HW № 4

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Variant (c)

Problem

Problem 1

Consider classical benchmark system in control theory - inverted pendulum on a cart (Figure 1). It is nonlinear under-actuated system that has the following dynamics.

$$(M + m)\ddot{x} - ml\cos(\theta)\ddot{\theta} + ml\sin(\theta)\dot{\theta}^2 = F$$

$$-\cos(\theta)\ddot{x} + l\ddot{\theta} - g\sin(\theta) = 0$$

where $g = 9.81$ is gravitational acceleration.

$$(c)M = 15.1, m = 1.2, l = 0.35$$

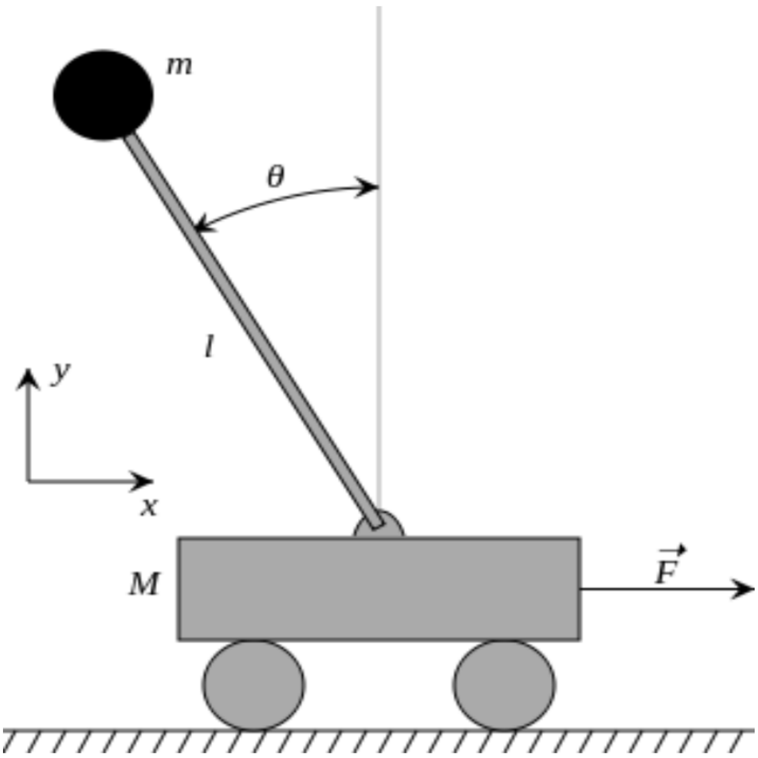


Figure 1: A schematic drawing of the inverted pendulum on a cart. The rod is considered massless. The mass of the cart and the point mass at the end of the rod are denoted by M and m. The rod has a length l.

part A

Description: Write equations of motion of the system in manipulator form

$$M(q)\ddot{q} + n(q, \dot{q}) = Bu$$

- where $u = F$, $q = \begin{bmatrix} x & \theta \end{bmatrix}^T$ is vector of generalized coordinates;

Solution:

$$\begin{bmatrix} M+m & -mlcos(\theta) \\ -cos(\theta) & l \end{bmatrix} \ddot{q} + \begin{bmatrix} mlsin(\theta)\dot{\theta}^2 \\ -gsin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$Answer: \begin{bmatrix} 16.3 & -0.42cos(\theta) \\ -cos(\theta) & 0.35 \end{bmatrix} \ddot{q} + \begin{bmatrix} 0.42sin(\theta)\dot{\theta}^2 \\ -gsin(\theta) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

part B

Description: Write dynamics of the system in control affine nonlinear form $\dot{z} = f(z) + g(z)u$ - where $z = \begin{bmatrix} x & \theta & \dot{x} & \dot{\theta} \end{bmatrix}^T$ is vector of states of the system;

Solution: $\ddot{x}$:

$$\ddot{Q} = \frac{1}{l}(gsin(\theta) + cos(\theta)\ddot{x})$$

$$\Rightarrow (M + m(1 - cos^2(\theta)))\ddot{x} - mgsin(\theta)cos(\theta) + mlsin(\theta)\dot{\theta}^2 = F$$

$$\ddot{x} = \frac{msin(\theta)(gcos(\theta) - l\dot{\theta}^2) + F}{(M + msin^2(\theta))}$$

