MATH UN1208, Honors Math B

Columbia University, Spring 2020

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Contents

1 January 22, 2020

1.1 Introduction

<u>Administrative Stuff</u>

- Webpage . . . /honorsmathB
- HW 1 due in a week (1/29)
- Office Hours: T 9 - 11, F 1-2
- Midterm in class last Wed before Spring Break
- Textbook: Vol II this semester
- Will cover almost all linear algebra, multivariable calculus with fund thm of calculus in dim ≤ 3

1.2 Linear Algebra Recap

- We defined a <u>field</u> (e.g. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$)
- We defined <u>vector spaces</u>. Has vector addition and scalar multiplication obeying various laws. (e.g. $\mathbb{R}^n := \{\text{functions } [n] \to \mathbb{R}\}$
  $= \{\text{n-tuples of elements in } \mathbb{R}\}$)
- We defined <u>linear maps</u> $V \to W$ (i.e. a function $F : V \to W$ such that $F(V_1 + V_2) = F(V_1) + F(V_2)$ and $F(cV_1) = cF(V_1)$)

- We defined a subspace of a vector space

$$W_1, W_2 \subseteq V \text{ subspaces} \Rightarrow W_1 \cap W_2 \text{ is a subspace}$$

1.3 Continuation

Definition. *Given a linear map $f : V \to W$,*

$$\ker f := \{v \in V \mid f(v) = 0\} \subseteq V$$

$$\operatorname{im} f := f(V) = \{w \in W \mid \exists v \in V, f(V) = W\}$$

Proposition. *Given a linear $f : V \to W$,*

1. *f injective* $\iff \ker f = \{0\}$
2. *f surjective* $\iff \operatorname{im} f = W$

"Pf". .

1. HW
2. Obvious

□

Definition. *A linear $f : V \to W$ is an isomorphism if it is bijective.*

Proposition. *A linear map $f : V \to W$ is bijective if and only if it has a linear inverse.*

Proof. Assume f is a bijection. Let $g : W \to V$ be the inverse. Need to check that g is linear. We know $f \circ g = Id_W$ and $g \circ f = Id_V$. Given $W_1, W_2 \in W$,

$$\begin{aligned} g(W_1 + W_2) &= g(f(g(W_1))) + f(g(W_2)) \\ &= g(f(g(W_1) + g(W_2))) \\ &= g(W_1) + g(W_2) \end{aligned}$$

$$\begin{aligned} g(cW_1) &= g(cf(g(W_1))) \\ &= g(f(cg(W_1))) \\ &= cg(W_1) \end{aligned}$$

□

Definition. *V, W vector spaces/ F*

$$\mathcal{L}(V, W) := \{\mathit{linear\ maps}\, V \to W\} \subseteq \{\mathit{functions}\, V \to W\}$$

On HW: Check that $\mathcal{L}(V, W)$ is a vector space.

Proposition. *If $f : U \to V$ and $g : V \to W$ are linear, then $g \circ f : V \to W$ is linear.*

Proof.

$$\begin{aligned}(g \circ f)(cV_1) &= g(f(cV_1)) \\ &= g(cf(V_1)) \\ &= cg(f(V_1)) \\ &= c(g \circ f)(V_1)\end{aligned}$$

and similar for addition. $\square$

1.4 Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition. *The standard basis vectors of $\mathbb{R}^n$ are $e_i = (0, 0, \ldots, 1, \ldots, 0)$.*

$$\begin{aligned}\textit{e.g.}\quad e_1 &= (1, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, \ldots, 0, 1)\end{aligned}$$

Notation: if $X \in \mathbb{R}^n$, then $x = (x_1, \ldots, x_n)$. Call x_i then i^{th} component of x.

$(e_i)_j = \delta_{ij}$ "Kronecker delta"

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Proposition. $\forall x \in \mathbb{R}^n$,

$$\forall i, x_i = a_i \iff x = \sum_{i=1}^{n} a_i e_i$$

In other words, $x = \sum_{i=1}^{n} x_i e_i$.

Proof.

$$\begin{aligned}\sum_{i=1}^{n} x_i e_i &= (1, 0, 0, \ldots, 0)x_1 + (0, 1, \ldots, 0)x_2 + \ldots + (0, 0, \ldots, 1)x_n \\ &= (x_1, 0, 0, \ldots, 0) + (0, x_2, 0, \ldots) + \ldots + (0, 0, \ldots, x_n) \\ &= (x_1, x_2, \ldots, x_n)\end{aligned}$$

$\square$

In other words, any vector in $\mathbb{R}^n$ can be uniquely written as a linear combinations of the e_i.

