MATH UN1208, Honors Math B

Columbia University, Spring 2020

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Contents

1 January 22, 2020

1.1 Introduction

Administrative Stuff

- Webpage .../honorsmathB
- HW 1 due in a week (1/29)
- Office Hours: T 9 - 11, F 1-2
- Midterm in class last Wed before Spring Break
- Textbook: Vol II this semester
- Will cover almost all linear algebra, multivariable calculus with fund thm of calculus in dim ≤ 3

1.2 Linear Algebra Recap

- We defined a field (e.g. $\mathbb{R}, \mathbb{Q}, \mathbb{C}$)
- We defined vector spaces. Has vector addition and scalar multiplication obeying various laws. (e.g. $\mathbb{R}^n := \{\text{functions } [n] \to \mathbb{R}\}$
  $= \{\text{n-tuples of elements in } \mathbb{R}\}$)
- We defined linear maps $V \to W$ (i.e. a function $F : V \to W$ such that $F(V_1 + V_2) = F(V_1) + F(V_2)$ and $F(cV_1) = cF(V_1)$)
- We defined a subspace of a vector space

$$W_1, W_2 \subseteq V \text{ subspaces} \Rightarrow W_1 \cap W_2 \text{ is a subspace}$$

1.3 Continuation

Definition. *Given a linear map $f : V \to W$,*

$$\ker f := \{v \in V \mid f(v) = 0\} \subseteq V$$

$$\operatorname{im} f := f(V) = \{w \in W \mid \exists v \in V, f(V) = W\}$$

Proposition. *Given a linear $f : V \to W$,*

1. *f injective $\iff \ker f = \{0\}$*
2. *f surjective $\iff \operatorname{im} f = W$*

"Pf". .

1. HW
2. Obvious

□

Definition. *A linear $f : V \to W$ is an* <u>*isomorphism*</u> *if it is bijective.*

Proposition. *A linear map $f : V \to W$ is bijective if and only if it has a linear inverse.*

Proof. Assume f is a bijection. Let $g : W \to V$ be the inverse. Need to check that g is linear. We know $f \circ g = Id_W$ and $g \circ f = Id_V$. Given $W_1, W_2 \in W$,

$$\begin{aligned} g(W_1 + W_2) &= g(f(g(W_1))) + f(g(W_2)) \\ &= g(f(g(W_1) + g(W_2))) \\ &= g(W_1) + g(W_2) \end{aligned}$$

$$\begin{aligned} g(cW_1) &= g(cf(g(W_1))) \\ &= g(f(cg(W_1))) \\ &= cg(W_1) \end{aligned}$$

□

Definition. *V, W vector spaces/ F*

$$\mathcal{L}(V, W) := \{\textit{linear maps}\, V \to W\} \subseteq \{\textit{functions}\, V \to W\}$$

On HW: Check that $\mathcal{L}(V, W)$ is a vector space.

Proposition. *If $f : U \to V$ and $g : V \to W$ are linear, then $g \circ f : V \to W$ is linear.*

Proof.

$$\begin{aligned} (g \circ f)(cV_1) &= g(f(cV_1)) \\ &= g(cf(V_1)) \\ &= cg(f(V_1)) \\ &= c(g \circ f)(V_1) \end{aligned}$$

and similar for addition. □

1.4 Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$

Definition. *The* *standard basis vectors* *of* $\mathbb{R}^n$ *are* $e_i = (0, 0, \ldots, 1, \ldots, 0)$*.*

$$\begin{aligned} e.g. \quad e_1 &= (1, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, \ldots, 0, 1) \end{aligned}$$

Notation: if $X \in \mathbb{R}^n$, then $x = (x_1, \ldots, x_n)$. Call x_i then i^{th} component of x.

$(e_i)_j = \delta_{ij}$ "Kronecker delta"

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Proposition. $\forall x \in \mathbb{R}^n$,

$$\forall i, x_i = a_i \iff x = \sum_{i=1}^{n} a_i e_i$$

In other words, $x = \sum_{i=1}^{n} x_i e_i$*.*

Proof.

$$\begin{aligned} \sum_{i=1}^{n} x_i e_i &= (1, 0, 0, \ldots, 0)x_1 + (0, 1, \ldots, 0)x_2 + \ldots + (0, 0, \ldots, 1)x_n \\ &= (x_1, 0, 0, \ldots, 0) + (0, x_2, 0, \ldots) + \ldots + (0, 0, \ldots, x_n) \\ &= (x_1, x_2, \ldots, x_n) \end{aligned}$$

$\square$

In other words, any vector in $\mathbb{R}^n$ can be uniquely written as a linear combinations of the e_i.

1.5 Matrices

Definition. *For* $m, n \in \mathbb{Z}_{\geq 0}$*, an* $m \times n$ *matrix over* F *is a* $m \times n$ *box of elements of* F*.*

e.g. $\begin{bmatrix} 0 & 3 \\ -3 & \pi \\ 0 & 4 \end{bmatrix}$

Better Definition. *An* $m \times n$ *matrix* A *over* F *is a function* $[m] \times [n] \to F$*.*

Notation: Write $A((i, j)) =: A_{ij}$

$B = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix}$ $B_{11} = 1, B_{12} = 0, B_{21} = 2, B_{22} = 5$

Set of $m \times n$ matrices over F is called $M_{m\times n}(F)$. It is a vector space!

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{pmatrix}$$

$$c \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} = \begin{pmatrix} cA_{11} \\ cA_{21} \end{pmatrix}$$

Overall Result $\Rightarrow$ $M_{m\times n}(F)$ is an F - vector space.

Will prove: $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is isomorphic to $M_{m\times n}(\mathbb{R})$.