Assignment 1

This assignment concerns the following planar model, proposed by Hindmarsh and Rose for neuronal spiking

$$\begin{aligned} \dot{v} &= f(v) - r + I \\ \dot{r} &= g(v) - r \end{aligned} \qquad \text{(HRM)}$$

In this dimensionless model, v represents the membrane voltage, I an external stimulus, and r a so-called recovery variable. **For this assignment there is no need to have a full biological understanding of what is a recovery variable: you can view r as a dynamical variable related to ionic currents:** it suffices to say that r accounts simultaneously for the activation of of K+ ionic currents and the inactivation of Na+ ionic currents, and it provides feedback to v, as seen from the equations above.

Analytical results of the HRM

We assume that the following hypotheses hold:

- The external current is null, $I = 0$.
- The functions f and g are defined as $f(v) = -v^3 + 3v^2$, and $g(v) = 5v^2 - 1$, respectively.

Under these assumptions, let us make some analytical progress on the HRM.

Q1. Show that a steady state $E = (v_*, r_*)$ of the HRM can be written as

$$E = \big(v_*, f(v_*)\big) \qquad \text{where } v_* \text{ satisfies } f(v_*) - g(v_*) = 0.$$

The main significance of the statement above is that equilibria of the HRM are in one-to-one correspondence with roots of the function $f(v) - g(v)$.

Q2. Prove that there exist 3 distinct equilibria E_1, E_2, E_3 to the HRM. **Hint:** you can do so by explicitly computing the corresponding equilibrium voltage values, $v_{*1} < v_{*2} < v_{*3}$, respectively. This will involve finding roots of a cubic function. It may be useful to note that -1 is a root of this cubic, that is, one equilibrium voltage value is at $v = -1$. Also, for what comes later it will be useful to write down the values v_{*1}, v_{*2}, v_{*3} explicitly.

Q3. Sketch the HRM nullclines and phase plane. Your picture need not be to scale, meaning that you can use pen and paper if you want. For the sketch you are also allowed to use software, or your own code. If you use software, let us know which one you used. If you write your own code, submit it together with the report. The final picture should contain a full partition of the phase space $\mathbb{R}^2$, as determined by the nullclines. It should also contain representative arrows within each region, and it should be clear how those arrows can be justified. Mark on the phase plane the equilibria E_1, E_2, E_3 of Q2.

Q4. Prove the following statements:

- S1: Let $E = (v_*, r_*)$ be an equilibrium of the HRM, and let $J(E)$ be the Jacobian of the HRM vectorfield evaluated at E. The trace and determinant of $J(E)$ depend on v_*, but not on r_*, namely one can find functions τ and Δ such that

$$\text{trace} J(E) = \tau(v_*), \qquad \det J(E) = \Delta(v_*),$$

- S2: Let $\tau(v)$ and $\Delta(v)$ be as in S1. The following holds

$$\tau(v) \begin{cases} \geq 0 & \text{if } 1 - \sqrt{2/3} \leq v \leq 1 + \sqrt{2/3} \\ < 0 & \text{otherwise} \end{cases}, \qquad \Delta(v) \begin{cases} < 0 & \text{if } -4/3 < v < 0 \\ \geq 0 & \text{ohterwise} \end{cases}$$

Q5. Determine the stability of each of the equilibria E_1, E_1, E_3 of the HRM, with corresponding equilibrium voltages given by $v_{*1} < v_{*2} < v_{*3}$, respectively. **Hint:** use S2. If, something goes wrong when proving S2, you can still obtain full marks here, by assuming S2 holds, and using it to infer stability of E_1, E_2, E_3.

Numerical results on the HRM

Let us now turn to some numerical exploration of the HRM, to verify the analytical prediction and experiment with parameters. Write code or use existing software to produce numerical evidence in support of the following claims:

Q6. The HRM can produce self-sustained, repeating spikes, even when there is no external current.

Q7. In the absence of an external current, the HRM exhibits bistability: orbits are attracted to the equilibrium E_1 or to a limit cycle, depending on initial conditions; the limit cycle surrounds E_3.

Q8. Upon applying an external current, it is possible to suppress periodic spikes. Give an estimate of the value of the current at which oscillations are suppressed.