Table 3.2 Applications for Public-Key Cryptosystems

Algorithm	Encryption/Decryption	Digital Signature	Key Exchange
RSA	Yes	Yes	Yes
Diffie–Hellman	No	No	Yes
DSS	No	Yes	No
Elliptic curve	Yes	Yes	Yes

Requirements for Public-Key Cryptography

The cryptosystem illustrated in Figure 3.9 depends on a cryptographic algorithm based on two related keys. Diffie and Hellman postulated this system without demonstrating that such algorithms exist. However, they did lay out the conditions that such algorithms must fulfill [DIFF76]:

1. It is computationally easy for a party B to generate a pair (public key PU_b, private key PR_b).
2. It is computationally easy for a sender A, knowing the public key and the message to be encrypted, M, to generate the corresponding ciphertext:

$$C = E(PU_b, M)$$

3. It is computationally easy for the receiver B to decrypt the resulting ciphertext using the private key to recover the original message:

$$M = D(PR_b, C) = D[PR_b, E(PU_b, M)]$$

4. It is computationally infeasible for an opponent, knowing the public key, PU_b, to determine the private key, PR_b.
5. It is computationally infeasible for an opponent, knowing the public key, PU_b, and a ciphertext, C, to recover the original message, M.

We can add a sixth requirement that, although useful, is not necessary for all public-key applications.

1. Either of the two related keys can be used for encryption, with the other used for decryption.

$$M = D[PU_b, E(PR_b, M)] = D[PR_b, E(PU_b, M)]$$

3.5 PUBLIC-KEY CRYPTOGRAPHY ALGORITHMS

Two widely used public-key algorithms are RSA and Diffie–Hellman. We look at both of these in this section and then briefly introduce two other algorithms.[4]

[4]This section uses some elementary concepts from number theory. For a review, see Appendix A.

The RSA Public-Key Encryption Algorithm

One of the first public-key schemes was developed in 1977 by Ron Rivest, Adi Shamir, and Len Adleman at MIT and first published in 1978 [RIVE78]. The RSA scheme has until recently reigned supreme as the most widely accepted and implemented approach to public-key encryption. Currently, both RSA and elliptic-curve cryptography are widely used. **RSA** is a block cipher in which the plaintext and ciphertext are integers between 0 and $n - 1$ for some n.

Basic RSA Encryption and Decryption Encryption and decryption are of the following form period for some plaintext block M and ciphertext block C:

$$C = M^e \bmod n$$
$$M = C^d \bmod n = (M^e)^d \bmod n = M^{ed} \bmod n$$

Both sender and receiver must know the values of n and e, and only the receiver knows the value of d. This is a public-key encryption algorithm with a public key of $KU = \{e, n\}$ and a private key of $KR = \{d, n\}$. For this algorithm to be satisfactory for public-key encryption, the following requirements must be met.

1. It is possible to find values of e, d, n such that $M^{ed} \bmod n = M$ for all $M < n$.
2. It is relatively easy to calculate M^e and C^d for all values of $M < n$.
3. It is infeasible to determine d given e and n.

The first two requirements are easily met. The third requirement can be met for large values of e and n.

Figure 3.10 summarizes the RSA algorithm. Begin by selecting two prime numbers p and q and calculating their product n, which is the modulus for encryption and decryption. Next, we need the quantity $\phi(n)$, referred to as the Euler totient of n, which is the number of positive integers less than n and relatively prime to n. Then select an integer e that is relatively prime to $\phi(n)$ [i.e., the greatest common divisor of e and $\phi(n)$ is 1]. Finally, calculate d as the multiplicative inverse of e, modulo $\phi(n)$. It can be shown that d and e have the desired properties.

Suppose that user A has published its public key and that user B wishes to send the message M to A. Then B calculates $C = M^e \pmod n$ and transmits C. On receipt of this ciphertext, user A decrypts by calculating $M = C^d \pmod n$.

