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# Palindromic Sequences of the Markov Spectrum*

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**Abstract**—We study the periods of Markov sequences, which are derived from the continued fraction expression of elements in the Markov spectrum. This spectrum is the set of minimal values of indefinite binary quadratic forms that are specially normalised. We show that the periods of these sequences are palindromic after a number of circular shifts, the number of shifts being given by Stern's diatomic sequence.

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## INTRODUCTION

In this paper, we state and prove a general result on the construction of palindromic sequences. These include sequences related to the Markov spectrum. The *Markov spectrum* is the set of numbers

$$\inf_{\mathbb{Z}^2\setminus\{(0,0)\}}\left|\frac{\sqrt{\Delta}}{f}\right| \tag{0.1}$$

for all binary quadratic forms $f$ with positive discriminant $\Delta(f)$.

Markov showed in his papers [7, 8] that, for any element of the Markov spectrum less than 3, there exists a sequence $(a_1,\ldots,a_{2n})$ of positive integers such that

$$\inf_{\mathbb{Z}^2\setminus\{(0,0)\}}\left|\frac{\sqrt{\Delta}}{f}\right| = [(a_1,a_2,\ldots,a_{2n})] + [0;\overline{(a_1,\ldots,a_{2n})}], \tag{0.2}$$

where $[(a_1,a_2,\ldots,a_{2n})]$ is the infinite continued fraction with period $(a_1,a_2,\ldots,a_{2n})$. In this paper, we denote the reverse of the sequence $(a_1,\ldots,a_{2n})$ by $\overline{(a_1,\ldots,a_{2n})}$. Equation (0.2) is known as the *Perron identity*, going back to [9].

It is known (see, e.g., the books [3] by Cusick and Flahive and [1] by Aigner) that the numbers $a_i$ satisfy the following conditions:

- $a_i \in \{1,2\}$
- $a_1 = a_2 = 2$, $a_{2n} = a_{2n-1} = 1$
- The subsequence $w=(a_3,\ldots,a_{2n-2})$ is palindromic, i.e. $w=\overline{w}$.

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One can find works on Markov numbers related to other branches of mathematics in the articles [4], [11], [10], [5], and [13].

The sequences for which expression (0.1) is less than 3, henceforth called *Markov sequences*, can be constructed as the concatenation of the sequences $(2,2)$ and $(1,1)$ (see Definition 4 below). This follows from results of Cohn's paper [2] and can be found in [1, Theorem 4.7].

We show in Theorem 1 that any sequence constructed in the same way as Markov sequences is *evenly palindromic*; that is, after some number of circular shifts, the sequence becomes palindromic. The number of circular shifts is given by Stern's diatomic sequence, an exposition of which can be found in Urbiha's paper [14]. Markov sequences are related to lower Christoffel words (see [10], [1]).

In the forthcoming paper [6], we use Theorem 1 to show that there is a generalization of Markov numbers coming from the graph structure in Definition 4.

### *Organization of the Paper*

In Sec. 1, we give a background for Markov sequences and the definitions necessary for our main result, Theorem 1.

In Sec. 2, we prove Theorem 1.

## 1. SOME HISTORY and BACKGROUND

In this section, we give the necessary definitions and background for the main result of the paper, Theorem 1.

### *1.1. The Markov Spectrum*

In this subsection, we define the Markov spectrum in terms of binary quadratic forms and sequences of positive integers. We start with the following definition.

**Definition 1.** Let $f$ be a binary quadratic form with positive discriminant $\Delta$. The *Markov element of* $f$ is defined as

$$M(f) = \inf_{\mathbb{Z}^2\setminus\{(0,0)\}} \left|\frac{f}{\sqrt{\Delta}}\right|.$$

The *Markov spectrum* is the set of values $1/M(f)$ for all such forms $f$.

For a sequence of positive integers $a_1, \dots, a_n$, let

$$[a_1; a_2 : \dots : a_n]$$

denote the continued fraction of $a_1, \dots, a_n$. Below we give an alternative definition of the Markov spectrum.

**Definition 2.** Let $A$ be a doubly infinite sequence of positive integers:

$$A = \dots, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots.$$

We define $M(A)$, the *Markov element of* $A$, by

$$\frac{1}{M(A)} = \sup_{i\in\mathbb{Z}} \bigl(a_i + [0; a_{i+1} : a_{i+2} : \dots] + [0; a_{i-1} : a_{i-2} : \dots]\bigr). \tag{1.1}$$

The right-hand side of Eq. (1.1) is known as the *Perron identity*. The set of values $1/M(A)$ for all such sequences $A$ is called the *Markov spectrum*.

