QF5210A Financial Time Series

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1st Lecture, 11 May 2019

Textbooks

1. Analysis of Financial Time Series, 3rd Edition, Ruey S. Tsay, Wiley, 2010. Chapters 1-4, 7, 8, 10. Software : S-Plus
(E-book available on the website of NUS Libraries here)

2. Statistics and Data Analysis for Financial Engineering, David Ruppert, Springer, 2010. Chapter 2, 4, 5, 7, 9, 10, 12, 14, 15, 18, 19. Software : R
(E-book available on the website of NUS Libraries)

3. Introductory Econometrics for Finance, 2nd Edition, Chris Brooks, Cambridge University Press, 2008. Chapters 2-8, 11. Software : EViews
(E-book available on the website of NUS Libraries)

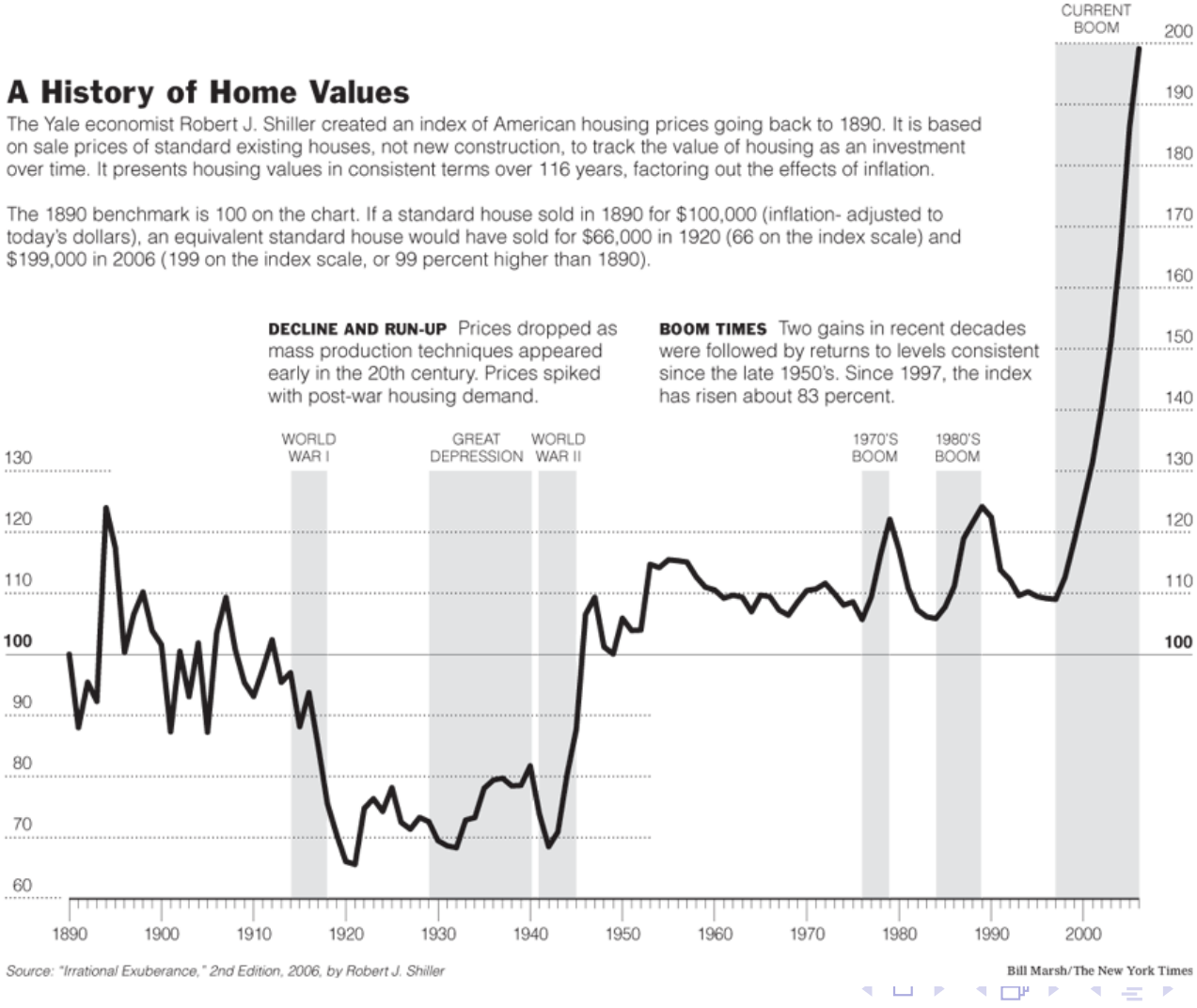
Outline

1 Financial Data and R Program

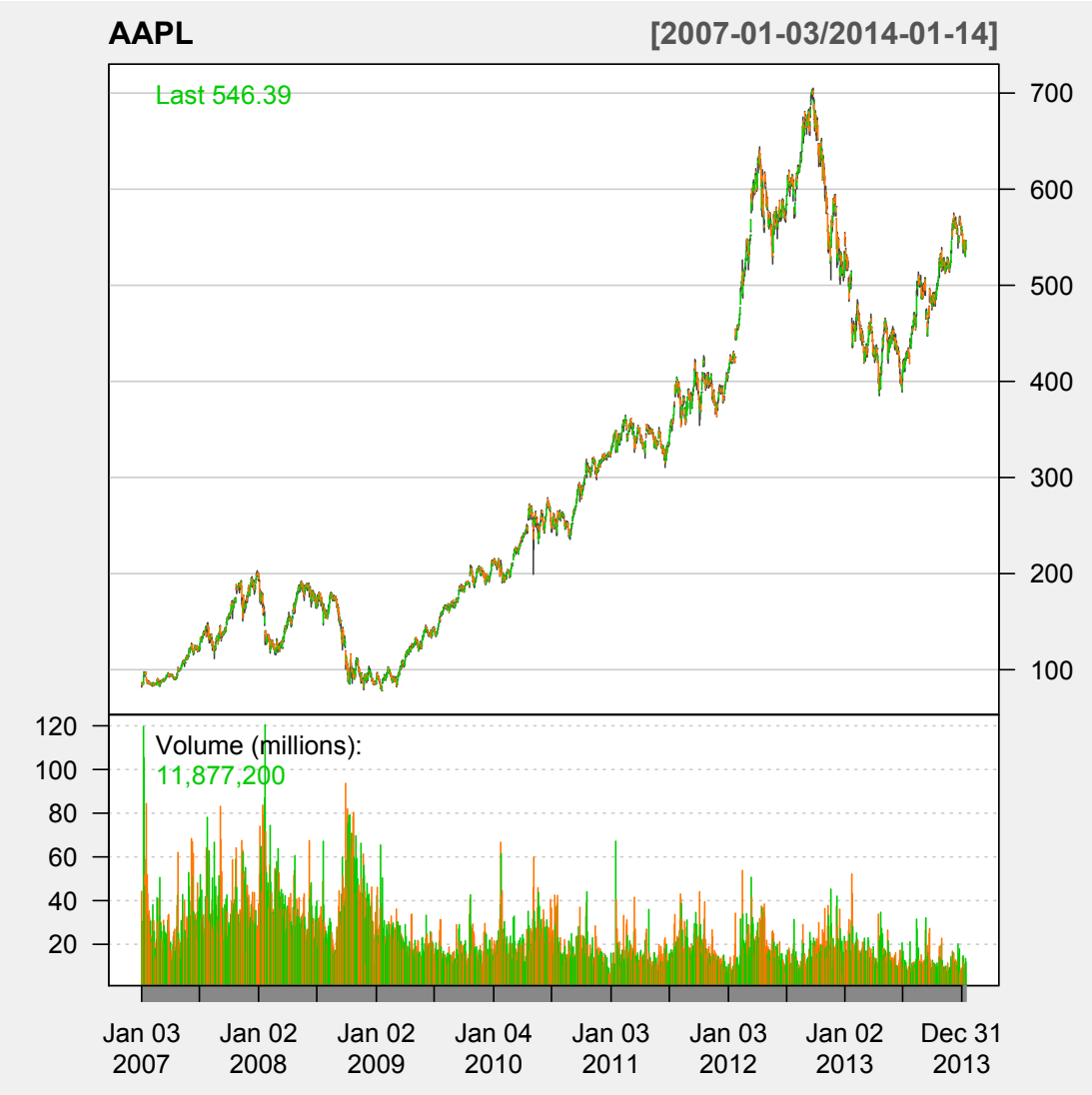
2 Review of probability and statistics

Financial data : US House Prices

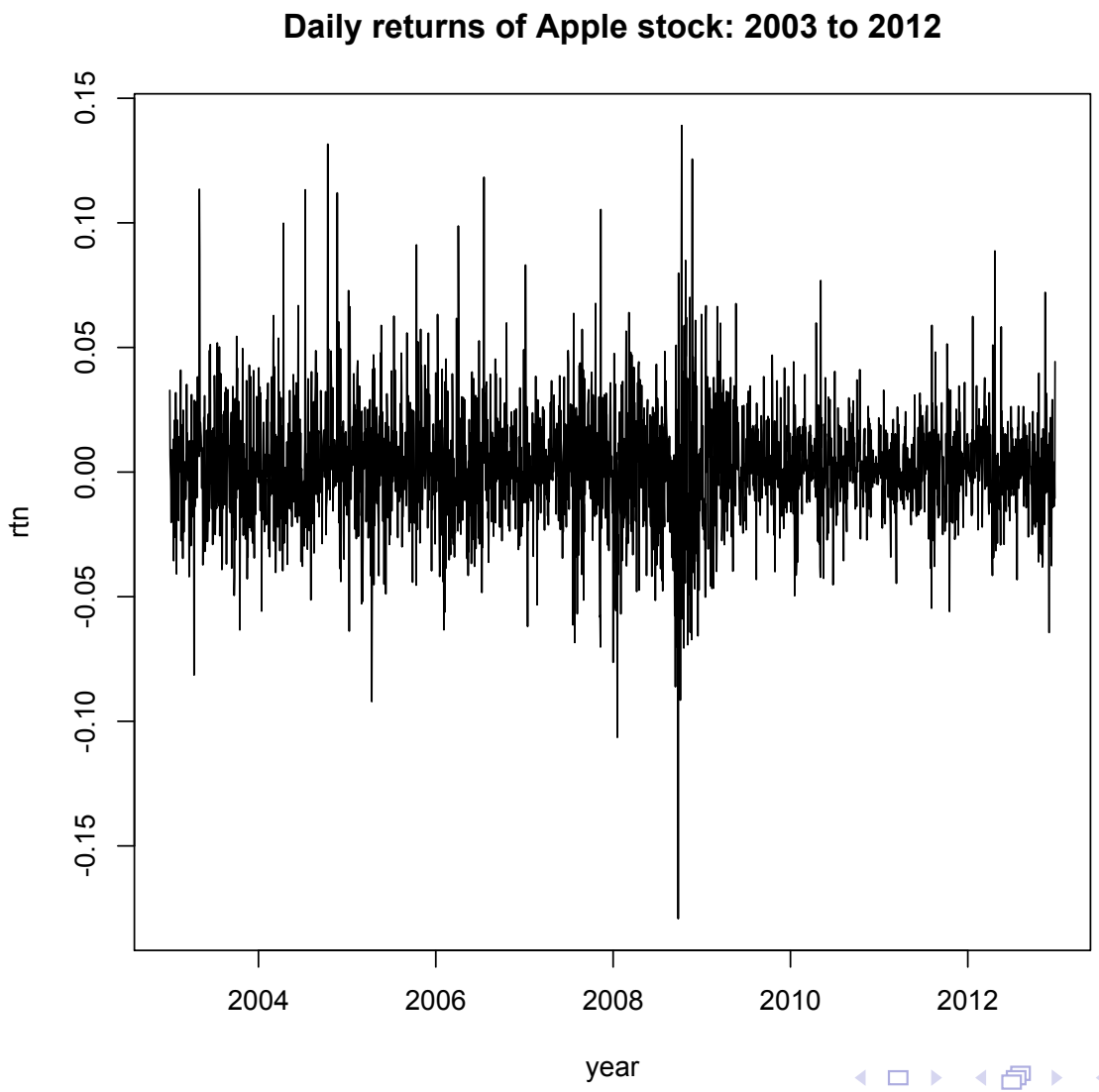
US House Prices Roller Coaster Ride



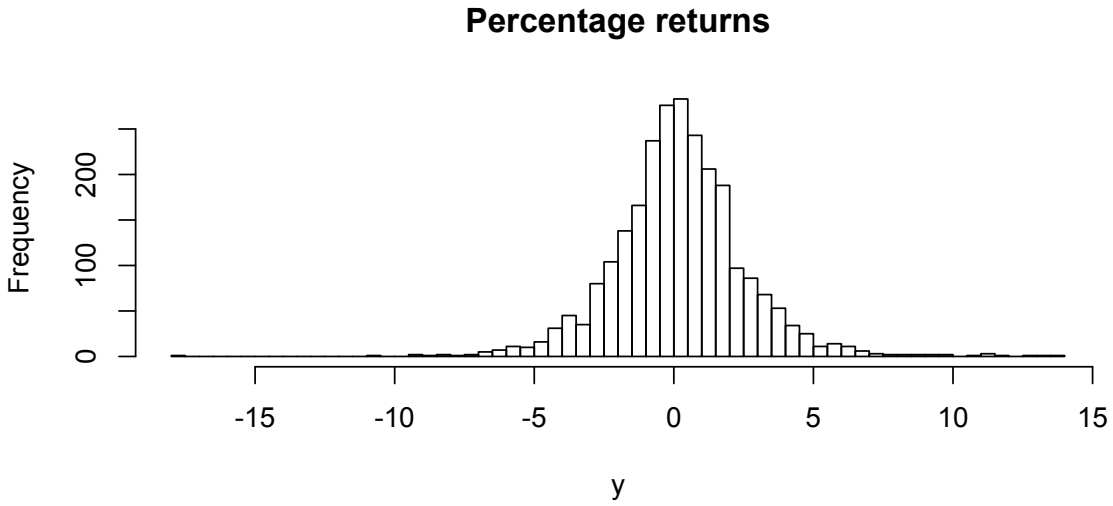
Financial data : Daily prices of Apple stock

