MATH UN1208, Honors Math B

Columbia University, Spring 2020

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Contents

1 January 22, 2020

1.1 Introduction

Administrative Stuff

- Webpage . . . /honorsmathB
- HW 1 due in a week (1/29)
- Office Hours: T 9 - 11, F 1-2
- Midterm in class last Wed before Spring Break
- Textbook: Vol II this semester
- Will cover almost all linear algebra, multivariable calculus with fund thm of calculus in dim ≤ 3

1.2 Linear Algebra Recap

- We defined a field (e.g. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$)
- We defined vector spaces. Has vector addition and scalar multiplication obeying various laws. (e.g. $\mathbb{R}^n := \{\text{functions } [n] \to \mathbb{R}\}$
  $= \{\text{n-tuples of elements in } \mathbb{R}\}$)
- We defined linear maps $V \to W$ (i.e. a function $F : V \to W$ such that $F(V_1 + V_2) = F(V_1) + F(V_2)$ and $F(cV_1) = cF(V_1)$)
- We defined a subspace of a vector space

$$W_1, W_2 \subseteq V \text{ subspaces} \Rightarrow W_1 \cap W_2 \text{ is a subspace}$$

1.3 Continuation

Definition. *Given a linear map $f : V \to W$,*

$$\ker f := \{v \in V \mid f(v) = 0\} \subseteq V$$

$$\operatorname{im} f := f(V) = \{w \in W \mid \exists v \in V, f(V) = W\}$$

Proposition. *Given a linear $f : V \to W$,*

1. *f injective $\iff \ker f = \{0\}$*
2. *f surjective $\iff \operatorname{im} f = W$*

"Pf". .

1. HW
2. Obvious

□

Definition. *A linear $f : V \to W$ is an isomorphism if it is bijective.*

Proposition. *A linear map $f : V \to W$ is bijective if and only if it has a linear inverse.*

Proof. Assume f is a bijection. Let $g : W \to V$ be the inverse. Need to check that g is linear. We know $f \circ g = Id_W$ and $g \circ f = Id_V$. Given $W_1, W_2 \in W$,

$$\begin{aligned} g(W_1 + W_2) &= g(f(g(W_1))) + f(g(W_2)) \\ &= g(f(g(W_1) + g(W_2))) \\ &= g(W_1) + g(W_2) \end{aligned}$$

$$\begin{aligned} g(cW_1) &= g(cf(g(W_1))) \\ &= g(f(cg(W_1))) \\ &= cg(W_1) \end{aligned}$$

□

Definition. *V, W vector spaces/ F*

$$\mathcal{L}(V,W) := \{\textit{linear maps}\, V \to W\} \subseteq \{\textit{functions}\, V \to W\}$$

On HW: Check that $\mathcal{L}(V,W)$ is a vector space.

Proposition. *If $f : U \to V$ and $g : V \to W$ are linear, then $g \circ f : V \to W$ is linear.*

Proof.

$$\begin{aligned}(g \circ f)(cV_1) &= g(f(cV_1)) \\ &= g(cf(V_1)) \\ &= cg(f(V_1)) \\ &= c(g \circ f)(V_1)\end{aligned}$$

and similar for addition. $\square$

1.4 Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition. *The <u>standard basis vectors</u> of $\mathbb{R}^n$ are $e_i = (0, 0, \ldots, 1, \ldots, 0)$.*

$$\begin{aligned} e.g. \quad e_1 &= (1, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, \ldots, 0, 1)\end{aligned}$$

Notation: if $X \in \mathbb{R}^n$, then $x = (x_1, \ldots, x_n)$. Call x_i then i$^{\text{th}}$ component of x.

$(e_i)_j = \delta_{ij}$ "Kronecker delta"

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Proposition. *$\forall x \in \mathbb{R}^n$,*

$$\forall i, x_i = a_i \iff x = \sum_{i=1}^{n} a_i e_i$$

In other words, $x = \sum_{i=1}^{n} x_i e_i$.

Proof.

$$\begin{aligned}\sum_{i=1}^{n} x_i e_i &= (1, 0, 0, \ldots, 0)x_1 + (0, 1, \ldots, 0)x_2 + \ldots + (0, 0, \ldots, 1)x_n \\ &= (x_1, 0, 0, \ldots, 0) + (0, x_2, 0, \ldots) + \ldots + (0, 0, \ldots, x_n) \\ &= (x_1, x_2, \ldots, x_n)\end{aligned}$$

$\square$

In other words, any vector in $\mathbb{R}^n$ can be uniquely written as a linear combinations of the e_i.