**Remark.** Definitions 1 and 1 are equivalent (see [9]). The sequences $A$ for which $M(A) > 1/3$ are purely periodic and consist solely of the integers 1 and 2. We refer to these sequences as *Markov sequences*.

### *1.2. Graph Structure of Markov Sequences*

Below we give an alternative definition of Markov sequences.

**Definition 3.** Let $\mathbb{Z}^\infty$ be the set of finite sequences of integer elements. Consider the binary operation $\oplus$ on $\mathbb{Z}^\infty$ defined as

$$(a_1,\dots,a_n)\oplus(b_1,\dots,b_m)=(a_1,\dots,a_n,b_1,\dots,b_m).$$

We call this the *concatenation* of the sequences $(a_1,\dots,a_n)$ and $(b_1,\dots,b_m)$. Also, for $A,B,C\in\mathbb{Z}^\infty$, we let

$$\mathcal{L}_\oplus(A,B,C)=(A,A\oplus B,B),\qquad \mathcal{R}_\oplus(A,B,C)=(B,B\oplus C,C).$$

**Definition 4.** We define $\mathcal{G}(\mathbb{Z}^\infty,\oplus,x)$ to be the directed graph whose vertices are the elements of $(\mathbb{Z}^\infty)^3$ and $x$. Vertices $v,w\in(\mathbb{Z}^\infty)^3$ are connected by an edge $(v,w)$ if either

$$w=\mathcal{L}_\oplus(v)\qquad\text{or}\qquad w=\mathcal{R}_\oplus(v).$$

For $A,B\in\mathbb{Z}^\infty$, we write

$$\mathcal{G}(\mathbb{Z}^\infty,\oplus,(A,A\oplus B,B))=\mathcal{G}_{A,B}.$$

**Remark.** The graph $\mathcal{G}_{(1,1),(2,2)}$ is called the *graph of general Markov sequences* and contains all Markov sequences; see [3, 6].

**Definition 5.** Let $v$ be the vertex $(A,A\oplus B,B)\in\mathcal{G}_{A,B}$, and let $w$ be another vertex in $\mathcal{G}_{A,B}$. We say that $(\alpha_1,\dots,\alpha_{2n})$ is a path from $v$ to $w$ if

$$w=\mathcal{L}_\oplus^{\alpha_{2n}}\mathcal{R}_\oplus^{\alpha_{2n-1}}\dots\mathcal{L}_\oplus^{\alpha_2}\mathcal{R}_\oplus^{\alpha_1}(v).$$

We define the $N$th level in $\mathcal{G}_{A,B}$ to be the set of all vertices $w$ such that the path $(\alpha_1,\dots,\alpha_{2n})$ from $v$ to $w$ satisfies $\sum_{i=1}^{2n}\alpha_i=N$.

We order the vertices at each level of $\mathcal{G}_{A,B}$ as follows.

**Definition 6.** For positive integers $n,m,\alpha_1,\dots,\alpha_{2n},\beta_1,\dots,\beta_{2m}$ satisfying $\sum_{i=1}^{2n}\alpha_i=\sum_{i=1}^{2m}\beta_i$, let $w_1$ and $w_2$ be two vertices in $\mathcal{G}_{A,B}$; suppose that

$$w_1=\mathcal{L}_\oplus^{\alpha_{2n}}\mathcal{R}_\oplus^{\alpha_{2n-1}}\dots\mathcal{L}_\oplus^{\alpha_2}\mathcal{R}_\oplus^{\alpha_1}(v)\qquad\text{and}\qquad w_2=\mathcal{L}_\oplus^{\beta_{2m}}\mathcal{R}_\oplus^{\beta_{2m-1}}\dots\mathcal{L}_\oplus^{\beta_2}\mathcal{R}_\oplus^{\beta_1}(v).$$

We define an ordering of vertices by setting

$$w_1\prec w_2$$

if either

$$\begin{aligned}&\alpha_i=\beta_i\quad\text{for } i=1,\dots,2k-1,\ k<m,n,\quad\text{and}\quad \alpha_{2k}>\beta_{2k}\qquad\text{or}\\ &\alpha_i=\beta_i\quad\text{for } i=1,\dots,2k,\ k<m,n,\qquad\text{and}\quad \alpha_{2k+1}<\beta_{2k+1}.\end{aligned}$$

**Definition 7.** The pair $(\mathcal{G}_{A,B},\prec)$ is the graph $\mathcal{G}_{A,B}$ in which each level $n$ is ordered:

$$w_1\prec w_2\prec\dots\prec w_{2^n}.$$

Now we define a sequence $(S(i))_{i=0}^\infty$.

