

ISM

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- Introduction: Observations of the ISM
- II Microscopic Processes.
- III Astrophysics of Gaseous Nebulae.

- IV Interstellar Dust.
- V Dynamics of the ISM.
- VI Selectec topics: protoplanetary disks, planetary nebulae, SNRs.



Part II

Microscopic Processes



$$n\frac{d}{dt}\left(\frac{3}{2}kT\right) - kT\frac{dn}{dt} = \Gamma - \Lambda$$
variation of internal E.

In steady state, dn/dt=dT/dt=0, so the temperature T is determined by $\Lambda(T)=\Gamma(T)$.

Outline





Cooling of the ISM

Radiative cooling (ref. Dyson & Williams, Chap. 3, Spitzer, Chap. 4 & 6, Osterbrock, Chap. 3).



$$A + B \rightarrow A + B^*$$

 $B^* \rightarrow B + h\nu$

Requisites for a mechanism of radiative cooling

- frequent encounters, i.e. A & B abundant
- $E(B^*) E(B) \stackrel{<}{\sim} kT$.
- large rate of B excitations: high 'collision strength'
- time scale for radiative decay of B^{\star} less than $au_{
 m col}$
- large escape probability of the output $h\nu$.

Detailed balance: abundance of excited species

In steady state, the density n_j of species \mathcal{B}^* in level j is determined by an equation of detailed balance, which models systems with a finite number of microscopic states:

$$\sum_{k} \left(\left\{ \begin{array}{c} n_{e} \\ n_{H_{2}} \\ n_{H_{1}} \end{array} \right\} n_{j} \gamma_{jk} + n_{j} B_{jk} J_{\nu} \right) + \sum_{k < j} (A_{jk} n_{j})$$

$$= \sum_{k} \left(n_{k} \left\{ \begin{array}{c} n_{e} \\ n_{H_{2}} \\ n_{H_{1}} \end{array} \right\} \gamma_{kj} + B_{kj} J_{\nu} n_{k} \right) + \sum_{k > j} (n_{k} A_{kj}),$$

which we may also write using a simplified notation:

$$\sum_{i \neq j} n_j C_{ji} + n_j B_{ji} U_{\nu_{ji}} + \sum_{i < j} n_j A_{ji} = \sum_{i \neq j} n_i C_{ij} + n_i B_{ji} U_{\nu_{ij}} + \sum_{i > j} n_i A_{ij}.$$

Detailed balance: rate of collisonal excitation



For any collision partner (i.e. either $n_{\rm H_2}$), the specific rate of $n_{\rm H_1}$

$$\gamma_{jk} = \langle u\sigma_{jk}(u) \rangle = rac{4}{\sqrt{\pi}} \left(rac{m_r}{2kT}
ight)^{3/2} \int_0^\infty du \ u^3\sigma_{jk}(u) \exp\left(-rac{m_r}{2kT}u^2
ight).$$

the specific rate (i.e. per pair of particles, in units of m^3 s⁻¹) of collisional de-excitations γ_{kj} must be related to γ_{jk} : in thermodynamic equilibrium any microscopic process must be balanced by its inverse (*principle of detailed balance*, otherwise the distribution functions would depend on t and there would be no equilibrium).

In **LTE** the rate of collisional excitations $j \to k$ with relative velocity $\vec{u} \in [\vec{u}, \vec{u} + d\vec{u}]$ must equal the rate of de-excitations $k \to j$ in the corresponding velocity range \vec{v} ,

$$n_{i}^{\text{LTE}}$$
 $n_{H_{2}}$ $\sigma_{jk}(u)f(u)ud\vec{u} = n_{k}^{\text{LTE}}$ $n_{H_{2}}$ $\sigma_{kj}(v)f(v)vd\vec{v}$, $n_{H_{1}}$ $n_{H_{1}}$

with
$$f(u) = \left(\frac{1}{\pi} \frac{m}{2kT}\right)^{3/2} \exp\left(-\frac{m}{2kT}u^2\right)$$
, and $\frac{1}{2}m_rv^2 = \frac{1}{2}m_ru^2 - E_{jk}^{-1}$, $g_ju^2\sigma_{jk}(u) = g_kv^2\sigma_{kj}(v)$, or $C_{kj} = C_{jk}\frac{g_j}{g_k}e^{E_{jk}/kT}$.