1.5 Matrices

Definition. *For $m, n \in \mathbb{Z}_{\geq 0}$, an $m \times n$ matrix over F is a $m \times n$ box of elements of F.*

e.g. $\begin{bmatrix} 0 & 3 \\ -3 & \pi \\ 0 & 4 \end{bmatrix}$

Better Definition. *An $m \times n$ matrix A over F is a function $[m] \times [n] \to F$.*

Notation: Write $A((i,j)) =: A_{ij}$

$B = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$ $B_{11} = 1, B_{12} = 0, B_{21} = 2, B_{22} = 5$

Set of $m \times n$ matrices over F is called $M_{m \times n}(F)$. It is a vector space!

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$$

$$c \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} cA_{11} \\ cA_{21} \end{pmatrix}$$

Overall Result $\Rightarrow M_{m \times n}(F)$ is an F - vector space.

Will prove: $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m \times n}(\mathbb{R})$.

2 January 27, 2020

2.1 TA Office Hours

M 9 -11 (Carson) T 12-2 (Ahmed) W 9 -12 (Sayan)

2.2 Classifying linear maps $\mathbb{R}^n \to \mathbb{R}^m$ by matrix

Want to defined a linear map $M : M_{m \times n}(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

$$A \mapsto (T : \mathbb{R}^n \to \mathbb{R}^m, T(x) = \begin{pmatrix} \sum_{j=1}^n A_{1j} x_j \\ \sum_{j=1}^n A_{2j}, x_j \\ \vdots \\ \sum_{j=1}^n A_{nj}, x_j \end{pmatrix})$$

We write

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n \end{bmatrix}$$

Check that $\mu(A)$ is linear ($T = \mu(A)$):

$$\begin{aligned}(T(x+y))_i &= \sum_{j=1}^{n} A_{ij}(x_j + y_j) \\ &= \sum_{j=1}^{n} A_{ij}x_j + \sum_{j=1}^{n} A_{ij}y_j \\ &= (T(x))_i + (T(y))_i \\ &= (T(x) + T(y))_i\end{aligned}$$

$$\begin{aligned}(T(cx))_i &= \sum_{j=1}^{n} A_{ij}cx_j \\ &= c\sum_{j=1}^{n} A_{ij}x_j \\ &= c(T(x))_i \\ &= (cT(x))_i\end{aligned}$$

2.3 Proof that matrices work as linear transformations of linear maps

Theorem 2.1. *M is a linear isomorphism.*

Proof. Let T_A denote $\mu(A)$.
Linearity: T_{A+B} sends x to the vector with i^{th} componen

$$\begin{aligned}\sum_{j=1}^{n}(A+B)_{ij}x_j &= \sum_{j=1}^{n}(A_{ij} + B_j)x_j \\ &= \sum A_{ij}x_j + \sum B_{ij}x_j\end{aligned}$$

$T_A + T_B$ sends x to vector w/ i^{th} component.

$$\sum A_{ij}x_j + \sum B_{ij}x_j \checkmark$$

Scalar Multiplication - similar

Injective: Suffices to show that $\ker\mu = \{0\}$. Suppose $T_A(x) = \widehat{0}$ for all x. By plugginh in $e_i, \ldots, e_n$ $\left[(e_i)_j = \delta_{ij}\right]$

$$(T_A(e_k))_i = \sum_{j=1}^{n} A_{ij}(e_k)_j = A_{ik} \cdot 1 = A_{ik} = 0 \Rightarrow A = 0$$

Surjective: Suppose we have a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Consider the vectors

$$T(e_k) = \begin{pmatrix} A_{1k} \\ \vdots \\ A_{mk} \end{pmatrix}$$

Now let $A \in M_{m\times n}(\mathbb{R})$ such that

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

We have to check $T_A = T$. Need to check that $T_A(x) = T(x)\, \forall x \in \mathbb{R}^n$. Write $x = (x_1, \ldots, x_n) = \sum_{i=1}^n x_i e_i$.

$$T_A(x) = T_A\left(\sum_{i=1}^{j} x_i e_i\right) = \sum_{i=1}^{n} x_i T_A(e_i)$$

$$T(x) = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T(e_i)$$

So it suffices to show that $T_A(e_i) = T(e_i)$ for each i.

$$T(e_i) = \begin{pmatrix} A_{1i} \\ \vdots \\ A_{mi} \end{pmatrix}$$

$$T_A(e_i) = \begin{pmatrix} \sum_{j=1}^n A_{1i}(e_i)_j \\ \sum_{j=1}^n A_{2j}(e_j)_j \\ \vdots \\ \sum_{j=1}^n A_{nj}(e_i)_j \end{pmatrix} = \begin{pmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{mi} \end{pmatrix}$$

□

Lesson: Standard basis vectors are very useful!