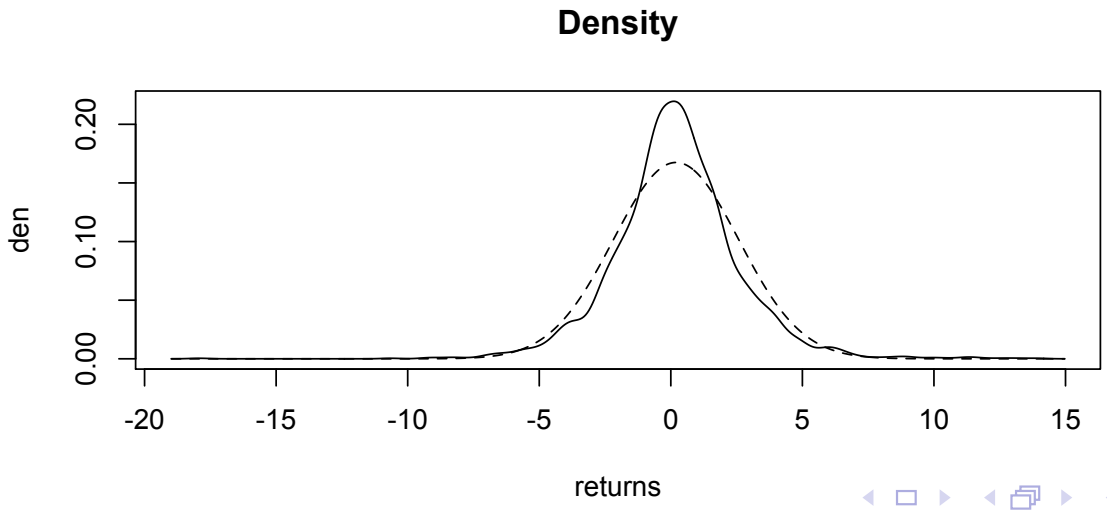


Financial data : Daily returns of Apple stock

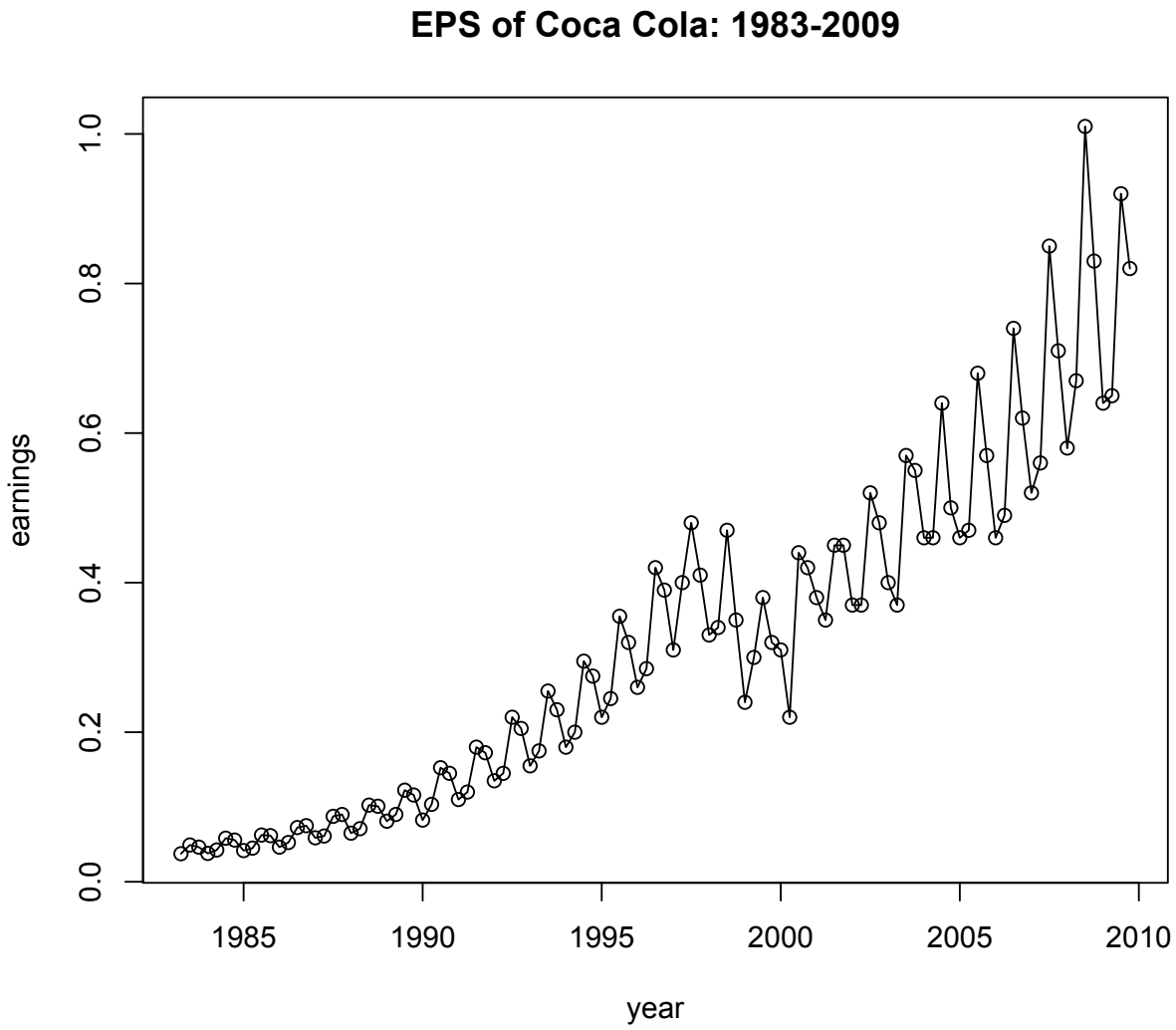


Financial data : Returns density

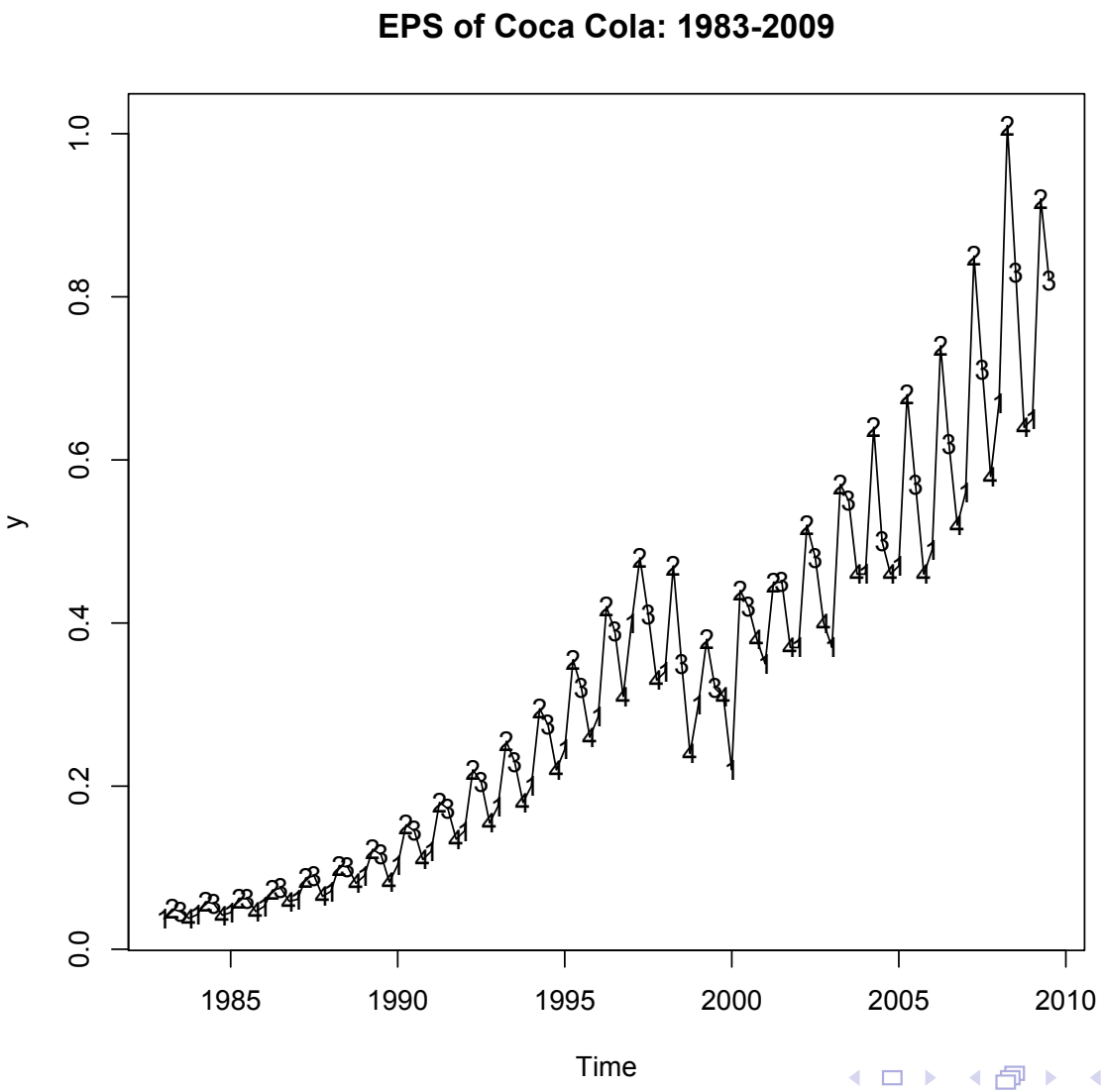




