MATH UN1207, Honors Math A

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Mateo Maturana

Contents

1 September 4, 2019

1.1 Statements

Definition. *A statement (or proposition) is an assertion that its true* <u>*or*</u> *false (but not both).*

<u>Ex</u>
P = "Ringo Starr is alive"
Q = "The earth is flat"

Definition. *The truth value of a statmeent is T (if true) and F (if false)*

Definition. *The negation of a statement P is the statement "P is false" ($\sim P$)*

Ex

$\sim$P = "Ringo Starr is dead"

$\sim$Q = "The earth is not flat"

Remark. $\sim(\sim P) = P$

Definition. *The conjunction of two statements P, Q is the statement "P and Q" ($P \wedge Q$)*
i.e. The truth value of $P \wedge Q$ is T if P is T and Q is T, and F otherwise.

Proposition. $P \wedge (\sim P) = F$

P	$\sim$P	P $\wedge$($\sim$P)
T	F	F
F	T	F

Definition (Disjunction)**.** *P or Q ($P \vee Q$) is F if P is F and Q is F, and T otherwise. "or" in math is always inclusive.*

Proposition. $(P \wedge Q) \vee (\sim P \vee \sim Q) = T$

P	Q	P $\wedge$ Q	$\sim$P $\vee \sim$ Q	(P $\wedge$ Q) $\vee$ ($\sim$P $\vee \sim$ Q)
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

1.2 Conditionals

Definition (Conditional)**.** *P implies Q or if P then Q, or $P \Rightarrow Q$, is $Q \vee (\sim P)$.*
i.e. if P is T, then Q is T.

Remark. *If P is F, then we say $P \Rightarrow Q$ is "vacuously true"*

Proposition. $(P \wedge Q) \Rightarrow P$, $(P \wedge Q) \Rightarrow Q$, $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$

Definition (Biconditional)**.** *P iff Q or P if and only if Q or $P \iff Q$ is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$*

P	Q	P $\iff$ Q
T	T	T
T	F	F
F	T	F
F	F	T

Proposition. *1 - Suppose P and* $P \Rightarrow Q$*. Then Q.*
2 - Suppose $P \Rightarrow Q$ *and* $\sim Q$*. Then* $\sim P$*.*
3 - Suppose $P \Rightarrow Q$ *and* $P \Rightarrow \sim Q$*. Then* $\sim P$*.*
4 - Suppose $P \vee Q, P \Rightarrow R$*, and* $Q \Rightarrow R$*. Then R.*

Proof of 3. (P ⇒ Q) ∧ (P ⇒∼ Q) ⇒ ∼ P

P	Q	P ⇒ Q	P ⇒∼ Q	(P ⇒ Q) ∧ (P ⇒∼Q)	Whole
T	T	T	F	F	T
T	F	F	T	F	T
F	T	T	T	T	T
F	F	T	T	T	T

□

$$\text{De Morgan's Laws} \begin{cases} \sim (P \wedge Q) \Longleftrightarrow \sim P \vee \sim Q \\ \sim (P \vee Q) \Longleftrightarrow \sim P \wedge \sim Q \end{cases}$$

Definition (Contrapositive)**.** *(P ⇒ Q) ⟺ (∼Q ⇒ ∼ P)*

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2.1 Predicates

Definition. *A predicate P(x) is a family of statements depending on a variable* x

Ex
$P(x)$ = "x is a banana"
$Q(x)$ = "$x > 7$ "

Existential Quantifier:
$\exists x \mid P(x)$: "there exists an x such that $P(x)$"
Universal Quantifier:
$\forall x\, P(x)$: "for all x, $P(x)$"

e.g. $\forall x, Q(x)$ is false ($Q(x)$ as above)
$\forall x(P(x) \vee \sim P(x)))$ is always true

e.g.

$$\sim (\forall x P(x)) \Longleftrightarrow \exists x(\sim P(x))$$

$$\sim (\exists P(x)) \Longleftrightarrow \forall x(\sim P(x))$$

Remark. *To prove* $\exists x P(x)$*, just find an x such that P(x). To prove* $\forall x P(x)$*, write something like "take an x..."*

