Definicja równania

$$\frac{d^2u}{dx^2}=4\pi G\rho(x)$$
$$u(0)=-5$$
$$u(3)=-4$$
$$I=[0,3)$$

$$\rho(x)=\begin{cases} 0 & \text{dla } x\in[0,1) \\ 10^{11} & \text{dla } x\in ¿ \\ 0 & \text{dla } x\in ¿ \end{cases}$$

Sformułowanie słabe

$$\int_0^3 u''v\,dx=\int_0^{34}\pi G\rho(x)v\,dx$$
$$u'v¿_0^3-\int_0^3 u'v'\,dx=\int_0^{34}\pi G\rho(x)v\,dx$$

$$u'(3)v(3)-u'(0)v(0)-\int_0^3 u'v'\,dx=\int_0^{34}\pi G\rho(x)v\,dx$$
$$u'(3)v(3)-u'(0)v(0)-\int_0^3 u'v'\,dx=4\pi G\left(\int_0^1\rho(x)v\,dx+\int_1^2\rho(x)v\,dx+\int_2^3\rho(x)v\,dx\right)$$
$$u'(3)v(3)-u'(0)v(0)-\int_0^3 u'v'\,dx=4\pi G\left(\int_0^{10}v\,dx+\int_1^2 10^{11}v\,dx+\int_2^{30}v\,dx\right)$$
$$u'(3)v(3)-u'(0)v(0)-\int_0^3 u'v'\,dx=4*10^{11}*\pi G\int_1^2 v\,dx$$
$$B(u,v)=L(v)$$

$$B(u,v)=u'(3)v(3)-u'(0)v(0)-\int_0^3 u'v'\,dx$$

$$L(v)=4*10^{11}*\pi G\int_1^2 v\,dx=k\int_1^2 v\,dx$$

$\stackrel{\sim}{u} = -5e_0 -4e_n $

$$B(w,v)=L(v)-B\left(\frac{x}{3}-5\right)$$

$$B(w,v) = -\int_0^3 u'v'\,dx$$

$$\tilde{L}(v) = k\int_1^2 v\,dx - 5B(e_0,v) - 4(e_n,v)$$

Parametry obliczeń

```
N = 20 # liczba elementów
left, right = 0.0, 3.0 # zakres
m = 1000 # liczba punktów użyta podczas wyliczania całek
```

Kod początkowy

```
import numpy as np
import matplotlib.pyplot as plt

k = 4 * np.pi * 6.67259
n = N # liczba elementów podprzestrzeni, po uwzględnieniu warunków brzegowych
w = (right - left) / (n+1) # szerokość elementu
```

Elementy skończone

$$e_i = \begin{cases} x - \frac{(p_i - p_{i-1}) * 1}{p_i - p_{i-1}} & \text{dla } x < p_i \\ x - \frac{p_i * -1}{p_i - p_{i-1}} & \text{dla } x > ¿\, p_i \end{cases}$$

```
def e(center, w, first=False, last=False):
    a = 1/w
    f = lambda x: (x - (center - w)) * a if x < center else (x - center) * (-a) + 1

    if first:
        return (lambda x: 0.0 if x < center or x > center + w else f(x))
    elif last:
        return (lambda x: 0.0 if x < center - w or x > center else f(x))
    else:
        return (lambda x: 0.0 if x < center - w or x > center + w else f(x))
```

Pochodne elementów skończonych

$$e'_i = \begin{cases} \frac{1}{p_i - p_{i-1}} & \text{dla } x < p_i \\ \frac{-1}{p_i - p_{i-1}} & \text{dla } x > ¿\, p_i \end{cases}$$

```
def de(center, w, first=False, last=False):
    a = 1/w
    f = lambda x: a if x < center else -a

    if first:
        return (lambda x: 0.0 if x < center or x > center+w else f(x))
    elif last:
        return (lambda x: 0.0 if x < center-w or x > center else f(x))
    else:
        return (lambda x: 0.0 if x < center-w or x > center+w else
f(x))
```

B(u, v)

```
def b(up, vp, m, left=0, right=3):
    xs = np.linspace(left, right, m)
    ys = np.array([up(x)*vp(x) for x in xs]).reshape(1, -1)[0]

    b = np.trapz(ys, x=xs)
    return -b
```

g(x)

$$\frac{d^2u}{dx^2}=4\pi G\rho(x)$$
$$u(0)=-5$$
$$u(3)=-4$$
$$I=[0,3)$$

$$\rho(x)=\begin{cases}4\pi G\rho(x)*10^{11} & \text{dla } x\in ¿\\ 0 & \text{dla } x\notin ¿\end{cases}$$