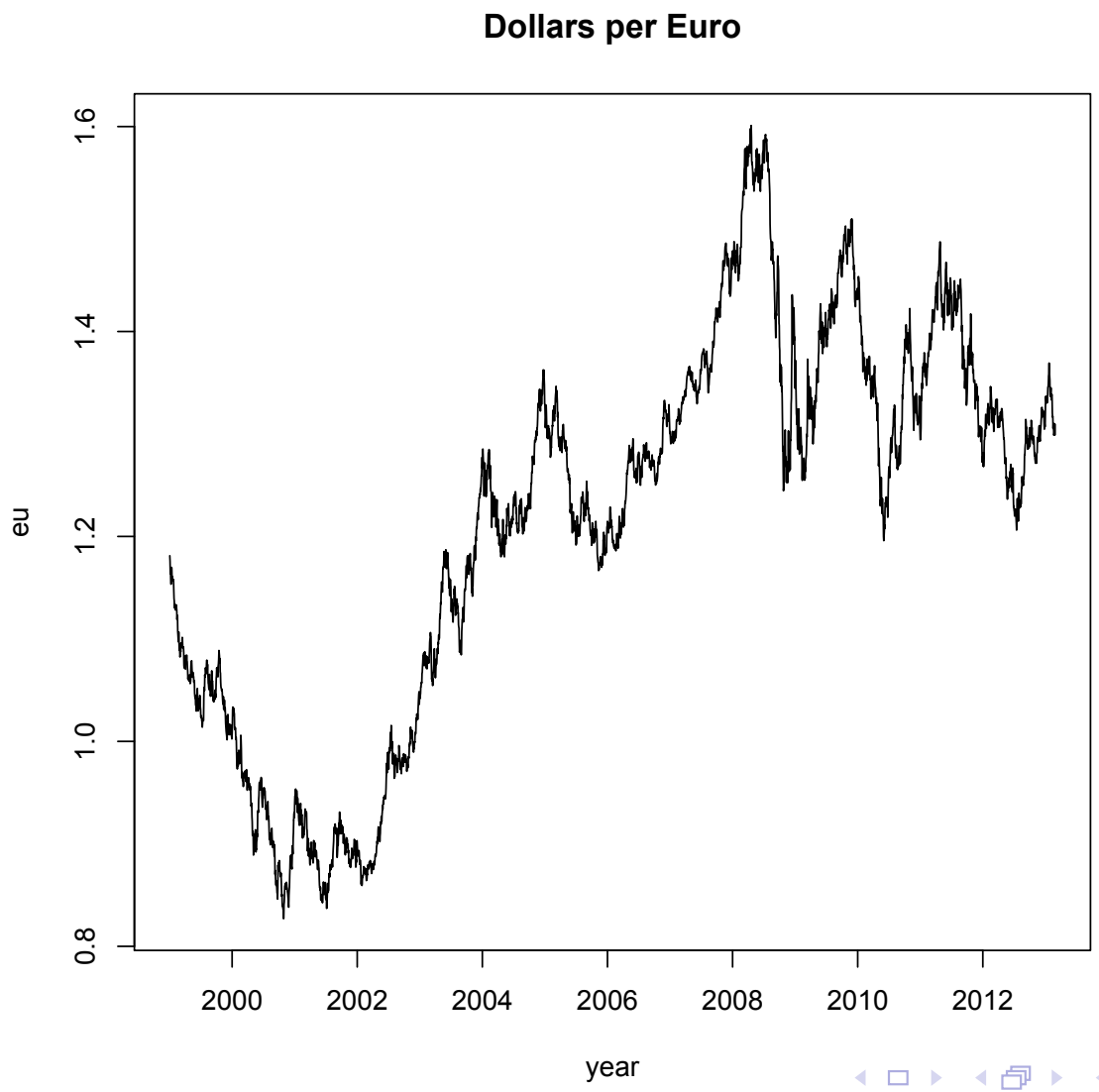
Financial data : Earnings per share of Coca Cola



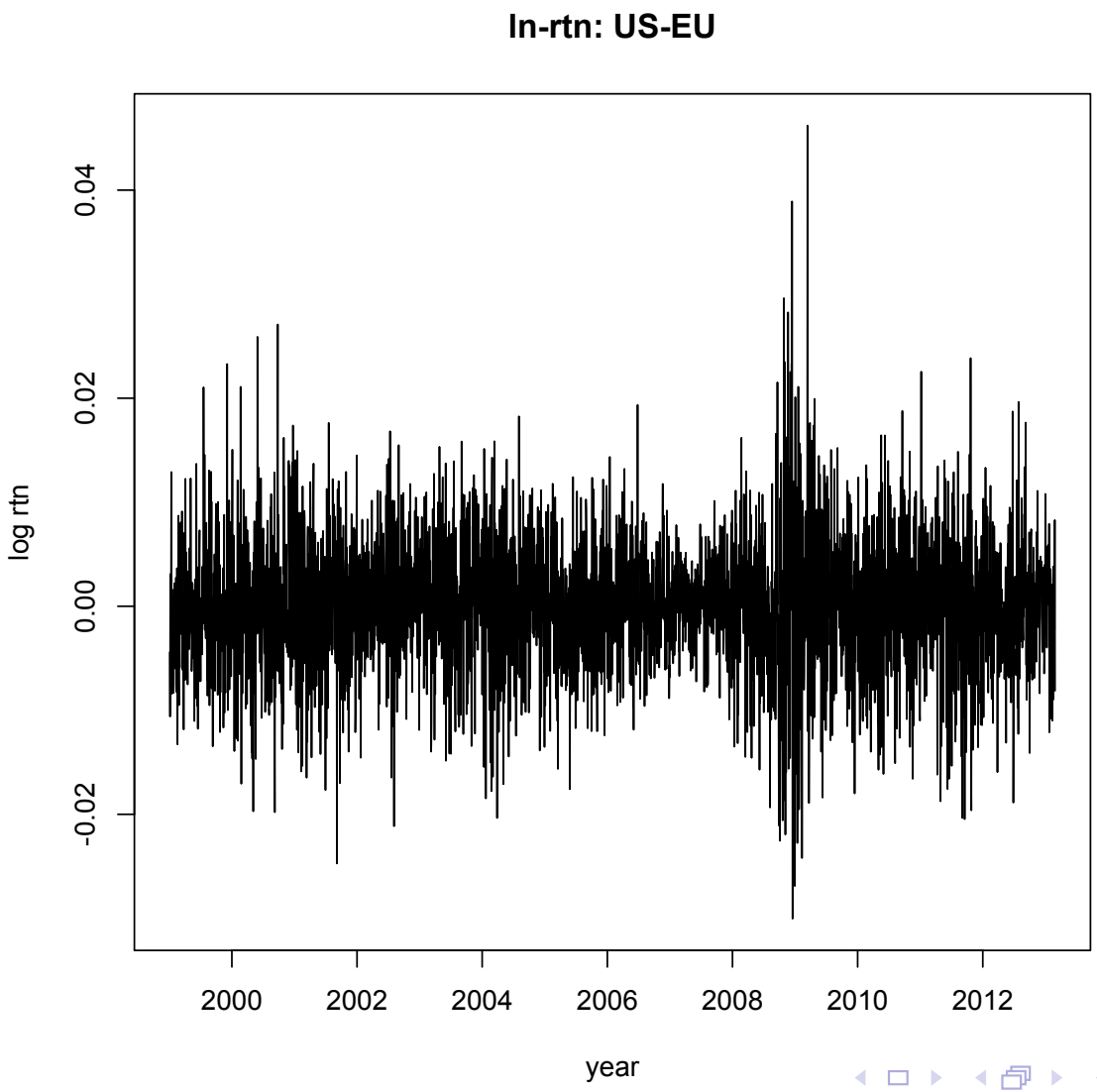
Financial data : EPS of Coca Cola - Quarterly