**Definition 8.** Given two sequences $A$ and $B$, we set

$$S(0)=A,\qquad S(1)=B,\qquad S(2)=A\oplus B.$$

For $n>1$ and $1\le i\le 2^{n-1}$, let $S(2^{n-1}+i)$ be the central element of the $i$th vertex in the $n$th level of the ordered graph $(\mathcal{G}_{A,B},\prec)$. We call $(S(i))$ the *ordered Markov sequence for $A$ and $B$*.

When we want to specify the sequences $A$ and $B$, we write $S_{A,B}(n)$.

**Example.** We have

$$S_{(a,a),(b,b)}(14) = (a, a, b, b, a, a, b, b, b, b, a, a, b, b, b, b).$$

For $(a, b) = (1, 2)$, $M(S_{(a,a),(b,b)}(i)) > 1/3$ for all $i \geq 1$.

**Definition 9.** Let $\Lambda = (\lambda_1, \dots, \lambda_{2n})$, and let $\Gamma = (\gamma_1, \dots, \gamma_{2n+1})$. We say that $\Lambda$ and $\Gamma$ are *evenly palindromic* or *oddly palindromic* if there exist $k_1, k_2 \in \mathbb{Z}$ such that, for all $i \in \mathbb{Z}$,

$$\lambda_{k_1+i \bmod 2n} = \lambda_{k_1-i-1 \bmod 2n}$$

or

$$\gamma_{k_2+i+1 \bmod 2n+1} = \gamma_{k_2-i-1 \bmod 2n+1},$$

respectively.

**Definition 10.** We set $d_0 = 0$, $d_1 = 1$, and

$$d_{2n} = d_n, \qquad d_{2n-1} = d_n + d_{n-1}$$

for all positive integers $n > 1$. The sequence $(d_n)_{n \geq 0}$ is called *Stern's diatomic sequence*.

We give the main theorem of this paper.

**Theorem 1.** *Let $A$ and $B$ be two palindromic sequences of positive integers. Let $n$ be a positive integer, and let $N$ be the sum of powers of $A$ and $B$ in $S_{A,B}(n)$. Let $\Lambda_i \in \{A, B\}$ be such that*

$$S_{A,B}(n) = \Lambda_1 \dots \Lambda_N.$$

*Then the following sequences are palindromic:*

$$\begin{cases} \Lambda_{\lceil d_n/2 \rceil} \Lambda_{\lceil d_n/2 \rceil + 1} \dots \Lambda_N \Lambda_1 \dots \Lambda_{\lceil d_n/2 \rceil - 1}, & d_n \text{ is even}, \\ \lceil \Lambda_{\lceil d_n/2 \rceil} \rceil \Lambda_{\lceil d_n/2 \rceil + 1} \dots \Lambda_N \Lambda_1 \dots \Lambda_{\lceil d_n/2 \rceil - 1} \lfloor \Lambda_{\lceil d_n/2 \rceil} \rfloor, & d_n \text{ is odd}. \end{cases}$$

## 2. Proof of Theorem 1

In this section, we prove Theorem 1. We start by stating Proposition 1, which gives a major part of the proof. Subsections 2.1 through 2.4 are devoted to the proof of this proposition; the proof of Theorem 1 is completed in Sec. 2.5.

### *2.1. Alternative Definition of $S(n)$*

We state Proposition 1, which is central to our proof of Theorem 1. Then we give an alternative definition of the sequences $(S(n))$ in Proposition 2, defining each $S(n)$ as the concatenation of the preceding terms in $(S(n))$. We start by defining circular shifts of Markov sequences.

**Definition 11.** Let $A = (a_0, \dots, a_n)$ be a sequence of positive integers. For each $0 \leq i < n$, we define the operation

$$C_i(A) = (a_i, \dots, a_n, a_1, \dots, a_{i-1}).$$

This operation $C_i(A)$ is called the *ith circular shift of $A$*.

**Proposition 1.** *Every sequence $S(i)$ is evenly palindromic. Moreover, the sequence $C_{d_i}(S(i))$, where each $d_i$ is the ith element in Stern's diatomic sequence, is palindromic.*

**Example.** Let $S_{(a,a),(b,b)}(3) = (a, a, a, a, b, b)$, and let $d_3 = 2$. Then

$$C_2(S_{(a,a),(b,b)}(3)) = (a, a, b, b, a, a).$$

Now we define a sequence $(a(j))_{j \geq 0}$, which simplifies the notation of $(S(j))$.