$\ddot{\theta}$:

$$\ddot{x} = \frac{gsin(\theta) - l\ddot{\theta}}{-cos(\theta)}$$

$$\Rightarrow -g(M+m)tan(\theta) + \frac{l(M+m)}{cos(\theta)}\ddot{\theta} - mlcos(\theta)\ddot{\theta} + mlsin(\theta)\dot{\theta}^2 = F$$

$$\ddot{\theta}(l\frac{M + msin^2(\theta)}{cos(\theta)}) = g(M+m)tan(\theta) - mlsin(\theta)\dot{\theta}^2 + F$$

$$\ddot{\theta} = \frac{g(M+m)sin(\theta) - mlsin(\theta)cos(\theta)\dot{\theta}^2 + cos(\theta)F}{l(M + msin^2(\theta))}$$

$\dot{z}$:

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \frac{msin(\theta)(gcos(\theta) - l\dot{\theta}^2)}{(M+msin^2(\theta))} \\ \frac{g(M+m)sin(\theta) - mlsin(\theta)cos(\theta)\dot{\theta}^2}{l(M+msin^2(\theta))} \end{bmatrix} + u \begin{bmatrix} 0 \\ 0 \\ \frac{1}{(M+msin^2(\theta))} \\ \frac{1}{l(M+msin^2(\theta))} \end{bmatrix}$$

part C

Description: Linearize nonlinear dynamics of the systems around equilibrium point $\bar{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$

$$\delta \dot{z} = A\delta z + B\delta u$$

Solution:

$$A = \frac{\delta f}{\delta z} = \begin{bmatrix} \frac{df}{dx} \\ \frac{df}{d\theta} \\ \frac{df}{d\dot{x}} \\ \frac{df}{d\dot{\theta}} \end{bmatrix}$$

$$f = \begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \frac{msin(\theta)(gcos(\theta)-l\dot{\theta}^2)}{(M+msin^2(\theta))} \\ \frac{g(M+m)sin(\theta)-mlsin(\theta)cos(\theta)\dot{\theta}^2}{l(M+msin^2(\theta))} \end{bmatrix}$$

$$\frac{df}{dx} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\frac{df}{d\theta} = \begin{bmatrix} 0 & 0 & \frac{gm}{M} & \frac{g(M+m)}{lM} \end{bmatrix}^T$$

$$\frac{df}{d\dot{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\frac{df}{d\dot{\theta}} = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}^T$$

$$A = \frac{\delta f}{\delta z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{gm}{M} & \frac{g(M+m)}{lM} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(M+m)}{lM} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ \frac{1}{lM} \end{bmatrix}$$

$$Answer : A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0.7796 & 0 & 0 \\ 0 & 30.256 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0.066 \\ 0.189 \end{bmatrix}$$

part D

Description: Check stability of the linearized system using any method you like;

Solution: Find the eighen-values of A matrix:

$$det(\begin{bmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & \frac{gm}{M} & -\lambda & 0 \\ 0 & \frac{g(M+m)}{lM} & 0 & -\lambda \end{bmatrix}) = -\lambda(-\lambda^3 + \lambda\frac{g(M+m)}{lm}) =$$

$$\lambda^4 - \lambda^2(\frac{gM+m}{lM}) = 0$$

$$\lambda = 0$$

$$\lambda = \sqrt{\frac{g(M+m)}{lM}}$$

$$\lambda = -\sqrt{\frac{g(M+m)}{lM}}$$

$$\lambda = \sqrt{\frac{9.81(16.3)}{5.285}}$$

$$\lambda = -\sqrt{\frac{9.81(16.3)}{5.285}}$$

$$\lambda = 5.5005461$$

$$\lambda = -5.5005461$$

$Answer$: Thus there exist lambda with module that greater than zero => system without control is not stable

part E

Description: check if linearized system is controllable; if not - try another variant or change values of your variant and find controllable.