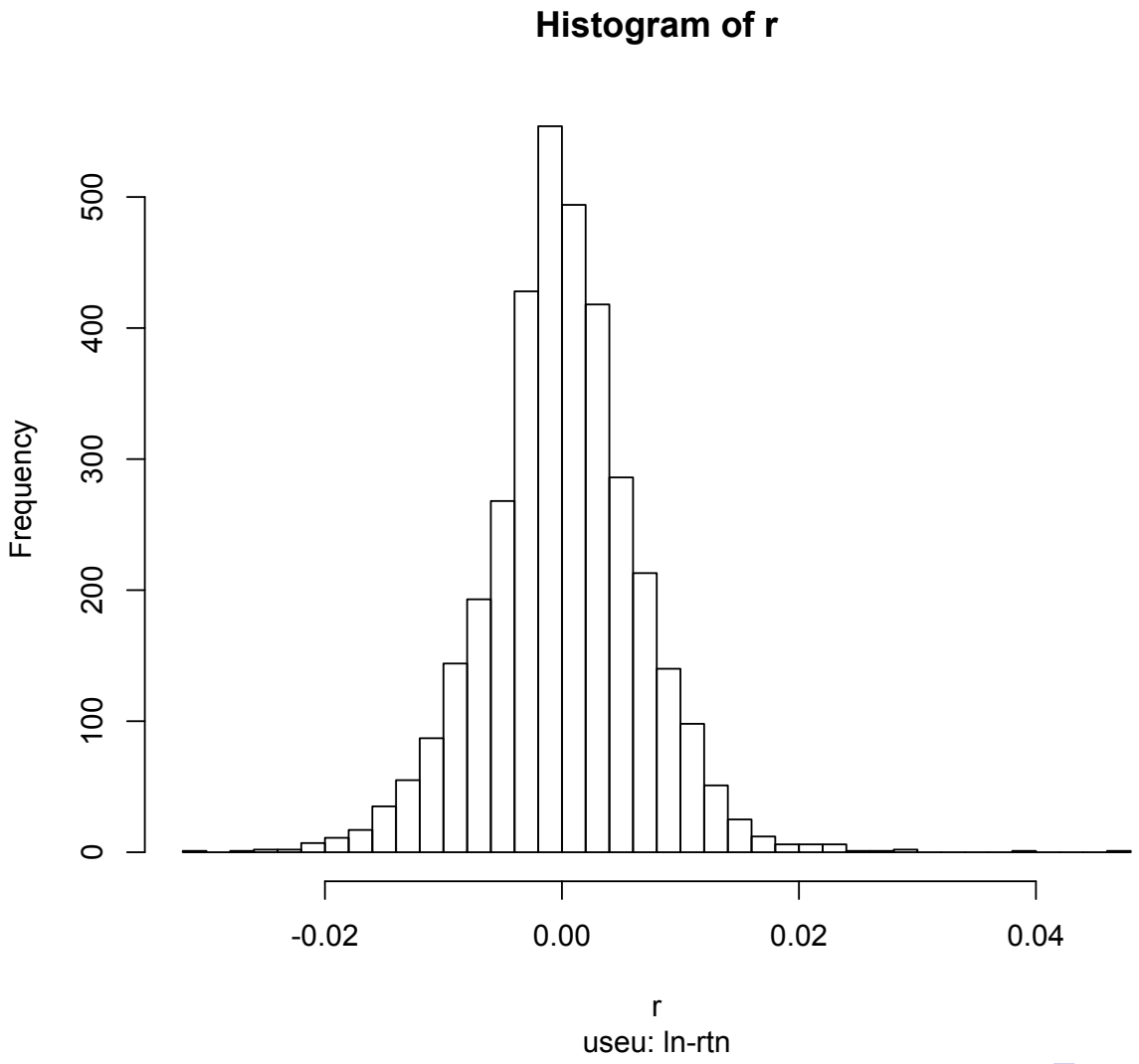
Financial data : US-EU exchange rates - prices



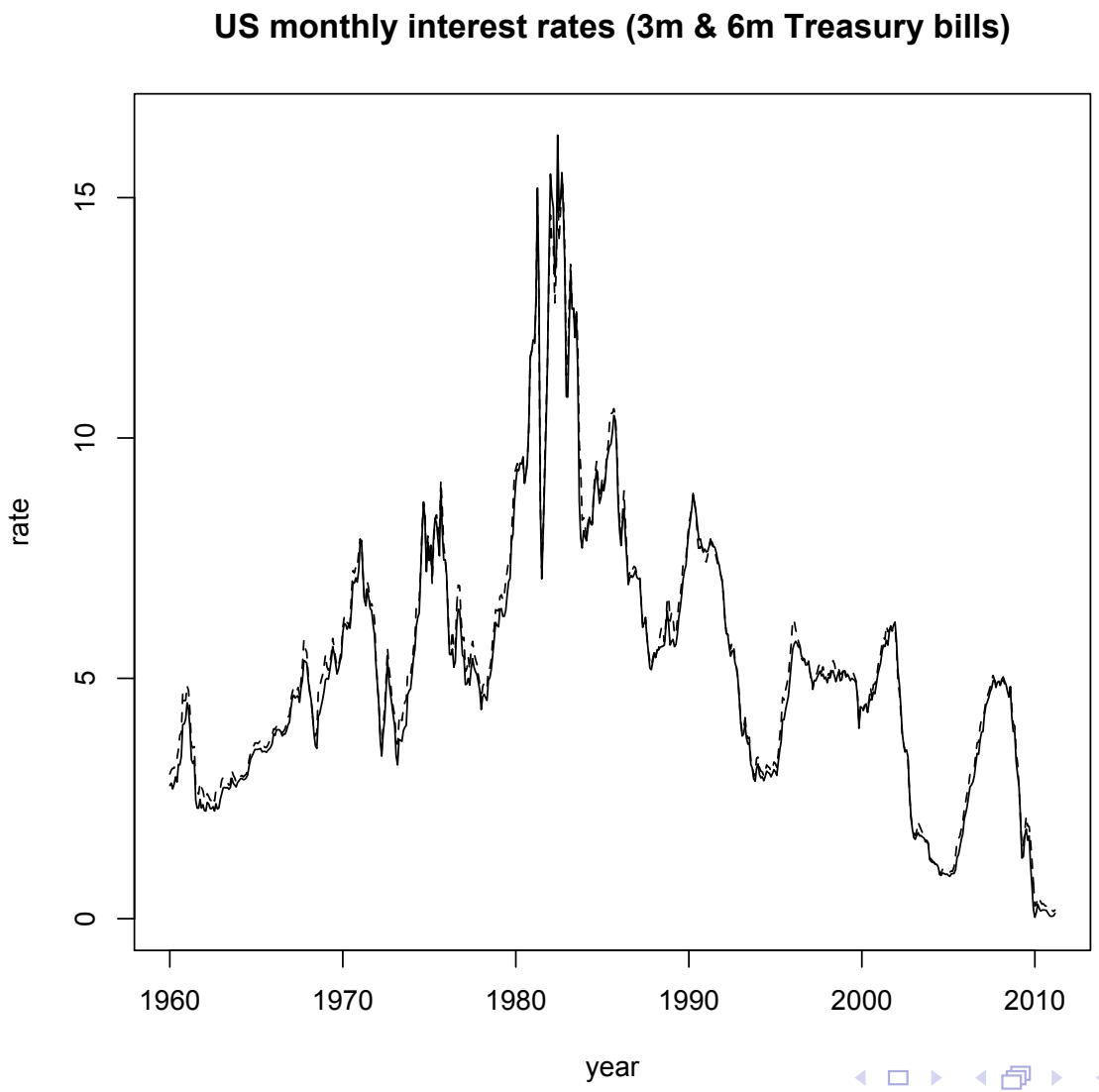
Financial data : US-EU exchange rates- returns



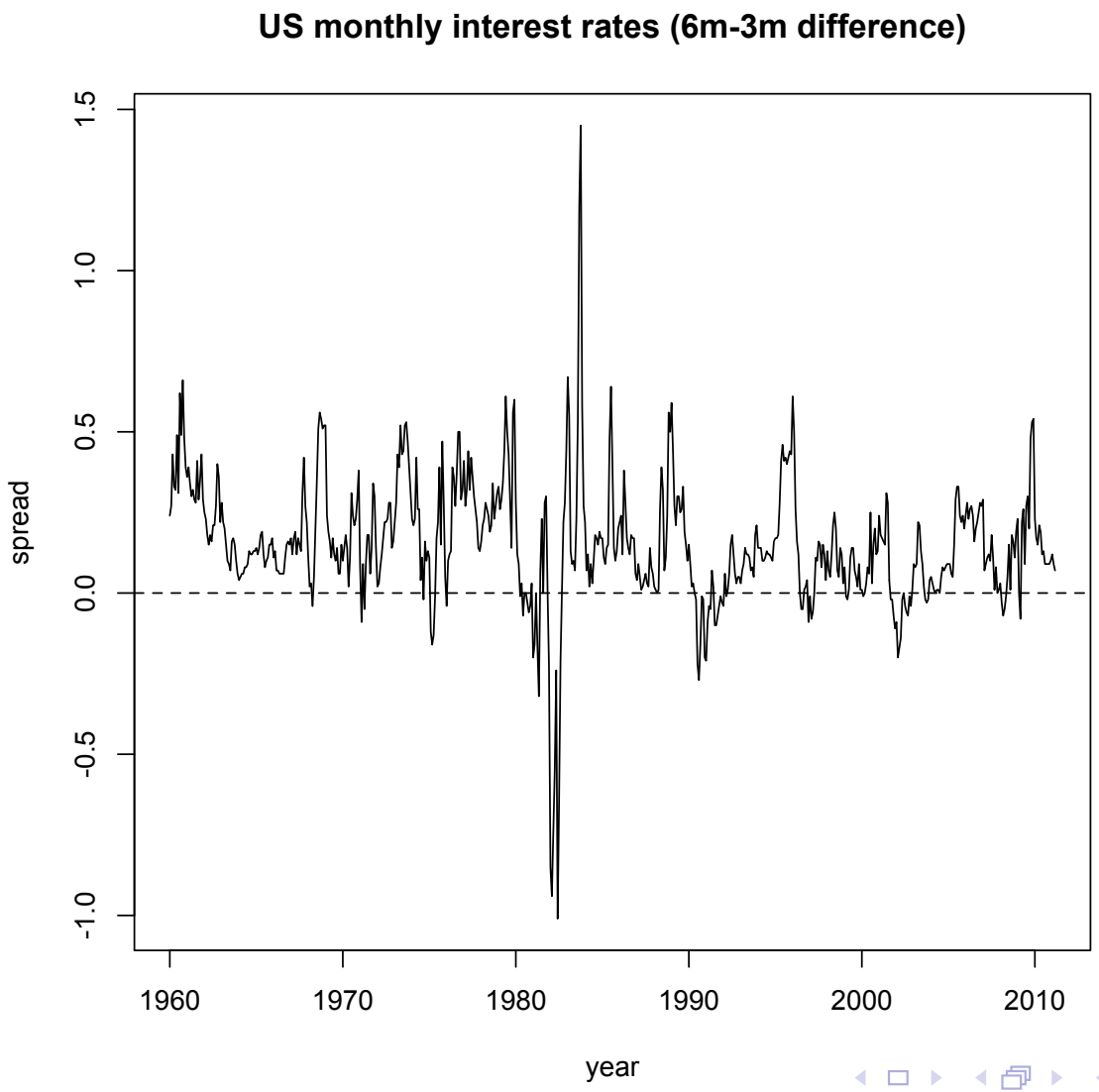
Financial data : Histogram of log returns



