# MATH UN1207, Honors Math A

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## Contents

# 1 September 4, 2019

## 1.1 Statements

**Definition.** *A statement (or proposition) is an assertion that its true <u>or</u> false (but not both).*

<u>Ex</u>
P = "Ringo Starr is alive"
Q = "The earth is flat"

**Definition.** *The truth value of a statmeent is T (if true) and F (if false)*

**Definition.** *The negation of a statement P is the statement "P is false" (∼P)*

<u>Ex</u>
∼P = "Ringo Starr is dead"
∼Q = "The earth is not flat"

**Remark.** *∼(∼P) = P*

**Definition.** *The conjunction of two statements P, Q is the statement "P and Q" (P ∧ Q)*
*i.e. The truth value of P∧Q is T if P is T and Q is T, and F otherwise.*

**Proposition.** *P ∧(∼P) = F*

| P | ∼P | P ∧(∼P) |
|---|---|---|
| T | F | F |
| F | T | F |

**Definition** (Disjunction)**.** *P or Q (P ∨ Q) is F if P is F and Q is F, and T otherwise. "or" in math is <u>always</u> inclusive.*

**Proposition.** $(P \wedge Q) \vee (\sim P \vee \sim Q) = T$

| P | Q | P ∧ Q | ∼P ∨ ∼ Q | (P ∧ Q) ∨ ( ∼P ∨ ∼ Q ) |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | T | T |
| F | T | F | T | T |
| F | F | F | T | T |

## 1.2 Conditionals

**Definition** (Conditional)**.** *P <u>implies</u> Q or <u>if P then</u> Q, or P ⇒ Q, is Q ∨ (∼P).*
*i.e. if P is T, then Q is T.*

**Remark.** *If $P$ is F, then we say $P \Rightarrow Q$ is "vacuously true"*

**Proposition.** *$(P \wedge Q) \Rightarrow P$, $(P \wedge Q) \Rightarrow Q$, $(P \wedge (P \Rightarrow Q)) \Rightarrow Q$*

**Definition** (Biconditional)**.** *$P$ iff $Q$ or $P$ if and only if $Q$ or $P \iff Q$ is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$*

| P | Q | P $\iff$ Q |
|---|---|---|
| T | T | T |
| T | F | F |
| F | T | F |
| F | F | T |

**Proposition.** *1 - Suppose $P$ and $P \Rightarrow Q$. Then $Q$.*
*2 - Suppose $P \Rightarrow Q$ and $\sim Q$. Then $\sim P$.*
*3 - Suppose $P \Rightarrow Q$ and $P \Rightarrow \sim Q$. Then $\sim P$.*
*4 - Suppose $P \vee Q, P \Rightarrow R$, and $Q \Rightarrow R$. Then $R$.*

*Proof of 3.* (P $\Rightarrow$ Q) $\wedge$ (P $\Rightarrow\sim$ Q) $\Rightarrow \sim$ P

| P | Q | P $\Rightarrow$ Q | P $\Rightarrow\sim$ Q | (P $\Rightarrow$ Q) $\wedge$ (P $\Rightarrow\sim$Q ) | Whole |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | T | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

□

$$\text{De Morgan's Laws} \begin{cases} \sim (P \wedge Q) \iff \sim P \vee \sim Q \\ \sim (P \vee Q) \iff \sim P \wedge \sim Q \end{cases}$$

**Definition** (Contrapositive)**.** *$(P \Rightarrow Q) \iff (\sim Q \Rightarrow \sim P)$*

# 2 September 9, 2019

## 2.1 Predicates

**Definition.** *A predicate $P(x)$ is a family of statements depending on a variable $x$*

Ex
$P(x)$ = "$x$ is a banana"
$Q(x)$ = "$x > 7$ "

Existential Quantifier:

$\exists x \mid P(x)$: "there exists an $x$ such that $P(x)$"

Universal Quantifier:

$\forall x\, P(x)$: "for all $x$, $P(x)$"

e.g. $\forall x, Q(x)$ is false ($Q(x)$ as above)
$\forall x(P(x) \vee \sim P(x)))$ is always true

e.g.

$$\sim (\forall x P(x)) \Longleftrightarrow \exists x(\sim P(x))$$

$$\sim (\exists P(x)) \Longleftrightarrow \forall x(\sim P(x))$$

**Remark.** *To prove $\exists x P(x)$, just find an $x$ such that P(x). To prove $\forall x P(x)$, write something like "take an x..."*

## 2.2 Sets

A set is a collection of objects.
e.g.

$$S = \{1, 2, 4\}, T = \{\{1\}, 2, \text{water}\}$$

$\{1\} \in T$, but $1 \notin T$. "1 is not an element of $T$"