¹note threshold excitation energy

Critical density



The *critical density* is defined as the density at which the rate of collisonal de-excitations is equal to the rate of radiative de-excitations:

$$\sum_{k < j} C_{kj} = \sum_{k < j} A_{jk}.$$

A good cooling mechanism is therefore one that involves abundant species, an effective collision strength $\stackrel{>}{\sim}$ 1, a critical density $n_{\rm crit}\gg n_{\rm e}$, and optically thin transport of the output $h\nu_{jk}$.

Outline



Ionic & atomic cooling of the ISM

For ionic collisions, the cross-section of collisional excitation highlights the Coulomb cross-section.

$$\sigma_{jk}(E) = \frac{\pi a_{\circ}^2 \Omega_{jk}(E)}{E/E_{\circ}},$$

where E_{\circ} is 1 Rydberg, a_{\circ} is the Bohr radius, and $\Omega_{jk}(E)$ (of order 1), is the *collision strength* as a function of kinetic energy E. Finally the rate of excitations $j \to k$ (with units of s⁻¹) is (tarea),

$$C_{jk} = \left\{ egin{array}{l} n_{
m H_2} \\ n_{
m H_1} \end{array}
ight\} \sqrt{rac{2\pi}{kT}} \left(rac{h}{2\pi}
ight)^2 rac{1}{m_r^{3/2}} rac{\Gamma_{jk}}{g_j} e^{-E_{jk}/kT},$$

where Γ_{jk} , the *effective collision strength*, is the Maxwellian average of $\Omega(E)$.



Fine structure cooling in the IR

In LS coupling energy levels are ordered according to the Hund rules (Shu I Chap. 27):

- Higher S → lower energy
- $\bullet \ \, \text{Higher L} \to \text{lower energy}$
- Higher J → higher energy if less than half-filled, lower energy if more than half-filled

Spectroscopic notation in spin-orbit coupling (ver Shu I,27):

 $^{2S+1}L_{J}\,$, with degeneracy (2 J+1)

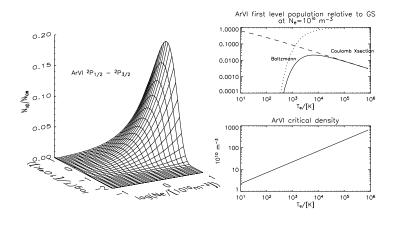
Examples:

- [Ar VI] $4.52\mu m$, ${}^{2}P_{\frac{1}{2}} \leftarrow {}^{2}P_{\frac{3}{2}}$, Z=18
- [Al VI] $3.65\mu\text{m}$, ${}^3\text{P}_2 \leftarrow {}^3\text{P}_1$, Z=13

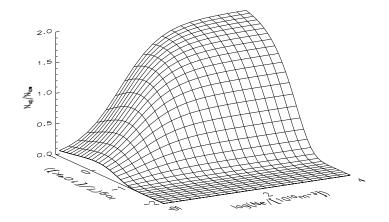


Ar VI ${}^2P_{\frac{3}{2}}$ population











- Collisional excitation of emission lines (case of a nebula photoionised by OB stars):
 - $[O II] \lambda \lambda 3726,3729$ ${}^4S_{3/2} \leftarrow {}^2D_{3/2,5/2}$.
 - $[O III] \lambda \lambda 4959,5007$ $^{3}P_{2,1} \leftarrow^{1} D_{2}$.
- continuum (free-free, bound-free): $A + B \rightarrow A + B + h\nu$ (postponed to chapter on photoionised nebulae)
- recombination (postponed to chapter on photoionised nebulae)

Cooling of the neutral ISM



examples: Table 3.1 in Dyson & Williams complemented by the list of lines in http://www.pa.uky.edu/~peter/atomic/