**Definition 12.** Let $a(1)=a(2)=1$. For all positive integers $j > 1$, we set

$$a(2j) = a(j), \qquad a(2j-1) = j.$$

The sequence $(a(j))_{j\geq 0}$ is A003602 in [12]. By $(a^*(j))$ we denote the sequence defined as

$$a^*(j) = \begin{cases} a(j) & \text{if } j > 1, \\ 0 & \text{if } j = 1. \end{cases}$$

For the $a(j)$th element in the sequence $(S(n))$, we write $S \circ a(j)$.

**Example.** The first 10 elements of $(a(j))$ and $(d_j)$ are

$$(a(j))_1^{10} = (1, 1, 2, 1, 3, 2, 4, 1, 5, 3),$$
$$(d_j)_0^9 = (0, 1, 1, 2, 1, 3, 2, 3, 1, 4).$$

The table given below shows the symmetry in Stern's diatomic sequence.

**Table.** The first entries in Stern's diatomic sequence. The list is read from left to right and from top to bottom. The numbers in brackets are used to construct the table, but they are not in the diatomic sequence

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | | | | | | | | 1 | | | | | | | | (1) |
| 1 | | | | | | | | 2 | | | | | | | | (1) |
| 1 | | | | 3 | | | | 2 | | | | 3 | | | | (1) |
| 1 | | 4 | | 3 | | 5 | | 2 | | 5 | | 3 | | 4 | | (1) |
| 1 | 5 | 4 | 7 | 3 | 8 | 5 | 7 | 2 | 7 | 5 | 8 | 3 | 7 | 4 | 5 | (1) |

**Proposition 2.** *Let $a > b$ be positive integers, and let*

$$S(0) = A, \qquad S(1) = B, \qquad S(2) = A \oplus B.$$

*The following definitions of the sequence $(S(j))_{j>2}$ are equivalent*:

*(i) for $n > 1$ and $1 \leq i \leq 2^{n-1}$, $S(2^{n-1}+i)$ is the central element of the $i$th vertex in the $n$th level of the graph*

$$\mathcal{G}(\mathbb{Z}^\infty, \oplus, (A, A \oplus B, B);$$

*(ii)*

$$S(2j) = S(j) \oplus S \circ a(j) \qquad \text{and}$$
$$S(2j-1) = S \circ a^*(j-1) \oplus S(j).$$

**Remark.** Proposition 2 gives us an alternative definition of the sequences $(S(n))$.

Before proving Proposition 2, we prove the following lemma.

**Lemma 1.** *For $m = 2k > 2$, the following relations hold:*

$$S(2^{n-2}+k) = S(a^*(2^{n-1}+m-1)), \; S(2^{n-1}+2k) = S(2^{n-1}+m),$$
$$S(2(2^{n-2}+k)) = S(2^{n-1}+m), \; S(a(2^{n-2}+k)) = S(a(2^{n-1}+m)).$$

*For $m = 2k-1 > 2$, the following relations hold*:

$$S(a^*(2^{n-2}+k-1)) = S(a^*(2^{n-1}+m-1)), \qquad S(2^{n-2}+k) = S(a(2^{n-1}+m)),$$
$$S(2(2^{n-2}+k)-1) = S(2^{n-1}+m), \qquad S(2(2^{n-2}+k)-1) = S(2^{n-1}+m),$$

**Proof.** In each case, the relations follow by the direct application of Definition 12. $\square$

*Proof of Proposition 2.* We prove the proposition by induction on the levels of the graph of Markov sequences. The base of induction is given by

$$\begin{aligned} S(3) &= S(0) \oplus S(2) = S \circ a^*(1) \oplus S(2) \qquad \text{and} \\ S(4) &= S(2) \oplus S(1) = S(2) \oplus S \circ a(2). \end{aligned}$$

Next we assume that the hypothesis is true for every $S(j)$ up to the $(n-1)$th level, that is, for $i = 1, \dots, 2^{n-1}$, we have

$$S(2^{n-1}+i) = \begin{cases} S(2(2^{n-2}+k)) & \text{if } i = 2k, \\ S(2(2^{n-2}+k)-1) & \text{if } i = 2k-1. \end{cases} \tag{2.1}$$