$\mathbb{N}$ = set of natural numbers $\{1, 2, 3, ...\}$
$\mathbb{Z}$ = set of integers $\{..., -2, -1, 0, 1, 2, ...\}$
$\mathbb{Q}$ = rational numbers
$\mathbb{R}$ = real numbers
$\mathbb{C}$ = complex numbers

**Definition.** $\forall x \in S, P(x)$ *just means* $\forall x(x \in S \Rightarrow P(x))$

**Definition** (Set Inclusion). $S \subseteq T$ *($S$ is a subset of $T$) if* $\forall x \in S, x \in T$

**Remark.** *In definitions, we write "if" instead of "if and only if" even though the latter is what we mean.*

**Definition.** $S = T$ *if* $S \subset T$ *and* $T \subset S$

Warning - Order matters with quantifiers!
$\forall n \in \mathbb{Z}, \exists m \in \mathbb{Z} \mid m + n = 0$ is TRUE
$\exists m \in \mathbb{Z} \mid \forall n \in \mathbb{Z}, m + n = 0$ is FALSE

**Proposition.** *IF* $S \subset T$ *and* $T \subset U$, *then* $S \subset U$

*Proof.* We know $\forall x \in S, x \in T$ and $\forall y in T, y \in U$. Therefore, $\forall x \in S, x \in U$. Thus, $S \subset U$. □

## 2.3 Axioms for Sets

1. There exists a set
2. "Axiom of Specification" (how to take subsets)
   Given set $S$ and any predicate $P(x)$, there exists a set $T$ such that
   (a) $T \subset S$
   (b) $\forall x, (x \in T \iff P(x))$

   We write $T = \{x \in S \mid P(x)\}$.

   Take $Q(x) =$ "$x \neq x$"(always false). Then, take $S$ any set $= \{x \in S \mid Q(x)\} = \emptyset$ (the empty set).

## 2.4 Russell's Paradox

If we take axiom of specification w/o picking $S$, we get a contradiction. Take $P(x) =$ "$x = x$"(always true). Stengthened axiom $\Rightarrow$ get a set of all sets $\mathcal{V}$. We will show a contradiction.
Let $T = \{S \in \mathcal{V} \mid S \notin S\}$. Is $T \in T$?. If $T \in T \Rightarrow T$ really bad $\Rightarrow T \notin T \Rightarrow\Leftarrow$. If $T \notin T \Rightarrow T$ is not really bad $\Rightarrow T \in T \Rightarrow\Leftarrow$. We conclude that there is no "set of all sets".

## 2.5 Axioms Cont.

3. Axiom of Unions
   Given sets $S, T$, there exists $S \cup T$ such that $\forall x, x \in S \cup T \iff x \in S$ or $x \in T$.

3'. Axiom of Intersection
   Given sets $S, T$, there eixsts $S \cap T$ such that $\forall x (x \in S \cap T) \iff x \in S$ and $x \in T$

**Theorem 2.1.** $A \cup B = B \cup A$

**Theorem 2.2.** $(A \cup b) \cup C = A \cup (B \cup C)$

**Theorem 2.3.** $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

*Proof of 2.3.* Need to show: $\forall x, x \in LHS \iff x \in RHS$

$$\begin{aligned} x \in LHS &\iff x \in A \cap (B \cup C) \\ &\iff x \in A \wedge x \in (B \cup C) \\ &\iff x \in A \wedge (x \in B \vee x \in C) \\ &\iff (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C) \\ &\iff x \in A \cap B \vee x \in A \cap C \\ &\iff x \in (A \cap B) \cup (A \cap C) \\ &\iff x \in RHS \end{aligned}$$

$\square$

# 3 September 11, 2019

## 3.1 More De Morgan

**Definition.** *Let S, A be sets. Then the complement S-A or S\ A is* $\{x \in S \mid x \notin A\}$

There are two more De Morgan laws.

1. $S \setminus (A \cup B) = (S \setminus A) \cap (S \setminus B)$
2. $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$

## 3.2 Power Sets

4. For all sets $A$, there exists a set $\mathcal{P}(A)$, the power set of $A$, such that its elements are precisely the subsets of $A$.

$$\forall B, B \subseteq A \Leftrightarrow B \in \mathcal{P}(A)$$

Ex $A = \{1, 2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

**Proposition.** $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$

*Proof.* Similar to other set equality proofs.

$$\begin{aligned}
S \in \mathcal{P}(A \cap B) &\Longleftrightarrow S \subseteq (A \cap B) \\
&\Longleftrightarrow \forall x \in S, (x \in A \cap B) \\
&\Longleftrightarrow \forall x \in S, (x \in A \text{ and } x \in B) \\
&\Longleftrightarrow \forall x \in S, x \in A \text{ and } \forall x \in S, s \in B \\
&\Longleftrightarrow S \subseteq A \text{ and } S \subseteq B \\
&\Longleftrightarrow S \in \mathcal{P}(A) \text{ and } S \in \mathcal{P}(B) \\
LHS &\Longleftrightarrow S \in \mathcal{P}(A) \cap \mathcal{P}(B)
\end{aligned}$$

