Mathematical Neuroscience Lecture 3

Key Points
- Codim 1 bifurcations at the onset & offset of oscillations.
- Class 1 & 2 neurons related to the period as I_{app} increases.
- Bits of (numerical) bifurcation theory illustrated using the Morris-Lecar Model.
  Saddle-Node, Hopf, Saddle-homoclinic, Saddle-node homoclinic
  → equilibria & periodic orbits (limit cycles)

Have a look at the two movies

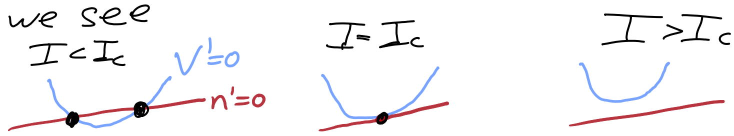
#1: Observation: the number of equilibria changes as I is increased.

Think of equilibria as zeros of $F(x,\alpha)=0$ as $x' = F(x,\alpha)$ (state, parameter)

Implicit Function Theorem:

Suppose (x_0,α_0) is a zero of a smooth function $F:\mathbb{R}^{n+1}\to\mathbb{R}^n$, i.e., $F(x_0,\alpha_0)=0$.
If the derivative $F_x(x_0,\alpha_0)$ is nonsingular, then there exists a smooth function $x(\alpha)$ with $x(\alpha_0)=x_0$ and $F(x(\alpha),\alpha)=0$ for $|\alpha-\alpha_0|$ sufficiently small.

Near $I\approx 40$ we see



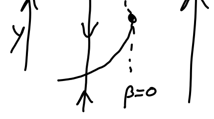
At the critical parameter value, the slope of the nullclines at the equilibrium is equal because of the tangency, hence F_x is singular → $\exists$ eigenvalue $\lambda=0$.

Saddle-Node bifurcation: Suppose $x_0\in\mathbb{R}^n$ is an equilibrium of a smooth ODE $\dot{x}=F(x,\alpha)$ with $F_x(x_0,\alpha_0)$ having a simple eigenvalue $\lambda=0$ (there are n_s eigenvalues with $Re(\lambda^s)<0$ & n_u eigenvalues with $Re(\lambda^u)>0$, $1+n_s+n_u=n$)

There are eigenvectors $F_x q=0$ & $F_x^T p=0$ (or $p^T F_x=0$)

If $\frac{d^2}{ds^2} p^T F(x_0+sq,\alpha_0)=a\neq 0$ and $p^T F_\alpha(x_0,\alpha_0)\neq 0$ then then the system is topologically equivalent to

$$\begin{cases}\dot{y}=\beta+ay^2\\ \dot{y}^s=-y^s \ \& \ \dot{y}^u=y^u\end{cases}$$



Local Bifurcations

$\dot{x} = f(x) \qquad x \in \mathbb{R}^2 \qquad f(x_0)=0$

$\lambda^2 - T\lambda + D = 0 \qquad T = \text{Trace}(f_x(x_0))$

$D = \det(f_x(x_0))$

Saddle-Node bifurcation $\longrightarrow D=0$.

Hopf bifurcation $\lambda = \pm i\omega \longrightarrow T=0 \quad (\& \ D>0)$

$$\begin{cases} \dot{x} = \alpha x - \omega y + l_1 x(x^2+y^2) \\ \dot{y} = \omega x + \alpha y + l_1 y (x^2+y^2) \end{cases} \xrightarrow{\qquad\qquad} \begin{cases} \dot{r} = r(\alpha + l_1 r^2) \\ \dot{\theta} = \omega \end{cases}$$

$\dot{r} = \frac{1}{r}(x\dot{x} + y\dot{y})$

$\dot{\theta} = \frac{1}{r^2}(x\dot{y} - y\dot{x})$

$r=0$ origin

$\dot{\theta}$ decouples ....

($\omega = 1$ wlog)

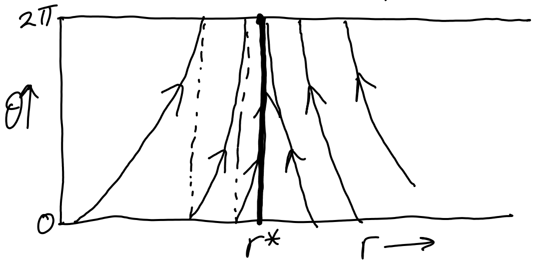
$(0,0)$ is an equilibrium for all values of α (Consider l_1 & ω fixed)

$DF = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} \rightarrow \lambda = \alpha \pm i\omega$

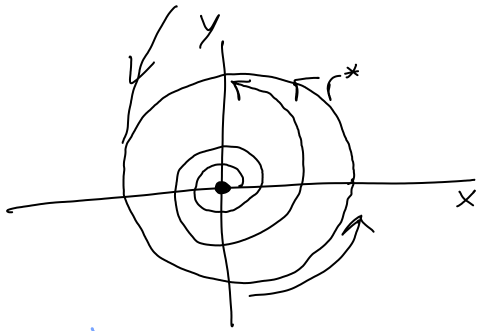
ⓐ Supercritical case $l_1 < 0$ Appearance of a stable cycle.