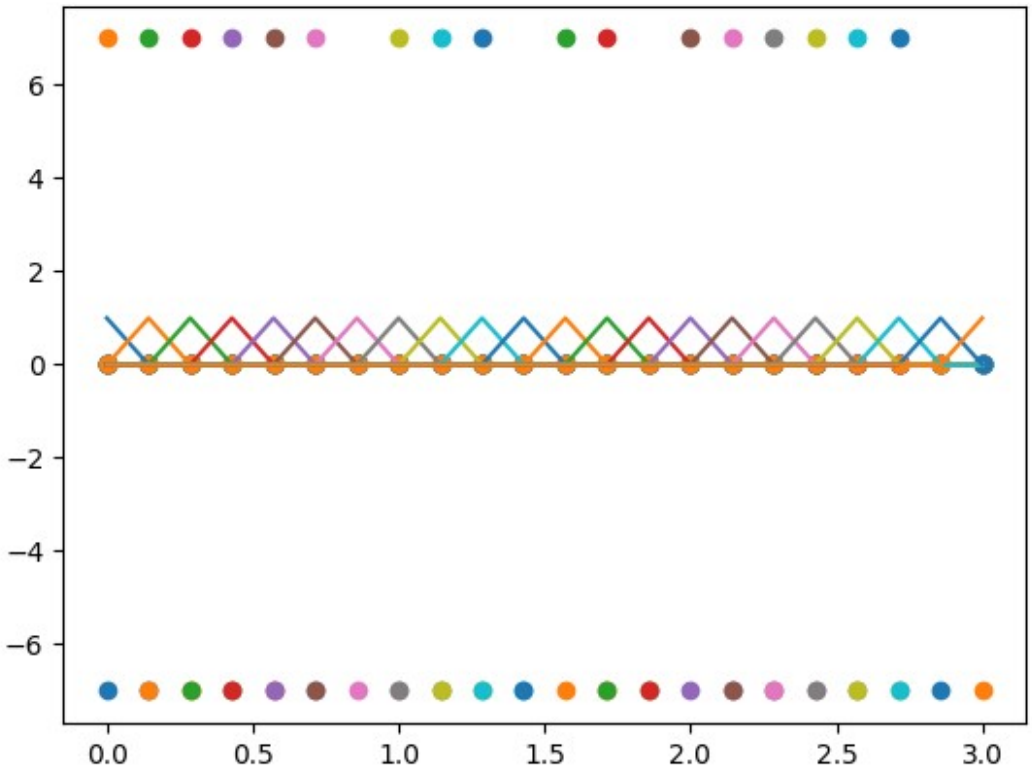
```
def g(x):
    if x > 1 and x <= 2:
        return k
    else:
        return 0
```

Matrix of $b_{i,j}=B(e_i,e_j)$

```
xs = np.linspace(0, 3, N + 2)
dV = np.array([de(xs[i], w, i==0, i==n-1) for i in range(0, N+2)])
V = np.array([e(xs[i], w, i==0, i==n-1) for i in range(0, N+2)])

for i in range(n+2):
    plt.plot(xs, [V[i](x) for x in xs ])
    plt.scatter(xs, [dV[i](x) for x in xs ])
plt.show()

B = [[b(dV[i], dV[j], m) for i in range(1, n+1)] for j in range(1,
n+1)]
```