By the definition of $(S(j))$, we have

$$\begin{aligned} S(2(2^{n-2}+k)) &= S(2^{n-2}+k) \oplus S \circ a(2^{n-2}+k) \qquad \text{and} \\ S(2(2^{n-2}+k)-1) &= S \circ a^*(2^{n-2}+k-1)) \oplus S(2^{n-2}+k). \end{aligned}$$

From the induction hypothesis we have

$$\begin{aligned} S(2(2^{n-1}+m)) &= S(2^{n-1}+m) \oplus S \circ a(2^{n-1}+m), \\ S(2(2^{n-1}+m)-1) &= S \circ a^*(2^{n-1}+m-1) \oplus S(2^{n-1}+m). \end{aligned}$$

We apply both operations $\mathcal{L}_\oplus$ and $\mathcal{R}_\oplus$ in either case in (2.1). For $i = 2k$, by Lemma 1 we have

$$\begin{aligned} \mathcal{L}_\oplus(S(2^{n-1}+i)) &= S(2^{n-2}+k) \oplus S(2^{n-1}+2k) \\ &= S \circ a^*(2^{n-1}+m-1) \oplus S(2^{n-1}+m), \\ \mathcal{R}_\oplus(S(2^{n-1}+i)) &= S(2(2^{n-2}+k)) \oplus S \circ a(2^{n-2}+k) \\ &= S(2^{n-1}+m) \oplus S \circ a(2^{n-1}+m). \end{aligned}$$

The case where $i = 2k-1$ is similar in view of Lemma 1. Thus, we have considered all cases for the element in the $n$th level coming from (2.1), which completes the proof. $\square$

**Definition 13.** The length of the sequence $S(n)$ is denoted by $|S(n)|$.

**Remark.** If follows from Proposition 2 and $|S(0)| = |S(1)|$ that

$$\begin{aligned} |S(2n)| &= |S(n)| + |S \circ a(n)| \qquad \text{and} \\ |S(2n-1)| &= |S(a(n-1))| + |S(n)|. \end{aligned}$$

### 2.2. *Symmetry of Construction of Sequences $S((n))$*

We use the symmetry of the graph $\mathcal{G}_{a,b}, \prec$ and of Stern's diatomic sequence to prove Lemma 5, which significantly shortens the proof of Proposition 1. For this, we need the following short lemmas.

**Lemma 2.** *For $k \geq 2$,*

$$|S(k)| = |S \circ a(k)| + |S \circ a(k-1)|.$$

**Proof.** We prove this lemma by induction. First, we have

$$|S(2)| = 4 = 2|S(1)| = |S \circ a(2)| + |S \circ a(1)|.$$

Assume that $|S(k)| = |S \circ a(k)| + |S \circ a(k-1)|$ for all $k = 2, \dots, N-1$, where $N \in \mathbb{Z}$ is some integer. We have two cases:

*N even*: If $N = 2m$, then

$$\begin{aligned}|S(2m)| &= |S(m)| + |S \circ a(m)| \\ &= |S \circ a(2m-1)| + |S \circ a(2m)| \\ &= |S \circ a(N-1)| + |S \circ a(N)|.\end{aligned}$$

*N odd*: The case $N = 2m - 1$ is similar. This concludes the proof.

□

**Lemma 3.** *For $k \geq 1$,*

$$|S \circ a(k)| = 2d_k.$$

**Proof.** We prove this lemma by induction again. First, we have

$$|S \circ a(2)| = |S \circ a(1)| = 2 = 2d_1 = 2d_2.$$

Assume that $|S \circ a(k)| = 2d_k$ for all $k = 1, \dots, N-1$, where $N$ is some positive integer. We have two cases:

*N even*: If $N = 2m$, then $d_{2m} = d_m$ and $|S \circ a(2m)| = |S \circ a(m)|$. Thus, $2d_N = |S \circ a(N)|$, which happens if and only if $2d_m = |S \circ a(m)|$.

*N odd*: The case $N = 2m-1$ is proved in a similar way. This concludes the proof.