An example, from [SING99], is shown in Figure 3.11. For this example, the keys were generated as follows:

1. Select two prime numbers, $p = 17$ and $q = 11$.
2. Calculate $n = pq = 17 \times 11 = 187$.
3. Calculate $\phi(n) = (p - 1)(q - 1) = 16 \times 10 = 160$.
4. Select e such that e is relatively prime to $\phi(n) = 160$ and less than $\phi(n)$; we choose $e = 7$.
5. Determine d such that $de \bmod 160 = 1$ and $d < 160$. The correct value is $d = 23$, because $23 \times 7 = 161 = (1 \times 160) + 1$.

The resulting keys are public key $PU = \{7, 187\}$ and private key $PR = \{23, 187\}$. The example shows the use of these keys for a plaintext input of $M = 88$.

Key Generation

Select p, q	p and q both prime, $p \neq q$
Calculate $n = p \times q$	
Calculate $\phi(n) = (p-1)(q-1)$	
Select integer e	$\gcd(\phi(n), e) = 1;\ 1 < e < \phi(n)$
Calculate d	$de \bmod \phi(n) = 1$
Public key	$KU = \{e, n\}$
Private key	$KR = \{d, n\}$

Encryption

Plaintext:	$M < n$
Ciphertext:	$C = M^e \pmod{n}$

Decryption

Ciphertext:	C
Plaintext:	$M = C^d \pmod{n}$

Figure 3.10 The RSA Algorithm

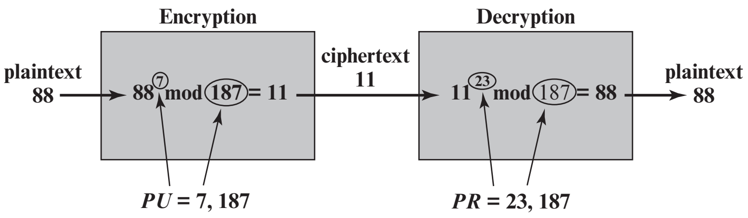


Figure 3.11 Example of RSA Algorithm

For encryption, we need to calculate $C = 88^7 \bmod 187$. Exploiting the properties of modular arithmetic, we can do this as follows:

$$88^7 \bmod 187 = [(88^4 \bmod 187) \times (88^2 \bmod 187) \times (88^1 \bmod 187)] \bmod 187$$

$$88^1 \bmod 187 = 88\ 88^2 \bmod 187 = 7744 \bmod 187 = 77$$

$$88^4 \bmod 187 = 59{,}969{,}536 \bmod 187 = 132$$

$$88^7 \bmod 187 = (88 \times 77 \times 132) \bmod 187 = 894{,}432 \bmod 187 = 11$$

For decryption, we calculate $M = 11^{23} \bmod 187$:

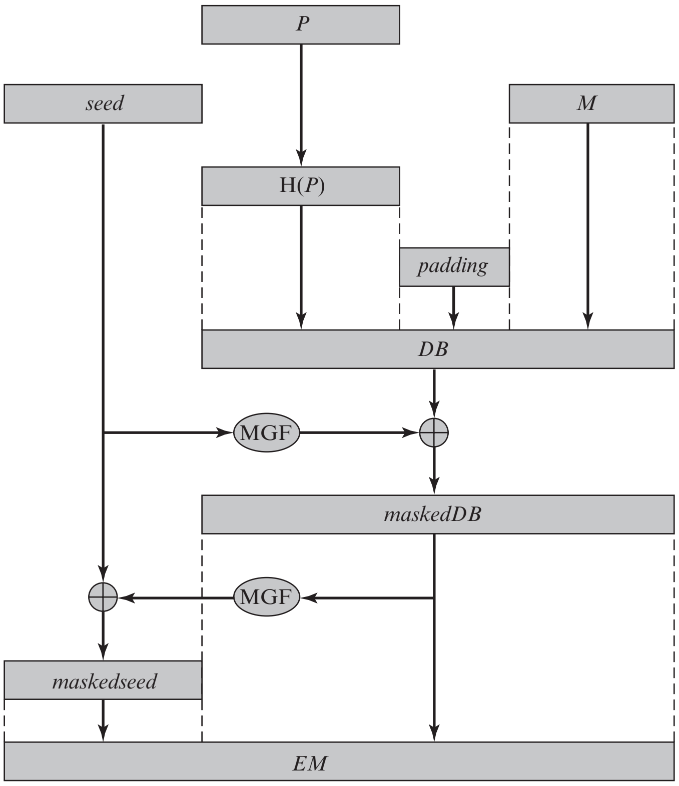
$$\begin{aligned}
11^{23} \bmod 187 &= [(11^1 \bmod 187) \times (11^2 \bmod 187) \times (11^4 \bmod 187) \times \\
&\qquad (11^8 \bmod 187) \times (11^8 \bmod 187)] \bmod 187 \\
11^1 \bmod 187 &= 11 \\
11^2 \bmod 187 &= 121 \\
11^4 \bmod 187 &= 14{,}641 \bmod 187 = 55 \\
11^8 \bmod 187 &= 214{,}358{,}881 \bmod 187 = 33 \\
11^{23} \bmod 187 &= (11 \times 121 \times 55 \times 33 \times 33) \bmod 187 \\
&= 79{,}720{,}245 \bmod 187 = 88
\end{aligned}$$

