DD MM AA

1) Determine si el enunciado en verdadero o falso. Si es verdadero, explique perqué Si es falso explique por qué o de un ejemplo que desapruebe el enunciado

a) $\int_{-1}^{2}\int_{0}^{6} x^2 \operatorname{sen}(x-y)\,dx\,dy = \int_{0}^{6}\int_{-1}^{2} x^2 \operatorname{sen}(x-y)\,dx\,dy$

Falso

$$\int_{-1}^{2}\int_{0}^{6} x^2 \operatorname{sen}(x-y)\,dx\,dy =$$

$$\int x^2 \operatorname{sen}(x-y)\,dx = u.v - \int v\,du$$

Por integración por parte:

$u = x^2 \qquad dv = \operatorname{sen}(x-y)\,dx$

$du = 2x\,dx \qquad v = -\cos(x-y)$

$$\int x^2 \operatorname{sen}(x-y)\,dx = -x^2\cos(x-y) + 2\int x\cos(x-y)\,dx$$

De nuevo por integración por partes:

$u = x \qquad dv = \cos(x-y)$

$du = dx \qquad v = \operatorname{sen}(x-y)$

$$\int x^2 \operatorname{sen}(x-y)\,dx = -x^2\cos(x-y) + 2\left[x\operatorname{sen}(x-y) - \int \operatorname{sen}(x-y)\,dx\right]$$

$$= -x^2\cos(x-y) + 2x\operatorname{sen}(x-y) + 2\cos(x-y)$$

Norma

Escaneado con CamScanner

DD MM AA

$$\int_{-1}^{2}\int_{0}^{6} x^2 \operatorname{sen}(x-y)\,dx\,dy =$$

$$\int_{-1}^{2} -x^2\cos(x-y)+2x\operatorname{sen}(x-y)+\cos(x-y)\Big]_0^6\,dy$$

$$= \int_{-1}^{2}\Big[-36\cos(6-y)+12\operatorname{sen}(6-y)+\cos(6-y)-\cos(-y)\Big]dy$$

$$= 36\operatorname{sen}(6-y)+12\cos(6-y)-\operatorname{sen}(6-y)+\operatorname{sen}(-y)\Big]_{-1}^{2}$$

$$= 36\operatorname{sen}(4)+12\cos(4)-\operatorname{sen}(4)+\operatorname{sen}(-2)-36\operatorname{sen}(7)$$
$$-12\cos(7)+\operatorname{sen}(7)-\operatorname{sen}(1)$$

$$= 35\operatorname{sen}(4)+12\cos(4)+\operatorname{sen}(-2)-35\operatorname{sen}(7)-12\cos(7)-\operatorname{sen}(1)$$

$$\int_{0}^{6}\int_{-1}^{2} x^2\operatorname{sen}(x-y)\,dx\,dy$$

$$= \int_{0}^{6} -x^2\cos(x-y)+2x\operatorname{sen}(x-y)+\cos(x-y)\Big]_{-1}^{2}\,dy$$

$$= \int_{0}^{6}\Big[-4\cos(2-y)+4\operatorname{sen}(2-y)+\cos(2-y)-(-\cos(-1-y)$$
$$+2\operatorname{sen}(-1-y)+\cos(-1-y))\Big]\,dy$$

$$= \int_{0}^{6}\Big[-4\cos(2-y)+4\operatorname{sen}(2-y)+\cos(2-y)+\cos(-1-y)$$
$$+2\operatorname{sen}(-1-y)-\cos(-1-y)\Big]\,dy$$

$$= 4\operatorname{sen}(2-y)+4\cos(2-y)-\operatorname{sen}(2-y)-\operatorname{sen}(-1-y)$$
$$+2\cos(-1-y)+\operatorname{sen}(-1-y)\Big]_0^6$$

$$= 4\operatorname{sen}(-4)+4\cos(-4)-\operatorname{sen}(-4)-\cancel{\operatorname{sen}(-7)}+\cos(-7)+\cancel{\operatorname{sen}(-7)}$$
$$+4\operatorname{sen}(2)+4\cos(2)-\operatorname{sen}(2)-\cancel{\operatorname{sen}(-1)}+2\cos(-1)+\cancel{\operatorname{sen}(-1)}$$