□

**Lemma 4.** *For a positive odd integer $k = k_1$, let $i \in \mathbb{Z}$ be such that the numbers*

$$k_j = \frac{k_{j-1} + 1}{2}, \ j = 1, \dots, i,$$

*are positive integers and $k_i$ is even. Let $k_{i+1} = k_i/2$. Then*

$$\frac{|S(k_{i+1})|}{2} = d_{k-1}.$$

**Proof.** From Definition 12 and Lemma 3, we obtain

$$a(2k_{i+1}-1) = k_{i+1}, \quad a(k_i-1) = k_{i+1}, \quad \text{and} \quad d_{k_i-1} = d_{k_2-1} = \dots = d_{k_i-1}.$$

Hence $S(k_{i+1}) = S \circ a(k_i-1)$, and, therefore,

$$|S \circ a(k_i-1)| = 2d_{k-1}.$$

□

**Lemma 5.** *Let $n > 1$, and let $i \in \{1, \dots, 2^{n-1}\}$. Then, for the integers $k' = 6 \cdot 2^{n-2}+i-1$ and $k'' = 6 \cdot 2^{n-2}-i+1$, the following equality holds*:

$$\frac{|S(k'+1)|}{2} - d_{k'+1} = d_{k''}.$$

**Remark.** Let $f$ be a function taking a sequence $S_{A,B}(k)$ to $S_{B,A}(k)$. Due to the symmetry of Definition 8, for every $k > 2$, there is a number $l$ such that

$$S(k) = \overline{f(S(m))}.$$

Lemma 5 says that if $k = 6 \cdot 2^{n-2} + i$ for positive integers $n$ and $i = 1, \dots, 2^{n-1}$, then

$$m = 6 \cdot 2^{n-2} - i + 1.$$

*of Lemma 5*. We have $k'+1 = a(2(k'+1)-1)$ by Definition 12. Using Lemma 3, we obtain

$$\frac{|S(k'+1)|}{2} = \frac{|S \circ a(2(k'+1)-1)|}{2} = d_{2(k'+1)-1} = d_{k'+1} + d_{k'}.$$

It remains to show that $d_{k'} = d_{k''}$, but this follows from the symmetry seen in the table. □

### *2.3. Alternative Form of Markov Sequences*

In the proof of Proposition 1, we shall use formulas for Markov sequences different from those in Proposition 2; we set them down in the following two lemmas.

**Lemma 6.** *Given a positive even integer $k = k_1$, let $i \in \mathbb{Z}$ be such that the numbers*

$$k_j = \frac{k_{j-1}}{2}, \qquad j = 1, \dots, i,$$

*are positive integers and $k_i$ is odd. Let $k_{i+1} = (k_i+1)/2$. Then*

$$S(k) = S \circ a^*(k_{i+1}-1) \oplus S(k_{i+1})^i,$$

*where the superscript $i$ denotes a sequence concatenated $i$ times.*

**Proof.** Applying Proposition 2, we obtain

$$S(k) = S(k_2) \oplus S \circ a(k_2) = S(k_i) \oplus S \circ a(k_i)^{i-1}.$$

Since $k_i = 2k_{i+1}-1$, we have

$$S(k_i) = S \circ a^*(k_{i+1}-1) \oplus S(k_{i+1}),$$

and, therefore,

$$S(k) = S \circ a^*(k_{i+1}-1) \oplus S(k_{i+1})^i,$$

which proves the lemma. □

**Lemma 7.** *For a positive odd integer $k = k_1$, let $i \in \mathbb{Z}$ be such that the numbers*

$$k_j = \frac{k_{j-1}+1}{2}, \qquad j = 1, \dots, i,$$

*are positive integers and $k_i = (k_{i-1}+1)/2$ is even. Let $k_{i+1} = k_i/2$. Then*

$$S(k) = \begin{cases} S(k_{i+1})^i \oplus S \circ a(k_{i+1}) & \text{if } k_i > 2, \\ S(0)^i \oplus S(1) & \text{if } k_i = 2. \end{cases}$$

**Proof.** This lemma, as well as Lemma 6, is proved by applying Proposition 2. □

**Remark.** The situation in which

$$S(k) = S(p)^\lambda \oplus S(q)^\rho,$$

where $\lambda = \rho = 1$, can never occur, since if $k$ is even and $k/2$ is odd, then $i = 2$ and

$$S(k) = S \circ a^*(k_{i+1}-1) \oplus S(k_{i+1})^2.$$

A similar statement holds if $k$ is odd.

Now we give the final lemma for Proposition 1.

