Assignment 3

Question 1. Firing Time Maps

For this question, you are advised to look at Section 8.2.3.1 in the book by Ermentrout and Terman.

Q2.1: Equation 8.22 is derived for two identical oscillators. Derive an analogue of equation 8.22 for 2 non-identical oscillators, with slightly different periods, say T_0 and $T_0 + \epsilon$, and phase-resetting curves given by Δ_1 and Δ_2, respectively.

Q2.2: Consider the oscillators as in Q2.1. Let $a \in (-1, 0)$ and $\Delta_1(\phi) = \Delta_2(\phi) = a\sin(2\pi\phi)$. Study the existence of fixed points as ϵ varies, while a remains fixed. Use a computer to find the fixed points and determine the magnitude of ϵ such that there exists a stable fixed point.

Question 2. Neural field analysis

Consider the following neural field model

$$
\begin{aligned}
\partial_t u(x,t) &= -u(x,t) + \int_{\mathbb{R}} w(x-y) f(u(y,t)-h)\,dy, && (x,t) \in \mathbb{R}\times\mathbb{R}_{>0}, \\
u(x,0) &= u_0(x), && x \in \mathbb{R},
\end{aligned} \tag{1}
$$

under the following hypotheses:

H1: The homogeneous synaptic kernel is given by $w(z) = (1-|z|)e^{-|z|}$.
H2: The firing rate is of Heaviside type, $f(u) = H(u)$.

Q2.1: Let $U(x)$ be a steady state of (1) that crosses h twice, at distinct points $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$. So, $U(x)$ has a nontrivial spatial profile. Show that $U(x)$ satisfies

$$U(x) = \int_{x_1}^{x_2} w(x-y)\,dy. \qquad (2)$$

Q2.2: Show that U is given by

$$U(x) = \operatorname{sgn}(x-x_1)\phi(|x-x_1|) + \operatorname{sgn}(x_2-x)\phi(|x_2-x|), \qquad \phi(x) = xe^{-x}.$$

[Hint] Use (2), write $\int_{x_1}^{x_2} = \int_{x_1}^{x} + \int_{x}^{x_2}$ for a generic $x \in \mathbb{R}$, show that $\int_a^x w(x-y)dy = (x-a)e^{-|x-a|}$.

Q2.3: Verify that $\lim_{x\to\pm\infty} U(x) = 0$, and that $U \in C^1(\mathbb{R})$. In addition, by plotting $U(x)$ for $x_1 = 0$, $x_2 = 1$, convince yourself that $U(x)$ is a "localised bump". Note that x_1 and x_2, as specified above, are arbitrary values, because this is a preliminary step for the next question, where you will determine x_1 and x_2.
[Hint] You want to visualize your solution for some choice of h. When plotting, distinguish the cases $x \le 0$, $x \in (0, \Delta)$ and $x \ge \Delta$.

Q2.4: Let $x_2 - x_1 = \Delta > 0$, which is a measure of the bump width. Prove that

$$h = \phi(\Delta) \qquad (3)$$

and hence conclude that there exist no bump if $h > 1/e$, and two bump solutions if $h < 1/e$. Use (3) to plot a bifurcation diagram for bump solutions, in the parameter h with solution measure Δ. [Hint: you can use MatCont or Matlab's fimplicit to draw the diagram.] You should see a branch of bump solutions, and a turning point at $h_* = 1/e$. Identify a branch with $\Delta < 1$ (a narrow bump) and a branch with $\Delta > 1$ (a wide branch). We have not yet computed stability of bump solutions, and in the next questions we will show that there is a saddle-node bifurcation at $h = h_*$.