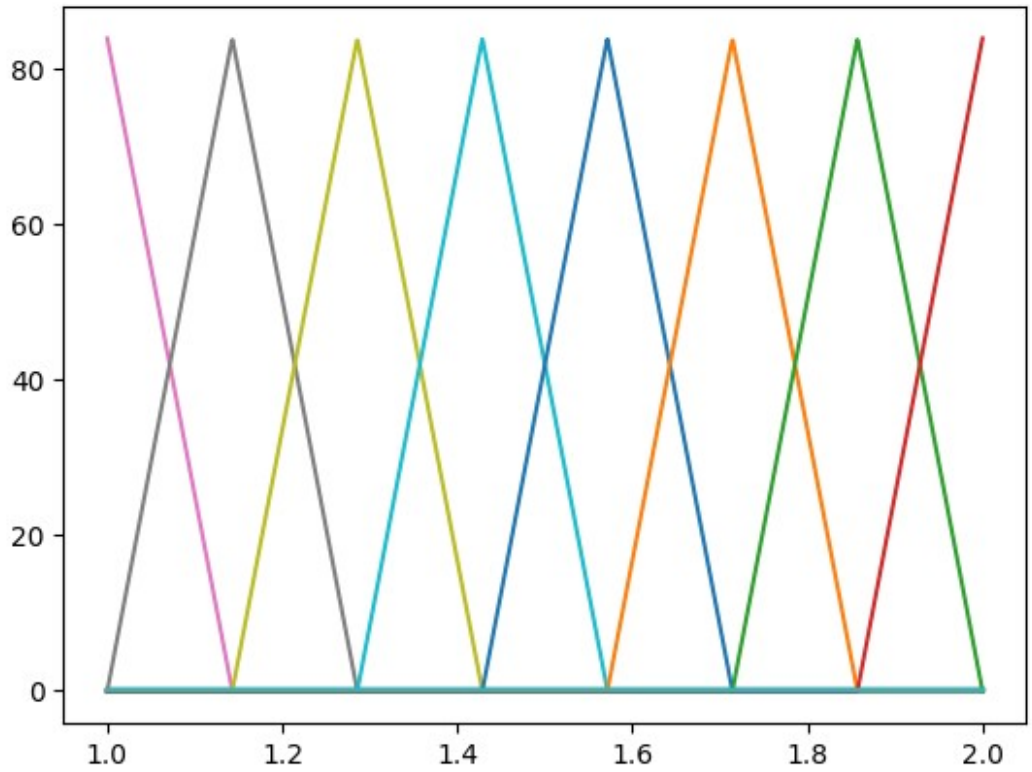


Vector of $l_i = L(e_i) + 5B(e_0, e_i) + 4B(e_n, e_i)$

$$\tilde{L}(v) = k\int_1^2 v\,dx + 5B(e_0, v) + 4B(e_n, v)$$

```
def l(v, m):
    xs = np.linspace(1,2,m)
    ys = np.array([v(x)*k for x in xs]).reshape(1, -1)[0]
    i = np.trapz(ys, x=xs)
    plt.plot(xs, ys)
    return i

L = np.array([l(V[i], m) + 5*b(dV[0], dV[i], m) + 4*b(dV[n+1], dV[i],
m) for i in range(1,n+1)])
plt.show()
```