Transition	collision partner	$\Delta E/k$
[C I] $\lambda 157.7 \mu\text{m}$ $^{2}\text{P}_{1/2} \leftarrow^{2}\text{P}_{3/2}$	H,e,H ₂	92 K
[Si II] $\lambda 34.8 \mu \text{m}^{-2} \text{P}_{1/2} \leftarrow^2 \text{P}_{3/2}$	е	92 K
$[O I] \lambda 63.2 \mu\text{m}$ $^{3}P_{1} \leftarrow ^{3}P_{2}$	H,e	228 K
$[OI] \lambda 145.5 \mu \text{m} ^{3}P_{0} \leftarrow^{3} P_{2}$	H,e	326 K

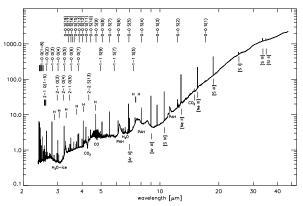
Outline





Molecular excitation - CO & H₂

ISO observations (Van Dishoeck, 2004, ARA&A, 42, 119) confirm that the contribution from H_2 cooling is more important than estimates based on IRAS data: H_2 is among the brightest emission lines in molecular clouds exposed to UV radiation, as for instance in Orion KL (note prominent dust continuuum):





fluorescent H₂: not a gas coolant



The rotational excitation of the ground vibrational state of H_2 is usually collisional (i.e. thermal), while higher vibrational states are excited by fluorescence (hence not a cooling process).

 \Rightarrow near-IR rovib H₂ is usually fluorescent, so not a coolant (except some of the lower-level rovib lines, in shocked regions).



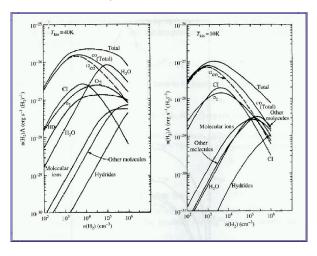
The most luminous spectral component in the molecular ISM is the mid-IR to far-IR continuum, due to dust. Dust is responsible for transporting the UV radiation of the exciting stars, but is not always coupled to the molecular gas. Collisions are efficient dust heating sources in the denser media only, such as dense SNRs or dense circumstellar disks.

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CO rotational-line cooling

The main cooling agents for cold molecular clouds are low-level rotational transitions of CO in the sub-mm. (Figure 2.11 from Tielens 2005).





Summary: diatomic molecules

In the Born-Oppenheimer approximation the Hamiltonian for a diatomic molecule AB separates, $H=H_{\rm AB}+H_{\rm el}$, where $H_{\rm AB}$ represents the kinetic energy of the nuclei, and $H_{\rm el}$ all the rest. It is also customary to factorize $\phi=\phi_{\rm el}\phi_{\rm AB}$ and obtain 2 :

$$H_{\rm el}\phi_{\rm el}=E_{\rm el}(R)\;\phi_{\rm el},$$

$$H_{AB}\phi_{AB} + E_{el}(R) \phi_{AB} = E \phi_{AB},$$

and using the centre of mass coordinates,

$$\phi_{AB} = \phi_{trans}(\vec{X}_{CM})\phi_{intern}(\vec{R}),$$

$$-\frac{\hbar^2}{2M}\nabla_{\rm CM}^2\phi_{\rm trans}=E_{\rm trans}\;\phi_{\rm trans}$$

$$-\frac{\hbar^2}{2\mu}\nabla_{\vec{R}}^2\phi_{\text{intern}} + E_{\text{el}}(R)\ \phi_{\text{intern}} = E_{\text{intern}}\ \phi_{\text{intern}},$$

with $E = E_{intern} + E_{trans}$.

²provided additional approximations according to Messiah, Vol. 2, XVIII.64 (p791), but exactly according to Shu I, 28. Probably valid only for processes slower than a rotation period.