Financial data : US monthly interest rates

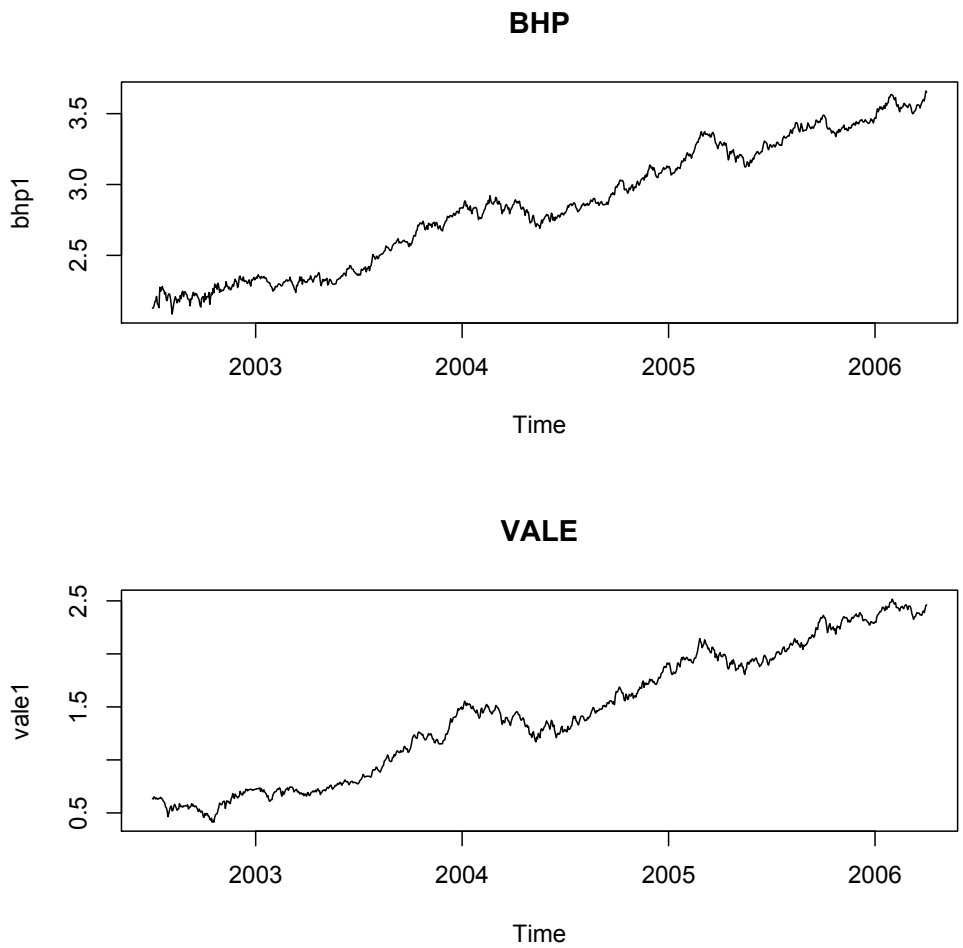


Financial data : US monthly IR - difference

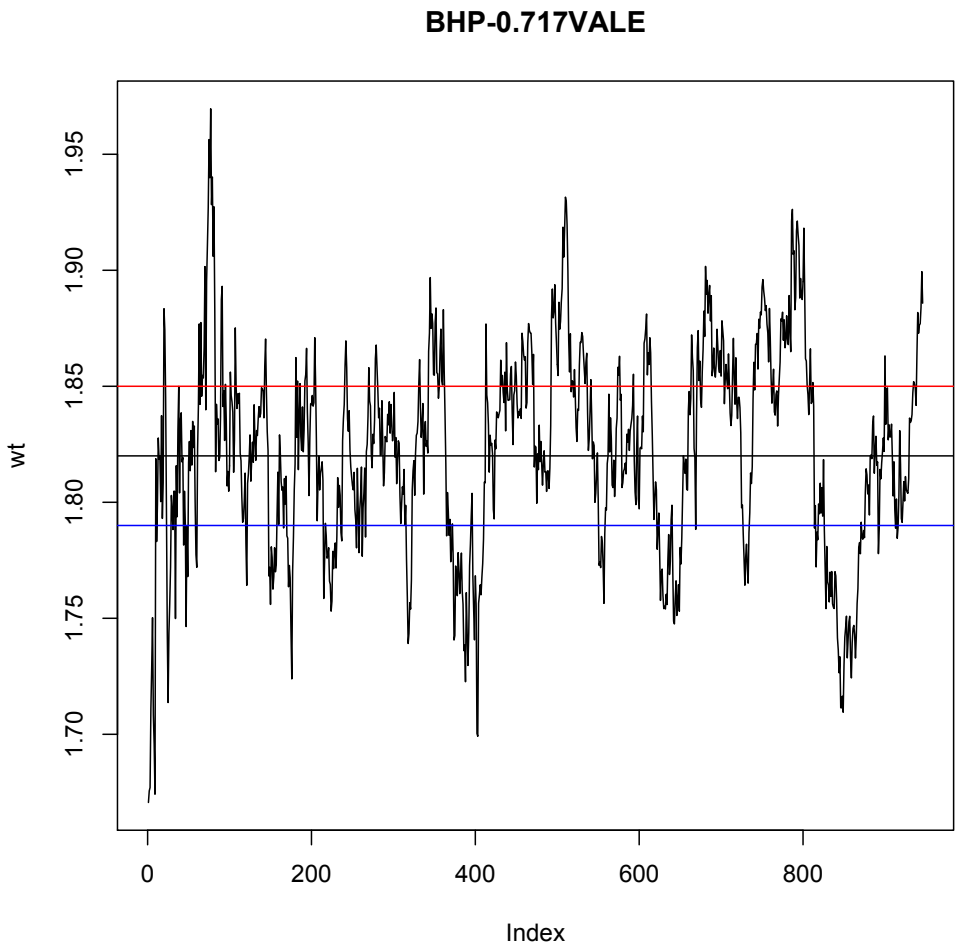


Financial data : BHP and VALE stocks

Log prices (daily) (01/07/2002 - 31/03/2006)