$\square$

## 3.3 Cartesian Products

"Def": An ordered pair is a list $(a, b)$ where $a, b$ are math objects. We say $(a, b) = (a', b')$ if $a = a'$ and $b = b'$.
Ex $(1, 2) \neq (2, 1)$

5. Axiom of Products: Givens sets $A, B$, there exists a set $A \times B$ whose elements are exactly the pais $(a, b)$ with $a \in A, b \in B$.

$$\forall x, x \in A \times B \iff x = (a,b) \text{ with } a \in A, b \in B$$

**Remark.** *This axiom is actually not necessary*

**Proposition.** $A \times (B \cup C) = A \times B \cup A \times C$

*Proof.* Same format as any other set equality proof.

$$\begin{aligned}(a,b) \in LHS &\iff a \in A \text{ and } b \in (B \cup C)\\ &\implies a \in A \text{ and } (b \in B \text{ or } b \in C)\\ &\iff (a \in A \text{ and } b \in B) \text{ or } (a \in A \text{ and } b \in C)\\ &\iff (a,b) \in A \times B \text{ or } (a,b) \in A \times C\\ &\iff (a,b) \in (A \times B) \cup (A \times C)\end{aligned}$$

□

## 3.4 Functions

<u>"Def"</u> Let $S, T$ be sets. A function $F : S \to T$ (from S to T) is a rule assigning an element of $T$ to each element of $S$. i.e. If $s \in S$, this element of $T$ is denoted $f(s)$.

**Remark.** *We can make a non-fake def of a function $S \to T$ by defining it as a subset of $S \times T$.*

**Remark.** *We write $f : S \to T$, "f is a function from S to T", "f maps S to T", and $x \mapsto x^2$, "x maps to $x^2$".*

**Definition.** *If $f : S \to T$, then $S$ is the domain and $T$ is the codomain.*

**Definition.** *The graph of $f : S \to T$, (sometimes denoted $\Gamma(f)$) is $\{(x,y) \in S \times T \mid y = f(x)\}$.*

**Definition.** *Two functions f: $S \to T$ and $g : S' \to T'$ are equal if $S = S'$, $T =$ T', and $\forall s \in S, f(s) = g(s)$.*

**Definition.** *If $f : S \to T$ and $g : T \to U$ are functions, their <u>composition</u> is the function $g \circ f : S \to U$ such that $(g \circ f)(s) = g(f(s)) \forall s \in S$.*

**Definition.** *If $S$ is set, then $id_s$: $S \to S$ is called the identity function*

## 3.5 Injectivity and Surjectivity

**Definition.** *$f : S \to T$ is injective if wheneber $f(s) = f(s'), s = s'$.*

**Definition.** *$f : S \to T$ is surjective if $\forall t \in T, \exists s \in S$ such that $f(s) = t$.*

**Proposition.** *$id_s$ is injective and surjective/*

*Proof.* If $\mathrm{id_s(s)} = \mathrm{id_s}(s'), s = s'$. This proves injectivity. $\forall s \in S$, we have $\mathrm{d_s(s)}$ = s. This proves surjectivity. Thus, we have both. $\square$

**Remark.** *$f : S \to T$ injective $\iff \forall t \in T$, there is at <u>most</u> one preimage(at most one $s \in S$ such that $f(s) = t$). $f$ surj $\iff \forall t \in T$, there is at least one element of preimage.*

**Definition.** *$f : S \to T$ is bijective if it is injective and surjecitve.*

<u>Ex/Prop</u>

1. $f : S \to T, g : T \to U$, both inj.
   (a) $\Rightarrow g \circ f$ inj.
   (b) $\Rightarrow f$ inj.
2. $f : S \to T, g : T \to U$ surj.
   (a) $g \circ f$ surj.
   (b) $g$ surj.

### 3.6 Inverses

**Definition.** *If $f : S \to T$ is a function, an <u>inverse</u> to $f$ is a function $g : T \to S$ such that*

1. $g \circ f = \mathrm{id_S}$: $S \to S$
2. $f \circ g = \mathrm{id_T}$: $T \to T$

## 4 September 16, 2019

### 4.1 Inverses Cont.

**Theorem 4.1.** *$f : S \to T$ has an inverse if and only if it is bijective.*

*Proof.* ($\Rightarrow$) by homework problem
($\Leftarrow$) We know by definition of bijectivity, $\forall t \in T, \exists s \in S \mid f(s) = t$ and $\forall s, s' \in S, f(s) = f(s') \Rightarrow s = s'$. Define $g : T \to S$.