2.2 Sets

A set is a collection of objects.
e.g.

$$S = \{1, 2, 4\}, T = \{\{1\}, 2, \text{water}\}$$

$\{1\} \in T$, but $1 \notin T$. "1 is not an element of T"

$\mathbb{N}$ = set of natural numbers $\{1, 2, 3, ...\}$
$\mathbb{Z}$ = set of integers $\{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{Q}$ = rational numbers
$\mathbb{R}$ = real numbers
$\mathbb{C}$ = complex numbers

Definition. $\forall x \in S, P(x)$ *just means* $\forall x(x \in S \Rightarrow P(x))$

Definition (Set Inclusion)**.** $S \subseteq T$ *(*S *is a subset of* T*) if* $\forall x \in S, x \in T$

Remark. *In definitions, we write "if" instead of "if and only if" even though the latter is what we mean.*

Definition. $S = T$ *if* $S \subset T$ *and* $T \subset S$

Warning - Order matters with quantifiers!
$\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \mid m + n = 0$ is TRUE
$\exists m \in \mathbb{Z} \mid \forall n \in \mathbb{Z}, m + n = 0$ is FALSE

Proposition. *IF* $S \subset T$ *and* $T \subset U$*, then* $S \subset U$

Proof. We know $\forall x \in S, x \in T$ and $\forall yinT, y \in U$. Therefore, $\forall x \in S, x \in U$. Thus, $S \subset U$. □

2.3 Axioms for Sets

1. There exists a set
2. "Axiom of Specification" (how to take subsets)
   Given set S and any predicate $P(x)$, there exists a set T such that

   (a) $T \subset S$

   (b) $\forall x, (x \in T \iff P(x))$

   We write $T = \{x \in S \mid P(x)\}$.

   Take $Q(x)$ = "$x \neq x$"(always false). Then, take S any set $= \{x \in S \mid Q(x)\} = \emptyset$ (the empty set).

2.4 Russell's Paradox

If we take axiom of specification w/o picking S, we get a contradiction. Take $P(x) =$ "$x = x$"(always true). Stengthened axiom $\Rightarrow$ get a set of all sets $\mathcal{V}$. We will show a contradiction.
Let $T = \{S \in \mathcal{V} \mid S \notin S\}$. Is $T \in T$?. If $T \in T \Rightarrow T$ really bad $\Rightarrow T \notin T \Rightarrow\Leftarrow$. If $T \notin T \Rightarrow T$ is not really bad $\Rightarrow T \in T \Rightarrow\Leftarrow$. We conclude that there is no "set of all sets".

2.5 Axioms Cont.

3. Axiom of Unions
   Given sets S, T, there exists $S \cup T$ such that $\forall x, x \in S \cup T \iff x \in S$ or $x \in T$.

3'. Axiom of Intersection
   Given sets S, T, there eixsts $S \cap T$ such that $\forall x(x \in S \cap T) \iff x \in S$ and $x \in T$

Theorem 2.1. $A \cup B = B \cup A$

Theorem 2.2. $(A \cup b) \cup C = A \cup (B \cup C)$

Theorem 2.3. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof of 2.3. Need to show: $\forall x, x \in LHS \iff x \in RHS$

$$\begin{aligned}
x \in LHS &\iff x \in A \cap (B \cup C) \\
&\iff x \in A \wedge x \in (B \cup C) \\
&\iff x \in A \wedge (x \in B \vee x \in C) \\
&\iff (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\
&\iff x \in A \cap B \vee x \in A \cap C \\
&\iff x \in (A \cap B) \cup (A \cap C) \\
&\iff x \in RHS
\end{aligned}$$

$\square$

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3.1 More De Morgan

Definition. *Let S, A be sets. Then the complement S-A or S$\setminus$ A is* $\{x \in S \mid x \notin A\}$

There are two more De Morgan laws.