Security Considerations The security of RSA depends on it being used in such a way as to counter potential attacks. Four possible attack approaches are as follows:

- **Mathematical attacks:** There are several approaches, all equivalent in effort to factoring the product of two primes. The defense against mathematical attacks is to use a large key size. Thus, the larger the number of bits in d, the better. However, because the calculations involved, both in key generation and in encryption/decryption, are complex, the larger the size of the key, the slower the system will run. SP 800-131A (*Transitions: Recommendation for Transitioning the Use of Cryptographic Algorithms and Key Lengths*, November 2015) recommends the use of a 2048-bit key size. A recent report from the European Union Agency for Network and Information Security (*Algorithms, key size and parameters report—2014*, November 2014) recommends a 3072-bit key length. Either of these lengths should provide adequate security for a considerable time into the future.
- **Timing attacks:** These depend on the running time of the decryption algorithm. Various approaches to mask the time required so as to thwart attempts to deduce key size have been suggested, such as introducing a random delay.
- **Chosen ciphertext attacks:** This type of attack exploits properties of the RSA algorithm by selecting blocks of data that, when processed using the target's private key, yield information needed for cryptanalysis. These attacks can be thwarted by suitable padding of the plaintext.

To counter sophisticated chosen ciphertext attacks, RSA Security Inc., a leading RSA vendor and former holder of the RSA patent, recommends modifying the plaintext using a procedure known as optimal asymmetric encryption padding (OAEP). A full discussion of the threats and OAEP are beyond our scope; see [POIN02] for an introduction and [BELL94a] for a thorough analysis. Here, we simply summarize the OAEP procedure.

Figure 3.12 depicts OAEP encryption. As a first step, the message M to be encrypted is padded. A set of optional parameters, P, is passed through a hash function, H. The output is then padded with zeros to get the desired length in the overall data block (DB). Next, a random seed is generated and passed through another hash function, called the mask generating function (MGF). The resulting hash value is bit-by-bit XORed with DB to produce a maskedDB. The maskedDB is in turn passed



P = encoding parameters
M = message to be encoded
H = hash function
DB = data block
MGF = mask generating function
EM = encoded message

Figure 3.12 Encryption Using Optimal Asymmetric Encryption Padding (OAEP)

through the MGF to form a hash that is XORed with the seed to produce the masked seed. The concatenation of the maskedseed and the maskedDB forms the encoded message *EM*. Note that the EM includes the padded message masked by the seed, and the seed masked by the maskedDB. The EM is then encrypted using RSA.

Diffie–Hellman Key Exchange

The first published public-key algorithm appeared in the seminal paper by Diffie and Hellman that defined public-key cryptography [DIFF76] and is generally referred to as the **Diffie–Hellman key exchange**. A number of commercial products employ this key exchange technique.

The purpose of the algorithm is to enable two users to securely exchange a secret key that then can be used for subsequent encryption of messages. The algorithm itself is limited to the exchange of the keys.