1.5 Matrices

Definition. *For $m, n \in \mathbb{Z}_{\geq 0}$, an $m \times n$ matrix over F is a $m \times n$ box of elements of F.*

e.g. $\begin{bmatrix} 0 & 3 \\ -3 & \pi \\ 0 & 4 \end{bmatrix}$

Better Definition. *An $m \times n$ matrix A over F is a function $[m] \times [n] \to F$.*

Notation: Write $A((i,j)) =: A_{ij}$

$B = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$ $B_{11} = 1, B_{12} = 0, B_{21} = 2, B_{22} = 5$

Set of $m \times n$ matrices over F is called $M_{m \times n}(F)$. It is a vector space!

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$$

$$c \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} cA_{11} \\ cA_{21} \end{pmatrix}$$

Overall Result $\Rightarrow M_{m \times n}(F)$ is an F - vector space.

Will prove: $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m \times n}(\mathbb{R})$.

2 January 27, 2020

2.1 TA Office Hours

M 9 -11 (Carson) T 12-2 (Ahmed) W 9 -12 (Sayan)

2.2 Classifying linear maps $\mathbb{R}^n \to \mathbb{R}^m$ by matrix

Want to defined a linear map $M : M_{m \times n}(\mathbb{R}) \to \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$.

$$A \mapsto (T : \mathbb{R}^n \to \mathbb{R}^m, T(x) = \begin{pmatrix} \sum_{j=1}^n A_{1j} x_j \\ \sum_{j=1}^n A_{2j}, x_j \\ \vdots \\ \sum_{j=1}^n A_{nj}, x_j \end{pmatrix})$$

We write

$$\begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + A_{12}x_2 + \ldots + A_{1n}x_n \\ \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \ldots + A_{mn}x_n \end{bmatrix}$$

Check that $\mu(A)$ is linear ($T = \mu(A)$):

$$\begin{aligned}(T(x+y))_i &= \sum_{j=1}^{n} A_{ij}(x_j + y_j) \\ &= \sum_{j=1}^{n} A_{ij}x_j + \sum_{j=1}^{n} A_{ij}y_j \\ &= (T(x))_i + (T(y))_i \\ &= (T(x) + T(y))_i\end{aligned}$$

$$\begin{aligned}(T(cx))_i &= \sum_{j=1}^{n} A_{ij}cx_j \\ &= c\sum_{j=1}^{n} A_{ij}x_j \\ &= c(T(x))_i \\ &= (cT(x))_i\end{aligned}$$

2.3 Proof that matrices work as linear transformations of linear maps

Theorem 2.1. *M is a linear isomorphism.*

Proof. Let T_A denote $\mu(A)$.
Linearity: T_{A+B} sends x to the vector with i^{th} componen

$$\begin{aligned}\sum_{j=1}^{n}(A+B)_{ij}x_j &= \sum_{j=1}^{n}(A_{ij} + B_j)x_j \\ &= \sum A_{ij}x_j + \sum B_{ij}x_j\end{aligned}$$

$T_A + T_B$ sends x to vector w/ i^{th} component.

$$\sum A_{ij}x_j + \sum B_{ij}x_j \checkmark$$

Scalar Multiplication - similar

Injective: Suffices to show that $\ker \mu = \{0\}$. Suppose $T_A(x) = \widehat{0}$ for all x. By plugginh in $e_i, \ldots, e_n$ $\left[(e_i)_j = \delta_{ij}\right]$

$$(T_A(e_k))_i = \sum_{j=1}^{n} A_{ij}(e_k)_j = A_{ik} \cdot 1 = A_{ik} = 0 \Rightarrow A = 0$$

Surjective: Suppose we have a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. Consider the vectors

$$T(e_k) = \begin{pmatrix} A_{1k} \\ \vdots \\ A_{mk} \end{pmatrix}$$

Now let $A \in M_{m\times n}(\mathbb{R})$ such that

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix}$$

We have to check $T_A = T$. Need to check that $T_A(x) = T(x)\,\forall x \in \mathbb{R}^n$. Write $x = (x_1, \dots, x_n) = \sum_{i=1}^n x_i e_i$.

$$T_A(x) = T_A\left(\sum_{i=1}^{j} x_i e_i\right) = \sum_{i=1}^{n} x_i T_A(e_i)$$

$$T(x) = T\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} x_i T(e_i)$$

So it suffices to show that $T_A(e_i) = T(e_i)$ for each i.

$$T(e_i) = \begin{pmatrix} A_{1i} \\ \vdots \\ A_{mi} \end{pmatrix}$$

$$T_A(e_i) = \begin{pmatrix} \sum_{j=1}^n A_{1i}(e_i)_j \\ \sum_{j=1}^n A_{2j}(e_j)_j \\ \vdots \\ \sum_{j=1}^n A_{nj}(e_i)_j \end{pmatrix} = \begin{pmatrix} A_{1i} \\ A_{2i} \\ \vdots \\ A_{mi} \end{pmatrix}$$

□

Lesson: Standard basis vectors are very useful!