2.4 Basis Vectors

Recall: A linear combo of $v_1, \ldots, v_n \in V$ is a vector of the form $\sum_{i=1}^n c_i v_i$ where F is a field and V is a vector space over F.

Definition. *A linear combo of a set of vectors S is any vector of the form $\sum_{i=1}^n c_i v_i$ where $c_i \in F$ and $v_i \in S$.*

Definition. *A set of vectors $S \subseteq V$ spans V if every $v \in V$ is a linear combo of S.*
i.e. $\forall v \in V, \exists v_i, \ldots, v_n \in S$ and $c_i, \ldots, c_n \in F$ such that $v = \sum_{i=1}^n c_i v_i$.
Write span$(S) = V$.

Ex $\text{span}(\{e_1, \dots, e_n\}) = \mathbb{R}^n$

Ex $\text{span}(\{(1,1), (-1,0)\}) = \mathbb{R}^2$
Why? $(a,b) = b(1,1) + (b-a)(-1,0)$

Ex $\{\}$ spans $\mathbb{R}^0 = \{0\}$

Ex $\text{span}(\{(1,0), (0,1), (3,2)\}) = \mathbb{R}^2$

Definition. *A set $S \subseteq V$ is linearly independent if whenever*

$$\sum_{i=0}^{n} c_i v_i = 0 \implies all\, c_i = 0$$

Note:

$$\text{Linear Independence} \iff \text{whenever } \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i, c_i = d_i$$

3 January 29, 2020

3.1 Definitions from Last Class

Recall: Let $S \subseteq V$ V a v.s. $/F$

1. S spans V if for every $v \in V$, $\exists s_1, \dots, s_n \in S$ and $c_1, \dots, c_n \in F$ such that $v = \sum_{i=1}^{n} c_i s_i$.
2. S is linearly independent if whenever $\sum_{i=1}^{n} c_i s_i = \vec{0}$ with $c_i \in F$ and $s_i \in S$, we have $\forall i, c_i = 0$.

3.2 Examples of Linear Dependence/Independence

1. If $\vec{0} \in S$, then S is dependent. $c\vec{0} = \vec{0}$ for any c, so c can be $\neq 0$. Uniqueness fails.
2. $V = \mathbb{R}^3$. Take 3 vectors. These are independent.

$$(1,0,0) \quad (1,1,0) \quad (1,0,1)$$

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,0,1) = (0,0,0)$$

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ \quad c_2 \qquad\quad = 0 \\ \qquad\qquad c_3 = 0 \end{cases}$$

$$\Rightarrow c_i \text{ all } = 0$$

3. $V = \mathbb{R}^2$. Take 3 vectors. These are dependent.

$$(1,0) \quad (0,1) \quad (1,01)$$

$$1 \cdot (1,0) + (-1) \cdot (0,1) + (-1)(1,-1) = (0,0).$$

   Thus, coefficients not all $= 0$.

4. $V = \mathbb{R}^n$, set of standard basis elements is independent.

$$\sum_{i=1}^{n} c_i e_i = \vec{0} \Rightarrow c_i = 0$$

5. $S = \varnothing$ is independent. No coefficients to choose, so vacuous.

6. $S = \{V\}$ is independent $\iff V \neq \vec{0}$

7. $S = \{V_1, V_2\}$ is independent $\iff$ there is no $c \in F$ such that $V_1 = CV_2$ or $V_2 = CV_1$.
   S dependent $\iff \exists a, b$ not both zero such that $aV_1 + bV_2 = \vec{0}$. If $a \neq 0$ then $V_1 = \frac{-b}{a} V_2$. If $b \neq 0$ then $V_2 = \frac{-a}{b} V_1$.

3.3 Basis

Definition. *$S \subseteq V$ is a <u>basis</u> if S is linearly independent and S spans V.*

<u>Ex</u> Let $V =$ all polynomials/ $\mathbb{R}$.

