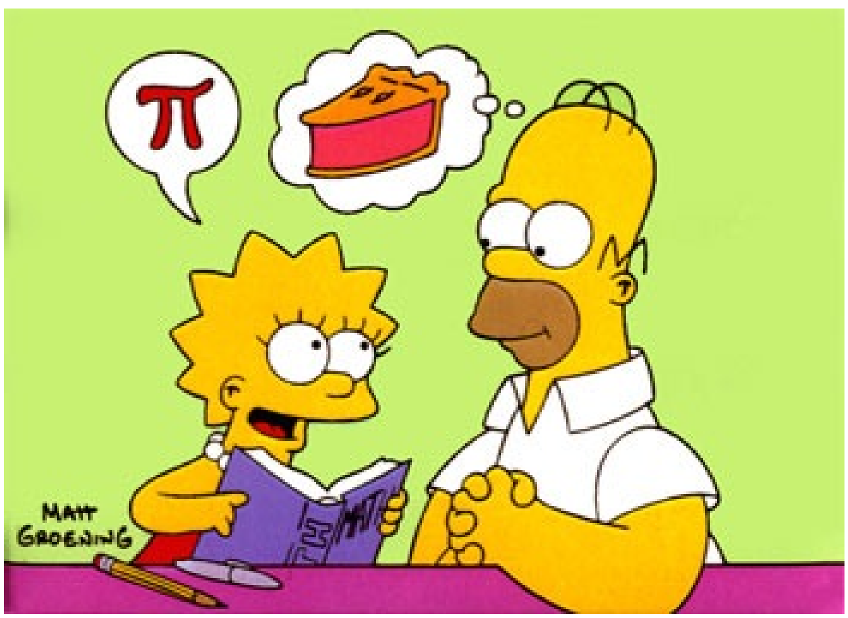

An Introduction to π (Sneak Preview!)

Like most sneak previews, I'm going to pick up somewhere totally random and leave you utterly confused. Enjoy![1]

Early Uses of π

The jury is usually out on whether ideas in math were invented or discovered. With π, however, people are often in agreement that the number is simply a "constant of nature," and exists whether we like it or not. Its original discoverers were the Egyptians and Babylonians, both of whom required impeccable measurements for agricultural purposes.[2] The number also makes brief appearance in the Bible,[3] before heading on its sold-out worldwide Approximation Tour, making acclaimed stops in Greece, India, and China before eventually settling down to spend time with the family.

Babylon

Babylonian Numbering

Before I get to the part where π makes its first appearances, let me talk a little bit about Babylonian numbering. Feel free to skip down a few paragraphs if you're an ancient numerical scholar or something and will be bored by this stuff.

Most of what we know about Babylonian mathematics comes from assorted clay tablets, like the one to the right.The little markings that resemble the flag at the end of each level in Super Mario Bros. actually represent numbers. They're written in cuneiform script, the Comic Sans of ancient Mesopotamia. The Babylonian numbering system was base-60, which can best be understood by comparing it to our familiar base-10 system.

In our current numbering system, each place value gets a digit between 0 and 9, and each place value represents a successive power of 10. So, for example, the number 1234 is actually a condensed way of writing $1 \cdot 1000 + 2 \cdot 100 + 3 \cdot 10 + 4 \cdot 1$. We're trained to think about the "ones place," the "tens place," etc. from a very young age, so it just seems natural, especially since we were practically *handed* a biologically convenient base 10 counter.

In base-60, each place value represents a different power of 60. So instead of the "tens" and the "hundreds" (respectively 10^1 and 10^2) we have the 60's and the 3600's place values. In a base 60 system, the number 1234 would be written as $20 \cdot 60 + 34 \cdot 1 = 20; 34$. The numerals "20" and "34" would be scribed using the symbols atop the next page.

As you can see, the list ends at 59, just as our counting system ends at 9 before moving to the next decimal place. It's from the Babylonians that we get the idea to break minutes into 60 seconds, years into 12 months of 30 days each, and circles into 360 degrees, because these are all numbers that are easy to work with in a base-60 system.

[1] That would be really rude, and I hope you know I'm not that kind of guy. You're gonna start by reading about π's first appearances in ancient Babylon. Now head back up to the text by clicking this arrow: ↑

[2] AND ALSO BECAUSE ALIENS VISITED THEM AND TOLD THEM ABOUT PI BEFORE BUILDING THE PYRAMIDS ↑

[3] Therefore definitively confirming its existence ↑

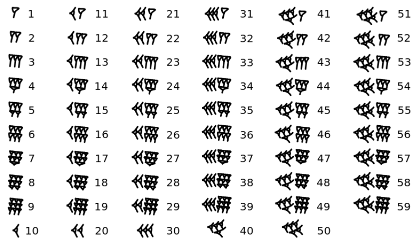

Figure 1: Babylonian Numerals In Base 60

The one problem with the Babylonian numbering system was that there was no notion of what we now call a **radix point**. That's probably not a term you've ever heard before, but is the formal name for the point that separates the integer part and fractional part of a number, which in base-10 would be called the "decimal point."[4] The problem with lacking a radix point is that it impossible to tell the difference between, for example, the numbers 400 and 0.4 (decimal shorthand for $4 \cdot 10^2$ and $\frac{4}{10}$, respectively). Similarly, in base 60, the numbers 4 and $\frac{4}{60}$ would be written the exact same way, and the correct interpretation would have to be inferred based on context.