1. $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$
2. $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$

3.2 Power Sets

4. For all sets A, there exists a set $\mathcal{P}(A)$, the power set of A, such that its elements are precisely the subsets of A.

$$\forall B, B \subseteq A \Leftrightarrow B \in \mathcal{P}(A)$$

<u>Ex</u> $A = \{1, 2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Proposition. $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

Proof. Similar to other set equality proofs.

$$\begin{aligned}
S \in \mathcal{P}(A \cap B) &\iff S \subseteq (A \cap B)\\
&\iff \forall x \in S, (x \in A \cap B)\\
&\iff \forall x \in S, (x \in A \text{ and } x \in B)\\
&\iff \forall x \in S, x \in A \text{ and } \forall x \in S, s \in B\\
&\iff S \subseteq A \text{ and } S \subseteq B\\
&\iff S \in \mathcal{P}(A) \text{ and } S \in \mathcal{P}(B)\\
LHS &\iff S \in \mathcal{P}(A) \cap \mathcal{P}(B)
\end{aligned}$$

□

3.3 Cartesian Products

<u>"Def":</u> An ordered pair is a list (a, b) where a, b are math objects. We say $(a, b) = (a', b')$ if $a = a'$ and $b = b'$.
<u>Ex</u> $(1, 2) \neq (2, 1)$

5. Axiom of Products: Givens sets A, B, there exists a set $A \times B$ whose elements are exactly the pais (a, b) with $a \in A, b \in B$.

$$\forall x, x \in A \times B \iff x = (a, b) \text{ with } a \in A, b \in B$$

Remark. *This axiom is actually not necessary*

Proposition. $A \times (B \cup C) = A \times B \cup A \times C$

Proof. Same format as any other set equality proof.

$$\begin{aligned}
(a, b) \in LHS &\iff a \in A \text{ and } b \in (B \cup C)\\
&\implies a \in A \text{ and } (b \in B \text{ or } b \in C)\\
&\iff (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C)\\
&\iff (a, b) \in A \times B \text{ or } (a, b) \in A \times C\\
&\iff (a, b) \in (A \times B) \cup (A \times C)
\end{aligned}$$

□

3.4 Functions

<u>"Def"</u> Let S, T be sets. A function $F : S \to T$ (from S to T) is a rule assigning an element of T to each element of S. i.e. If $s \in S$, this element of T is denoted $f(s)$.

Remark. *We can make a non-fake def of a function $S \to T$ by defining it as a subset of $S \times T$.*

Remark. *We write $f : S \to T$, "f is a function from S to T", "f maps S to T", and $x \mapsto x^2$, "x maps to x^2".*

Definition. *If $f : S \to T$, then S is the domain and T is the codomain.*

Definition. *The graph of $f : S \to T$, (sometimes denoted $\Gamma(f)$) is $\{(x, y) \in S \times T \mid y = f(x)\}$.*

Definition. *Two functions f: $S \to T$ and $g : S' \to T'$ are equal if $S = S'$, $T =$ T', and $\forall s \in S, f(s) = g(s)$.*

Definition. *If $f : S \to T$ and $g : T \to U$ are functions, their <u>composition</u> is the function $g \circ f : S \to U$ such that $(g \circ f)(s) = g(f(s)) \forall s \in S$.*

Definition. *If S is set, then id_s: $S \to S$ is called the identity function*

3.5 Injectivity and Surjectivity

Definition. *$f : S \to T$ is injective if wheneber $f(s) = f(s'), s = s'$.*

Definition. *$f : S \to T$ is surjective if $\forall t \in T, \exists s \in S$ such that $f(s) = t$.*

Proposition. *id_s is injective and surjective/*

Proof. If $\mathrm{id_s(s)} = \mathrm{id_s}(s'), s = s'$. This proves injectivity. $\forall s \in S$, we have $\mathrm{d_s(s)}$ = s. This proves surjectivity. Thus, we have both. ☐

Remark. *$f : S \to T$ injective $\iff \forall t \in T$, there is at <u>most</u> one preimage(at most one $s \in S$ such that $f(s) = t$). f surj $\iff \forall t \in T$, there is at least one element of preimage.*

Definition. *$f : S \to T$ is bijective if it is injective and surjecitve.*

<u>Ex/Prop</u>

1. $f : S \to T, g : T \to U$, both inj.
   (a) $\Rightarrow g \circ f$ inj.
   (b) $\Rightarrow f$ inj.
2. $f : S \to T, g : T \to U$ surj.
   (a) $g \circ f$ surj.
   (b) g surj.