The Laplacian in spherical coordinates,

$$\nabla_{\vec{R}}^2 = \frac{1}{R^2} \left[\frac{\partial}{\partial R} \left(R^2 \frac{\partial}{\partial R} \right) - \frac{L^2}{\hbar^2} \right],$$

inspires the decomposition of the wave function for the internal degrees of freedom in spherical armonics,

$$\phi_{\mathrm{intern}}(\vec{R}) = \frac{1}{R} Z_{\mathrm{vib}}(R) Y_{Jm}(\theta, \varphi),$$

The 1-D equation for $Z_{vib}(R)$ is a function of $E_{el}(R)$, the energy eigenvalue of the electronic wave function:

$$-\frac{\hbar^2}{2\mu}\frac{d^2 Z_{\rm vib}}{dR^2} + E_{\rm el}(R)Z_{\rm vib} = \left[E_{\rm intern} - \frac{J(J+1)\hbar^2}{2\mu R^2}\right]Z_{\rm vib}.$$



In the armonic approximation we expand the 1-D equation for $Z_{\rm vib}$ about $R_{\rm o}$, the AB separation in the fundamental state, highlighting the natural frequency $\mu\omega_{\circ}\equiv E_{el}^{"}(R_{\circ})$:

$$R_{\circ}$$
, the AB separation in the fundamental state, the natural frequency $\mu\omega_{\circ}\equiv E_{el}^{\prime\prime}(R_{\circ})$:
$$-\frac{\hbar^{2}}{2\nu}\frac{d^{2}Z_{\mathrm{vib}}}{dx^{2}}+\frac{\mu}{2}\omega_{\circ}^{2}x^{2}Z_{\mathrm{vib}}=E_{\mathrm{vib}}Z_{\mathrm{vib}},$$

$$-\frac{1}{2\mu}\frac{1}{dx^{2}} + \frac{1}{2}\omega_{\circ}^{2}X^{2}Z_{\text{vib}} = E_{\text{vib}}Z_{\text{vib}},$$

$$x \equiv R - R_{\circ}, \text{ and}$$

$$\boxed{E_{\text{intern}} = E_{\text{el}}(R_{\circ}) + E_{\text{vib}} + E_{\text{rot}},}$$

$$E_{\text{rot}} = J(J+1)\frac{\hbar}{2\mu R_{\circ}^{2}} \equiv J(J+1)B, \quad B: \text{ rotational constant},$$

$$E_{\text{vib}} = (v + \frac{1}{2})\hbar\omega_{\circ}.$$





The quantum numbers that determine the nuclear state of the molecule are J, m, v. Note the rigid body assumption: $\phi_{\rm el}$ is independent of J, which is the fundamental hypothesis implicit in the decomposition $\phi = \phi_{\rm el}\phi_{\rm AB}$. The quantum numbers that characterise the electronic state of the molecule are Λ , and S, where Λ corresponds to the projection of $L_{\rm tot}$ along the internuclear axis³, and S is the total spin.

³the only component of \vec{L} that is conserved in diatomic molecules

Outline



Interaction with electromagnetic radiation

The coupling term between charged particles and the electromagnetic field, $\vec{p_i} \cdot \vec{A}(\vec{k} \cdot \vec{x} - wt)^4$, can be expressed through an expansion in $\vec{k} \cdot \vec{x}$ as $H_{\text{int}} = H_{\text{d}} + H_{\text{M}} + H_{\text{Q}}$ (see Shu I,24), for which

$$H_{\rm d} = -\vec{E} \cdot \vec{d}$$
 (zeroth order)

where, for a molecule, $\vec{d} = \vec{d}_{\rm el} + \vec{d}_{\rm nuc}.$

$$H_{\rm M} = -\vec{B} \cdot \vec{M}$$
 (first order)

where the magnetic dipole moment $\vec{M} \propto \vec{L}$, and

$$H_{\rm Q} = -\frac{e}{6} \vec{\nabla} \vec{E} : (3\vec{x}\vec{x} - |\vec{x}|^2 \mathbb{I})$$
 (also order one).

In general $H_{\rm M} > H_{\rm O}$.