The Diffie–Hellman algorithm depends for its effectiveness on the difficulty of computing discrete logarithms. Briefly, we can define the discrete logarithm in the following way. First, we define a primitive root of a prime number p as one whose powers generate all the integers from 1 to $p - 1$. That is, if a is a primitive root of the prime number p, then the numbers

$$a \bmod p, a^2 \bmod p, \ldots, ap^{-1} \bmod p$$

are distinct and consist of the integers from 1 through $p - 1$ in some permutation.

For any integer b less than p and a primitive root a of prime number p, one can find a unique exponent i such that

$$b = a^i \bmod p \quad 0 \leq i \leq (p - 1)$$

The exponent i is referred to as the discrete logarithm, or index, of b for the base a, mod p. We denote this value as $\mathrm{dlog}_{a,p}(b)$.[5]

The Algorithm With this background, we can define the Diffie–Hellman key exchange, which is summarized in Figure 3.13. For this scheme, there are two publicly known numbers: a prime number q and an integer α that is a primitive root of q. Suppose the users A and B wish to exchange a key. User A selects a random integer $X_A < q$ and computes $Y_A = \alpha^{X_A} \bmod q$. Similarly, user B independently selects a random integer $X_B < q$ and computes $Y_B = \alpha^{X_B} \bmod q$. Each side keeps the X value private and makes the Y value available publicly to the other side. User A computes the key as $K = (Y_B)^{X_A} \bmod q$ and user B computes the key as $K = (Y_A)^{X_B} \bmod q$. These two calculations produce identical results:

$$\begin{aligned}
K &= (Y_B)^{X_A} \bmod q \\
&= (\alpha^{X_B} \bmod q)^{X_A} \bmod q \\
&= (\alpha^{X_B})^{X_A} \bmod q \\
&= \alpha^{X_B X_A} \bmod q \\
&= (\alpha^{X_A})^{X_B} \bmod q \\
&= (\alpha^{X_A} \bmod q)^{X_B} \bmod q \\
&= (Y_A)^{X_B} \bmod q
\end{aligned}$$

The result is that the two sides have exchanged a secret value. Furthermore, because X_A and X_B are private, an adversary only has the following ingredients to work with: q, α, Y_A, and Y_B. Thus, the adversary is forced to take a discrete logarithm to determine the key. For example, to determine the private key of user B, an adversary must compute

$$X_B = \mathrm{dlog}_{\alpha,q}(Y_B)$$

The adversary can then calculate the key K in the same manner as user B does.

The security of the Diffie–Hellman key exchange lies in the fact that, while it is relatively easy to calculate exponentials modulo a prime, it is very difficult to calculate discrete logarithms. For large primes, the latter task is considered infeasible.

[5]Many texts refer to the discrete logarithm as the *index*. There is no generally agreed notation for this concept, much less an agreed name.