**Lemma 8.** *Assume that the $d_n$th circular shift of $S(n)$ is palindromic for all $n = 1, \dots, k_1-1 \in \mathbb{Z}$ and let $L = d_{k_2}$.*

*(i) Let $k = k_1, \dots, k_{i+1}$ be as in Lemma 6 for some positive integer $i$, and let*

$$R = L + \frac{|S \circ a(k_{i+1}-1)| + (i-1)|S(k_{i+1})|}{2}.$$

*Then $R > |S \circ a(k_{i+1}-1)|$.*

*(ii) Let $k = k_1, \dots, k_{j+1}$ be as in Lemma 7 for some positive integer $j$, and let*

$$R = L + \frac{|S(k_2)|}{2}.$$

*Then $L < (j-1)|S(k_{j+1})|$.*

**Proof.** (i) If $R > |S \circ a(k_{i+1}-1)|$ for $i = 2$, then this is also true for all $i > 2$. Thus, let $i = 2$. We want to show that

$$d_{k_2} + \frac{|S \circ a(k_3-1)| + |S(k_2)|}{2} > |S \circ a(k_3-1)|. \tag{2.2}$$

Since $|S \circ a(k_2)|/2 = d_{k_2}$ by Lemmas 2 and 3, it follows that (2.2) becomes

$$2|S \circ a(k_2)| + |S \circ a(k_2-1)| > |S \circ a(k_3-1)|,$$

which is true, since

$$a(k_3-1) = a\left(\frac{k_2+1}{2}-1\right) = a\left(\frac{k_2-1}{2}\right) = a(k_2-1).$$

(ii) Consider the case were $k = k_1, \dots, k_{j+1}$ are as in Lemma 7. Let $k_1^*,\dots,k_j^*$ be as in Lemma 7 with $k_j^* = k_2$ and $k_{j-1}^* = k$. Since $k$ is odd, it follows that if $L < |S(k_2/2)|$, then

$$L < \left|S\left(\frac{k_2}{2}\right)\right| < (j-1)\left|S\left(\frac{k_2}{2}\right)\right| = (j-1)\left|S\left(\frac{k_j^*}{2}\right)\right|.$$

Let $k_3 = k_2/2$. By Lemmas 2 and 3 we have

$$L = d_{k_2} = \frac{|S \circ a(k_3)|}{2} \quad \text{and} \quad |S(k_3)| = |S \circ a(k_3)| + |S \circ a(k_3-1)|.$$

These two facts imply

$$L = \frac{|S \circ a(k_3)|}{2} < |S \circ a(k_3)| < |S \circ a(k_3)| + |S \circ a(k_3-1)| = |S(k_3)|,$$

as required. $\square$

### *2.4. Proof of Proposition 1*

In this section, we give the proof of Proposition 1.

*Proof of Proposition 1.* We prove the proposition by induction on $n$. We must show two things: firstly, that $S_{(a,,a),(b,b)}(n)$ is evenly palindromic, and secondly, that $C_{d_n}(S(n))$ is palindromic.

It is clear that the statement holds for $n = 0, 1, 2$. Assume that $k > 3$ and the statement is true for all $n = 3, \dots, k-1$. We have two cases, in which $k$ is either even or odd. In either case, we denote the elements of the sequence $S(k)$ by $\lambda$'s:

$$S(k) = (\lambda_1, \dots, \lambda_{|S(k)|}).$$

(i) Let $k = k_1$ be even, and let $i \in \mathbb{Z}$ be such that the numbers

$$k_j = \frac{k_{j-1}}{2}, \qquad j = 1, \dots, i,$$

are positive integers and $k_i$ is odd. Let $k_{i+1} = (k_i + 1)/2$. Let $N$ and $M$ be the lengths of the sequences $S \circ a^*(k_{i+1}-1)$ and $S(k_{i+1})$, respectively. Then $S(k) = S \circ a^*(k_{i+1}-1) \oplus S(k_{i+1})^i = (\lambda_1, \dots, \lambda_{N+iM})$.

By the induction hypothesis, $C_{d_{k_2}}(S(k_2))$ is palindromic. Recall that $S(k) = S(k_2) \oplus S(k_{i+1})$.

Let $L = d_{k_2}$. We

$$R = L + \frac{N + (i-1)M}{2}.$$

Using the fact that $R > N$ by Lemma 8, we obtain the following relations for the elements of $S(k_2)$:

$$\lambda_L = \lambda_{L+1}, \ldots, \lambda_R = \lambda_{R+1}.$$

They imply the following relations for the elements of $S(k)$:

$$\lambda_L = \lambda_{L+1}, \ldots, \lambda_R = \lambda_{R+M+1}.$$

We must show that the following condition holds:

$$\lambda_R = \lambda_{R+M+1}, \ldots, \lambda_{R+M/2} = \lambda_{R+M/2+1}.$$

To do this, we note that, since $R > N$, we can ignore the sequence $S \circ a^*(k_{i+1}-1)$ at the beginning of $S(k)$, remove any excess copies of the sequence $S(k_{i+1})$, and conclude that this condition is equivalent to

$$R+\frac{M}{2}-N \mod M = \begin{cases} d_{k_{i+1}} & \text{if } i \text{ is even,} \\ d_{k_{i+1}} + \frac{|S(k_{i+1})|}{2} & \text{if } i \text{ is odd.} \end{cases} \tag{2.3}$$