3.6 Inverses

Definition. *If $f : S \to T$ is a function, an* <u>*inverse*</u> *to f is a function $g : T \to S$ such that*

1. $g \circ f = \mathrm{id_S}\colon\ S \to S$
2. $f \circ g = \mathrm{id_T}\colon\ T \to T$

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4.1 Inverses Cont.

Theorem 4.1. *$f : S \to T$ has an inverse if and only if it is bijective.*

Proof. ($\Rightarrow$) by homework problem
($\Leftarrow$) We know by definition of bijectivity, $\forall t \in T, \exists s \in S \mid f(s) = t$ and $\forall s, s' \in S, f(s) = f(s') \Rightarrow s = s'$. Define $g : T \to S$.

Lemma 4.2. *$f(s) = t \iff g(t) = s$*

Lemma can be proven by definition of g. Lemma $\Rightarrow$ if $g(t) = s'$, then $f(s') = t$. Thus, $t = f(s) = f(s')$. By injectivity of $f, s = s'$.

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) \\ &= f(s) \\ &= t\end{aligned}$$

$$\begin{aligned}(g \circ f)(s) &= g(f(s)) \\ &= g(t) \\ &= s\end{aligned}$$

□

Remark. *By definition, if g is inverse to f, then f is inverse to g.*

Proposition. *If g, g' are inverses to f, then $g = g'$. (Inverses are unique)*

Proof. Take any $t \in T$.

$$\begin{aligned}g'(t) &= (g' \circ id_t)(t) \\ &= (g' \circ (f \circ g))(t) \\ &= ((g' \circ f) \circ g)(t) \\ &= (id_s \circ g)(t) \\ &= g(t)\end{aligned}$$

□

Definition. *If $f : S \to T$ and $U \subseteq S$, then the image of U (under f), denoted $f(U)$, is $\{t \in T \mid \exists s \in U \text{ with } f(s) = t\} = \{f(s) \mid \overline{s \in U}\}$.*

Definition. *If $f : S \to T$ and $V \subseteq T$, the preimage of V under f, denoted $f^{-1}(V$, is $\{s \in S \mid f(s) \in V\}$.*

4.2 Numbers

We will axiomatize $\mathbb{R}$.

Definition. *A binary operation on a set S is a function $S \times S \to S$.*

Definition. *A relation on S is a subset of $S \times S$.*

Assumption: There exists a set $\mathbb{R}$, equippied with two binary relations $+$, $\bullet$, one relation $>$, and two elements 0, 1 satisfying the following axioms.

4.3 Axioms

All of the axioms are $\forall x, y, z \in \mathbb{R}$, unless otherwise noted.

1. Commutativity
   $x + y = y + x \qquad x \cdot y = y \cdot x$

2. Associativity
   $x + (y + z) = (x + y) + z \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z$

3. Distributivity
   $x \cdot (y + z) = x \cdot y + x \cdot z$

4. Identity Elements
   $0 \neq 1 \qquad 0 + x = x \qquad 1 \cdot x = x$

5. Additive Inverse
   $\forall x \in \mathbb{R}, \exists w \in \mathbb{R} \mid w + x = 0$. We can denote w by $-x$.

Note: Axioms 1-5 are a commutative ring.

6. Multiplicative Inverse
   $\forall x \neq 0 \in \mathbb{R}, \exists w \in \mathbb{R} \mid w \cdot x = 1$. We can denote w by $\frac{1}{x}, x^{-1}$, etc.