Q2.5: Show formally that, if $\tilde{u}(x,t) = e^{\lambda t}V(x)$ is a small perturbation to the steady state $U(x)$, then to leading order we have

$$(\lambda+1)V(x) = \sum_{j=1}^{2} \frac{w(x-x_i)}{|U'(x_i)|}V(x_j). \qquad (4)$$

To obtain (4), linearise (1) around $U(x)$ for a generic smooth firing rate function $f \in C^\infty$, which involves the derivative f' of f. You can then use the following results:

- R1: The distributional derivative of the Heaviside function H is the Dirac distribution δ: even if H is not differentiable at 0, it is possible to attribute meaning to the following identity

  $$\int_{\mathbb{R}} g(x)\frac{d}{dx}H(x)dx = \int_{\mathbb{R}} g(x)\delta(x)dx = g(0) \qquad (5)$$

  for all functions g in a suitable function space. To address this question, it is not necessary to delve in distribution theory: we can linearise (4) using f, f', realise that f' occurs under an integral, and use (5) when $f = H$. Distribution theory makes the passages above rigorous, without ever writing $H'(x)$, the derivative of H, which is not defined at 0.
- R2: If a function $g \in C^1(\mathbb{R})$ has n distinct roots $x_1, \dots, x_n$, then

  $$\delta(g(x)) = \sum_{j=1}^{n} \frac{\delta(x-x_j)}{|g'(x_j)|}.$$

Q2.6: In this question, we are going to find two eigenpairs $(\lambda_-, v_-(x))$ and $(\lambda_+, v_+(x))$ associated with the eigenvalue problem (4). This, in turn will determine the stability of the bump.

Q2.6.1: Let $v_i = V(x_i)$ for $i = 1, 2$. Show that

$$\lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} := M \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

where

$$a = -1 + \frac{w(0)}{|w(0)-w(\Delta)|}, \qquad b = \frac{w(\Delta)}{|w(0)-w(\Delta)|}$$

[Hint: use (4) in conjunction with (2); the latter is useful to find $|U'(x)|$, and hence the denominators of (4).]

Q2.6.2: Show that M has eigenpairs

$$
\begin{aligned}
\lambda_+ &= a+b, & v_+ &= (1,1), \\
\lambda_- &= a-b, & v_- &= (-1,1).
\end{aligned}
$$

Q2.6.3: Conclude that this leads to the following eigenpairs for the eigenvalue problem (4)

$$
\begin{aligned}
\lambda_+ &= 0, & V_+(x) &= C[w(x-x_2) - w(x-x_1)], \\
\lambda_- &= -1 + \frac{w(0)+w(\Delta)}{|w(0)-w(\Delta)|}, & V_-(x) &= C[w(x-x_2) + w(x-x_1)],
\end{aligned}
$$

lambda_+&- and V_+&-(x) should be swapped

where C is a constant.

Q2.6.4: Plot $V_+(x)$, $V_-(x)$, when $C = 1$ and discuss their symmetries. Note that $V_+(x) = U'(x)$, as expected.

Q2.6.5: Show that if $w(\Delta) < 0$ then $U(x)$ is linearly stable. For this analysis, you can use the fact that the 0 eigenvalue corresponding to $V_+(x)$ does not contribute stability, so for the stability condition you can simply concentrate on the sign of λ_-.

Q2.6.6: Return to the bifurcation diagram of Q2.4, and recall that, for the particular kernel under consideration, $w(x) = (1-|x|)e^{-|x|}$, and for sufficiently low h, there is a branch of stationary bumps with $\Delta > 1$, and one with $\Delta < 1$. Which one of these branches is stable, which one unstable? Justify your answer. The simulation below may guide you.

Question 3. Numerical field simulation

Perform a numerical simulation of the neural field

$$
\begin{aligned}
\partial_t u(x,t) &= -u(x,t) + \int_{\mathbb{R}} w(x-y) f(u(y,t)-h)\,dy, && (x,t) \in \Omega\times(0,T], \\
u(x,0) &= u_0(x), && x \in \Omega,
\end{aligned}
$$

where

- The domain Ω is a ring of width 20, that is, the interval $[-10, 10)$, where -10 and 10 are identified.
- The synaptic kernel is given by $w(x) = (1-|x|)e^{-|x|}$.
- The firing rate is sigmoidal $f(u) = \frac{1}{1+\exp(-10u)}$.

The code distributed on ELO, in last week's lecture (week–12-neural-fields.zip) performs this task on a generic ring of width L_x, with a kernel different to the one we will use below. You can adapt that code.

Q3.1: Set $u_0(x) = \frac{1}{\cosh^2(0.5x)}$. Plot initial conditions and the solution $u(x,t)$. Produce numerical evidence that a stable localised bump is formed. By varing the initial condition (so as to change its "width" Δ, provide evidence of the theory found in Question 2.