**Lemma 4.2.** *$f(s) = t \iff g(t) = s$*

Lemma can be proven by definition of $g$. Lemma $\Rightarrow$ if $g(t) = s'$, then $f(s') = t$. Thus, $t = f(s) = f(s')$. By injectivity of $f, s = s'$.

$$\begin{aligned}(f \circ g)(t) &= f(g(t)) \\ &= f(s) \\ &= t\end{aligned}$$

$$
\begin{aligned}
(g \circ f)(s) &= g(f(s)) \\
&= g(t) \\
&= s
\end{aligned}
$$

□

**Remark.** *By definition, if g is inverse to f, then f is inverse to g.*

**Proposition.** *If $g, g'$ are inverses to $f$, then $g = g'$. (Inverses are unique)*

*Proof.* Take any $t \in T$.

$$
\begin{aligned}
g'(t) &= (g' \circ id_t)(t) \\
&= (g' \circ (f \circ g))(t) \\
&= ((g' \circ f) \circ g)(t) \\
&= (id_s \circ g)(t) \\
&= g(t)
\end{aligned}
$$

□

**Definition.** *If $f : S \to T$ and $U \subseteq S$, then the image of $U$ (under $f$), denoted $f(U)$, is $\{t \in T \mid \exists s \in U \text{ with } f(s) = t\} = \{f(s) \mid \overline{s \in U}\}$.*

**Definition.** *If $f : S \to T$ and $V \subseteq T$, the preimage of $V$ under $f$, denoted $f^{-1}(V$, is $\{s \in S \mid f(s) \in V\}$.*

## 4.2 Numbers

We will axiomatize $\mathbb{R}$.

**Definition.** *A binary operation on a set $S$ is a function $S \times S \to S$.*

**Definition.** *A relation on $S$ is a subset of $S \times S$.*

Assumption: There exists a set $\mathbb{R}$, equippied with two binary relations $+$, $\bullet$, one relation $>$, and two elements 0, 1 satisfying the following axioms.

## 4.3 Axioms

All of the axioms are $\forall x, y, z \in \mathbb{R}$, unless otherwise noted.

1. Commutativity
   $x + y = y + x$ $\qquad$ $x \cdot y = y \cdot x$

2. Associativity
   $x + (y + z) = (x + y) + z$ $\qquad$ $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

3. Distributivity
   $x \cdot (y + z) = x \cdot y + x \cdot z$

4. Identity Elements
   $0 \neq 1 \qquad 0 + x = x \qquad 1 \cdot x = x$

5. Additive Inverse
   $\forall x \in \mathbb{R}, \exists w \in \mathbb{R} \mid w + x = 0$. We can denote $w$ by $-x$.

Note: Axioms 1-5 are a commutative ring.

6. Multiplicative Inverse
   $\forall x \neq 0 \in \mathbb{R}, \exists w \in \mathbb{R} \mid w \cdot x = 1$. We can denote $w$ by $\frac{1}{x}, x^{-1}$, etc.

Note: Axioms 1-6 are a field.

e.g. $\frac{\mathbb{Z}}{n\mathbb{Z}} = \{0, 1, ..., n-1\}$ with modular arithmetic is a commutative ring and is a field iff $n$ is prime.
Notation: $x + (-y) =: x - y$
$x \cdot \frac{1}{y} = \frac{x}{y}$

7. Order Axiom 1
   If $x > 0$ and $y > 0$, then $x + y > 0$ and $x \cdot y > 0$. ("add + mult preserve the order").

8. Order Axiom 2
   If $x \neq 0$, then $x < 0$ or $x > 0$ but not both. ("trichotomy")

9. Order Axiom 3
   not $0 > 0$ ($0 \not> 0$)

10. Order Axiom 4
    If $x > y$, then $x + z > y + z$.

Note: Axioms 1 - 10 are an ordered field. (e.g. $\mathbb{Q}, \mathbb{R}$)

**Proposition.** *High School Algebra*

*Proof.* exercise $\square$

# 5 September 18, 2019

## 5.1 Application of Axioms

**Proposition** (Multiplicative Cancellation)**.** *Given an ordered field* $\mathbb{R}, \forall a, b, x \in \mathbb{R}$

$$xa = xb, x \neq 0 \Longrightarrow a = b$$

*Proof.* By axiom of multiplicative inverse, $\exists w \in \mathbb{R}$ with $wx = 1$. Since $xa = xb$, we can multiply both sides by $w$ to obtain $wxa = wxb$. This statement is equal to $(wx)a = (wx)b$ by associativity. Then, $1a = 1b$. Thus, $a = b$ by the axiom of identity. $\square$