2.4 Basis Vectors

Recall: A <u>linear combo</u> of $v_1, \dots, v_n \in V$ is a vector of the form $\sum_{i=1}^n c_i v_i$ where F is a field and V is a vector space over F.

Definition. *A <u>linear combo</u> of a set of vectors S is any vector of the form $\sum_{i=1}^n c_i v_i$ where $c_i \in F$ and $v_i \in S$.*

Definition. *A set of vectors $S \subseteq V$ <u>spans V</u> if every $v \in V$ is a linear combo of S.*
i.e. $\forall v \in V, \exists v_i, \dots, v_n \in S$ and $c_i, \dots, c_n \in F$ such that $v = \sum_{i=1}^n c_i v_i$.
Write span$(S) = V$.

Ex $\quad \text{span}(\{e_1, \ldots, e_n\}) = \mathbb{R}^n$

Ex $\quad \text{span}(\{(1,1), (-1,0)\}) = \mathbb{R}^2$
Why? $\quad (a,b) = b(1,1) + (b-a)(-1,0)$

Ex $\quad \{\}$ spans $\mathbb{R}^0 = \{0\}$

Ex $\quad \text{span}(\{(1,0), (0,1), (3,2)\}) = \mathbb{R}^2$

Definition. *A set $S \subseteq V$ is linearly independent if whenever*

$$\sum_{i=0}^{n} c_i v_i = 0 \Longrightarrow \text{ all } c_i = 0$$

Note:

$$\text{Linear Independence} \iff \text{whenever } \sum_{i=1}^{n} c_i v_i = \sum_{i=1}^{n} d_i v_i, c_i = d_i$$

3 January 29, 2020

3.1 Definitions from Last Class

Recall: Let $S \subseteq V$ V a v.s. $/F$

1. S spans V if for every $v \in V$, $\exists s_1, \ldots, s_n \in S$ and $c_1, \ldots, c_n \in F$ such that $v = \sum_{i=1}^{n} c_i s_i$.
2. S is linearly independent if whenever $\sum_{i=1}^{n} c_i s_i = \vec{0}$ with $c_i \in F$ and $s_i \in S$, we have $\forall i, c_i = 0$.

3.2 Examples of Linear Dependence/Independence

1. If $\vec{0} \in S$, then S is dependent. $c\vec{0} = \vec{0}$ for any c, so c can be $\neq 0$. Uniqueness fails.
2. $V = \mathbb{R}^3$. Take 3 vectors. These are independent.

$$(1,0,0) \quad (1,1,0) \quad (1,0,1)$$

$$c_1(1,0,0) + c_2(1,1,0) + c_3(1,0,1) = (0,0,0)$$

$$\begin{cases} c_1 + c_2 + c_3 = 0 \\ \quad\; c_2 \quad\quad\; = 0 \\ \quad\quad\quad\;\; c_3 = 0 \end{cases}$$

$$\Rightarrow c_i \text{ all } = 0$$

3. $V = \mathbb{R}^2$. Take 3 vectors. These are dependent.

$$(1,0) \quad (0,1) \quad (1,01)$$

$$1 \cdot (1,0) + (-1) \cdot (0,1) + (-1)(1,-1) = (0,0).$$

Thus, coefficients not all $= 0$.

4. $V = \mathbb{R}^n$, set of standard basis elements is independent.

$$\sum_{i=1}^{n} c_i e_i = \vec{0} \Rightarrow c_i = 0$$

5. $S = \varnothing$ is independent. No coefficients to choose, so vacuous.

6. $S = \{V\}$ is independent $\iff V \neq \vec{0}$

7. $S = \{V_1, V_2\}$ is independent $\iff$ there is no $c \in F$ such that $V_1 = CV_2$ or $V_2 = CV_1$.
S dependent $\iff \exists a, b$ not both zero such that $aV_1 + bV_2 = \vec{0}$. If $a \neq 0$ then $V_1 = \frac{-b}{a} V_2$. If $b \neq 0$ then $V_2 = \frac{-a}{b} V_1$.

3.3 Basis

Definition. *$S \subseteq V$ is a <u>basis</u> if S is linearly independent and S spans V.*

<u>Ex</u> Let $V =$ all polynomials/ $\mathbb{R}$.

Claim: $S = \{1, x, x^2, x^3, x^4, \ldots\}$
$\quad = \{x^n \mid n \in \mathbb{Z}_{\geq 0}\}$

S spans V by definition: every polynomial is $\sum_{i=1}^{n} c_i x_i =$ linear combo of x^i.