Solution: Sysem is controllable, when rank of matrix C is equal to 4, where $C = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & \frac{gm}{M} & 0 & 0 \\ 0 & \frac{g(M+m)}{lM} & 0 & 0 \\ 0 & 0 & 0 & \frac{gm}{M} \\ 0 & 0 & 0 & \frac{g(M+m)}{lM} \end{bmatrix}, A^2B = \begin{bmatrix} 0 \\ 0 \\ \frac{gm}{M^2} \\ \frac{g(M+m)}{(lM)^2} \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & \frac{gm}{M} \\ 0 & 0 & 0 & \frac{g(M+m)}{lM} \\ 0 & \frac{g^2m(M+m)}{lM^2} & 0 & 0 \\ 0 & (\frac{g(M+m)}{lM})^2 & 0 & 0 \end{bmatrix}, A^3B = \begin{bmatrix} \frac{gm}{lM^2} \\ \frac{g(M+m)}{(lM)^2} \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & \frac{1}{M} & 0 & \frac{gm}{lM^2} \\ 0 & \frac{1}{lM} & 0 & \frac{g(M+m)}{l^2M^2} \\ \frac{1}{M} & 0 & \frac{gm}{M^2} & 0 \\ \frac{1}{lM} & 0 & \frac{g(M+m)}{l^2M^2} & 0 \end{bmatrix}$$

If ranc of C is less, then 4, then $\lambda = 0$ - one of its eig values, thus, the determinant of matrix C is zero itself.

$$det(C) = -\frac{1}{M}(\frac{1}{M}(\frac{g(M+m)}{l^2M^2})^2 - \frac{g^2m(M+m)}{(lM)^3M^2}) - \frac{gm}{lM^2}(\frac{gm}{(lM)^2M^2} - \frac{g(M+m)}{M(lM)^3}) =$$

$$= \frac{g^2}{M^4l^2}(-(M+m)^2 + m(M+m) - m^2 + m(M+m)) = \frac{g^2}{M^4l^2}((M+m)(-M+m) - m^2) =$$

$$= \frac{g^2}{M^4l^2}(-M^2) = \frac{-g^2}{M^2l^2} \neq 0$$

$Answer$: all eig != 0 and rank(C) = 4 => system is controllable

part F

Description: (for the controllable system) design state feedback controller for lin- earized system using pole placement method. Assess the performance of the controller for variety of initial conditions. Justify the choice of initial conditions. Solve the task by two ways: using root-locus and with python. Compare them;

Solution: Initial conditions were chosen such that system will converge with different speed to demonstrate how controller will act

Using root-locus method

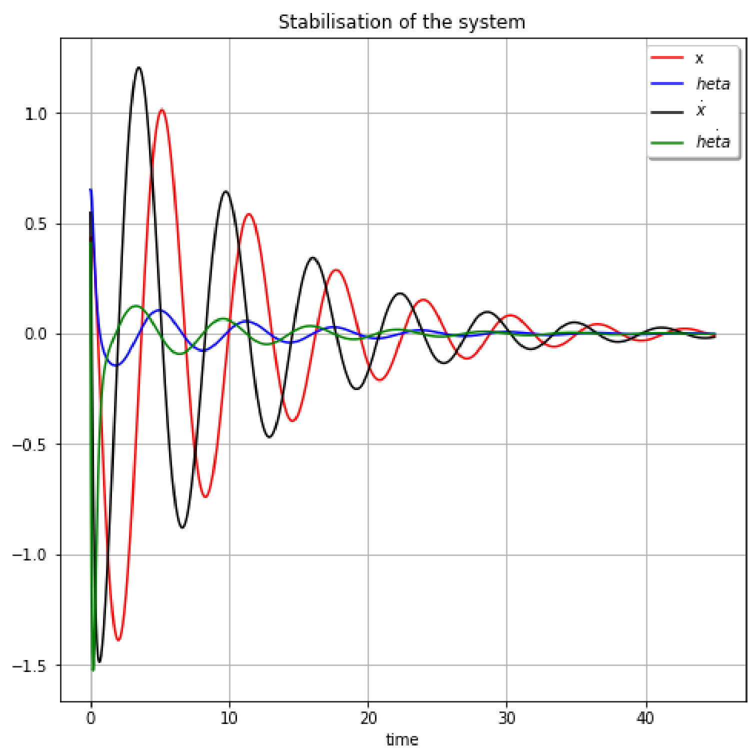
Link to the Matlab method

```
>> g = 9.81;
M = 0.2;
m = 1;
l = 0.1;
>> A = [[0, 0, 1, 0]; [0, 0, 0, 1]; [0, g*m/M, 0, 0]; [0, g*(M+m)/l/M,
    0, 0]];
B = [0; 0; 1/M; 1/l/M];
eig_vals = [-0.1 + 1j, -0.1 - 1j, -6 + 1j, -6 - 1j];
>> [K,prec,message] = place(A,B,eig_vals);
>> K

K =

   -0.0762   12.5878   -0.0398    0.2480

>> eig(A - B*K)

ans =

  -6.0000 + 1.0000i
  -6.0000 - 1.0000i
  -0.1000 + 1.0000i
  -0.1000 - 1.0000i
```