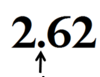

Figure 2: A Radix Point, For When Pretentious People Don't Like Saying "Decimal Point"

π in Babylonian Mathematics

The Babylonians and π go waaaaaay back. By 2000 BCE there is evidence that the Babylonians used both 3 and the number $\frac{25}{8} = 3.125$ for the ratio of circumference to diameter, certainly not bad approximations. Most Babylonian math was done on clay tablets and has been preserved for a few millennia. The picture below depicts a circle, and so ought to provide some information about π.

Figure 3: Either A Babylonian Tablet Or A Random Rock

Let's break this tablet down a little bit.[5] The number 3 seems to denote the circumference

[4] "Well how 'bout that!" you think to yourself, acknowledging the fact that within the hour you'll have completely forgotten what that darned word was ↑

[5] Though natural exogenic processes would have done that already, am I right or am I right?!? ↑

of the circle. The symbol for 45 is in the interior, so it would make sense that it represents the area. But given that the circumference is only 3 it would make sense to interpret the area as $\frac{45}{60}$. Using this information, we can get a useful result:

$$C = 3 = 2\pi r \implies r = \frac{3}{2\pi}$$
$$A = \frac{45}{60} = \pi r^2 = \pi \left(\frac{3}{2\pi}\right)^2$$
$$\frac{45}{60} = \cancel{\pi}\left(\frac{9}{4\pi^{\cancel{2}}}\right) = \frac{9}{4\pi}$$
$$\frac{45}{60} = \frac{9}{4\pi} \implies \pi = \frac{9 \cdot 60}{4 \cdot 45} = 3$$

Yippee! That's exactly what we were looking for. We can then conclude that, in this particular tablet, the value of 3 was being used for π.

Another formulation for π comes from a tablet which provides a formula for the perimeter of a hexagon inscribed in a circle, as in this picture:[6]

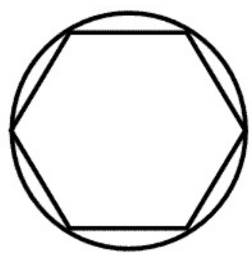

Figure 4: Hexagon In A Circle: Too Difficult For The Illuminati To Draw

The formula says this:

$$\text{Perimeter of Hexagon} = \frac{24}{25} \cdot \text{Circumference of circle} = \frac{24}{25} \cdot 2\pi r$$

We're going to need a little bit of background knowledge to make these numbers werk. First off, let's get some intuition on the geometry of a hexagon inscribed in a circle. This shape will prove to be super important later on in our journey, so it's best we get this out of the way now. I would highly recommend making this part fun and interactive by drawing your own hexagon and following along![7]

The first thing to do is to draw line segments that connect each vertex of the hexagon to the center of the circle. You'll see that this creates 6 isoceles triangles, where the base is the side of the hexagon and the two other sides of the triangle have length equal to the radius of the circle.

[6]The word "tabloid" actually has the same etymological root as "tablet," which makes sense because I think Perez Hilton got his start coming up with formulae for the perimeter of inscribed polygons ↑

[7]You've made it this far, so it's time to concede that math is indeed a blast when it's fun and interactive ↑

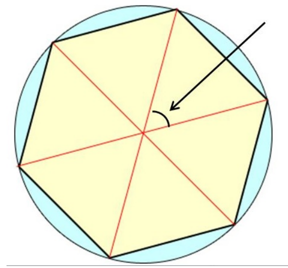

Figure 5: I Saw Sleeze Triangles In A Circle

Let's call the angle with the arrow θ, and ask ourselves how many θ's there are in the whole hexagon. That's not too tough: just count 'em around the circle and see that there are 6. Well, all these θ's need to add up to 360, so:

$$6 \cdot \theta = 360 \implies \theta = 60^\circ$$

That means that those triangles are more than just isosceles: they're equilateral. This is a big deal!! It means that the radius of the circle and the side length of the hexagon are equal.

With that in mind, let's revisit the formula that's going to lead us to π (henceforth called "The Recipe"). The recipe says that:

$$\text{Perimeter of Hexagon} = \frac{24}{25} \cdot \text{Circumference of circle} = \frac{24}{25} \cdot 2\pi r$$

Since we now know that the side length of the hexagon is equal to the radius of the circle, we can set the perimeter of the hexagon to $6r$ in the recipe:

$$6r = \frac{24}{25} \cdot 2\pi r$$

$$6\cancel{r} = \frac{24}{25} \cdot 2\pi\cancel{r}$$[8]

And finally, the epic result:

$$\pi = \frac{25}{8} = 3.125$$

Not half bad, I'd say.