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$= 5\,\text{sen}(-4) + 4\cos(-4) + \cos(-7) + 3\,\text{sen}(-2)$
$+ 4\cos(2) + 2\cos(-1)$

b) $\int_0^1 \int_0^x \sqrt{x+y^2}\, dy\, dx = \int_0^x \int_0^1 \sqrt{x+y^2}\, dy\, dx$

Falso: La segunda expresión no tiene sentido

c) Si f es continua en el intervalo $[0,1]$, entonces $\int_0^1 \int_0^1 f(x) f(y)\, dy\, dx$

$= \left[\int_0^1 f(x)\, dx\right]^2$

Verdadero. Usamos $\int k f(x)\, dx = k \int f(x)\, dx$

$$\int_0^1 \int_0^1 f(x) f(y)\, dy\, dx = \int_0^1 f(x) \left[\int_0^1 f(y)\, dy\right] dx$$

$$= \left[\int_0^1 f(y)\, dy\right] \cdot \int_0^1 f(x)\, dx.$$

Podemos $x = y$

$$= \int_0^1 f(x)\, dx \cdot \int_0^1 f(x)\, dx$$

$$= \left[\int_0^1 f(x)\, dx\right]^2$$

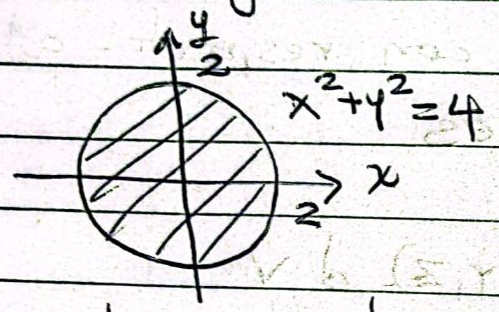
Norma

Escaneado con CamScanner

d) Si D es el disco dado por $x^2+y^2 \leq 4$, entonces $\iint_D \sqrt{4-x^2-y^2}\, dA = \frac{16\pi}{3}$

Verdadero

Si dibujamos el disco D



usando coordenadas polares:

$$\iint_D \sqrt{4-(x^2+y^2)}\, dA = \int_0^{2\pi}\int_0^2 \sqrt{4-r^2}\cdot r\, dr\, d\theta$$

Si hacemos $u=4-r^2$, $du=-2r\,dr$ y $\frac{du}{-2} = r\,dr$ ademas si $r=0$ entonces $u=4-0^2=4$ y si $r=2$, $u=4-2^2=0$

$$\iint_D \sqrt{4-(x^2+y^2)}\, dA = \int_0^{2\pi}\int_4^0 \sqrt{u}\,\frac{du}{-2}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\int_0^4 u^{1/2}\, du\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{2u^{3/2}}{3}\Big]_0^4 d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \left(4^{3/2} - 0^{3/2}\right) d\theta = \frac{8}{3}\,\theta\Big]_0^{2\pi} = \frac{16\pi}{3}$$

Escaneado con CamScanner

e) La integral $\iiint_E k r^3 dz\, dr\, d\theta$ representa el momento de inercia respecto al eje z de un sólido E con densidad constante k.

Verdadero.

El momento de masa con respecto al eje z del solido E es

$$I_z = \iiint_E (x^2+y^2)\, \rho(x,y,z)\, dV$$

dende $\rho(x,y,z)$ es la densidad.

Así al hacer cambio a coordenadas polares

$$I_z = \iiint_E r^2 \cdot k \cdot r\, dz\, dr\, d\theta$$

$$= \iiint_E r^3 k\, dz\, dr\, d\theta$$

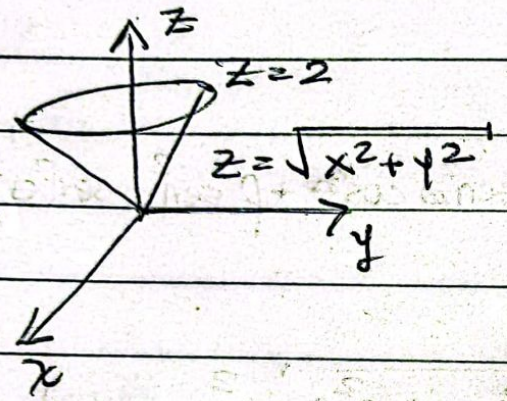
Norma

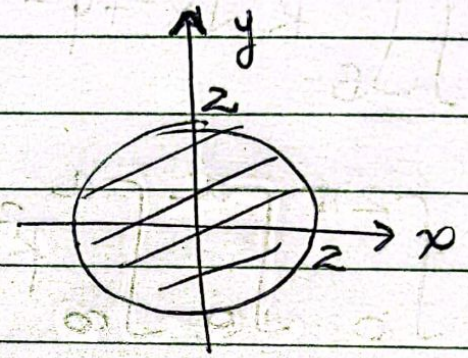
Escaneado con CamScanner

f) La integral $\int_0^{2\pi}\int_0^2\int_r^2 dz\,dr\,d\theta$

representa el volumen encerrado por el cono $z=\sqrt{x^2+y^2}$ y el plano $z=2$

Falso:





Proyección plano XY

En coordenadas cilíndricas:

$$V=\int_0^{2\pi}\int_0^2\int_r^2 r\,dz\,dr\,d\theta$$

Falta el factor integrante r.

g) La masa de una pelota B dada por $x^2+y^2+z^2\leq a^2$ si la densidad en cualquier punto es proporcional a su distancia con el eje z es igual a $2a$

Falso

La densidad es $\rho(x,y,z)=k\sqrt{(x-0)^2+(y-0)^2+(z-2)^2}$
$=k\sqrt{x^2+y^2}$

Escaneado con CamScanner

La masa es

$$m = \iiint_B \rho(x, y, z)\, dV$$

$$m = \iiint_B k\sqrt{x^2+y^2}\, dV$$

$$m = k\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sqrt{\rho^2 \operatorname{sen}^2\phi \cos^2\theta + \rho^2 \operatorname{sen}^2\phi \operatorname{sen}^2\theta}\, \operatorname{sen}\phi$$

$$m = k\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho \operatorname{sen}\phi \sqrt{\cos^2\theta + \operatorname{sen}^2\theta}\, \rho^2 \operatorname{sen}\phi\, d\rho\, d\phi\, d\theta$$

$$m = k\int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^3 \operatorname{sen}^2\phi\, d\rho\, d\phi\, d\theta$$

$$m = k\int_0^{2\pi}\int_0^{\pi} \operatorname{sen}^2\phi \left.\frac{\rho^4}{4}\right]_0^a d\rho\, d\phi\, d\theta$$

$$m = \frac{k}{4}\int_0^{2\pi}\int_0^{\pi} \frac{1-\cos 2\phi}{2}\, a^4\, d\phi\, d\theta$$

$$m = \frac{a^4 k}{8}\int_0^{2\pi} \left.\phi - \frac{\operatorname{sen} 2\phi}{2}\right]_0^{\pi} d\theta = \frac{a^4 k}{8}\pi\, \theta\Big]_0^{2\pi}$$

$$= \frac{a^4 k \pi^2}{4} \neq 2a$$

Norma

Escaneado con CamScanner

2) Considere la integral $\iint_D \frac{y}{1+x^2}\, dA$

donde D está acotada por $y=\sqrt{x}$ $y=0$ y $x=1$

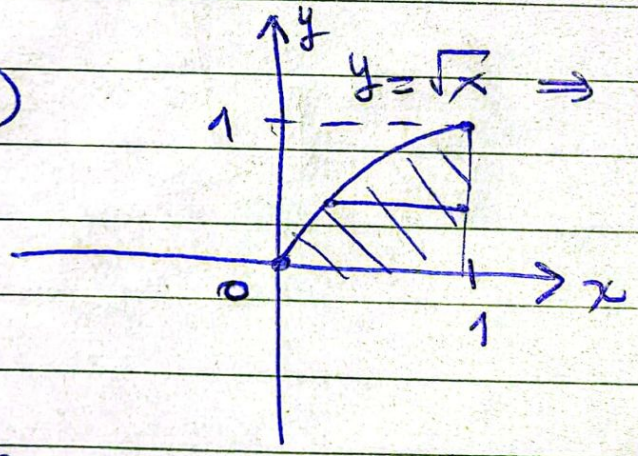
a) Realice una representación de D

b) Plantee las integrales iteradas para resolver la integral en los dos ordenes

c) Resuelva la más sencilla de calcular

Solución:

a)



b) $$\iint_D \frac{y}{1+x^2}\, dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx$$