After the substitution of the expressions for $R$, $M$, and $N$, the left-hand side of (2.3) becomes

$$d_{k_2}+\frac{|S(k_2)|}{2}+\frac{|S(k_{i+1})|}{2}-|S \circ a(k_{i+1}-1)| \mod |S(k_{i+1})|.$$

Recall that $d_{k_2} = \ldots = d_{2k_i-1} = d_{k_{i+1}}+d_{k_{i+1}-1}$ and

$$|S(k_2)| = |S \circ a(k_{i+1}-1)| + (i-1)|S(k_{i+1})|,$$

and, therefore,

$$R+\frac{M}{2}-N \mod M = d_{k_{i+1}} + d_{k_{i+1}-1} + \frac{i|S(k_{i+1})|}{2} - \frac{|S \circ a(k_{i+1}-1)|}{2} \mod M.$$

If $i$ is even, then this becomes

$$d_{k_{i+1}} + d_{k_{i+1}-1} - \frac{|S \circ a(k_{i+1}-1)|}{2} \equiv d_{k_{i+1}} \mod |S(k_{i+1})|,$$

which is true by Lemma 3.

If $i$ is odd, then we have

$$d_{k_{i+1}} + d_{k_{i+1}-1} + \frac{|S(k_{i+1})|}{2} - \frac{|S \circ a(k_{i+1}-1)|}{2} \equiv d_{k_{i+1}} + \frac{|S(k_{i+1})|}{2} \mod |S(k_{i+1})|,$$

which is again true by Lemma 3. Thus, the $d_k$th circular shift of $S(k)$ is palindromic, which completes the induction.

$(ii)$ The proof in the case where $k = k_1$ is odd is equivalent to that in the even case by Lemma 5 and Remark 2.2.

This concludes the proof of Proposition 1.

□

### 2.5. Completion of the Proof of Theorem 1

In this subsection, we complete the proof of Theorem 1. We start with the following Lemma, coming from [3].

**Lemma 9.** *For each $k > 1$, there is an $n$ such that*

$$S_{A,B}(k) = A^{\alpha_1} B^{\beta_1} \dots A^{\alpha_n} B^{\beta_n}.$$

*According to [3], if $A = (a, a)$ and $B = (b, b)$, then either $\alpha_i = 1$ and $\beta_i \geq 1$ for all $i$ and at least one $\beta_i$ is greater that 1 or vice versa.*

**Definition 14.** Let $\Lambda = \overline{\Lambda} = (\lambda_1, \dots, \lambda_m)$ for some positive $m$. The half sequences $\lfloor \Lambda \rfloor$ and $\lceil \Lambda \rceil$ are defined by

$$\lfloor \Lambda \rfloor = (\lambda_1, \dots, \lambda_{\lfloor m/2 \rfloor}) \qquad \text{and} \qquad \lceil \Lambda \rceil = (\lambda_{\lceil m/2 \rceil}, \dots, \lambda_m).$$

**Remark.** Clearly, $\overline{\lfloor \Lambda \rfloor} = \lceil \Lambda \rceil$.

*of Theorem 1.* Let $d_n$ be odd. For $A = (a, a)$, $B = (b, b)$, and $\lambda_1 = \lambda_2 \in \{a, b\}$, the sequence $(\lambda_1)\Lambda_{\lceil d_n/2 \rceil + 1} \dots \Lambda_{\lfloor d_n/2 \rfloor - 1}(\lambda_2)$ is palindromic by Lemma 9 and Proposition 1. Replacing $A$ and $B$ in $\Lambda_{\lceil d_n/2 \rceil + 1} \dots \Lambda_{\lfloor d_n/2 \rfloor - 1}$ by any two palindromic sequences does not affect this fact, neither does setting $\lambda_1 = \lceil \Lambda \rceil$ and $\lambda_2 = \lfloor \Lambda \rfloor$ for any palindromic $\Lambda$.

If $d_n$ is even, then $\Lambda_{d_n/2} = \Lambda_{-1+d_n/2}$ by Lemma 9 and Proposition 1. The theorem follows as a corollary of Lemma 9. ☐

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