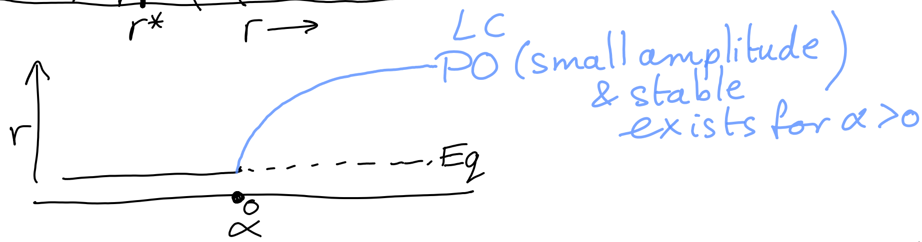
For $r^* = \sqrt{-\alpha/l_1}$ we have a periodic orbit.

$\alpha > 0$



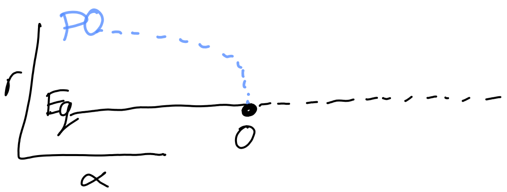
$(r,\theta) \longrightarrow (x,y)$





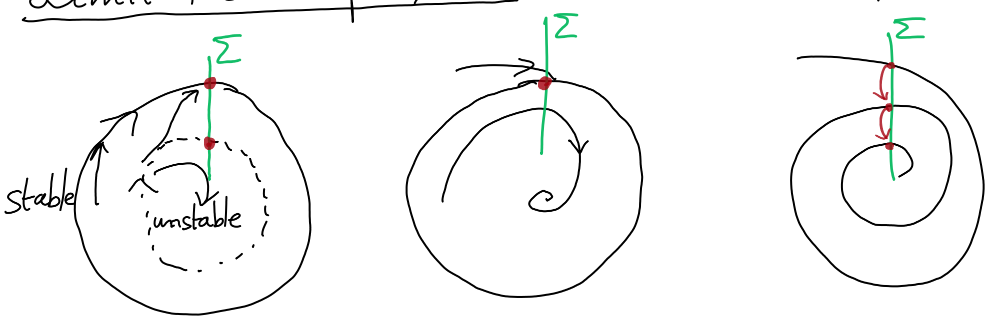
ⓑ Subcritical case $l_1 > 0$ unstable Periodic orbit shrinks as α increases and disappears

"Hard loss of stability"



Q The two Hopf bifurcations in tutorial 2, what is their criticality?

Limit Point of Cycles (or saddle-node of POs)



On the Poincaré section Σ we take a coordinate ξ

The return map $P: \xi_0 \longmapsto \xi_1 = \alpha + \xi_0 + a\xi_0^2$ $(a \neq 0)$ giving a sequence ξ_0, ξ_1, ξ_2

- Fixed points ξ^* of P ($P(\xi) - \xi = 0$) correspond to POs of the original continuous-time system.
- $P(\xi^* + u) = P(\xi^*) + P'(\xi^*)u \longrightarrow \boxed{u \longmapsto P'(\xi^*)u}$

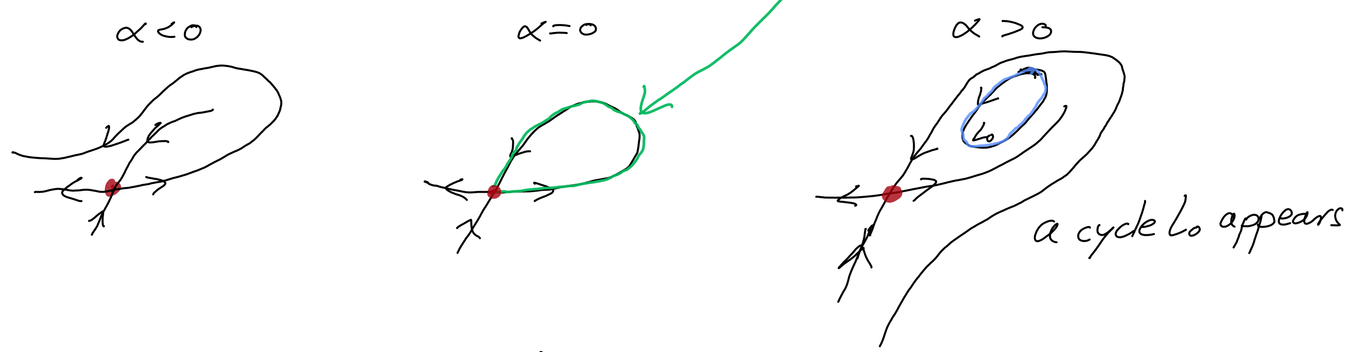
Let x_0 be a saddle equilibrium, i.e., x_0 has eigenvalues with positive and negative real part.

If $\lim_{t\to\pm\infty} x(t) = x_0$, then a nontrivial solution $x(t)$ is a homoclinic orbit.

Let x_1 be another saddle

If $\lim_{t\to-\infty} x(t) = x_0$ & $\lim_{t\to+\infty} x(t) = x_1$, then $x(t)$ is a heteroclinic orbit.

Saddle-homoclinic bifurcation



Saddle-Node homoclinic bifurcation