**Theorem 5.1** (Trichotomy). $\forall a, b \in \mathbb{R}$, *exactly one of the following is true:*

1. $a > b$
2. $b > a$
3. $a = b$

**Lemma 5.2.** $\forall a, b \in \mathbb{R}, a > b \iff a - b > 0$

*Proof.* Do each direction separately.
($\Rightarrow$)

$$\begin{aligned} a > b &\Rightarrow a + (-b) > b + (-b) \text{ by axiom of add. inverse and axiom 10} \\ &= a - b > 0 \text{ by def of "-"} \checkmark \end{aligned}$$

($\Leftarrow$)

$$\begin{aligned} a - b > 0 &\Rightarrow (a-b) + b > 0 + b \text{ by axiom 10} \\ &\Rightarrow a + (-b + b) > b \text{ by associativity and add. identity} \\ &\Rightarrow a + (b + -b) > b \text{ by commutativity} \\ &\Rightarrow a + 0 > b \text{ by add. inverse} \\ &\Rightarrow 0 + a > b \text{ by commutativity} \\ &\Rightarrow a > b \text{ by add. identity} \end{aligned}$$

□

**Lemma 5.3.** $\forall a, b \in \mathbb{R}, b > a \iff 0 > a - b$

*Proof.* Either copy Lemma 5.2 (almost) or $b > a \iff b - a > 0$ by Lemma 5.2 then show $b - a > 0 \iff 0 > a - b$. □

**Lemma 5.4.** $\forall a, b \in \mathbb{R}, a = b \iff a - b = 0$

*Proof.* Simple

$$\begin{aligned} a = b &\Rightarrow a + (-b) = b + (-b) \\ &\Rightarrow a - b = 0 \\ &\Rightarrow (a - b) + b = 0 + b \\ &\Rightarrow (a) + (-b + b) = b \\ &\Rightarrow a + (b + -b) = b \\ &\Rightarrow a + 0 = b \\ &\Rightarrow a = b \end{aligned}$$

□

*Proof of 5.1.* Let $x = a - b$. Either $x = 0$ or $x \neq 0$ but not both. By Axiom 8, if $x \neq 0$, either $x > 0$ or $0 > x$ but not both. By Axiom 9, if $x = 0$ then NOT $x > 0, 0 > x$. This implies that exactly one of $x = 0, x > 0, 0 > x$ is true. Lemmas 5.2, 5.3, 5.4 tell us that $x = 0 \iff 5.4, x > 0 \iff 5.2, 0 > x \iff 5.3$. □

<u>Exercises to try on own</u>

- $\forall x, 0 \cdot x = 0$
- $1 > 0$ (tricky)

## 5.2 Natural Numbers

**Definition.** *$S \subseteq \mathbb{R}$ is an <u>inductive set</u> if*

1. $0 \in S$
2. $\forall x \in S, x + 1 \in S$

<u>Ex</u> $\{x \in \mathbb{R} \mid x \geq 0\}$ is inductive (by above exercise)

**Definition.** *A <u>natural number</u> is an $x \in \mathbb{R}$ such that $x$ is a number of every inductive set.*

The set of natural numbers is called $\mathbb{Z}_{\geq 0}$.

**Theorem 5.5** (Mathematical Induction)**. ??** *Let $P(n)$ be a predicate defined on $\mathbb{Z}_{\geq 0}$ such that*

1. *$P(0)$ is true (base case)*
2. *$\forall n \in \mathbb{Z}, P(n) \Rightarrow P(n+1)$*

*Then, $\forall n \in \mathbb{Z}_{\geq 0}, P(n)$ is true.*

*Proof.* $\{n \in \mathbb{R} \mid P(n)\}$ is an inductive set. Hence $\mathbb{Z}_{\geq 0} \subseteq S$ by definition of the natural numbers. □

**Proposition.** $\forall n \in \mathbb{Z}_{\geq 0}, n^2 > n$

*Proof.* Base Case $n = 0 : 0 \geq 0$✓. Inductive step: Assume $n^2 \geq n$.

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &\geq n + 2n + 1 \\ &\geq n + n + 1 \\ &\geq n + 1\end{aligned}$$

□

**Proposition.** $\forall n \in \mathbb{Z}_{\geq 0}, 0 + 1 + 2 + ... + n = \frac{n(n+1)}{2}$

*Proof.* Base Case $n = 0 : 0 = 0\checkmark$. Inductive step: Assume $0 + ... + n = \frac{n(n+1)}{2}$. Add $n + 1$ to both sides.

$$\begin{aligned} 0 + ... + n + n + 1 &= \frac{n(n+1)}{2} + n + 1 \\ &= n + 1\left(\frac{n}{2} + 1\right) \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

□

**Definition.** *A positive integer is a natural number that is not 0.*

**Definition.** *An integer is the difference of two natural numbers (called $\mathbb{Z}$).*

**Definition.** *A rational number is a quotient of two integers($\mathbb{Q}$).*

"Principle" of recursive definition: For any set $S$, a function $f : \mathbb{Z}_{\geq 0} \to S$ may be specified by a choice of $f(0)$ and a function expressing $f(n)$ in terms of $f(m), m < n$.

