# ICS 2022 Problem Sheet #2

## **Problem 2.1**: proof by contrapositive

Let x and y be real numbers, i.e., $x, y \in \mathbb{R}$. If $y^3 + yx^2 \leq x^3 + xy^2$, then y ≤ x.

Proof:

We prove the contrapositive, if y > x , then $y^3 + yx^2 > x^3 + xy^2$.
Assume y > x. Since $x, y \in \mathbb{R}$, then $y^2 + x^2 \geq 0$, which follows that $y(y^2 + x^2) > x(y^2 + x^2)$. This finally leads to $y^3 + yx^2 > x^3 + xy^2$. □

## **Problem 2.2**: proof by induction

Let n be a natural number with n ≥ 1. Prove that the following holds:

$$1^2 + 3^2 + 5^2 + \cdots (2n-1)^2 = \sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

Proof:

We prove $1^2 + 3^2 + 5^2 + \cdots (2n-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$ by induction

*Base case*:

We show that the equation is true for $n = 1$. Setting $n = 1$, the equation becomes

$$1^2 = \frac{2(2-1)(2+1)}{6} = \frac{6}{6} = 1$$

and hence the equation holds for $n = 1$.

*Induction step*:

Assume that the equation holds for some $n \in \mathbb{N}$. Let's consider the case $n+1$:

$$1^2 + 3^2 + 5^2 + \cdots (2n-1)^2 + (2n+1)^2 = \frac{2n(2n-1)(2n+1)}{6} + (2n+1)^2$$

$$= \frac{2n(2n-1)(2n+1) + 6(2n+1)(2n+1)}{6}$$

$$= \frac{2(2n+1)[n(2n-1) + 3(2n+1)]}{6}$$

$$= \frac{2(2n+1)(2n^2 - n + 6n + 3)}{6}$$

$$= \frac{2(2n+1)(2n^2 + 5n + 3)}{6}$$

$$= \frac{2(2n \mp 1)(2n+3)(n+1)}{6}$$

$$= \frac{2(n+1)(2n+1)(2n+3)}{6}$$

This shows that the equation holds for $n+1$.

It follows by induction that

$$1^2 + 3^2 + 5^2 + \cdots (2n-1)^2 = \sum_{k=1}^{n} (2k-1)^2 = \frac{2n(2n-1)(2n+1)}{6}$$

holds for arbitrary integers n. □

**Problem 2.3**: sum of divisors in Haskell

a) Write a function divisors :: Int -> [Int] that returns the list of divisors of a given positive integer n. The list of divisors includes 1 and the number n itself.

```haskell
divisors :: Int -> [Int]
divisors n = [ d | d <- [1..n], n `mod` d == 0 ]
```

b) Write a function sigma :: Int -> Int -> Int that takes the two arguments z and n and returns the sum of the $z^{th}$ powers of the positive divisors of n.

I.
```haskell
sigma :: Int -> Int -> Int
sigma z n = sum [ x^z | x <- [1..n], n `mod` x == 0 ]
```

or, using the divisors function:

II.
```haskell
sigma :: Int -> Int -> Int
sigma z n = sum [ x^z | x <- divisors n]
```