Claim: $S = \{1, x, x^2, x^3, x^4, \dots\}$
$= \{x^n \mid n \in \mathbb{Z}_{\geq 0}\}$

S spans V by definition: every polynomial is $\sum_{i=1}^{n} c_i x_i =$ linear combo of x^i.

S linearly independent: Check that $\sum_{i=1}^{n} c_i x^i = 0 \Rightarrow c_i = 0$ (from one long problem on final exam!)

Plug in $x = 0 \Rightarrow c_0 = 0$

$$\sum_{i=1}^{n} c_i i x^{i-1} = 0$$

Plug in $x = 0 \Rightarrow c_1 = 0$

etc... Keep differentiating

<u>Ex</u>

$$f(x) = x(x-1)$$
$$g(x) = (x-1)(x-2)$$
$$h(x) = x(x-2)$$

Claim: These are independent.

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

Plug in $x = 0 \Rightarrow c_2 \cdot 2 = 0 \Rightarrow c_2 = 0$
Plug in $x = 1 \Rightarrow c_3(-1) = 0 \Rightarrow c_3 = 0$
Plug in $x = 2 \Rightarrow c_1 \cdot 2 = 0 \Rightarrow c_1 = 0$

Lemma 3.1 (Lemma of Preceding Elements)**.** *V v.s./F $\quad v_1, \dots v_n$ is a sequence of elements of V. Suppose that $\{v_1, \dots, v_n\}$ is dependent.*
$\Rightarrow$ Some v_k may be written as a linear combo of $v_1, \dots, v_{k-1}$.

Proof. $\{v_1, \dots, v_n\}$ dependent $\Rightarrow \exists c_1 \in F$ not all $= 0$ such that $\sum_{i=1}^{n} c_i v_i = \vec{0}$. Let c_k be the last nonzero coefficient.

$$\text{Then } \sum_{i=1}^{k} c_i v_i = 0 \Rightarrow c_k v_k = -\sum_{i=1}^{k-1} c_i v_i$$
$$\Rightarrow v_k = -\frac{1}{c_k} \sum_{i=1}^{k-1} c_i v_i$$

□

Definition. *A vector space is finite dimensional if it has a finite basis. It is infinite-dimensional if not.*

Ex $\quad \mathbb{R}^n$ is a finite-dimensional vector space. (standard basis have n elements)

Theorem 3.2. *Let $\{v_1, \dots, v_n\}$ be a basis of V. Suppose $\{u_1, \dots, u_k\}$ is independent. Then $k \leq n$. (i.e. a basis has max size over all independent sets)*

Proof. Equivent: if $k > n$, then $\{u_1, \dots, u_k\}$ are dependent.

$u_1 \in \operatorname{span}\{v_1, \dots, v_n\}$

$$(v_1, \dots, v_n, u_1) \text{ is linearly dep.}$$

$$(u_1, v_1, \dots, v_n) \text{ is linearly dep.}$$

Lemma $\Rightarrow$ some element is linear combination of previous elements.

If it's u_1, then $v_1 = 0 \Rightarrow$ we're done.
If not, then $v_i \in \operatorname{span}(u_1, v_1, \dots v_{i-1})$.

$$\text{Therefore } \{u_1, v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \text{ spans } V$$

$\Rightarrow \{u_1, v_1, \dots, \hat{v_i}(\text{hat} := \text{remove element}), \dots, v_n, u_2\}$ is dependent.
$\Rightarrow \{u_1, u_2, v_1, \dots, \hat{v_i}, \dots, v_n\}$ is dependent.

Lemma $\Rightarrow$ some element is combo of previous ones. If u_1 or u_2 is said element, we're done.

Proceed by induction: get that, since $k > n$, $\{u_1, u_2, \ldots, u_n\}$ spans V
$\Rightarrow u_{n+1}$ is linear combo of $\{u_1, \ldots, u_n\}$
$\Rightarrow$ we're done. $\square$

Corollary 3.2.1. *Any two finite bases have some number of elements.*

Proof.
$\{v_1, \ldots v_n\}$ Thm $\Rightarrow k \leq n$
$\{u_1, \ldots, u_k\}$ Thm applied to Switching them $\Rightarrow n \leq k \Rightarrow k = n$. $\square$