$$= \int_0^1 \int_{y^2}^1 \frac{y}{1+x^2}\, dx\, dy$$

DD MM AA

c) $\int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\, dy\, dx$

$\int_0^1 \frac{1}{1+x^2} \int_0^{\sqrt{x}} y\, dy\, dx$

$= \int_0^1 \frac{1}{1+x^2} \left. \frac{y^2}{2} \right]_0^{\sqrt{x}} dx$

$= \frac{1}{2} \int_0^1 \frac{(\sqrt{x})^2}{1+x^2}\, dx$

$= \frac{1}{2} \int_0^1 \frac{x}{1+x^2}\, dx$

$u = 1+x^2 \quad du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$= \frac{1}{2} \int_1^2 \frac{\frac{du}{2}}{u}$

si $x=1 \Rightarrow u=2$

si $x=0 \Rightarrow u=1$

$= \frac{1}{4} \int_1^2 \frac{du}{u} = \frac{1}{4} \left. \ln|u| \right]_1^2$

$= \frac{1}{4} \left(\ln(2) - \cancel{\ln|1|}^{\,0} \right) = \frac{\ln(2)}{4}$

Escaneado con CamScanner

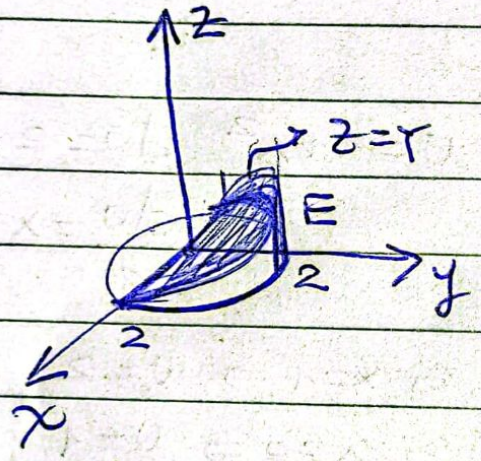
3) Sea $\iiint_E yz\,dV$ donde E está sobre el plano $z=0$ debajo de $z=y$ y dentro del cilindro $x^2+y^2=y$

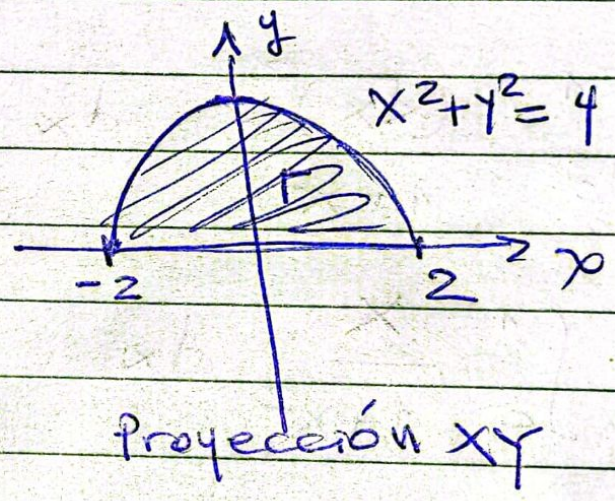
a) Realice una representación gráfica de la región E

b) Plantee y resuelva la integral

Solución

a)





b) En coordenadas rectangulares:

$$\iiint_E yz\,dV = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{y} yz\,dz\,dy\,dx$$

$$= \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \left.\frac{yz^2}{2}\right]_0^y dy\,dx$$

Norma

Escaneado con CamScanner

$$= \frac{1}{2}\int_{-2}^{2}\int_{0}^{\sqrt{1-x^2}} y \cdot y^2 \, dy \, dx$$

$$= \frac{1}{2}\int_{-2}^{2}\int_{0}^{\sqrt{1-x^2}} y^3 \, dy \, dx$$

$$= \frac{1}{2}\int_{-2}^{2} \left.\frac{y^4}{4}\right]_{0}^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{8}\int_{-2}^{2} \left(\sqrt{1-x^2}\right)^4 dx$$

$$= \frac{1}{8}\int_{-2}^{2} (1-x^2)^2 \, dx$$

$$= \frac{1}{8}\int_{-2}^{2} (1-2x^2+x^4) \, dx$$

$$= \frac{1}{8}\left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)_{-2}^{2}$$

$$= \frac{1}{8}\left[\left(2 - \frac{16}{3} + \frac{32}{5}\right) - \left(-2 + \frac{16}{3} - \frac{32}{5}\right)\right]$$

$$= \frac{1}{8}\left[4 - \frac{32}{3} + \frac{64}{5}\right] = \frac{1}{8}\left(\frac{60 - 160 + 192}{15}\right)$$