Note: Axioms 1-6 are a field.

e.g. $\frac{\mathbb{Z}}{n\mathbb{Z}} = \{0, 1, ..., n-1\}$ with modular arithmetic is a commutative ring and is a field iff n is prime.
Notation: $x + (-y) =: x - y$
$x \cdot \frac{1}{y} = \frac{x}{y}$

7. Order Axiom 1
   If $x > 0$ and $y > 0$, then $x + y > 0$ and $x \cdot y > 0$. ("add + mult preserve the order").

8. Order Axiom 2
   If $x \neq 0$, then $x < 0$ or $x > 0$ but not both. ("trichotomy")

9. Order Axiom 3
   not $0 > 0$ ($0 \not> 0$)

10. Order Axiom 4
    If $x > y$, then $x + z > y + z$.

Note: Axioms 1 - 10 are an ordered field. (e.g. $\mathbb{Q}, \mathbb{R}$)

Proposition. *High School Algebra*

Proof. exercise ☐

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5.1 Application of Axioms

Proposition (Multiplicative Cancellation)**.** *Given an ordered field* $\mathbb{R}, \forall a, b, x \in \mathbb{R}$

$$xa = xb, x \neq 0 \Longrightarrow a = b$$

Proof. By axiom of multiplicative inverse, $\exists w \in \mathbb{R}$ with $wx = 1$. Since $xa = xb$, we can multiply both sides by w to obtain $wxa = wxb$. This statement is equal to $(wx)a = (wx)b$ by associativity. Then, $1a = 1b$. Thus, $a = b$ by the axiom of identity. ☐

Theorem 5.1 (Trichotomy)**.** $\forall a, b \in \mathbb{R}$, *exactly one of the following is true:*

1. $a > b$
2. $b > a$
3. $a = b$

Lemma 5.2. $\forall a, b \in \mathbb{R}, a > b \Longleftrightarrow a - b > 0$

Proof. Do each direction separately.
($\Rightarrow$)

$$\begin{aligned} a > b &\Rightarrow a + (-b) > b + (-b) \text{ by axiom of add. inverse and axiom 10} \\ &= a - b > 0 \text{ by def of "-"} \checkmark \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} a - b > 0 &\Rightarrow (a - b) + b > 0 + b \text{ by axiom 10} \\ &\Rightarrow a + (-b + b) > b \text{ by associativity and add. identity} \\ &\Rightarrow a + (b + -b) > b \text{ by commutativity} \\ &\Rightarrow a + 0 > b \text{ by add. inverse} \\ &\Rightarrow 0 + a > b \text{ by commutativity} \\ &\Rightarrow a > b \text{ by add. identity} \end{aligned}$$

□

Lemma 5.3. $\forall a, b \in \mathbb{R}, b > a \iff 0 > a - b$

Proof. Either copy Lemma 5.2 (almost) or $b > a \iff b - a > 0$ by Lemma 5.2 then show $b - a > 0 \iff 0 > a - b$. □

Lemma 5.4. $\forall a, b \in \mathbb{R}, a = b \iff a - b = 0$

Proof. Simple

$$\begin{aligned} a = b &\Rightarrow a + (-b) = b + (-b) \\ &\Rightarrow a - b = 0 \\ &\Rightarrow (a - b) + b = 0 + b \\ &\Rightarrow (a) + (-b + b) = b \\ &\Rightarrow a + (b + -b) = b \\ &\Rightarrow a + 0 = b \\ &\Rightarrow a = b \end{aligned}$$

□

Proof of 5.1. Let $x = a - b$. Either $x = 0$ or $x \neq 0$ but not both. By Axiom 8, if $x \neq 0$, either $x > 0$ or $0 > x$ but not both. By Axiom 9, if $x = 0$ then NOT $x > 0, 0 > x$. This implies that exactly one of $x = 0, x > 0, 0 > x$ is true. Lemmas 5.2, 5.3, 5.4 tell us that $x = 0 \iff 5.4, x > 0 \iff 5.2, 0 > x \iff 5.3$. □

Exercises to try on own

- $\forall x, 0 \cdot x = 0$
- $1 > 0$ (tricky)

5.2 Natural Numbers

Definition. *$S \subseteq \mathbb{R}$ is an inductive set if*

1. $0 \in S$
2. $\forall x \in S, x + 1 \in S$

Ex $\{x \in \mathbb{R} \mid x \geq 0\}$ is inductive (by above exercise)

Definition. *A natural number is an $x \in \mathbb{R}$ such that x is a number of every inductive set.*

The set of natural numbers is called $\mathbb{Z}_{\geq 0}$.