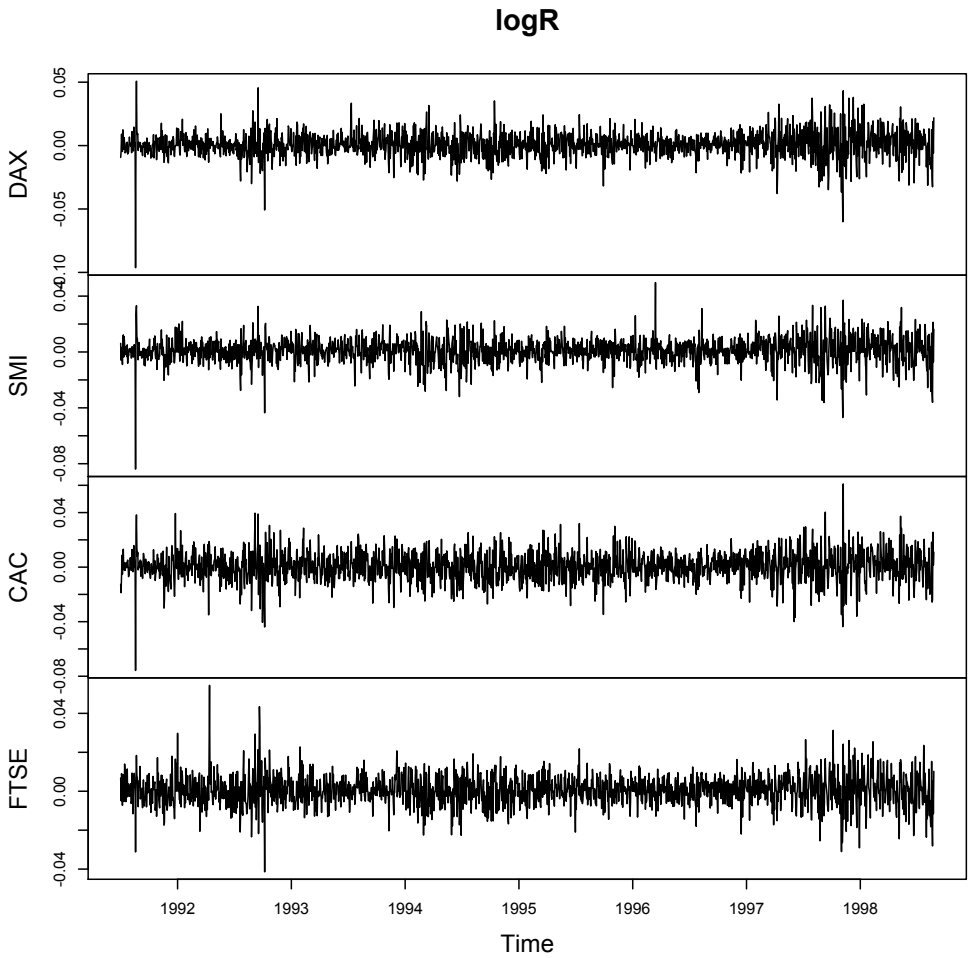


Pairs trading of BHP and VALE : Linear regression



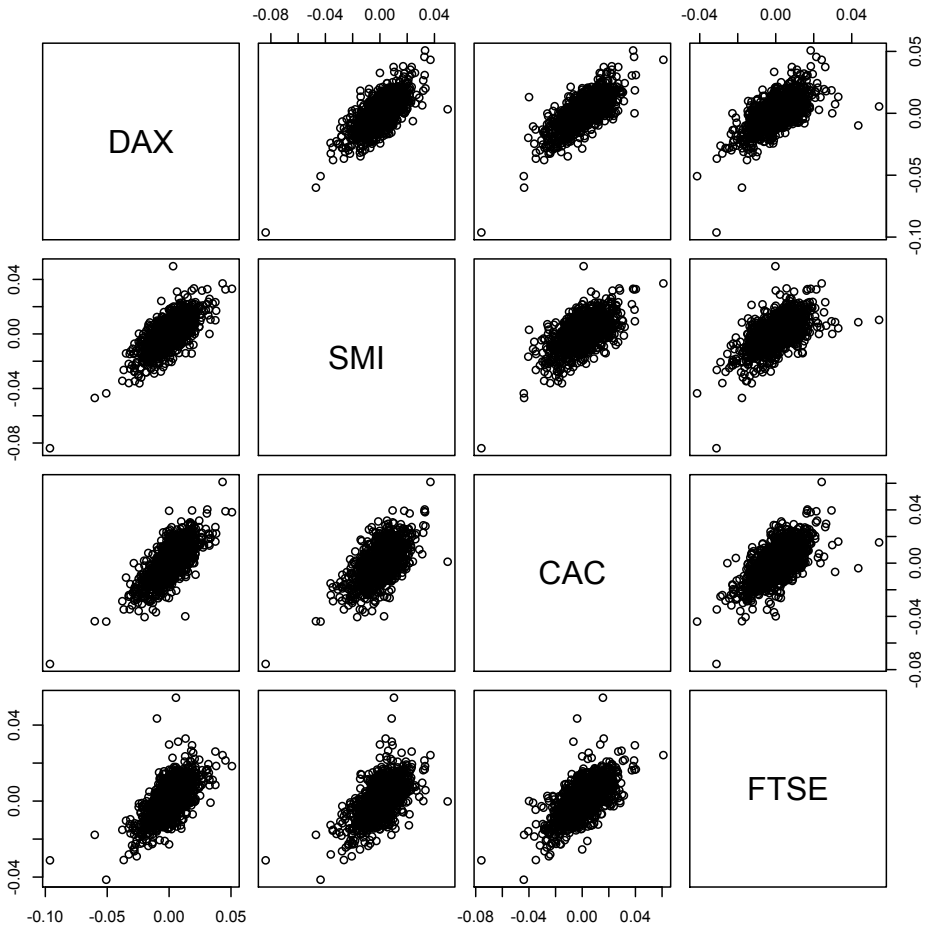
Financial data : EU stock markets

Log returns (daily)

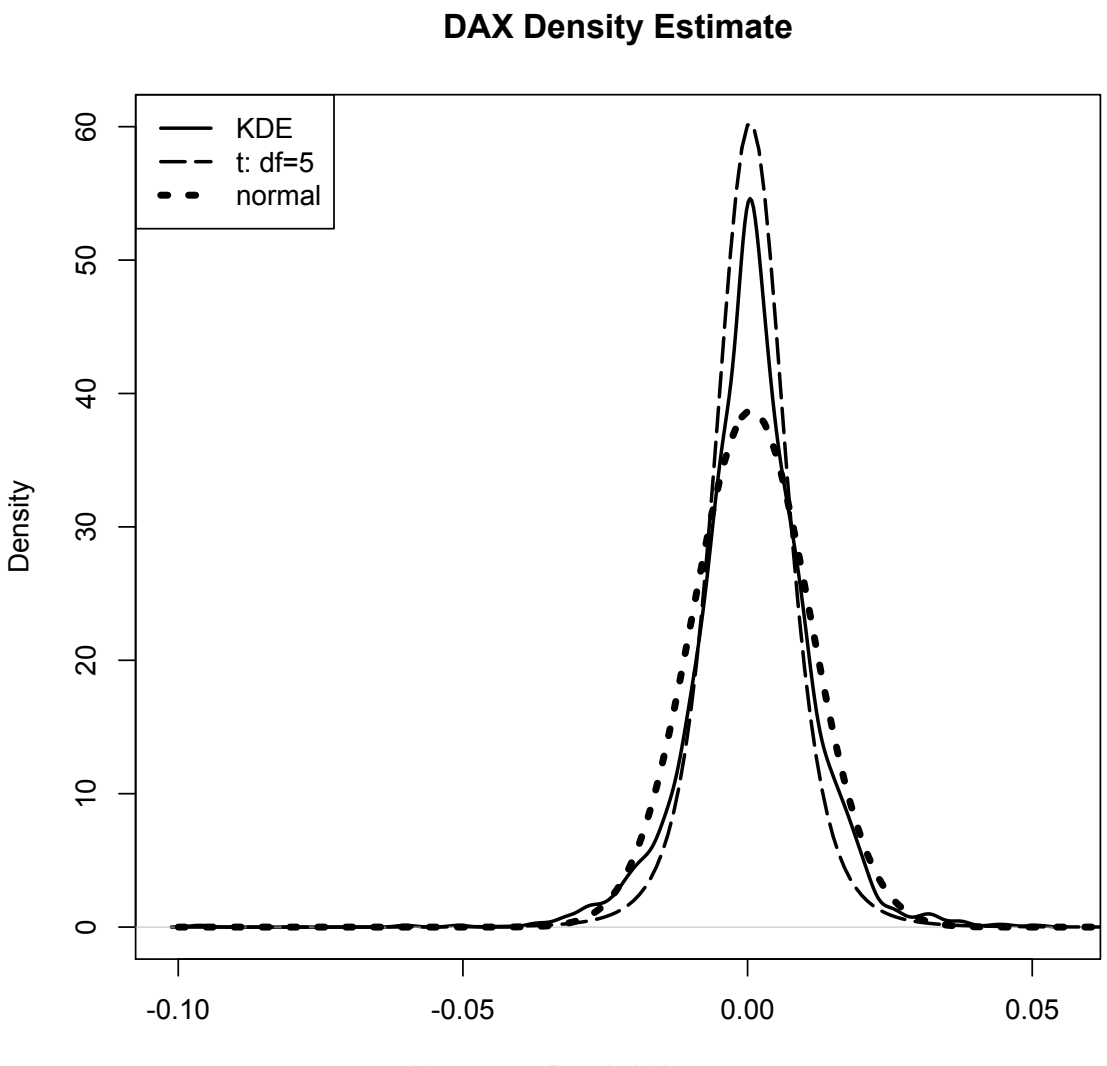


Financial data : EU stock markets

Log returns (daily) : Scatterplot matrix



Financial data : DAX Log returns (daily)



Asset returns : Simple return

Let P_t be the price of an asset at time t, and assume no dividend.
One-period simple return :

$$R_t = \frac{P_t}{P_{t-1}} - 1 = \frac{P_t - P_{t-1}}{P_{t-1}}$$

Multiperiod simple return :

$$1 + R_t(k) = \frac{P_t}{P_{t-k}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t-1}}{P_{t-2}} \times \cdots \times \frac{P_{t-k+1}}{P_{t-k}}$$

$$= (1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})$$

$$R_t(k) = \frac{P_t}{P_{t-k}} - 1$$

Asset returns : Log return

Continuously compounded (or log) return :

$$r_t = ln(1 + R_t) = ln\frac{P_t}{P_{t-1}} = p_t - p_{t-1}, \text{ where } p_t = lnP_t$$

Multiperiod log return :

$$\begin{aligned} r_t(k) &= ln[1 + R_t(k)] \\ &= ln[(1 + R_t)(1 + R_{t-1}) \cdots (1 + R_{t-k+1})] \\ &= ln(1 + R_t) + ln(1 + R_{t-1}) + \cdots + ln(1 + R_{t-k+1}) \\ &= r_t + r_{t-1} + \cdots + r_{t-k+1}. \end{aligned}$$

Relationship :

$$r_t = ln(1 + R_t), \quad R_t = e^{r_t} - 1.$$

R program 1

- R is a free software available from http ://www.r-project.org (click here)
- For financial time series analysis, the **Rmetrics** of Dr. Diethelm Wuertz and his associates has many useful packages, including **fBasics**, **fGarch** and **fPortfolio**.
- To install packages, one can click on the command "Packages" to select "Install packages".
- With packages installed, one can load them into R by clicking on the command "Packages" followed by clicking "Load packages".

R program 2

- An alternative approach to load a package is to use the command *library* or *require*.
- The **quantmod** Package : a useful R package for downloading financial data directly from some open sources, Yahoo Finance, Google Finance, and the Federal Reserve Economic Data (FRED) of Federal Reserve Bank of St. Louis.
- To access data : *getSymbols*
- To obtain time series plots : *chartSeries*
- To set the working directory : click on the command "File" (Windows) or "Misc" (Mac) ; using the command *setwd*
- To load data from files : the command is *read.table* for simple text data (.txt), the command is *read.csv* for .csv files.

Outline

1 Financial Data and R Program

2 Review of probability and statistics

We will learn how to use mathematical and statistical tools to study financial data.

Mathematics will be used includes :

1. Probability and statistics, with calculus
2. Some optimization tools.
3. Basic analysis and linear algebra

Statistical modeling vs Machine learning

	Statistical modeling	Machine learning
School	Mathematics	Computer science
Problem	Output = function(Inputs) + error	Input & Output
Assump.	Multiple assumptions concerning distributions	No major assumptions concerning data
Power	Feature engineering less complex relationships	Less human implication more complex relationships

Univariate Random Variables

Discrete : binomial, Poisson,...
Continuous : normal, exponential, student-t, χ^2, ...

Density functions for continuous random variables

$$P(X \in A) = \int_A f_X(x)dx,$$

where $f_X(x)$ is the density function of X.
Normal density : the density of N(0,1) is given by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{x^2}{2}\}.$$

the density of $N(\mu, \sigma^2)$ is given by

$$\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma}).$$

Cumulative distribution function

Cumulative distribution function (cdf) :

$$F_X(x) = P(X \leq x).$$

If X is continuous, then

$$F_X(x) = \int_{-\infty}^{x} f_X(v)dv.$$

The cdf of N(0,1) is given by

$$\Phi(x) = \int_{-\infty}^{x} \phi(u)du.$$

Moments

Expectation :
discrete : $E(X) = \sum_x xP(X = x)$.
continuous : $E(X) = \int_{-\infty}^{\infty} xf_X(x)dx$.