⁴when substituting $\vec{p}\to\vec{p}-\frac{q}{c}\vec{A}$, and neglecting terms in A^2 (OK for the ISM) see Shu I, 21

Bound-bound transition probabilities and cross-sections

In time-dependent perturbation theory, the rate of radiative excitations $i \rightarrow f$ is:

$$rac{dP_{if}}{dt} \propto \mathfrak{N}(\omega_{if}) \left| \langle \phi_f | \mathcal{H}_{ ext{int}}(\omega) | \phi_i
angle
ight|^2.$$

In a cubic box where the ocupation number of state \vec{n} is \mathfrak{N} , the density of states is $d^3\vec{n} = Vd^3\omega/(2\pi c)^3$, with a volume $V \to \infty$. The absorption cross-section σ_{if} derives from $P_{if} = N_f/N_i \propto \#$ of absorbed photons 5 :

$$P_{if} = \int d^3\vec{n} \, \frac{\mathfrak{N}(\vec{n})}{V} c \, t \, \frac{\sigma_{if}}{V} \Rightarrow \frac{dP_{if}}{dt} = \int_0^\infty \frac{\sigma_{if}}{\sigma_{if}} \, c \, \mathfrak{N}(\omega) \, \frac{4\pi\omega^2}{(2\pi)^3 c^2} d\omega$$

Identifying (see Shu I, 22, 23), we obtain

$$\sigma_{if} \propto \left| \langle \phi_f | \mathcal{H}_{int}(\mathbf{w}) | \phi_i \rangle \right|^2 \delta(\omega - \omega_{if}).$$



⁵note optically thin case: $dN_f = -\Gamma N_f dt + \frac{dP_{if}}{dt} N_i dt$.

Oscilator strength

For the electric dipole Hamiltonian, one gets

$$\sigma_{\mathit{if}} = rac{4\pi^2}{3\hbar c} \left| \langle \phi_{\mathit{f}} | ec{m{d}} | \phi_{\mathit{i}}
angle
ight|^2 \delta(\omega - \omega_{\mathit{if}}),$$

which is usually expressed in terms of the oscilator strength f_{if} ,

$$\sigma_{if} = \frac{\pi e^2}{m_e c} f_{if} \delta(\nu - \nu_{if}), \text{ con } f_{if} \equiv \frac{4\pi m_e}{3e^2 \hbar} \nu_{if} \left| \langle \phi_f | \vec{d} | \phi_i \rangle \right|^2.$$

For a single electron with position \vec{x} ,

$$f_{if} = \frac{2m_e(\omega_{if}\langle f|\vec{x}|i\rangle)^2}{3\hbar w_{if}},$$

which is roughly the ratio between the vibrational potential energy of the electron and that of the radiated photon.



Relationship with Einstein coeficients

The equation of detailed balance,

$$n_i B_{if} J_{\nu_{if}} = n_f A_{fi} + n_f B_{fi} J_{\nu_{fi}},$$

and the LTE relationships,

$$rac{n_f}{n_i} = rac{g_f}{g_i} \exp(-rac{h
u_{if}}{kT}), \quad ext{and} \quad J_
u = B_
u(T), \quad ext{lead to}$$

$$A_{fi} = rac{g_i}{g_f} (2h
u^3/c^2) B_{if}, \qquad B_{fi} = rac{c^2}{2h
u^3} A_{fi} = (g_i/g_f) B_{if},$$

where the rate of stimulated excitations is related to the oscilator strength⁶:

$$B_{\it if} = rac{4\pi^2}{h
u}rac{e^2}{m_e c}f_{\it if}.$$



Selection rules

- Electric dipole
 - atoms: $\Delta I = 1$, $\Delta m = 0$.
 - molecules:
 - vibrational-rotational transitions, or rovibrational, $\Delta J=\pm 1$, $\Delta m=0, \Delta v=\pm 1$, allowed when $\Lambda \neq 0, \Delta J=0^{7}$.
 - electronic transitions, $\Delta \Lambda = 0, \pm 1, \Delta S = 0$
 - electronic-vibrational-rotational transitions (i.e. *vibronic* transitions): $\Delta J = 0, \pm 1, \Delta m = 0, \pm 1$ and $\Delta J \neq 0$ si $\Delta J = 0$ and if J = 0.

