Practical 3
Optimisation

Part A

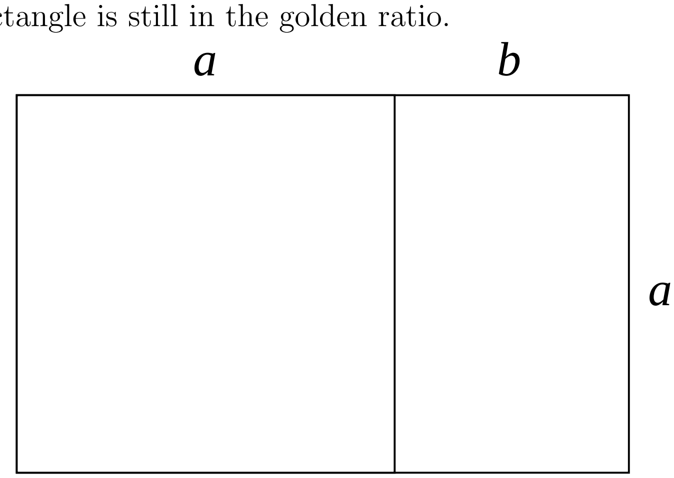
Fixed point iteration

A *fixed point iteration* is an iterative algorithm of the form

$$x_{i+1} = g(x_i)$$

such that the fixed point (when $x_{i+1} = x_i$) is the solution to our problem.

The golden ratio can be defined geometrically as the ratio of the sides of a rectangle which is such that when a square is cut from one end, the remaining rectangle is still in the golden ratio.



Algebraically, this means

$$\frac{a+b}{a} = \frac{a}{b}$$

where $\phi = \frac{a}{b}$ is the golden ratio.

Find an equation of the form $\phi = g(\phi)$ for the golden ratio, and use this to create a fixed point iteration. Implement this function at the marked point in function `partA`.

Call `partA` to check you method and see a plot the the iteration.

Part B

Non-linear optimisation

In lectures 9 and 10 we looked at the problem of carrying a ladder around a corner. For corridors of widths a and b, and joining angle θ, we showed that there was a critical angle:

$$\alpha^* = \arg\min_{\alpha} \left(\frac{a}{\sin\alpha} + \frac{b}{\sin(\pi - \theta - \alpha)} \right)$$

which leads to a maximum length

$$l^* = \frac{a}{\sin\alpha^*} + \frac{b}{\sin(\pi - \theta - \alpha^*)}$$

We will now write a method to solve this problem for any allowed a, b and θ using two methods.

Gradient Descent

We will start by using the method of gradient descent to find the minimum. Firstly, we must set up our functions. We want to minimise the length with respect to α

$$l(\alpha) = \frac{a}{\sin\alpha} + \frac{b}{\sin(\pi - \theta - \alpha)}$$

To do this, we find the gradient of the function

$$f(\alpha) = l'(\alpha) = \frac{dl}{d\alpha} = -\frac{a\cos\alpha}{\sin^2\alpha} + \frac{b\cos(\pi - \theta - \alpha)}{\sin^2(\pi - \theta - \alpha)}$$

and try to solve the equation $f(\alpha) = 0$ to find the stationary point. In the gradient descent method, the solution is found using the fixed-point iteration (lecture 9, slide 9)

$$\alpha_{i+1} = \alpha_i - f(\alpha_i)$$

Implement this method in the function `partB(a,b,theta,startpoint)`. Run `partB` with the parameters $a = 2, b = 3, \theta = 1.2$,startpoint=0.9. 'startpoint' is the initial value to use for α. You should see that the iteration diverges rapidly, even though we start close to the solution.

We can fix this (temporarily) by changing the step size. Introduce a step size of 0.1 into the iteration:

$$\alpha_{i+1} = \alpha_i - 0.1 f(\alpha_i)$$

Now run with the same parameters, and then with startpoint=0.5. You should now see that it converges in both cases. However, if you start with 0.4, it will not, so this is not a very satisfactory solution.

Newton's method

Newton's method is a very similar idea to gradient descent, except that we calculate an optimal step size by assuming that locally the function is a quadratic (via a Taylor expansion). This give Newtons method (Lecture 10, slide 6):

$$\alpha_{i+1} = \alpha_i - \frac{f(\alpha_i)}{f'(\alpha)}$$

The drawback is that we have to compute the derivative:

$$f'(\alpha) = \frac{df}{d\alpha} = a\frac{\sin^2\alpha + 2\cos^2\alpha}{\sin^2\alpha} + b\frac{\sin^2(\pi - \theta - \alpha) + 2\cos^2(\pi - \theta - \alpha)}{\sin^3(\pi - \theta - \alpha)}$$

Modify the function to use the Newton method, and plot the results. You should now find the method converges quickly.

Part C

A Multivariate Problem

Use Newton's method (Lecture 11, slide 5) to find the maximum value of the function

$$f(x, y) = x^2 + y^2 + 3\exp[\sin(x) + \cos(y - 1)]$$

A appropriate starting point is (1.3,0.6).

Demonstrate that this is a maximum using the properties of the Hessian matrix (Lecture 11, slide 9).