$$= \frac{23}{30}$$

Escaneado con CamScanner

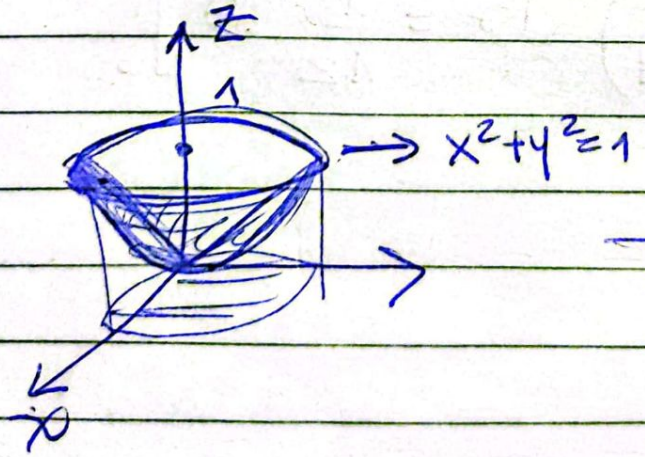
4) Considere el sólido arriba del paraboloide $z = x^2 + y^2$ y debajo del semi cono $z = \sqrt{x^2+y^2}$

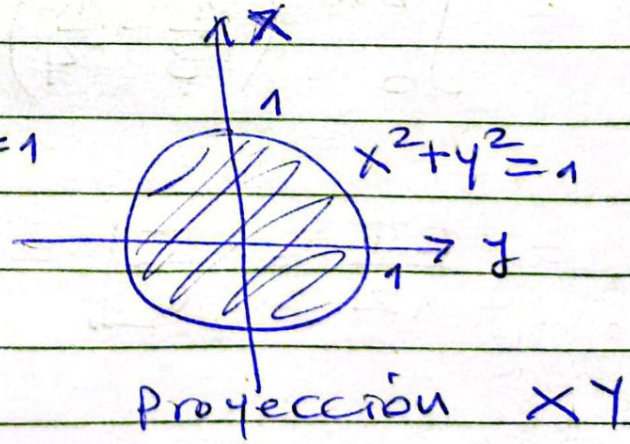
a) Represente gráficamente el sólido
b) Plantee una integral triple para hallar el volumen del sólido
c) Consiga el volumen del sólido calculando la integral planteada.

Solución Hallamos intersección entre las superficies

$$z = \sqrt{z} \Rightarrow z^2 = z \Rightarrow z^2 - z = 0$$
$$\Rightarrow z(z-1) = 0$$
$$\Rightarrow z = 0 \text{ ó } z = 1$$

Cuando $z=0$, $x^2+y^2=0$. es el origen
$z=1$, $x^2+y^2=1$ es un círculo





Escaneado con CamScanner

b) Por coordenadas cilíndricas

$$V = \int_0^{2\pi} \int_0^1 \int_{r^2}^{r} r \, dz \, dr \, d\theta$$

$$V = \int_0^{2\pi} \int_0^1 r z \Big]_{r^2}^{r} dr \, d\theta$$

$$V = \int_0^{2\pi} \int_0^1 r(r - r^2) \, dr \, d\theta$$

$$V = \int_0^{2\pi} \int_0^1 (r^2 - r^3) \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3} - \frac{r^4}{4} \Big]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{1}{3} - \frac{1}{4}\right) d\theta = \frac{1}{12} \theta \Big]_0^{2\pi}$$

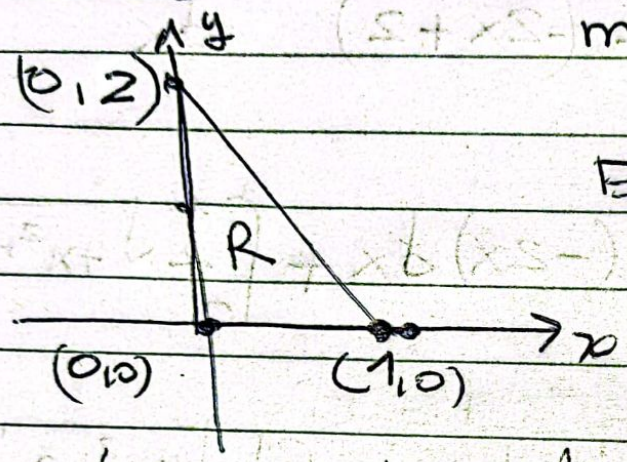
$$= \frac{2\pi}{12} = \frac{\pi}{6}$$

Norma

Escaneado con CamScanner

5) Determine el área de la parte de la superficie $z = x^2 + y$ que está por encima del triángulo con vértices $(0,0)$, $(1,0)$ y $(0,2)$