$$\Delta J = \left\{ \begin{array}{l} +1 \rightarrow R \text{ branch} \\ 0 \rightarrow Q \text{ branch} \\ -1 \rightarrow P \text{ branch} \end{array} \right.$$

- magnetic dipole, atoms: $\Delta I = 0$, $\Delta m = 0, \pm 1$.
- electric quadrupole, atoms: $\Delta I = 0, \pm 2, \Delta m = 0, \pm 1, \pm 2,$ rotational transitions in molecules $\Delta J = 0, \pm 1, \pm 2.$

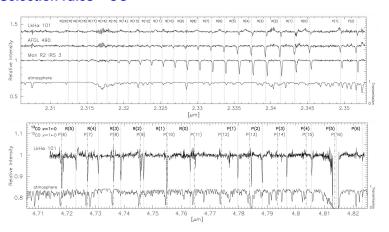


 $^{^7}$ Lambda doubling, two states $\pm \Lambda$ for each J. Example: hyperfine structure of the OH Λ doublet at ~ 1.7 GHz.

Selection rules - CO⁸



ISM



⁸Subaru - IRCS + echelle & X-disperser, Goto et al. 2003, ApJ, 598, 1038



In the case of H_2 we have $\vec{d}=0$. Moreover the fundamental state is $\vec{L}=0$, and $\vec{S}=0$, so that $\vec{M}=0$ and all low-energy transitions for H_2 are quadrupolar.

Note that the antisymmetry of the nuclear wave function implies that the state J=1 (J odd) is triplet (**Ortho** H₂, $I=S_{\text{nuclear}}=1$), while J=0 (J even) is singlet (**Para** H₂, J=0). In H₂ the exclusion principle ⁹ forbids $\Delta J=1$, unless the

transition involves a change in spin state. The spin transitions can only occur through the exchange of protons in collisions. Radiative transitions between spin states can occur, but at a rate corresponding to the quadrupolar transitions in the Hamiltonian of the deviations to the Born-Oppenheimer approximations.

⁹the requirement that the wave function be antisymmetric



In the ISM, **Ortho** and **Para** H_2 are effectively different molecules. The distinction extends to all molecules that contain H_2 radicals.

Rovibrational transitions between an upper level ¹ and a lower level ² are written $(v_1 - v_2)O(J_2)$, when $J_2 - J_1 = -2$, $(v_1 - v_2)Q(J_2)$ when $J_2 - J_1 = 0$, $(v_1 - v_2)S(J_2)$ when $J_2 - J_1 = +2$.

H₂ energy levels

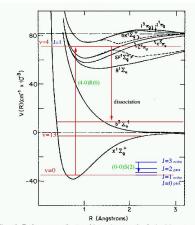


Figure 1. Total energy as a function of internuclear separation for low-lying states of H₂. Horizontal dashed lines denote total energy of two H atoms at infinite separation (both atoms in ground state; one atom in ground state and the other in the first excited states).

 $\begin{array}{l} \Lambda = 0, 1, 2, ... \\ \Leftrightarrow \Sigma, \Pi, \Delta \\ \text{levels in } E_{el}(R) \text{:} \\ X, B, C, D. \end{array}$

The energy separation between levels B and C with respect to X are \sim 11 eV and 14 eV.



Dissociation of H₂

The collisional dissociation of H₂,

$$\begin{array}{cccc} He & He \\ H_2 + & H & \rightarrow 2H + & H \\ e & & e \end{array} ,$$

requires temperatures of order the binding energy of H_2 , 4.48 eV/k = 52000 K 10 .

But an H_2 cloud is more likely to be photo-dissociated before it reaches 50 000 K. The dissociation continuum of H_2 (i.e. free states of H+H) starts 14.7 eV above the fundamental level of H_2 (i.e. $X^1\Sigma$, para- H_2), while the ionization continuum (the ejection of one electron) starts at 15.4 eV. Both continua, ionising and dissociating, lie above 13.6 eV. In the context of a molecular cloud surrounded by H I, the H_2 dissociating radiation is completely absorbed by the Lyman continuum of H I.