Definition. *V finite-dimensional v.s.*

$$\dim(V) = \textit{ number of elements in any basis of } V$$

Ex $\dim(\mathbb{R}^n) = n$

Proposition. *If V is finite-dimensional, then any independent set $\{v_1, \ldots, v_k\}$ may be extended to a basis. (i.e. $\exists \{v_1, \ldots, v_n\}$ that is a basis)*

Proof. If $\{v_1, \ldots, v_k\}$ spans V, we're done. Otherwise, pick some $v_{k+1} \in V -$ span$\{v_1, \ldots, v_k\}$. By lemma of preceding element, $\{v_1, \ldots, v_{k+1}\}$ still independent. Continue inductively. $\square$

4 February 3, 2020

4.1 Basis (Cont.)

Proposition. *If V is finite-dimensional and $S = \{v_1, \ldots, v_n\}$ spans V, then there is a subset of S that is a basis for V.*

Proof. If no v_i is a linear combination of $v_1, \ldots, v_{i-1}$, then we're done by lemma of preceding elements. Otherwise, consider the first v_j such that $v_j \in$ span $\{v_1, \ldots, v_{j-1}\}$. So throw out v_j and $\{v_1, \ldots, \hat{v_j}, \ldots, v_n\}$ still spans V. Continue inductively. $\square$

Proposition. *$S = \{v_1, \ldots, v_n\} \subseteq V$ finite-dimensional. Any two of the following implies the third.*

1. *S is independent*
2. *S spans V*
3. $\dim V = n$

Proof.
1) + 2) $\Rightarrow$ 3) by definition of basis + dim.
1) + 3) $\Rightarrow$ 2) We can fill out $\{v_i\}$ to be a basis by proposition last class. Such a basis has $\dim V = n$ elements, so we added nothing.
2) + 3) $\Rightarrow$ 1) By previous proposition we can take away elements to form a basis, but a basis has $\dim V = n$ elements, so we did nothing. $\square$

4.2 Construction Principle

Say we have two vector spaces V, W.

Theorem 4.1 (Construction Principle)**.** *Suppose $v_1, \ldots, v_n$ is an ordered basis for V and $w_1, \ldots, w_m \in W$ Then $\exists!$ linear map $F : V \to W$ such that $F(v_i) = w_i \, \forall i$.*

Proof. Define F as follows: if $v = \sum_{i=1}^n c_i v_i$, then let $F(v) = \sum_{i=1}^n c_i w_i$. Well-defined since expression for v in terms of v_i is unique. Also $F(v_i) = w_i$.

Linear: if $v = \sum c_i v_i, v' = \sum c_i' v_i$, then $F(v + v') = \sum (c_i + c_i') w_i = \sum c_i w_i + \sum c_i' w_i = F(v) + F(v')$. If $v = \sum c_i v_i$ and $c \in F$, then $cv = \sum (cc_i) v_i$. So $F(cv) = \sum cc_i w_i = c \sum c_i w_i = cF(v)$.

Uniqueness: Suppose G is another such function (i.e. G is linear and $G(v_i) = w_i \forall i$). Let $v \in V$ and write $v = \sum c_i v_i$. Then $G(v) = G(\sum c_i v_i) = \sum c_i G(v_i) = \sum c_i w_i = F(v)$. Works for any $v \in V$, therefore, $F = G$. $\square$

<u>Ex</u> $V = \text{span}\,\{1, x, x^2, \ldots, x^n\} \subset \mathcal{F}(\mathbb{R}\mathbb{R})$
$\quad = \text{polynomials of degree} \leq n$
Let $W = \mathbb{R}^{n+1}$. Thm $\Rightarrow \exists!$ a linear map sending x^i to e_{i+1}. Also, $\Rightarrow \exists!$ a linear map sending e_{i+1} to x^i. These are inverse maps! In particular $V \simeq W$. $\simeq$ means isomorphism.

In general:

Corollary 4.1.1. *Any finite dimensional vector space V with $\dim V = n$ is isomorphic to F^n.*

Proof. Choose a basis for V and do the same as above example. $\square$

4.3 Rank-Nullity Theorem

Theorem 4.2 (Rank-Nullity)**.** *Let $T : V \to W$ linear, V finite dimensional. Then,* $\text{im}(T)$ *is finite dimensional and*

$$\dim(\ker T) + \dim(\text{im}\, T) = \dim V$$

Proof. Let $\{v_1, \ldots, v_j\}$ be a basis of $\ker T$. Extend to $\{v_1, \ldots, v_n\}$ a basis of V.