"Proof"
$\{n \in \mathbb{Z}_{\geq 0} \mid \forall m < n$, the above data determine $f(n)$ uniquely$\}$ is an inductive set.
Ex
Define $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. $f(0) = 0, f(1) = 0, f(n) = f(n-1) + f(n-2)$.

### 5.3 Sums

Define for $n \in \mathbb{Z}_{\geq 0}, f : \mathbb{Z}_{\geq 0} \to \mathbb{R}$.

$$\sum_{i=0}^{n} f(i)$$

**Remark.** *A few rules*

1. *"i" is a dummy variable so we can use anything.*
2. $$\sum_{i=m}^{n} f(i) = \sum_{i=1}^{n} f(i) - \sum_{i=1}^{m-1} f(i)$$

## 6 September 23, 2019

### 6.1 Supremum

Recall: Both $\mathbb{Q}$ and $\mathbb{R}$ should be ordered fields (satisfy axioms 1-10)
e.g. $\sqrt{2} \in \mathbb{R}, \notin \mathbb{Q}$

**Definition.** *A subset $S \subseteq \mathbb{R}$ is* <u>*bounded above*</u> *if $\exists y \in \mathbb{R}$ such that $\forall x \in S, x \leq y$.*

e.g. $S = \{3, 3.1, 3.14, 3.141, ...\}$ is bounded above by 4 (or $\pi$).

**Definition.** *An upper bound $y$ for $S \subseteq \mathbb{R}$ is a least upper bound or supremum if $y$ is an upper bound for $S$ and if $z$ is an upper bound for $S$, then $y \leq z$.*

11. Axiom(completeness): every nonempty bounded above set $S \subseteq \mathbb{R}$ has a supremum.

**Proposition.** *If $y$ and $y'$ are suprema of $S$, then $y = y'$. (suprema are unique)*

*Proof.* $y$ is a supremum, and $y'$ is an upper bound $\Rightarrow y \leq y'$ by definition of supremum. $y'$ is a supremum, and y is an upper bound $\Rightarrow y' \leq y$ by definition of supremum. So $y = y'$ by trichotomy. $\square$

Notation: $\sup S$ is the supremum of $S$.

## 6.2 Infimum

**Definition.** *$S$ is bounded below if existsy $\mid \forall x \in S, y \leq x.y$ is a lower bound for $S$ if $\forall x \in S, y \leq x$. $y$ is the greatest lower bound or* <u>*infimum*</u> *if $y$ is a lower bound and if $z$ is any other lower bound, then $x \leq y$. $\inf \overline{S = \textit{infimum}}$ of $S$.*

**Definition.** *If $S \subseteq \mathbb{R}$ is nonempty and bdd below, then $\exists!$ infimum $\inf S$.*

*Proof.* Let $-S = \{x \mid x \in S\}$. $-S$ is nonempty and bounded above. Thus, $\sup(-S)$ exists and is unique.
Claim: $-\sup(-S)$ is an infimum for $S$. Uniqueness same as for sup. $\square$

**Theorem 6.1** (Approximation Property)**.** *Let $S \subseteq \mathbb{R}$ be nonempty and bounded above. $\forall \epsilon > 0, \exists x \in S \mid \sup S - \epsilon \leq x$.*

*Proof by Contradiction.* Assume $\exists \epsilon > 9. \forall x \in S, \sup S - \epsilon > x$. Then, $\sup S - \epsilon$ is an upper bound for $S$. Then, by definition of $\sup S, \sup S \leq \sup S - \epsilon$. Contradiction of $\epsilon > 0$. $\square$

**Theorem 6.2** (Additivity of Supremum)**.** *If $S, T \subseteq \mathbb{R}$ nonempty, bounded above, let $S + T := \{s + t \mid s \in S, t \in T\}$. Then $\sup(S + T)$ exists and equals $\sup(S) + \sup(T)$.*

*Proof.* Let $s = \sup S, t = \sup T$. $\forall x \in S, x \leq s, \forall y \in T, y \leq t \Rightarrow \forall x \in S, \forall y \in T, x + y \leq s + t \Rightarrow s + t$ is an upper bound for $S + T$. We want to show that $s + t$ is the <u>least</u> upper bound. Suppose not. Then $\exists \delta > 0 \mid s + t - \delta$ is an upper bound for $S + T$. Let $\epsilon = \frac{\delta}{2}$.

$$\begin{aligned} \text{Approximation Property} \Longrightarrow & \exists x \in S \mid s - \epsilon < x \\ & \exists y \in T \mid t - \epsilon < y \end{aligned}$$

Then $x + y \in S + T$. $s + t - \delta = s + t - 2\epsilon < x + y$. Contradiction! $\square$