Using python

```
import numpy as np
import matplotlib.pyplot as plt
import scipy.signal as sig
from scipy.integrate import odeint

g = 9.81
M = 0.2
m = 1
l = 0.1

eig_1 = [-0.25, -0.5, -1, -2]
eig_2 = [-1.1, -1.2, -1.3, -1.4]
eig_3 = [-0.1 + 1j, -0.1 - 1j, -6 + 1j, -6 - 1j]

A = np.array([[0, 0, 1, 0], [0, 0, 0, 1], [0, g*m/M, 0, 0], [0, g*(M+m
    )/l/M, 0, 0]])
B = np.array([0, 0, 1/M, 1/l/M]).reshape(-1, 1)
```

```python
res_pole = sig.place_poles(A, B, eig_3)
K = res_pole.gain_matrix

# visualization
def control(x, t):
    return np.dot(A - np.dot(B, K), x)

time = np.linspace(0, 45, 1000)
x0 = (np.random.random(4))
res = odeint(control, x0, time).T

fig = plt.figure(figsize=(8, 8))
plt.title("Stabilisation of the system")
plt.xlabel("time")
plt.plot(time, res[0], "r-", label="x")
plt.plot(time, res[1], "b-", label="$\theta$")
plt.plot(time, res[2], "k-", label="$\dot{x}$")
plt.plot(time, res[3], "g-", label="$\dot{\theta}$")
plt.grid()
plt.legend(shadow=True)
plt.show()
```



part G

Description: (for the controllable system) design linear quadratic regulator for linearized system. Assess the performance of the controller for variety of initial conditions. Justify the choice of initial conditions;

Solution: Initial conditions were chosen such that system will behave differently

```
import scipy.linalg as lin
#3
M = 0.2
m = 1
l = 0.1
#2
# M = 0.02
# m = 0.01
# l = 0.5

#1
# M = 0.01
# m = 0.001
# l = 0.1

# A, B - the same

A = np.array([[0, 0, 1, 0], [0, 0, 0, 1], [0, g*m/M, 0, 0], [0, g*(M+m)/l/M, 0, 0]])
B = np.array([0, 0, 1/M, 1/l/M]).reshape(-1, 1)

# Q, R - random, but appropriate
Q = np.array([[1, 0, 0, 0],
              [0, 1, 0, 0],
              [0, 0, 1, 0],
              [0, 0, 0, 1]])
# R = np.array([-0.4]) #1
R = np.array([4]) #1

S = lin.solve_continuous_are(A, B, Q, R)

k = np.array(1/R[0]).dot(B.T).dot(S)

def simulator(x, t):
    return (A - np.dot(B, k)).dot(x)

time = np.linspace(0, 15, 1000)
res = odeint(simulator, x0, time).T
```

```python

fig = plt.figure(figsize=(8, 8))
plt.title("Stabilisation of the system")
plt.xlabel("time")
plt.plot(time, res[0], "r-", label="x")
plt.plot(time, res[1], "b-", label="$\theta$")
plt.plot(time, res[2], "k-", label="$\dot{x}$")
plt.plot(time, res[3], "g-", label="$\dot{\theta}$")
plt.grid()
plt.legend(shadow=True)
plt.show()
```