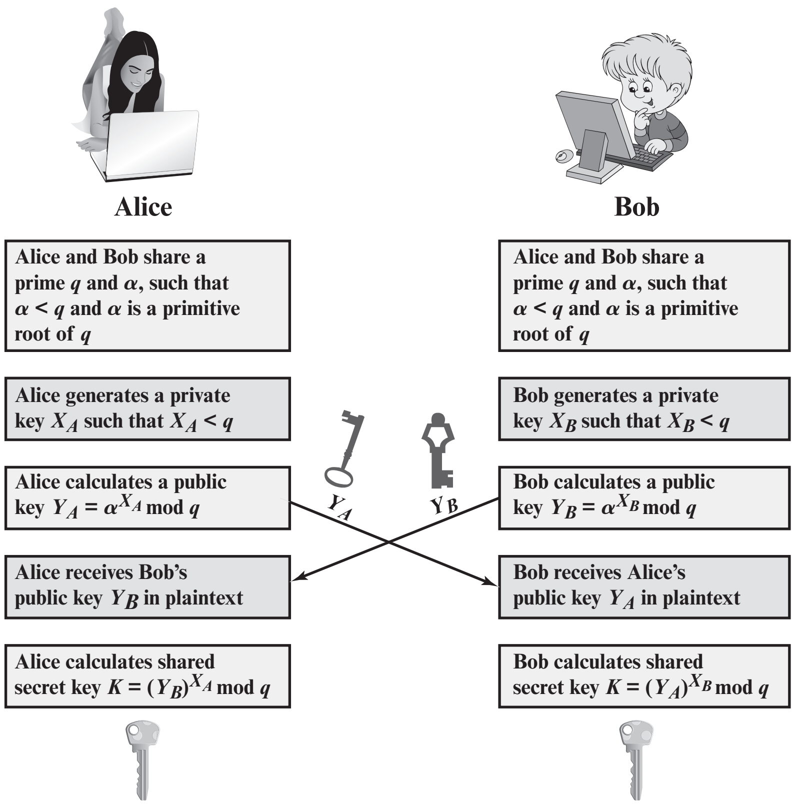

Alice
Bob
Alice and Bob share a prime q and α, such that $\alpha < q$ and α is a primitive root of q
Alice and Bob share a prime q and α, such that $\alpha < q$ and α is a primitive root of q
Alice generates a private key X_A such that $X_A < q$
Bob generates a private key X_B such that $X_B < q$
Alice calculates a public key $Y_A = \alpha^{X_A} \bmod q$
Bob calculates a public key $Y_B = \alpha^{X_B} \bmod q$
Y_A
Y_B
Alice receives Bob's public key Y_B in plaintext
Bob receives Alice's public key Y_A in plaintext
Alice calculates shared secret key $K = (Y_B)^{X_A} \bmod q$
Bob calculates shared secret key $K = (Y_A)^{X_B} \bmod q$

Figure 3.13 The Diffie–Hellman Key Exchange

Here is an example. Key exchange is based on the use of the prime number $q = 353$ and a primitive root of 353, in this case $\alpha = 3$. A and B select secret keys $X_A = 97$ and $X_B = 233$, respectively. Each computes its public key:

$$\text{A computes } Y_A = 3^{97} \bmod 353 = 40. \text{ B computes } Y_B = 3^{233} \bmod 353 = 248.$$

After they exchange public keys, each can compute the common secret key:

$$\text{A computes } K = (Y_B)^{X_A} \bmod 353 = 248^{97} \bmod 353 = 160.$$
$$\text{B computes } K = (Y_A)^{X_B} \bmod 353 = 40^{233} \bmod 353 = 160.$$

We assume an attacker would have available the following information:

$$q = 353; \quad a = 3; \quad Y_A = 40; \quad Y_B = 248$$

In this simple example, it would be possible to determine the secret key 160 by brute force. In particular, an attacker E can determine the common key by discovering a solution to the equation $3^a \bmod 353 = 40$ or the equation $3^b \bmod 353 = 248$. The brute-force approach is to calculate powers of 3 modulo 353, stopping when the result equals either 40 or 248. The desired answer is reached with the exponent value of 97, which provides $3^{97} \bmod 353 = 40$.

With larger numbers, the problem becomes impractical.

KEY EXCHANGE PROTOCOLS Figure 3.13 shows a simple protocol that makes use of the Diffie–Hellman calculation. Suppose that user A wishes to set up a connection with user B and use a secret key to encrypt messages on that connection. User A can generate a one-time private key X_A, calculate Y_A, and send that to user B. User B responds by generating a private value X_B, calculating Y_B, and sending Y_B to user A. Both users can now calculate the key. The necessary public values q and α would need to be known ahead of time. Alternatively, user A could pick values for q and include those in the first message.

As an example of another use of the Diffie–Hellman algorithm, suppose that a group of users (e.g., all users on a LAN) each generate a long-lasting private value X_A and calculate a public value Y_A. These public values, together with global public values for q and α, are stored in some central directory. At any time, user B can access user A's public value, calculate a secret key, and use that to send an encrypted message to user A. If the central directory is trusted, then this form of communication provides both confidentiality and a degree of authentication. Because only A and B can determine the key, no other user can read the message (confidentiality). Recipient A knows that only user B could have created a message using this key (authentication). However, the technique does not protect against replay attacks.