Claim: $T(v_{k+1}), \ldots, T(v_n)$ is a <u>basis</u> of $\text{im}\, T$ (this suffices).

Span: Let $y \in \text{im}\, T$, so $y = T(x)$ for some $x \in V$. Write $x = \sum_{i=1}^n c_i v_i, y = T(x) = T(\sum_{i=1}^n c_i v_i) = \sum_{i=1}^n c_i T(v_i) = \sum_{i=k+1}^n c_i T(v_i)$ since $T(v_i) = \vec{0}$ when $1 \leq i \leq k$.

Linear Independent:

$$\sum_{i=k+1}^{n} c_i T(v_i) = \vec{0} \Rightarrow T\left(\sum_{i=k+1}^{n} c_i v_i\right) = \vec{0}$$
$$\Rightarrow x = \sum_{i=k+1}^{n} c_i v_i \in \ker T$$

If $x \in kerT$, then we can write $x = \sum_{i=1}^{k} d_i v_i$

$$\vec{0} = x - x = \sum_{i=1}^{k} d_i v_i - \sum_{i=k+1}^{n} c_i v_i$$

$\{v_1, \ldots, v_n\}$ linearly independent $\Rightarrow c_i = 0$ and $d_i = 0$. ☐

Corollary 4.2.1. *If $\dim V = \dim W$ where V, W finite dimensional, and $T : V \to W$ linear, then T injective $\Leftrightarrow T$ surjective.*

Proof. V injective $\Leftrightarrow \ker T = \{0\} \Leftrightarrow \dim\ker T = 0 \Leftrightarrow \dim \operatorname{im} T = \dim V = \dim W \Leftrightarrow \operatorname{im} T = W \Leftrightarrow T$ surjective. ☐

Similar: if S and T are finite sets and $|S| = |T|$ and $f : S \to T$, then f injective $\Leftrightarrow f$ surjective.

Ex Let $V = W = \mathcal{F}(\mathbb{Z}_{>0}, \mathbb{R})$

$T : V \to V$
$T(\{a_1, a_2, a_3, \ldots\}) = \{0, a_1, a_2, a_3, \ldots\}$

Claim T is linear, injective, not surjective.

$T' : V \to V$
$T'(\{a_1, a_2, \ldots\}) = \{a_2, a_3, a_4, \ldots\}$

T' is linear, surjective, not injective.

Note: Rank-nullity and corollary really only useful in finite dimensional case

5 February 5, 2020

5.1 Name

Proposition. *Let V, W be finite dimensional over F and choose ordered bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots w_m\}$. Then $m : \mathcal{L}(V, M) \to M_{m\times n}(F)$ defined by $m(T) = A$ where $T(v_j) = \sum_{i=1}^{m} A_{ij} w_i$ is an isomorphism of vector spaces.*

Proof.
Linearity: If $m(T) = A, m(S) = B$, then we know

$$T(v_j) = \sum_{i=1}^{m} A_{ij} w_i \quad S(v_j) = \sum_{i=1}^{m} B_{ij} w_i$$

$$\begin{aligned}(T+S)(v_j) &= T(v_j) + S(v_j) \\ &= \sum_{i=1}^{m} A_{ij} w_i + \sum_{i=1}^{m} B_{ij} w_i \\ &= \sum_{i=1}^{m} (A_{ij} + B_{ij}) w_i \\ &= \sum_{i=1}^{m} (A+B)_{ij} w_i\end{aligned}$$

So by definition of $m, m(T+S) = A+B = m(T)+m(S)$. Scalar multiplication is similar.

Injectivity: If $m(T) = m(S)$, then $T(v_j) = S(v_j)$ for all j. So by uniqueness part of construction theorem, $T = S$.

Surjectivity: Given A, let T be the linear map (given by construction theorem) taking v_j to $\sum_{i=1}^{m} A_{ij} w_i$. Then $m(T) = A$. $\square$

Ex $\quad V = \mathbb{R}^2 \;\; W = \mathbb{R}^3$
Let $T((x,y)) = (x+2y, -y, -x)$. Pick standard bases on $\mathbb{R}^2$ and $\mathbb{R}^3$. What is $m(T)$?
$(1,0) = e_1 \mapsto (1,0,-1) = 1 \cdot e_1 + 0 \cdot e_2 - 1 \cdot e_3$
$(0,1) = e_2 \mapsto (2,-1,0) = 2 \cdot e_1 - 1 \cdot e_2 + 0 \cdot e_3$

$$m(T) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Ex $\quad V = W = \{\text{polynomials over } \mathbb{R} \text{ of degree} \leq 3\}$
Pick basis $\{1, x, x^2, x^3\}$. Consider $D : V \to V$. What is the matrix $m(D)$?

$$D(1) = 0 \quad D(x) = 1 \quad D(x^2) = 2x \quad D(x^3) = 3x^2$$

$$m(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

If $m(T) = A$, then how to compute $T(v)$ in terms of A?