Solución:



Pendiente entre $(1,0)$ y $(0,2)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-0}{0-1} = -2$$

Ecuación recta:

$$y = -2(x-1)$$

$$y = -2x + 2$$

El área de la superficie $z = x^2 + y$ se calcula:

$$A = \iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA$$

$$f(x,y) = x^2 + y$$

$$f_x(x,y) = 2x$$

$$f_y(x,y) = 1$$

Luego

$$A = \int_0^1 \int_0^{-2x+2} \sqrt{(2x)^2 + (1)^2 + 1}\, dy\, dx$$

$$A = \int_0^1 \int_0^{-2x+2} \sqrt{4x^2 + 2}\, dy\, dx$$

Escaneado con CamScanner

$$A = \int_0^1 \sqrt{4x^2+2}\; Y \Big]_0^{-2x+2} dx$$

$$A = \int_0^1 \sqrt{4x^2+2}\,(-2x+2)\; dx$$

$$A = \int_0^1 \sqrt{4x^2+2}\,(-2x)\,dx + \int_0^1 2\sqrt{4x^2+2}\,dx$$

$$A = -2\int_0^1 \sqrt{4x^2+2}\; x\,dx + 2\int_0^1 \sqrt{4x^2+2}\,dx$$

$$A = -2\int_2^6 u^{1/2}\,\frac{du}{8} + 2\int_0^1 \sqrt{4x^2+2}\,dx$$

$u = 4x^2+2 \Rightarrow du = 8x\,dx$

$\Rightarrow \frac{du}{8} = x\,dx$

Si $x = 1 \Rightarrow u = 6$

$x = 0 \Rightarrow u = 2$

Sustitución trigonométrica

$$\int \sqrt{4x^2+2}\; dx$$

$2x = \sqrt{2}\tan\theta$

$2\,dx = \sqrt{2}\sec^2\theta\, d\theta$

$\sqrt{4x^2+2} = \sqrt{2}\sec\theta$

$$\int \sqrt{4x^2+2}\,dx = \int \sqrt{2}\sec\theta\,\sqrt{2}\sec^2\theta\,d\theta$$

Norma

Escaneado con CamScanner

$$= 2\int \sec^3\theta \, d\theta$$

$$= \frac{\cancel{2}\sec(\theta).\tan\theta + \ln|\sec\theta + \tan\theta|}{\cancel{2}} + C$$

$$= \frac{\sqrt{4x^2+2}}{\cancel{\sqrt{2}}} \cdot \frac{\cancel{2}x}{\cancel{\sqrt{2}}} + \ln\left|\frac{\sqrt{4x^2+2}}{\sqrt{2}} + \frac{2x}{\sqrt{2}}\right| + C$$

$$= x\sqrt{4x^2+2} + \ln\left|\frac{\sqrt{4x^2+2}+2x}{\sqrt{2}}\right| + C$$

Luego

$$A = -\frac{2}{\cancel{8}}\left(\frac{2u^{3/2}}{3}\right)\Big]_2^6 + x\sqrt{4x^2+2} + \ln\left|\frac{\sqrt{4x^2+2}+2x}{\sqrt{2}}\right|\Big]_0^1$$

$$A = -\frac{2}{12}\left[6^{3/2} - 2^{3/2}\right] + \sqrt{6} + \ln\left(\frac{\sqrt{6}+2}{\sqrt{2}}\right) - \ln(\sqrt{2})$$

$$A = -\frac{1}{6}\left(6\sqrt{6} - 2\sqrt{2}\right) + \sqrt{6} + \ln\left(\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}\sqrt{2}}\right)$$

$$A = -\cancel{\sqrt{6}} + \frac{\sqrt{2}}{3} + \cancel{\sqrt{6}} + \ln\left(\frac{\sqrt{3}+\sqrt{2}}{(\sqrt{2})^2}\right)$$

$$A = \frac{\sqrt{2}}{3} + \ln\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$$