MAN-IN-THE-MIDDLE ATTACK The protocol depicted in Figure 3.13 is insecure against a man-in-the-middle attack. Suppose Alice and Bob wish to exchange keys, and Darth is the adversary. The attack proceeds as follows (Figure 3.14):

1. Darth prepares for the attack by generating two random private keys X_{D1} and X_{D2}, and then computing the corresponding public keys Y_{D1} and Y_{D2}.
2. Alice transmits Y_A to Bob.
3. Darth intercepts Y_A and transmits Y_{D1} to Bob. Darth also calculates $K2 = (Y_A)^{X_{D2}} \bmod q$.
4. Bob receives Y_{D1} and calculates $K1 = (Y_{D1})^{X_B} \bmod q$.
5. Bob transmits Y_B to Alice.
6. Darth intercepts Y_B and transmits Y_{D2} to Alice. Darth calculates $K1 = (Y_B)^{X_{D1}} \bmod q$.
7. Alice receives Y_{D2} and calculates $K2 = (Y_{D2})^{X_A} \bmod q$.

At this point, Bob and Alice think that they share a secret key. Instead Bob and Darth share secret key $K1$, and Alice and Darth share secret key $K2$. All future communication between Bob and Alice is compromised in the following way:

1. Alice sends an encrypted message M: $E(K2, M)$.
2. Darth intercepts the encrypted message and decrypts it to recover M.
3. Darth sends Bob $E(K1, M)$ or $E(K1, M')$, where M' is any message. In the first case, Darth simply wants to eavesdrop on the communication without altering it. In the second case, Darth wants to modify the message going to Bob.