Kalkulacja wektora u_i

```
Bm = np.matrix(B)
Bmi = Bm.getI()
Lm = np.matrix([L]).getT()
U = np.array(Bm * Lm).reshape(1, -1)[0]

print(Bm)
print(Bmi)
print(Lm)
print(U)
```

```
[[-14.05255255   7.06306306  -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.        ]
 [  7.06306306 -13.97897898   6.91591592  -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.        ]
 [ -0.           6.91591592 -13.97897898   7.06306306  -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.        ]
 [ -0.          -0.           7.06306306 -13.97897898   6.91591592
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
```

```
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.           6.91591592 -13.97897898
   7.06306306  -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.           7.06306306
 -14.12612613   6.76876877   0.14714715  -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
   6.76876877 -13.97897898   7.06306306  -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
   0.14714715   7.06306306 -14.12612613   7.06306306  -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.           7.06306306 -13.97897898   6.91591592
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.           6.91591592 -13.97897898
   7.06306306  -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.           7.06306306
 -13.97897898   6.91591592  -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.
   6.91591592 -13.97897898   7.06306306  -0.          -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.           7.06306306 -13.97897898   6.91591592  -0.
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.           6.91591592 -13.97897898   7.06306306
  -0.          -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.           7.06306306 -14.12612613
   7.06306306  -0.          -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.          -0.
  -0.          -0.          -0.          -0.           7.06306306
 -13.97897898   6.91591592  -0.          -0.          -0.        ]
[ -0.          -0.          -0.          -0.          -0.
```

```
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
    6.91591592 -13.97897898   7.06306306  -0.          -0.        ]
 [ -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.           7.06306306 -13.97897898   6.91591592  -0.        ]
 [ -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.           6.91591592  -6.91591592  -0.        ]
 [ -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.          -0.
   -0.          -0.          -0.          -0.         -13.97897898]]
[[-0.14044682 -0.13784903 -0.13519596 -0.13259816 -0.12994509 -
0.1273473
  -0.1273473  -0.13000037 -0.13000037 -0.13000037 -0.13000037 -
0.13000037
  -0.13000037 -0.13000037 -0.13000037 -0.13000037 -0.13000037 -
0.13000037
  -0.13000037 -0.        ]
 [-0.13784903 -0.27426213 -0.26898363 -0.26381509 -0.25853659 -
0.25336806
  -0.25336806 -0.25864656 -0.25864656 -0.25864656 -0.25864656 -
0.25864656
  -0.25864656 -0.25864656 -0.25864656 -0.25864656 -0.25864656 -
0.25864656
  -0.25864656 -0.        ]
 [-0.13519596 -0.26898363 -0.40561784 -0.39782388 -0.38986408 -
0.38207012
  -0.38207012 -0.39002991 -0.39002991 -0.39002991 -0.39002991 -
0.39002991
  -0.39002991 -0.39002991 -0.39002991 -0.39002991 -0.39002991 -
0.39002991
  -0.39002991 -0.        ]
 [-0.13259816 -0.26381509 -0.39782388 -0.52904081 -0.51845558 -
0.50809088
  -0.50809088 -0.51867611 -0.51867611 -0.51867611 -0.51867611 -
0.51867611
  -0.51867611 -0.51867611 -0.51867611 -0.51867611 -0.51867611 -
0.51867611
  -0.51867611 -0.        ]
 [-0.12994509 -0.25853659 -0.38986408 -0.51845558 -0.64978307 -
0.63679293
  -0.63679293 -0.65005945 -0.65005945 -0.65005945 -0.65005945 -
0.65005945
  -0.65005945 -0.65005945 -0.65005945 -0.65005945 -0.65005945 -
0.65005945
  -0.65005945 -0.        ]
```

```
 [-0.1273473  -0.25336806 -0.38207012 -0.50809088 -0.63679293 -
0.7628137
  -0.7628137  -0.77870565 -0.77870565 -0.77870565 -0.77870565 -
0.77870565
  -0.77870565 -0.77870565 -0.77870565 -0.77870565 -0.77870565 -
0.77870565
  -0.77870565 -0.        ]
 [-0.1273473  -0.25336806 -0.38207012 -0.50809088 -0.63679293 -
0.7628137
  -0.90740771 -0.92329966 -0.92329966 -0.92329966 -0.92329966 -
0.92329966
  -0.92329966 -0.92329966 -0.92329966 -0.92329966 -0.92329966 -
0.92329966
  -0.92329966 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.08110432 -1.08110432 -1.08110432 -
1.08110432
  -1.08110432 -1.08110432 -1.08110432 -1.08110432 -1.08110432 -
1.08110432
  -1.08110432 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.22268596 -1.22268596 -
1.22268596
  -1.22268596 -1.22268596 -1.22268596 -1.22268596 -1.22268596 -
1.22268596
  -1.22268596 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.36727997 -
1.36727997
  -1.36727997 -1.36727997 -1.36727997 -1.36727997 -1.36727997 -
1.36727997
  -1.36727997 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.5088616
  -1.5088616  -1.5088616  -1.5088616  -1.5088616  -1.5088616  -
1.5088616
  -1.5088616  -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.65345561 -1.65345561 -1.65345561 -1.65345561 -1.65345561 -
1.65345561
  -1.65345561 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
```

```
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.79503724 -1.79503724 -1.79503724 -1.79503724 -
1.79503724
  -1.79503724 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.93963125 -1.93963125 -1.93963125 -1.93963125 -
1.93963125
  -1.93963125 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.93963125 -2.08121288 -2.08121288 -2.08121288 -
2.08121288
  -2.08121288 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.93963125 -2.08121288 -2.22279451 -2.22279451 -
2.22279451
  -2.22279451 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.93963125 -2.08121288 -2.22279451 -2.36738852 -
2.36738852
  -2.36738852 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.93963125 -2.08121288 -2.22279451 -2.36738852 -
2.50897015
  -2.50897015 -0.        ]
 [-0.13000037 -0.25864656 -0.39002991 -0.51867611 -0.65005945 -
0.77870565
  -0.92329966 -1.08110432 -1.22268596 -1.36727997 -1.5088616  -
1.65345561
  -1.79503724 -1.93963125 -2.08121288 -2.22279451 -2.36738852 -
2.50897015
  -2.65356416 -0.        ]
 [-0.         -0.         -0.         -0.         -0.         -0.
  -0.         -0.         -0.         -0.         -0.         -0.
```