¹⁰the law of mass-action does not change this result because the number of continuum states accessible to both products and reactants is similar, in contrast with ionisation, which is described by the Saha equation

Dissociation of H₂



The likeliest mechanism for H_2 dissociation is absorption to the robrivational bands of excited electronic states (e.g. B o C), and

- subsequent de-excitation to dissociated levels of lower energy (e.g. b³Σ), or
- ② decay to the vibrational continuum of electronic state X (with v ≥ 14), or
- 3 decay to excited levels of the fundamental state and subsequent absorption to the dissociating continuum (one order of magnitude slower than mechanisms 1) and 2), Stecher& Williams 1967, ApJ 149, L29)

IR emission from H₂

Since the rate of radiative transitions within the ground state electronic level of H_2 is quadrupolar, the distribution of ro-vibrational population is dominated by collisions between H_2 molecules.

If $kT\gg \Delta E_{FIR}$, the typical energy difference between rotational levels of ground state H₂, the occupation numbers of the robrivational levels in the ground state electronic state follow LTE:

$$N(v, J) \propto \omega(J) \exp(-B(J(J+1))/kT),$$

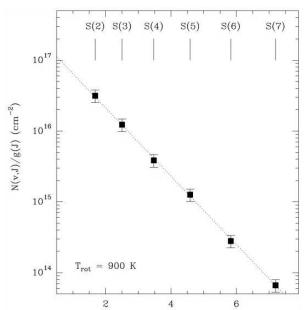
in which $\omega(J)=3\times(2J+1)$ for ortho- H_2 , and $\omega(J)=1\times(2J+1)$ for para- H_2 . This situation (i.e. hot molecular gas) is typical of shocks in the ISM. In the presence of an intense UV field the IR emission of H_2 is fluorescent (Black & Van Dishoeck, 1987, ApJ, 322, 412). Fluorescence dominates if $kT\ll\Delta E_{\rm FIR}$. The absorption of one UV photon excites H_2 to an electronic state, leading to photodissociation (with a 15% probability), and to an IR cascade in all other cases.



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IR emission from H₂

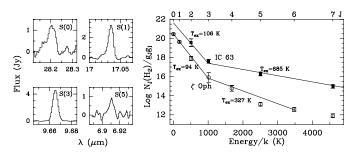
Example excitation diagram for the Helix nebula.





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IR emission from H_2 , ζ Oph







Outline



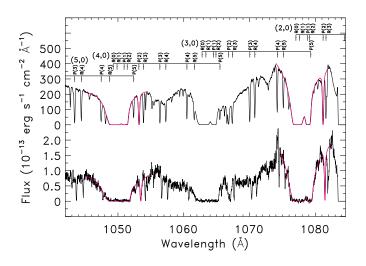


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Heating of the neutral ISM

 $\rm H_2$ is an important source of UV opacity, as shown in the *FUSE* spectra of HD210839 and HD154368 (Rachford et al., 2002, ApJ, 577, 221).





heating by photodissociation and photoionisation:

$$AB + h\nu \rightarrow A + B$$
,

dominated by H_2 photodissociation.

- collisional de-excitation (in particular for H_2 , when $n(H_2) > 10^4 \ cm^{-3}$).
- heating by photoionisation:

$$A + h\nu \rightarrow A^+ + e$$

in the case of the neutral ISM, photoionisation heating is dominated by C⁺, with an ionization potential of 11.3 eV. The average kinetic enery of the ejected electron is

$$\frac{3}{2}kT_{i} = \frac{\int_{\nu_{i}}^{\infty} (J(\nu)/h\nu)\alpha_{i}(\nu)h(\nu-\nu_{i})d\nu}{\int_{\nu_{i}}^{\infty} (J(\nu)/h\nu)\alpha_{i}(\nu)d\nu}.$$