Escaneado con CamScanner

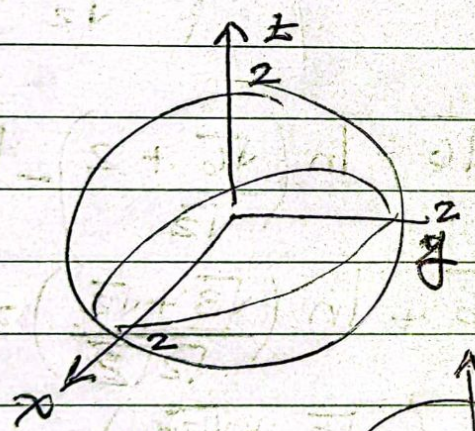
6) Use coordenadas esféricas o cilíndricas para calcular la siguiente integral.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

<u>Solución</u> La región de la integral es

$-2 \leq y \leq 2$, $0 \leq x \leq \sqrt{4-y^2}$

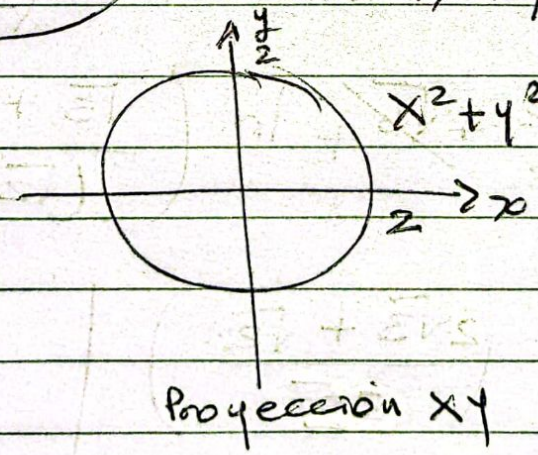
$-\sqrt{4-x^2-y^2} \leq z \leq \sqrt{4-x^2-y^2}$



Esfera de radio 2

$z = \sqrt{4-x^2-y^2}$

$z^2+x^2+y^2 = 4$



$x^2+y^2=4$

Proyección XY

En cilíndricas:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

$$=\int_{0}^{2\pi}\int_{0}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^2\operatorname{sen}^2\theta\sqrt{r^2+z^2}\,r\,dz\,dr\,d\theta$$

$$=\int_{0}^{2\pi}\int_{0}^{2}\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r^3\operatorname{sen}^2\theta\sqrt{r^2+z^2}\,dz\,dr\,d\theta$$

En Esféricas

$$\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{2} \rho^2\operatorname{sen}^2\phi\operatorname{sen}^2\theta\cdot\rho\cdot\rho^2\operatorname{sen}\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_{0}^{2\pi}\int_{0}^{\pi}\int_{0}^{2} \rho^4\operatorname{sen}^3\phi\operatorname{sen}^2\theta\,d\rho\,d\phi\,d\theta$$

$$=\int_{0}^{2\pi}\int_{0}^{\pi}\left.\frac{\rho^5}{5}\right]_0^2\operatorname{sen}^3\phi\operatorname{sen}^2\theta\,d\phi\,d\theta$$

$$=\frac{32}{5}\int_{0}^{2\pi}\operatorname{sen}^2\theta\,d\theta\cdot\int_{0}^{\pi}\operatorname{sen}^3\phi\,d\phi$$

$$=\frac{32}{5}\int_{0}^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta\cdot\int_{0}^{\pi}\operatorname{sen}^2\phi\cdot\operatorname{sen}\phi\,d\phi$$

Escaneado con CamScanner

$$= \frac{32}{8}\left(\theta - \frac{\operatorname{sen} 2\theta}{2}\right)\Bigg]_0^{2\pi}\left(\int_0^{\pi}\left(1-\cos^2\phi\right)\operatorname{sen}\phi\, d\phi\right)$$

$$= \frac{32}{8}\left(2\pi\right)\cdot\int_0^{\pi}\left(\operatorname{sen}\phi - \cos^2\phi\cdot\operatorname{sen}\phi\right)d\phi$$

$$= 8\pi\left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Bigg]_0^{\pi}$$

$$= 8\pi\left[\left(1-\frac{1}{3}\right)-\left(-1+\frac{1}{3}\right)\right]$$

$$= 8\pi\left(2-\frac{2}{3}\right) =$$

$$= \frac{32\pi}{3}$$

Escaneado con CamScanner