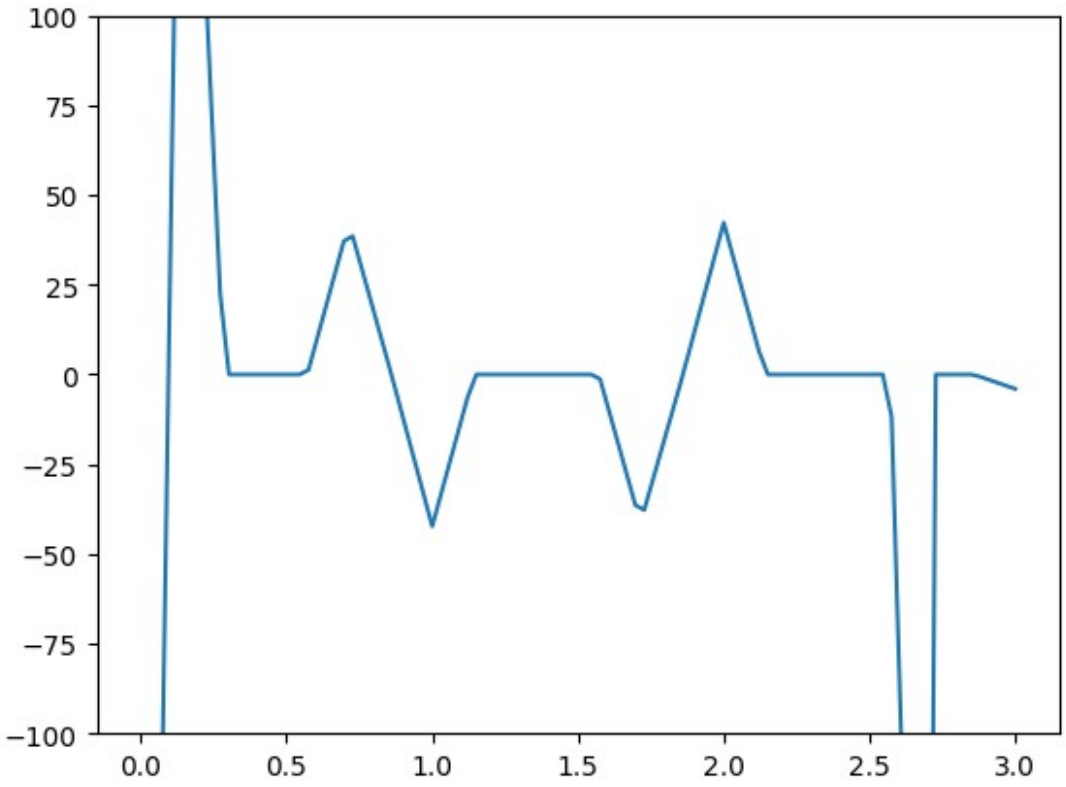
```
  -0.         -0.         -0.         -0.         -0.         -0.
  -0.         -0.07153598]]
[[ 3.49474474e+01]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [-1.86371303e-17]
 [ 5.98936279e+00]
 [ 1.19785575e+01]
 [ 1.19785575e+01]
 [ 1.19786416e+01]
 [ 1.19786416e+01]
 [ 1.19785575e+01]
 [ 1.19785575e+01]
 [ 5.98936279e+00]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [ 0.00000000e+00]
 [ 2.76636637e+01]]
[-4.91100842e+02  2.46836025e+02  0.00000000e+00  0.00000000e+00
 -1.31635227e-16  4.23032224e+01  8.80130796e-01 -4.23020603e+01
  5.81062746e-04 -5.81062746e-04 -5.81062746e-04  5.81062746e-04
 -4.14207673e+01 -8.82479773e-01  4.23032471e+01  0.00000000e+00
  0.00000000e+00  0.00000000e+00  0.00000000e+00 -3.86709773e+02]
```

Prezentacja funkcji $u(x)$

```
u = lambda x: sum([U[i] * V[i](x) for i in range(n)]) - 5 * V[0](x) -
4 * V[n+1](x)

xs = np.linspace(0, 3, 100)
ys = np.array([u(x) for x in xs]).reshape(1, -1)[0]

# plt.scatter
plt.plot(xs, ys)
ys = plt.gca()
ys.set_ylim([-100, 100])
plt.savefig('plot_n'+ str(N) +'.png')
plt.show()
```


100
75
50
25
0
−25
−50
−75
−100
0.0
0.5
1.0
1.5
2.0
2.5
3.0