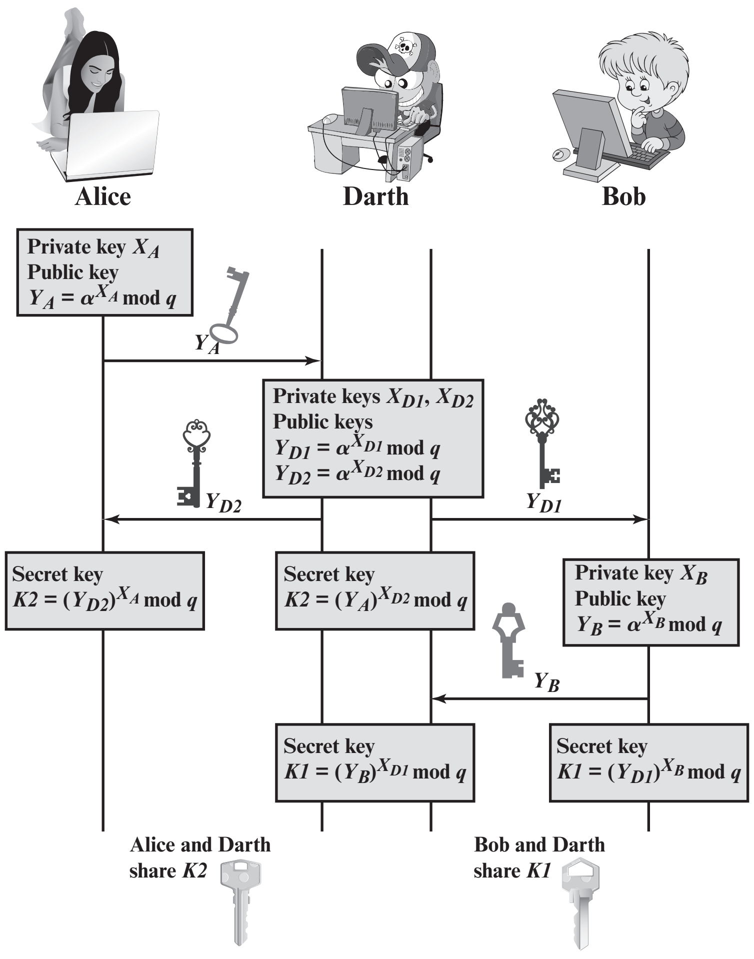

Alice
Darth
Bob
Private key X_A
Public key
$Y_A = \alpha^{X_A} \bmod q$
Y_A
Private keys X_{D1}, X_{D2}
Public keys
$Y_{D1} = \alpha^{X_{D1}} \bmod q$
$Y_{D2} = \alpha^{X_{D2}} \bmod q$
Y_{D2}
Y_{D1}
Secret key
$K2 = (Y_{D2})^{X_A} \bmod q$
Secret key
$K2 = (Y_A)^{X_{D2}} \bmod q$
Private key X_B
Public key
$Y_B = \alpha^{X_B} \bmod q$
Y_B
Secret key
$K1 = (Y_B)^{X_{D1}} \bmod q$
Secret key
$K1 = (Y_{D1})^{X_B} \bmod q$
Alice and Darth share $K2$
Bob and Darth share $K1$

Figure 3.14 Man-in-the-Middle Attack

The key exchange protocol is vulnerable to such an attack because it does not authenticate the participants. This vulnerability can be overcome with the use of digital signatures and public-key certificates; these topics are explored later in this chapter and in Chapter 4.

Other Public-Key Cryptography Algorithms

Two other public-key algorithms have found commercial acceptance: DSS and elliptic-curve cryptography.

Digital Signature Standard The National Institute of Standards and Technology (NIST) has published Federal Information Processing Standard FIPS PUB 186, known as the **Digital Signature Standard (DSS)**. The DSS makes use of the SHA-1 and presents a new digital signature technique, the Digital Signature Algorithm

(DSA). The DSS was originally proposed in 1991 and revised in 1993 in response to public feedback concerning the security of the scheme. There was a further minor revision in 1996. The DSS uses an algorithm that is designed to provide only the digital signature function. Unlike RSA, it cannot be used for encryption or key exchange.

Elliptic-Curve Cryptography The vast majority of the products and standards that use public-key cryptography for encryption and digital signatures use RSA. The bit length for secure RSA use has increased over recent years, and this has put a heavier processing load on applications using RSA. This burden has ramifications, especially for electronic commerce sites that conduct large numbers of secure transactions. Recently, a competing system has begun to challenge RSA: **elliptic curve cryptography (ECC)**. Already, ECC is showing up in standardization efforts, including the IEEE P1363 Standard for Public-Key Cryptography.

The principal attraction of ECC compared to RSA is that it appears to offer equal security for a far smaller bit size, thereby reducing processing overhead. On the other hand, although the theory of ECC has been around for some time, it is only recently that products have begun to appear and that there has been sustained cryptanalytic interest in probing for weaknesses. Thus, the confidence level in ECC is not yet as high as that in RSA.

ECC is fundamentally more difficult to explain than either RSA or Diffie–Hellman, and a full mathematical description is beyond the scope of this book. The technique is based on the use of a mathematical construct known as the elliptic curve.

3.6 DIGITAL SIGNATURES

NIST FIPS PUB 186-4 [*Digital Signature Standard (DSS)*, July 2013] defines a digital signature as follows: The result of a cryptographic transformation of data that, when properly implemented, provides a mechanism for verifying origin authentication, data integrity, and signatory non-repudiation.

Thus, a digital signature is a data-dependent bit pattern, generated by an agent as a function of a file, message, or other form of data block. Another agent can access the data block and its associated signature and verify that (1) the data block has been signed by the alleged signer and that (2) the data block has not been altered since the signing. Further, the signer cannot repudiate the signature.

FIPS 186-4 specifies the use of one of three digital signature algorithms:

- **Digital Signature Algorithm (DSA):** The original NIST-approved algorithm, which is based on the difficulty of computing discrete logarithms.
- **RSA Digital Signature Algorithm:** Based on the RSA public-key algorithm.
- **Elliptic Curve Digital Signature Algorithm (ECDSA):** Based on elliptic-curve cryptography.

In this section, we provide a brief overview of the digital signature process, then describe the RSA digital signature algorithm.