1. Write $v = \sum_{j=1}^{n} c_j v_j$

2. $$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} c_1 A_{11} + c_2 A_{12} + \ldots + c_n A_{1n} \\ c_1 A_{21} + c_2 A_{22} + \ldots + c_n A_{2n} \\ \vdots \\ c_1 A_{m1} + c_2 A_{m2} + \ldots + c_n A_{mn} \end{bmatrix}$$

This is $T(v)$ expressed in temrs of $\{w_1, \ldots, w_n\}$.

$$\begin{aligned} T(v) &= \left(\sum_{j=1}^{n} c_j A_{i1} \right) \cdot w_1 + \left(\sum_{j=1}^{n} c_j A_{2j} \right) w_2 + \ldots + \left(\sum_{j=1}^{n} c_j c_{mj} \right) w_m \\ &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} c_j A_{ij} \right) w_i \\ &= \sum_{i=1}^{n} c_j \left(\sum_{i=1}^{m} A_{ij} w_i \right) \end{aligned}$$

Notation: Same if $m(T) = A \begin{cases} T(v) = w \\ Av = w \end{cases}$

5.2 Matrix Multiplication

$$U, V, W \ f.d./F \qquad T : U \to V \qquad S : V \to W$$

Pick ordered bases $\{u_1, \ldots u_m\}, \{v_1, \ldots, v_n\}$, and $\{w_1, \ldots, w_p\}$.
Q: What is the relationship between $m(S), m(T), m(S \circ T)$?
We know $m(S)$ is $M_{p \times n}$, $m(T)$ is $M_{n \times m}$, and $m(S \circ T)$ is $M_{p \times m}$. Let $A = m(S), B = m(T)$, and $C = m(S \circ T)$.

$$S(v_k) = \sum_{i=1}^{n} A_{ik} w_i \quad 1 \le k \le n$$

$$T(u_j) = \sum_{k=1}^{n} B_{kj} v_k \quad 1 \le j \le m$$

$$\begin{aligned} (S \circ T)(u_j) &= S(T(u_j)) \\ &= S \left(\sum_{k=1}^{n} B_{kj} v_k \right) \\ &= \sum_{k=1}^{n} B_{kj} S(v_k) \\ &= \sum_{k=1}^{n} B_{kj} \sum_{i=1}^{p} A_{ik} w_i \\ &= \sum_{i=1}^{p} \left(\sum_{k=1}^{n} A_{ik} B_{kj} \right) w_i \end{aligned}$$

$$i.e. \quad C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

This defines matrix multiplication $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$.

Theorem 5.1. *With notation as above, $m(S)m(T) = m(ST)$.*

Ex

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad AB = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \quad AB = \begin{bmatrix} 3 & 4 \\ 3 & 8 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 2 \\ -1 & 10 \end{bmatrix}$$

Proposition. *Let A, B, C be matrices/F.*

1. *(Associativity) If the produces make sense, $(AB)C = A(BC)$.*
2. *(Right Distributivity) If A, B are same size and AC makes sense, then $(A+B)C = AC + BC$.*
3. *(Left Distributivity) If B, C are same size and AB makes sense, then $A(B+C) = AB + AC$.*

Proof (Idea for 1). Use the fact that matrices classify linear maps. Pick v.s. U, V, W, X of correct dimension and ordered bases. Let $A = m(R), B = m(S), C = m(T)$. We know $(R \circ S) \circ T = R \circ (S \circ T)$ because function compositions are associative. Apply m and then you get $(AB)(C) = A(BC)$. $\square$

5.3 Preview for Next Class

$$\begin{cases} 3x + 2y = 0 \\ x - y = 0 \end{cases} \qquad (x, y) = (0, 0)$$

$$\begin{cases} 3x + 2y = 0 \\ 6x + 4y = 0 \end{cases} \qquad \text{Entire line } (2t, -3t), t \in \mathbb{R}$$