[8] Notice that by crossing our r we're essentially asserting that π is the same no matter what the radius of the circle ↑

This next preview section is an excerpt from a discussion on places in math where π shows up "unannounced" (i.e. in contexts you wouldn't expect)

Bouffon's Needle

Our final sojourn in this adventure of unexpected appearances of π takes us to France, into the presumably lavish residence of one Georges-Louis Leclerc, Comte de Bouffon (1707-1788). Unfortunately, not much is really known about his life, but this painting pretty much sums up everything we need to know about him.

Figure 6: OK Mr. Bouffon, Now One Where You're Scratching Your Belly

Bouffon is best known as a botanist and naturalist and early proponent of evolution, though he did plenty of work in math and astronomy. He also published a fantastic book that attempted to calculate the probability that the sun would continue to rise after being observed n days in a row. He was unable to complete the work, unfortunately, after being sent to the guillotine, but as far as I can tell we're still looking pretty good as far as sunrise, sunset is concerned.[9]

In math, Bouffon is best known for an experiment that on the surface seems incredibly simplistic. Before I tell you what it was, let me provide some context, via some remarkably broad-stroked historical generalizations. During Bouffon's lifetime, probability theory was all the rage. Frenchmen with pretentious-sounding names like Poisson, Fermat, and Pascal, along the Bernoullis – the Kardashians of 17$^{\text{th}}$ century Switzerland – invented the subject in the late 1600s and early 1700s to make some fast cash playing dice and cards. Their only real goal was to answer the question "what is the likeliness that a specific event will occur?" without having to consult some sort of oracle.

So what was the question of likeliness that Bouffon wanted to answer? It's a pretty easy one. He asked (and I quote):

> Suppose we have a floor made of parallel strips of wood, each with the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?

The reason Bouffon was thinking about this in the first place is an absolute mystery. Also, the fact that anyone took this seriously as a scientific experiment speaks volumes about the

[9] Now I'm a little verklempt ↑

added prestige that comes from wearing a puffy shirt.[10] Fortunately for us Bouffon was a pretty distinguished gentleman, so people probably didn't bat an eyelash when they found out he spent most of his time tossing needles onto his newly refurbished parallel-striped hardwood floor.

Here's a picture of the experiment Bouffon's talking about, with 10 needles as a demonstration:

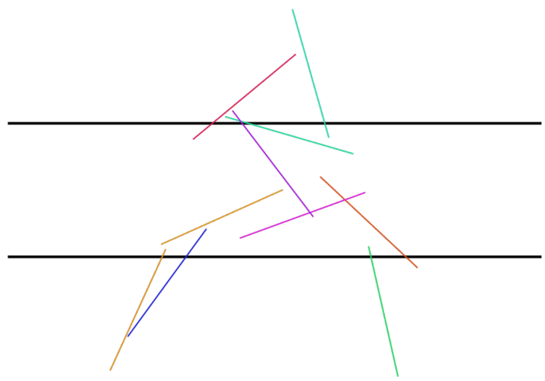

Figure 7: Math Experiment Or EDM Album Cover?

So, the question is simple. What is the probability that a needle will cross the line? The answer, it turns out, is kinda complicated, and is much more nuanced than one might expect from a seemingly easy experiment. Now that you know (or at least you will when the full book is released!)all about probability distributions, though, things shouldn't be too bad. This particular question involves something called a **geometric probability**, a name which you'll better understand after I take you through the solution of Bouffon's needle problem.[11]

In order to answer this question, there are three cases to consider:

1. Length of needle is **equal** to width of floor strips.
2. Length of needle is **shorter** than width of floor strips.
3. Length of needle is **longer** than width floor strips.

The first case is definitely the simplest, so I'm going to start with that one. The second case isn't too bad either, so I'll go over that one too in Appendix B (Not in this sneak preview!). The third case, however, is pretty darn difficult if I do say so myself, so we'll leave that one to the pretentious-sounding Frenchmen, who definitely deserve some credit for their math skillz despite the poofy shirts.

Before we start, I'm going to introduce some terminology that is entirely my own. I'm going to call the edges of the strips of wood (i.e. the pieces that the needle can cross) the **walls**. I'm going to call the length of the needle **N**, for needle. The distance between two walls I'm going to call **D**, for distance. I've never been an advocate for symbols replacing words, but I think in this case things will just get too jumbled if I don't use letters to represent these things. Alrighty. Everything's in place now. You might want to call this an intervention, because we're about to solve Bouffon's needle problem **MIC DROP**.

[10] I wonder if Bouffon was forced into it by a low-talker? ↑

[11] Also what friends and family called Bouffon's rather tragic late-life foray into the drug business ↑

Well there you have it! A sneak preview of my book about π, which I hope to finish by September. I hope you learned something from this and at least had a good chuckle in the process. Thanks for taking a look!