S linearly independent: Check that $\sum_{i=1}^{n} c_i x^i = 0 \Rightarrow c_i = 0$ (from one long problem on final exam!)

Plug in $x = 0 \Rightarrow c_0 = 0$

$$\sum_{i=1}^{n} c_i i x^{i-1} = 0$$

Plug in $x = 0 \Rightarrow c_1 = 0$

etc... Keep differentiating

<u>Ex</u>

$$f(x) = x(x-1)$$
$$g(x) = (x-1)(x-2)$$
$$h(x) = x(x-2)$$

Claim: These are <u>independent</u>.

$$c_1 f(x) + c_2 g(x) + c_3 h(x) = 0$$

Plug in $x = 0 \Rightarrow c_2 \cdot 2 = 0 \Rightarrow c_2 = 0$
Plug in $x = 1 \Rightarrow c_3(-1) = 0 \Rightarrow c_3 = 0$
Plug in $x = 2 \Rightarrow c_1 \cdot 2 = 0 \Rightarrow c_1 = 0$

Lemma 3.1 (Lemma of Preceding Elements)**.** *V v.s./F $\quad v_1, \dots v_n$ is a sequence of elements of V. Suppose that $\{v_1, \dots, v_n\}$ is dependent.*
$\Rightarrow$ Some v_k may be written as a linear combo of $v_1, \dots, v_{k-1}$.

Proof. $\{v_1, \dots, v_n\}$ dependent $\Rightarrow \exists c_1 \in F$ not all $= 0$ such that $\sum_{i=1}^{n} c_i v_i = \vec{0}$. Let c_k be the last nonzero coefficient.

$$\text{Then } \sum_{i=1}^{k} c_i v_i = 0 \Rightarrow c_k v_k = -\sum_{i=1}^{k-1} c_i v_i$$
$$\Rightarrow v_k = -\frac{1}{c_k} \sum_{i=1}^{k-1} c_i v_i$$

□

Definition. *A vector space is <u>finite dimensional</u> if it has a finite basis. It is <u>infinite-dimensional</u> if not.*

<u>Ex</u> $\quad \mathbb{R}^n$ is a finite-dimensional vector space. (standard basis have n elements)

Theorem 3.2. *Let $\{v_1, \dots, v_n\}$ be a basis of V. Suppose $\{u_1, \dots, u_k\}$ is independent. Then $k \leq n$. (i.e. a basis has max size over all independent sets)*

Proof. Equivent: if $k > n$, then $\{u_1, \dots, u_k\}$ are dependent.

$u_1 \in \text{span}\{v_1, \dots, v_n\}$

$$(v_1, \dots, v_n, u_1) \text{ is linearly dep.}$$

$$(u_1, v_1, \dots, v_n) \text{ is linearly dep.}$$

Lemma $\Rightarrow$ some element is linear combination of previous elements.

If it's u_1, then $v_1 = 0 \Rightarrow$ we're done.
If not, then $v_i \in \text{span}(u_1, v_1, \dots v_{i-1})$.

$$\text{Therefore } \{u_1, v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_n\} \text{ spans } V$$

$\Rightarrow \{u_1, v_1, \dots, \hat{v_i}(\text{hat} := \text{remove element}), \dots, v_n, u_2\}$ is dependent.
$\Rightarrow \{u_1, u_2, v_1, \dots, \hat{v_i}, \dots, v_n\}$ is dependent.

Lemma $\Rightarrow$ some element is combo of previous ones. If u_1 or u_2 is said element, we're done.

Proceed by induction: get that, since $k > n$, $\{u_1, u_2, \ldots, u_n\}$ spans V
$\Rightarrow u_{n+1}$ is linear combo of $\{u_1, \ldots, u_n\}$
$\Rightarrow$ we're done. $\square$

Corollary 3.2.1. *Any two finite bases have some number of elements.*

Proof.
$\{v_1, \ldots v_n\}$ Thm $\Rightarrow k \leq n$
$\{u_1, \ldots, u_k\}$ Thm applied to Switching them $\Rightarrow n \leq k \Rightarrow k = n$. $\square$

Definition. *V finite-dimensional v.s.*

$$\dim(V) = \text{ number of elements in any basis of } V$$

<u>Ex</u> $\dim(\mathbb{R}^n) = n$

Proposition. *If V is finite-dimensional, then any independent set $\{v_1, \ldots, v_k\}$ may be extended to a basis. (i.e. $\exists\{v_1, \ldots, v_n\}$ that is a basis)*

Proof. If $\{v_1, \ldots, v_k\}$ spans V, we're done. Otherwise, pick some $v_{k+1} \in V - \text{span}\{v_1, \ldots, v_k\}$. By lemma of preceding element, $\{v_1, \ldots, v_{k+1}\}$ still independent. Continue inductively. $\square$