ICS 2022 Problem Sheet #3

Problem 3.1: cartesian products

Prove or disprove the following two propositions.

a) (A ∩ B) × (C ∩ D) = (A × C) ∩ (B × D)

$(x,y) \in (A \cap B) \times (C \cap D)$

$\Rightarrow (x \in (A \cap B)) \wedge (y \in (C \cap D))$

$\Rightarrow ((x \in A) \wedge (x \in B)) \wedge ((y \in C) \wedge (y \in D))$

$\Rightarrow ((x \in A) \wedge (y \in C)) \wedge ((x \in B) \wedge (y \in D))$

$\Rightarrow (x \in (A \times C)) \wedge (y \in (B \times D))$

$\Rightarrow (x,y) \in (A \times C) \cap (C \times D)$, which proves the propositions.

b) (A ∪ B) × (C ∪ D) = (A × C) ∪ (B × D)

$(x,y) \in (A \cup B) \times (C \cup D)$

$\Rightarrow (x \in (A \cup B)) \wedge (y \in (C \cup D))$

$\Rightarrow ((x \in A) \vee (x \in B)) \wedge ((y \in C) \vee (y \in D))$

Using the distributivity of $\wedge$:

$\Rightarrow ((x \in A) \wedge (y \in C)) \vee ((x \in B) \wedge (y \in D))$

$\Rightarrow (x,y) \in (A \times C) \vee (x,y) \in (B \times D)$

$\Rightarrow (x,y) \in (A \times C) \cup (B \times D)$, which proves the propositions.

Problem 3.2: reflexive, symmetric, transitive

For each of the following relations, determine whether they are reflexive, symmetric, or transitive. Provide a reasoning.

a) The absolute difference of the integer numbers a and b is less than or equal to 3.
$R = \{ (a, b) \mid a, b \in \mathbb{Z} \wedge |a - b| \leq 3 \}$

- Reflexive iff $\forall a \in \mathbb{Z}.\ (a, a) \in R$
  Let $a \in \mathbb{Z} \Rightarrow (a, a) \in R \Leftrightarrow |a - a| \leq 3$ that is equal to $|0| \leq 3$, which is true $\forall a \in \mathbb{Z}$.
  $\Rightarrow$ R is reflexive $\forall a \in \mathbb{Z}$
- Symmetric iff $\forall a, b \in \mathbb{Z}.\ (a, b) \in R \Rightarrow (b, a) \in R$
  Let $(a, b) \in R \Rightarrow a, b \in \mathbb{Z}$ and $|a - b| \leq 3$
  $(b, a) \in R \Leftrightarrow b, a \in \mathbb{Z}$, true as proved before, and $|b - a| \leq 3 \Leftrightarrow |-(-b + a)| \leq 3 \Leftrightarrow$ $|a - b| \leq 3$, true as proved before $\Rightarrow (b, a) \in R$
  $\Rightarrow$ R is symmetric $\forall a, b \in \mathbb{Z}$
- Transitive iff $\forall a, b, c \in \mathbb{Z}.\ ((a, b) \in R \wedge (b, c) \in R) \Rightarrow (a, c) \in R$
  Let $(a, b) \in R \wedge (b, c) \in R \Rightarrow a, b, c \in \mathbb{Z}$ and $|a - b| \leq 3$ and $|b - c| \leq 3$
  Let $(a, b) = (5, 2)$. $5, 2 \in \mathbb{Z}$ and $|5 - 2| \leq 3 \Rightarrow (5, 2) \in R$
  $(b, c) = (2, 1)$. $2, 1 \in \mathbb{Z}$ and $|2 - 1| \leq 3 \Rightarrow (2, 1) \in R$
  But $|5 - 1| \nleq 3$ so $(5, 1) = (a, c) \notin R \Rightarrow$ R is not transitive $\forall a, b, c \in \mathbb{Z}$

b) The last digit of the decimal representation of the integer numbers a and b is the same.
R = { (a, b) | a, b ∈ ℤ ∧ (a mod 10) = (b mod 10) }