Theorem 5.5 (Mathematical Induction)**. ??** *Let $P(n)$ be a predicate defined on $\mathbb{Z}_{\geq 0}$ such that*

1. *$P(0)$ is true (base case)*
2. *$\forall n \in \mathbb{Z}, P(n) \Rightarrow P(n+1)$*

Then, $\forall n \in \mathbb{Z}_{\geq 0}, P(n)$ is true.

Proof. $\{n \in \mathbb{R} \mid P(n)\}$ is an inductive set. Hence $\mathbb{Z}_{\geq 0} \subseteq S$ by definition of the natural numbers. $\square$

Proposition. $\forall n \in \mathbb{Z}_{\geq 0}, n^2 > n$

Proof. Base Case $n = 0 : 0 \geq 0\checkmark$. Inductive step: Assume $n^2 \geq n$.

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 2n + 1 \\ &\geq n + n + 1 \\ &\geq n + 1\end{aligned}$$

$\square$

Proposition. $\forall n \in \mathbb{Z}_{\geq 0}, 0 + 1 + 2 + ... + n = \frac{n(n+1)}{2}$

Proof. Base Case $n = 0 : 0 = 0\checkmark$. Inductive step: Assume $0 + ... + n = \frac{n(n+1)}{2}$. Add $n + 1$ to both sides.

$$\begin{aligned}0 + ... + n + n + 1 &= \frac{n(n+1)}{2} + n + 1 \\ &= n + 1\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

$\square$

Definition. *A positive integer is a natural number that is not 0.*

Definition. *An integer is the difference of two natural numbers (called $\mathbb{Z}$).*

Definition. *A rational number is a quotient of two integers($\mathbb{Q}$).*

"Principle" of recursive definition: For any set S, a function $f : \mathbb{Z}_{\geq 0} \to S$ may be specified by a choice of $f(0)$ and a function expressing $f(n)$ in terms of $f(m), m < n$.

"Proof"
$\{n \in \mathbb{Z}_{\geq 0} \mid \forall m < n$, the above data determine $f(n)$ uniquely$\}$ is an inductive set.
Ex
Define $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. $f(0) = 0, f(1) = 0, f(n) = f(n-1) + f(n-2)$.

5.3 Sums

Define for $n \in \mathbb{Z}_{\geq 0}, f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$.

$$\sum_{i=0}^{n} f(i)$$

Remark. *A few rules*

1. *"i" is a dummy variable so we can use anything.*
2. $$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

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6.1 Supremum

Recall: Both $\mathbb{Q}$ and $\mathbb{R}$ should be ordered fields (satisfy axioms 1-10)
e.g. $\sqrt{2} \in \mathbb{R}, \notin \mathbb{Q}$

Definition. *A subset $S \subseteq \mathbb{R}$ is bounded above if $\exists y \in \mathbb{R}$ such that $\forall x \in S, x \leq y$.*

e.g. $S = \{3, 3.1, 3.14, 3.141, ...\}$ is bounded above by 4 (or π).

Definition. *An upper bound y for $S \subseteq \mathbb{R}$ is a least upper bound or supremum if y is an upper bound for S and if z is an upper bound for S, then $y \leq z$.*

11. Axiom(completeness): every nonempty bounded above set $S \subseteq \mathbb{R}$ has a supremum.

Proposition. *If y and y' are suprema of S, then $y = y'$. (suprema are unique)*

Proof. y is a supremum, and y' is an upper bound $\Rightarrow y \leq y'$ by definition of supremum. y' is a supremum, and y is an upper bound $\Rightarrow y' \leq y$ by definition of supremum. So $y = y'$ by trichotomy. $\square$

Notation: $\sup S$ is the supremum of S.

6.2 Infimum

Definition. *S is bounded below if existsy $\mid \forall x \in S, y \leq x.y$ is a lower bound for S if $\forall x \in S, y \leq x$. y is the greatest lower bound or infimum if y is a lower bound and if z is any other lower bound, then $x \leq y$.* $\inf S$ *= infimum of S.*

Definition. *If $S \subseteq \mathbb{R}$ is nonempty and bdd below, then $\exists!$ infimum* $\inf S$.