Variance : $Var(X) = E[(X - E(X))^2] = E[X^2] - (E[X])^2$.
Standard deviation : $SD(X) = \sqrt{Var(X)}$
Sample mean : $E(X)$ can be estimated from the sample by using

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

Similarly, sample variance is defined as

$$s_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Skewness

Skewness :

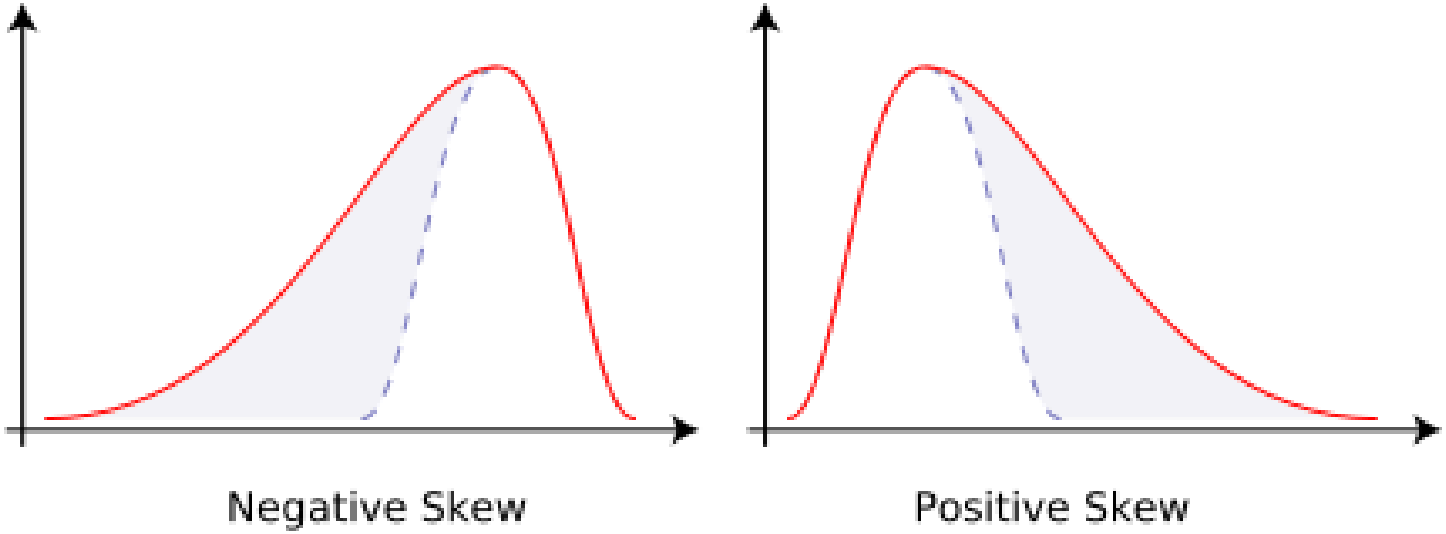
$$E\left[\left(\frac{X-\mu}{\sigma}\right)^3\right]$$

and the corresponding sample skewness

$$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^3}{s_X^3(n-1)}.$$

For a symmetric distribution (e.g. normal), skewness is zero. Thus, intuitively, skewness is a measure of the degree of symmetry for distributions.

Skewness



(Source : Wikipedia)

Kurtosis

Kurtosis and Excess Kurtosis :

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] \text{ and } E\left[\left(\frac{X-\mu}{\sigma}\right)^4\right]-3$$

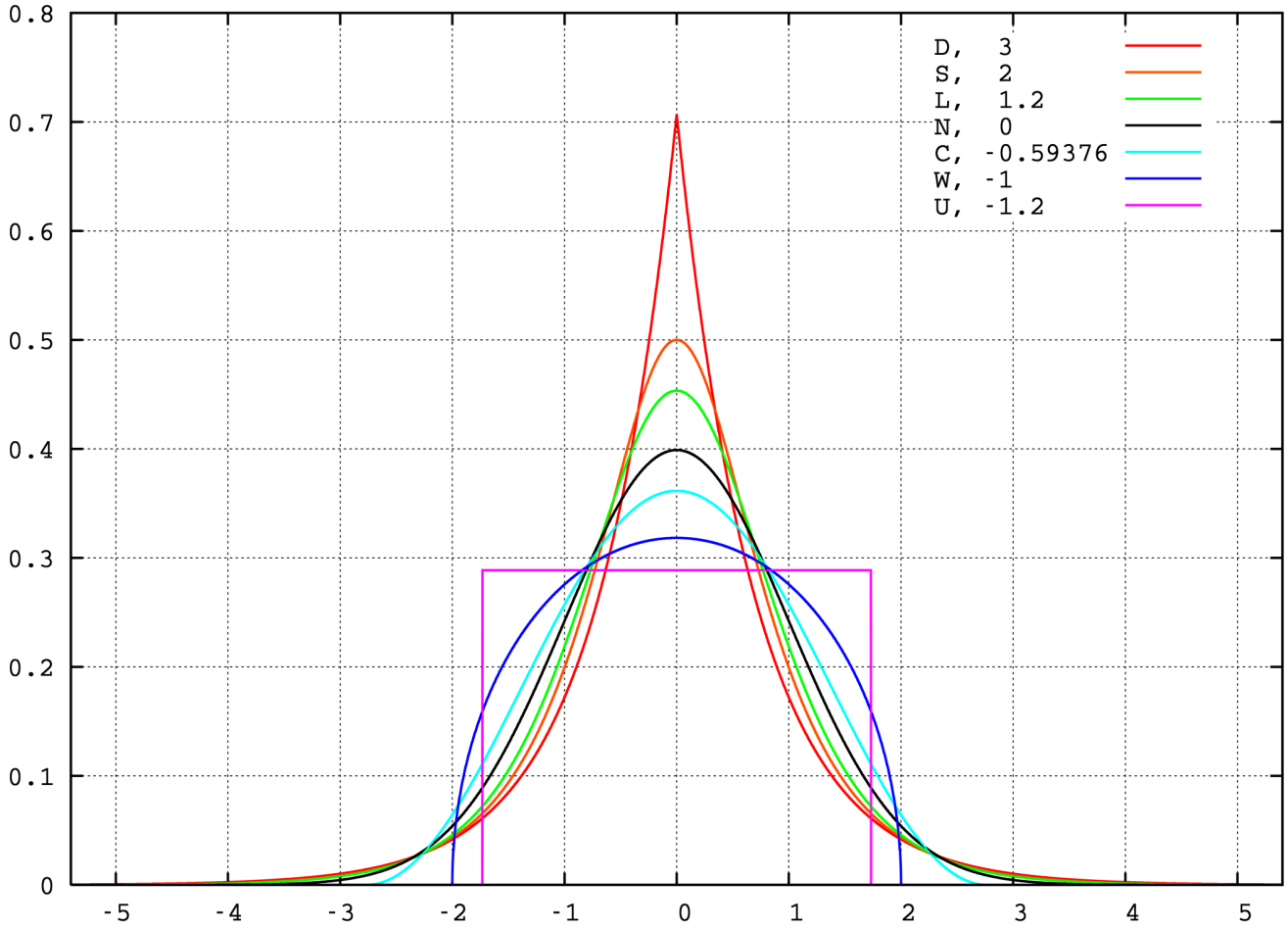
and the corresponding sample kurtosis

$$\frac{\sum_{i=1}^{n}(X_i-\bar{X})^4}{s_X^4(n-1)}.$$

For a normal distribution the kurtosis is equal to 3. Any kurtosis bigger than 3 means a density with heavier tail(s) and a higher peak than those of normal distribution.

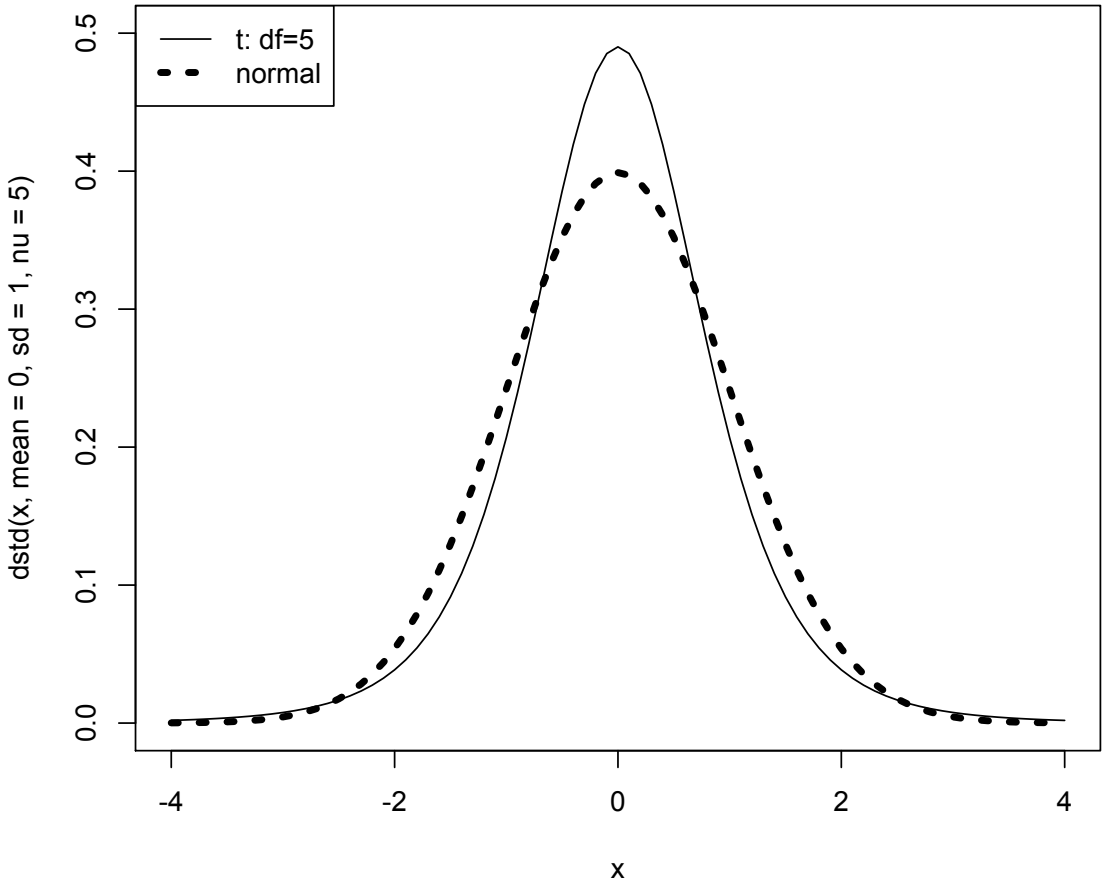
Leptokurtic distribution : distributions with heavier tails and a higher peak than those of normal distribution.