**Proposition.** *Suppose $S$, $T \subseteq \mathbb{R}$ such that $\forall x \in S, \forall y \in T, x \leq y$. Then $\sup S$ exists and $\inf T$ exists, and $\sup S \leq \inf T$.*

*Proof.* Any $x \in S$ is a lower bound for $T$
$\Rightarrow \inf T$ exists
Any $y \in T$ is upper bd for $S$
$\Rightarrow \sup S$ exists
Suppose $\sup S > \inf T$. Then $S - \sup S - \inf T$. Let $\epsilon = \frac{\delta}{2}$.
Approx $\Rightarrow \exists x \in S \mid \sup S - \epsilon < x$. Similar approx. result $\Rightarrow \exists y \in T \mid \inf T + \epsilon > y$.

$$y < \inf T + \epsilon = \sup S - \epsilon < x$$

Contradiction! $\sup S \leq \inf T$ ☐

**Theorem 6.3.** *$\mathbb{Z}_{\geq 0} \subseteq \mathbb{R}$ has no upper bound.*

*Proof.* By contradiction. If it did, let $\Psi = \sup \mathbb{Z}_{\geq 0}$. Approx w/ $\epsilon = \frac{1}{2} \Rightarrow \exists n \in \mathbb{Z}_{\geq 0}$ such that $\Psi - \frac{1}{2} < n$. Then $n + 1 \in \mathbb{Z}_{\geq 0}$ and $n + 1 > \Psi + \frac{1}{2} > \Psi$. Contradiction. ☐

### 6.3 Absolute Value

$|\cdot| : \mathbb{R} \to \mathbb{R}$
$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$ "Distance" between $x, y \in \mathbb{R}$ is $|x - y|$. $|x - y| < \epsilon$ means $x, y$ are "$\epsilon$-close".

**Theorem 6.4** (Triangle Inequality)**.**

$$|x + y| \leq |x| + |y|$$

$$|x - z| \leq |x - y| + |y - z|$$

*Proof.* Easy if $x = 0$ or $y = 0$. If both $> 0$, then LHS $= x + y =$ RHS. If both $< 0$, then LHS $= -x - y =$ RHS. If $y < 0 < x$, RHS $= x - y$
$\left.\begin{array}{l} x - y > -x - y \\ x - y > x + y \end{array}\right\} \implies x - y \geq |x + y|$
Similar if $x < 0 < y$. ☐

# 7 September 25, 2019

### 7.1 Archimedean Prop.

**Proposition** (Archimedean Property)**.** *Let $x > 0$ and $y \in \mathbb{R}$. Then $\exists n \in \mathbb{Z} \mid nx > y$.*

*Proof.* Consider $\frac{y}{x}$. From earlier result, $\exists n \in \mathbb{Z}_{\geq 0}$ with

$$n > \frac{y}{x} \iff nx > y$$

$\square$

**Corollary 7.0.1.** *If $a, x, y \in \mathbb{R}$ with $a \leq x \leq a + \frac{y}{n}$ for every $n \in \mathbb{Z}_{\geq 0}$, then $a = x$.*

*Proof.* Assume otherwise, so $a < x \iff x - a > 0$.

$$\begin{aligned} \text{Prop} \Rightarrow \exists n \in \mathbb{Z}_{\geq 0} \text{ such that} n(x-a) &> y \\ x - a &> \frac{y}{n} \\ x &> \frac{y}{n} + a \end{aligned}$$

$\square$

## 7.2 Finite Sets

Notation $[n] = \{m \in \mathbb{Z}_{\geq 0} \mid 0 < m \leq n\}$
$= \{1, 2, 3, ..., n\}$

**Theorem 7.1.** $\forall m, n \in \mathbb{Z}_{\geq 0}$

1. $\exists$ *inj.* $f : [m] \to [n] \iff m \leq n$
2. $\exists$ *surj.* $f : [m] \to [n] \iff m \geq n$

*Proof.* Proof of both.

1. $(\Leftarrow)$ $m \leq n$ so define $f : [m] \to [n]$. $f(i) = i$ so $f$ is clearly injective.

   $(\Rightarrow)$ Induct on $m$ (fix $n$). Assume for given $m$, all $n$. Suppose $f : [m+1] \to [n]$ is injective. $\forall i \in [m+1]$, if $i \neq m+1 \Rightarrow f(m+1)$. Define $\bar{f} : [m] \to [n] \backslash \{f(m+1)\}$. $\bar{f}(i) = f(i)$ so it is stilll injective. Define $h : [n] \backslash \{f(m+1)\} \to [n-1]$.
   $h(i) = \begin{cases} i \text{ if } i < f(m+1) \\ i - 1 \text{ if } i > f(m+1) \end{cases}$ Easy to show that $h$ is injective. Then $h \circ \bar{f} : [m] \to [n-1]$ is injective (composition of injective functions). Inductive hypothesis $\Rightarrow m \leq n - 1 \Rightarrow m + 1 \leq n$