Proof. Let $-S = \{x \mid x \in S\}$. $-S$ is nonempty and bounded above. Thus, $\sup(-S)$ exists and is unique.
Claim: $-\sup(-S)$ is an infimum for S. Uniqueness same as for sup. $\square$

Theorem 6.1 (Approximation Property). *Let $S \subseteq \mathbb{R}$ be nonempty and bounded above.* $\forall \epsilon > 0, \exists x \in S \mid \sup S - \epsilon \leq x$.

Proof by Contradiction. Assume $\exists \epsilon > 9. \forall x \in S, \sup S - \epsilon > x$. Then, $\sup S - \epsilon$ is an upper bound for S. Then, by definition of $\sup S, \sup S \leq \sup S - \epsilon$. Contradiction of $\epsilon > 0$. $\blacksquare$ $\square$

Theorem 6.2 (Additivity of Supremum). *If $S, T \subseteq \mathbb{R}$ nonempty, bounded above, let $S + T := \{s + t \mid s \in S, t \in T\}$. Then $\sup(S+T)$ exists and equals $\sup(S) + \sup(T)$.*

Proof. Let $s = \sup S, t = \sup T$. $\forall x \in S, x \leq s, \forall y \in T, y \leq t \Rightarrow \forall x \in S, \forall y \in T, x + y \leq s + t \Rightarrow s + t$ is an upper bound for $S + T$. We want to show that $s + t$ is the <u>least</u> upper bound. Suppose not. Then $\exists \delta > 0 \mid s + t - \delta$ is an upper bound for $S + T$. Let $\epsilon = \frac{\delta}{2}$.

$$\text{Approximation Property} \Longrightarrow \exists x \in S \mid s - \epsilon < x$$
$$\exists y \in T \mid t - \epsilon < y$$

Then $x + y \in S + T$. $s + t - \delta = s + t - 2\epsilon < x + y$. Contradiction! $\square$

Proposition. *Suppose S, $T \subseteq \mathbb{R}$ such that $\forall x \in S, \forall y \in T, x \leq y$. Then $\sup S$ exists and $\inf T$ exists, and $\sup S \leq \inf T$.*

Proof. Any $x \in S$ is a lower bound for T
$\Rightarrow \inf T$ exists
Any $y \in T$ is upper bd for S
$\Rightarrow \sup S$ exists
Suppose $\sup S > \inf T$. Then $S - \sup S - \inf T$. Let $\epsilon = \frac{\delta}{2}$.
Approx $\Rightarrow \exists x \in S \mid \sup S - \epsilon < x$. Similar approx. result $\Rightarrow \exists y \in T \mid \inf T + \epsilon > y$.

$$y < \inf T + \epsilon = \sup S - \epsilon < x$$

Contradiction! $\sup S \leq \inf T$ $\square$

Theorem 6.3. $\mathbb{Z}_{\geq 0} \subseteq \mathbb{R}$ *has no upper bound.*

Proof. By contradiction. If it did, let $\Psi = \sup \mathbb{Z}_{\geq 0}$. Approx w/ $\epsilon = \frac{1}{2} \Rightarrow \exists n \in \mathbb{Z}_{\geq 0}$ such that $\Psi - \frac{1}{2} < n$. Then $n + 1 \in \mathbb{Z}_{\geq 0}$ and $n + 1 > \Psi + \frac{1}{2} > \Psi$. Contradiction. $\square$

6.3 Absolute Value

$|\cdot|: \mathbb{R} \to \mathbb{R}$

$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ "Distance" between $x, y \in \mathbb{R}$ is $|x-y|$. $|x-y| < \epsilon$ means x, y are "ϵ-close".

Theorem 6.4 (Triangle Inequality)**.**

$$|x+y| \leq |x| + |y|$$

$$|x-z| \leq |x-y| + |y-z|$$

Proof. Easy if $x = 0$ or $y = 0$. If both > 0, then LHS $= x + y =$ RHS. If both < 0, then LHS $= -x - y =$ RHS. If $y < 0 < x$, RHS $= x - y$

$\left.\begin{array}{l} x - y > -x - y \\ x - y > x + y \end{array}\right\} \implies x - y \geq |x+y|$

Similar if $x < 0 < y$. □