- Reflexive iff ∀a ∈ ℤ. (a, a) ∈ R
  Let a ∈ ℤ ⇒ (a, a) ∈ R ⇔ (a mod 10) = (a mod 10), which is true ∀a ∈ ℤ.
  ⇒ R is reflexive ∀a ∈ ℤ
- Symmetric iff ∀a, b ∈ ℤ. (a, b) ∈ R ⇒ (b, a) ∈ R
  Let (a, b) ∈ R ⇒ a, b ∈ ℤ and (a mod 10) = (b mod 10)
  (b, a) ∈ R ⇔ b, a ∈ ℤ, true as proved before, and (a mod 10) = (b mod 10), also true as proved before ⇒ (b, a) ∈ R
  ⇒ R is symmetric ∀a, b ∈ ℤ
- Transitive iff ∀a, b, c ∈ ℤ. ((a, b) ∈ R ∧ (b, c) ∈ R) ⇒ (a, c) ∈ R
  Let (a, b) ∈ R ∧ (b, c) ∈ R ⇒ a, b, c ∈ ℤ and (a mod 10) = (b mod 10) ∧ (b mod 10) = (c mod 10) ⇒ (a mod 10) = (c mod 10) (a, c ∈ ℤ) ⇒ (a, c) ∈ R
  ⇒ R is transitive ∀a, b, c ∈ ℤ

Problem 3.3: total, injective, surjective, bijective functions

Are the following functions total, injective, surjective, or bijective? Explain why or why not.

a) f : ℕ → : ℕ with f(x) = $2x^2$
total: ∀x ∈ ℕ, ∃! y ∈ ℕ with (x, y) ∈ f .
x ∈ ℕ ⇒ there is exactly one f(x) = $2x^2$ = y ∈ ℕ ⇒ f is a total function
injective: ∀x, y ∈ ℕ. f (x) = f (y) ⇒ x = y
Let x, y ∈ ℕ and f (x) = f (y) ⇒ $2x^2 = 2y^2 \Rightarrow x^2 = y^2 \Rightarrow |x| = |y|$ and x, y ∈ ℕ
⇒ x = y ⇒ f is an injective function
surjective: ∀y ∈ ℕ. ∃x ∈ ℕ. f (x) = y
Let y ∈ ℕ and f (x) = y ⇒ $2x^2 = y \Rightarrow x^2 = \frac{y}{2} \Rightarrow x = \sqrt{\frac{y}{2}}$ which doesn't always ∈ ℕ
⇒ f is not an surjective function ⇒ f is also not a bijective function

b) f : ℝ → ℝ with f(x) = $x^2 + 6$
total: ∀x ∈ ℝ, ∃! y ∈ ℝ with (x, y) ∈ f .
x ∈ ℝ ⇒ there is exactly one f(x) = $x^2 + 6$ = y ∈ ℝ ⇒ f is a total function
injective: ∀x, y ∈ ℝ. f (x) = f (y) ⇒ x = y
Let x, y ∈ ℝ and f (x) = f (y) ⇒ $x^2 + 6 = y^2 + 6 \Rightarrow x^2 = y^2 \Rightarrow |x| = |y|$ and x, y ∈ ℝ
⇒ x = - y or x = y ⇒ f is not an injective function
surjective: ∀y ∈ ℝ. ∃x ∈ℝ. f (x) = y
Let y ∈ ℝ and f (x) = y ⇒ $x^2 + 6 = y \Rightarrow x^2 = y - 6 \Rightarrow x = \sqrt{y-6} \in (6, +\infty) \neq \mathbb{R}$
⇒ f is not an surjective function ⇒ f is also not a bijective function

Problem 3.4: types (Haskell)

a) What is the type signature of the zip function? How many type variables appear in the type signature? Could it be more or less? Explain

The type signature of the zip function is: **zip :: [a] -> [b] -> [(a, b)]**

There are 2 type variables in the type signature: **Num ([a], [b])** and **(Num, Num) ([(a, b)])**

There can't be more nor less type variables because the zip function itself has to have exactly 2 arguments, same type variable, and returns one different type variable, so it can only have exactly two type variables.

b) What are the types of the following expressions? Explain why!

2 + 3 **Num a => a because both 2 and 3 can be considered any number type**

2 + 9 \`div\` 3 **Integral a => a because the result of \`div\` is an integral**

2 + 9 / 3 **Rational (Fractional) a => a because the result of \`/\` is a rational number**

2 + sqrt 9 **Double a => a because the result of \`sqrt\` is a double/ floating point number**