2. Similar to 1

$\square$

**Definition.** *A set is finite if $\exists$ bijection $f : [n] \to S$ for some $n \in \mathbb{Z}_{\geq 0}$. If not, $S$ is <u>infinite</u>.*

**Proposition.** *Given finite set $S$, the $n \in \mathbb{Z}_{\geq 0}$ as above is <u>unique</u>.*

*Proof.* Suppose $f : [n] \to S.g : [m] \to S$ are bijections. Then let $h : S \to [n]$ be inverse of $f$. Then $h \circ g : [m] \to [n]$ is bijective.

$$\begin{aligned} \text{Thm applied to } h \circ g \Longrightarrow & m \leq n \\ & n \leq m \\ \Longrightarrow & n = m \end{aligned}$$

$\square$

<u>Ex</u>
\# of elemenets: $|\dot{S}|$ (like magnitude). If $S, T$ are finite sets, then $S \cup T, S \cap T, S \times T, S^T$ are finite. Any subset of $S$ is finite. If $S \subseteq T, |\dot{S}| \leq |\dot{T}|$.

<u>Ex</u>
$\mathbb{Z}_{\geq 0}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are infinite.

Notation: Suppose $f : S \to T, U \subseteq S$. Then, $f|_U : U \to T.f|_U(s) = f(s), s \in U$.

**Definition.** *The inclusion of $S \subseteq T$ is $id_T|_S$.*

<u>Ex</u>
$f$ injective $\Rightarrow f|_U$ injective. Does <u>not</u> hold for sujectivity.

**Definition.** *For $a \leq b \in \mathbb{R}$, an <u>interval</u> is one of*

$$\begin{aligned} [a, b] &= \{a \leq x \leq b \mid x \in \mathbb{R}\} \\ (a, b) &= \{a < x < b \mid x \in \mathbb{R}\} \\ [a, b) &= \{a \leq x < b \mid x \in \mathbb{R}\} \\ (a, b] &= \{a < x \leq b \mid x \in \mathbb{R}\} \end{aligned}$$

*<u>or</u> allow $a$ or $b$ to be "$\infty$" or "$-\infty$" for open intervals w/ $-\infty < x < \infty \forall x \in \mathbb{R}$.*

<u>Ex</u>
If $a \neq b$, these are infinite sets.

**Definition.** *Some function definitions*

$$\begin{array}{lcl} \text{If } f : [a, b] \to \mathbb{R} & \text{then} & f + g : [a, b] \to \mathbb{R} \\ \quad g : [a, b] \to \mathbb{R} & & fg : [a, b] \to \mathbb{R} \end{array}$$

*If $c \in \mathbb{R}, cf : [a, b] \to \mathbb{R}$. $(cf)(x) = cf(x)$.*

**Definition.** *If* $f_1, \dots, f_n : [a,b] \to \mathbb{R}, c_1, \dots, c_n \in \mathbb{R}$, *then the corresponding linear combination is*

$$\sum_{i=1}^{n} c_i f_i = c_1 f_1 + c_2 f_2 + \dots c_n f_n$$

## 7.3 Step Functions

**Definition.** $f : [a,b] \to \mathbb{R}$ *is a step function if* $\exists$ *a finite set of real numbers* $S = \{x_0, \dots, x_n\} \subset \mathbb{R}$, *called a partition for* $f$, *with* $a = x_0 < x_1 < x_2 < \dots < x_n = b$ *and* $c_1 \dots c_2 \in \mathbb{R}$ *such that* $\forall i \in [n], \forall x \in (x_{i-1}, x_i), f(x) = c_i$.

$$(i.e\, f|_{(x_{i-1},x_i)} = c_i)$$

**Proposition.** *If* $f, g : [a,b] \to \mathbb{R}$, *so are* $f + g, fg$.

*Proof.* Let $S$ be partition for $f$ and let $T$ be partition for $g$. Idea: Then $S \cup T$ is a partition for $f+g$ and $fg$. Let $S = \{x_0, \dots, x_m\}, T = \{y_o, \dots, y_n\}$. $S \cup T = \{z_0, \dots, z_p\}$ (is finite by earlier exercise). For any $z_k \in S \cup T$ with $k > 0$, let $k_i$ = greatest element of $S \mid x_i < z_k$. Let $y_j$ = greatest element of $T \mid y_j < z_k$. Then $z_{k-1}$ = a maximum of $x_i, y_j$ and $z_k$ = a minimum of $x_{i+1}, y_{j+1}$.
Hence $(z_{k-1}, z_k) \subset (x_i, x_{i+1}) \cap (y_j, y_{j+1})$
$\Rightarrow f+g, fg$, constant on $(z_{k-1}, z_k)$ $\square$