Excess Kurtosis

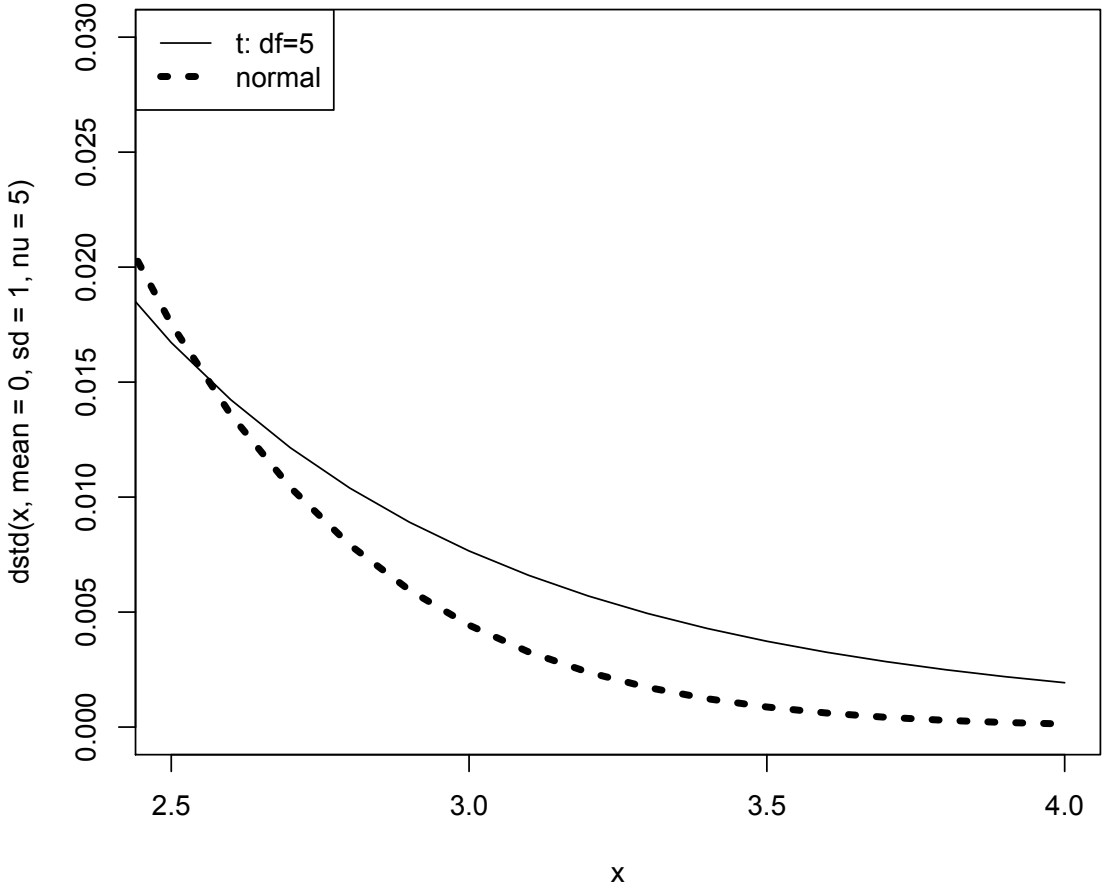


Probability density functions for selected distributions with mean 0, variance 1 and different excess kurtosis (Source : Wikipedia)

Comparison : normal density and t-density



Tail comparison : normal density and t-density



Pencentiles

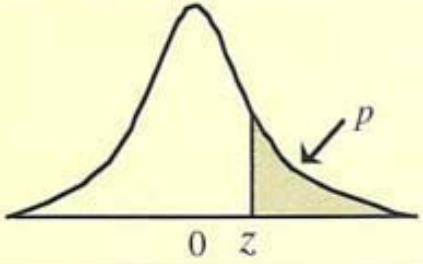
Pencentiles (quantiles) : the qth percentile of a continuous r.v. X, denoted by x_q, is a number such that the probability being smaller than it is q, i.e. $P(X \leq x_q) = q$

Median is the 50% percentile.
25% and 75% percentiles are called the first and third quartiles.
10% and 90% percentiles are called the first and ninth quantiles.

For standard normal distribution, $q = 5\%, x_q = -1.644854$;
$q = 2.5\%, x_q = -1.959964$.

With R :
$> qnorm(0.05)$
$[1] - 1.644854$
$> qnorm(0.025)$
$[1] - 1.959964$

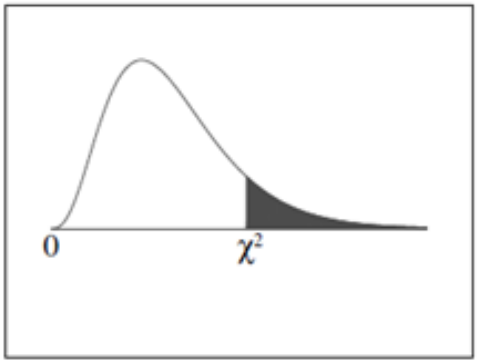
Normal distribution



	Second decimal place of z									
z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
0.7	.2420	.2389	.2358	.2327	.2297	.2266	.2236	.2206	.2177	.2148
0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233

Chi-squared distribution

Chi-Square Distribution Table



The shaded area is equal to α for $\chi^2 = \chi^2_\alpha$.

df	$\chi^2_{.995}$	$\chi^2_{.990}$	$\chi^2_{.975}$	$\chi^2_{.950}$	$\chi^2_{.900}$	$\chi^2_{.100}$	$\chi^2_{.050}$	$\chi^2_{.025}$	$\chi^2_{.010}$	$\chi^2_{.005}$
1	0.000	0.000	0.001	0.004	0.016	2.706	3.841	5.024	6.635	7.879
2	0.010	0.020	0.051	0.103	0.211	4.605	5.991	7.378	9.210	10.597
3	0.072	0.115	0.216	0.352	0.584	6.251	7.815	9.348	11.345	12.838
4	0.207	0.297	0.484	0.711	1.064	7.779	9.488	11.143	13.277	14.860
5	0.412	0.554	0.831	1.145	1.610	9.236	11.070	12.833	15.086	16.750
6	0.676	0.872	1.237	1.635	2.204	10.645	12.592	14.449	16.812	18.548
7	0.989	1.239	1.690	2.167	2.833	12.017	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	3.490	13.362	15.507	17.535	20.090	21.955

Independence

Independence of univariate random variables X and Y

$$P(X \in A, Y \in B) = P(X \in A)P(Y \in B).$$

$$E[h(X)g(Y)] = E[h(X)]E[g(Y)].$$

Law of Large Numbers and Central Limit Theorem.
Suppose X_1, X_2, X_2, ... are independent and identically distributed (i.i.d. for short) random variables with mean μ and variance σ^2.

$$\frac{\sum_{i=1}^{n} X_i}{n} \to E[X], \text{ (LLN)}$$

$$\frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \to N(0,1) \text{ (CLT)}$$

in distribution.

χ^2, t and F distributions

Z_i, $i = 1, 2, ...,$ are independent standard normal random variables.
Chi-squared (χ^2) distribution with degree of freedom (d.f.) n.

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2,$$

t-distribution (with d.f. n)

$$t_n = \frac{Z}{\sqrt{\chi_n^2/n}}.$$

F-distribution (with d.f. n_1 and n_2)

$$F_{n_1,n_2} = \frac{\chi_{n_1}^2/n_1}{\chi_{n_2}^2/n_2}$$

where $\chi_{n_1}^2$ and $\chi_{n_2}^2$ are two independent chi-squared distributions with d.f. n_1 and n_2, respectively.

χ^2, t and F distributions

There are some interesting relations between these distributions. For instance,

$$F_{n_1,\infty} = \chi^2_{n_1}/n_1,$$

because

$$\chi^2_{n_2}/n_2 = \frac{\sum_{i=1}^{n_2} Z_i^2}{n_2} \to E[Z^2] = 1,$$

as $n_2 \to \infty$.

t-distribution is a special case of the F-distribution. In particular,

$$\text{If } Y \sim t_n, \text{ then } Y^2 \sim F_{1,n}$$

Multivariate random variables

Bivariate distribution :

$$P((X,Y) \in C) = \int\int_C f_{X,Y}(x,y)dxdy \text{ with } C \in R \times R.$$

Covariance and Correlation

$$\begin{aligned} Cov(X,Y) &= E[\{X - E(X)\}\{Y - E(Y)\}] \\ &= E[XY] - E[X]E[Y]. \end{aligned}$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}, \ -1 \leq \rho(X,Y) \leq 1.$$

Sample correlation coefficient

$$\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{s_X s_Y}.$$

Correlation

Correlation is important as all time series data typically has non-zero correlations.

For example, today's stock return may be correlated with yesterday's stock return.

In fact, to certain extent, most of the time series analysis attempts to understand the correlation between the past, present, and future.

Conditional distribution

Conditional density

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \ f_Y(y) = \int f_{X,Y}(x,y)dx.$$

Conditional expectation and conditional variance

$$E[X|Y=y] = \int x f_{X|Y}(x|y)dx,$$

$$Var[X|Y=y] = \int (x - E[X|Y=y])^2 f_{X|Y}(x|y)dx.$$

For the bivariate normal distribution with correlation ρ_{XY} :

$$E[X|Y] = E[X] + \frac{Cov(X,Y)}{Var(Y)}\{Y - E(Y)\}$$

$$Var(X|Y) = Var(X)(1 - \rho_{XY}^2).$$

Linear combinations

For w_1 and w_2 two constants

$$E[w_1 X + w_2 Y] = w_1 E[X] + w_2 E[Y].$$

$$\begin{aligned} & Var[w_1 X + w_2 Y] \\ = \; & (w_1, w_2) \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\ = \; & w_1^2 Var(X) + 2 w_1 w_2 Cov(X,Y) + w_2^2 Var(Y). \end{aligned}$$

Linear combinations

In general, if $\mathbf{X} = \begin{pmatrix} X_1 \\ \vdots \\ X_d \end{pmatrix}$,then

$$E[w^T \mathbf{X}] = w^T \begin{pmatrix} E[X_1] \\ \vdots \\ E[X_d] \end{pmatrix} = \sum_{i=1}^{d} w_i E[X_i],$$

where w is a d-dimensional vector and T means transpose.

Covariance matrix

Covariance matrix :

$$Cov(\mathbf{X}) = \begin{pmatrix} Var(X_1) & \cdots & Cov(X_1, X_k) & \cdots & Cov(X_1, X_d) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Cov(X_k, X_1) & \cdots & Var(X_k) & \cdots & Cov(X_k, X_d) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Cov(X_d, X_1) & \cdots & Cov(X_d X_k) & \cdots & Var(X_d) \end{pmatrix}$$

$$Var(w^T \mathbf{X}) = w^T Cov(\mathbf{X}) w = \sum_{i=1}^{d} \sum_{j=1}^{d} w_i w_j Cov(X_i, X_j)$$

$$Cov(w_1^T \mathbf{X}, w_2^T \mathbf{X}) = w_1^T Cov(\mathbf{X}) w_2 = \sum_{i=1}^{d} \sum_{j=1}^{d} w_{1,i} w_{2,j} Cov(X_i, X_j)$$

Example

Suppose

$$Var(X_1) = 2, \;\; Var(X_2) = 3,$$

$$Var(X_3) = 4, \;\; \rho_{X_1 X_2} = 0.5,$$

and X_1 and X_2 are independent of X_3. Then

$$\begin{aligned} & Cov(\frac{X_1 + X_2 + X_3}{3}, \frac{X_2 + X_3}{2}) \\ = \; & (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \begin{pmatrix} 2 & 0.5\sqrt{2*3} & 0 \\ 0.5\sqrt{2*3} & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} (0, \frac{1}{2}, \frac{1}{2})^T \\ = \; & 1.370791 \end{aligned}$$

Thank you for your attention.