Consider a standard baseline New Keynesian model described in. New Perspectives on Monetary Policy, Inflation, and the Business Cycle., by Jordi Galí.

In this model, the nominal interest rate (notice that in this paper it is denoted by r_t and not i_t as in the class notes. Also, notice that ρ, the steady state real interest rate, is set to 0) can be written in the following way:

$$\boldsymbol{r_t = \left(\frac{\sigma-1}{1+\eta}\right)\sum_{k=1}^{\infty}\left(\left(\frac{\eta}{1+\eta}\right)^{k-1} E_t[\Delta y_{t+k}]\right) + \Delta m_t \frac{1+\eta}{1+\eta(1-\rho_m)}} \quad \textbf{(1)}$$

The above equation can be derivate as follow: using the money demand and the output gap equation:

$m_t - p_t = y_t - \eta r_t$ (5)

$X_t = E_t\{X_{t+1}\} - \frac{1}{\sigma}(r_t - E_t\{\pi_{t+1}\} - \overline{rr}_t)$ (6)

Where $X_t = y_t - \bar{y}_t$, $\bar{y}_t$ is the natural product and $\overline{rr}_t$ is the expected real rate. Also, using the assumption that growth rate of the money supply follows an exogenous stationary process AR(1) as:

$\Delta m_t = \rho_m \Delta m_{t-1} + \varepsilon_t^m$ (7)

Where the expected representation of (7) is:

$E(\Delta m_{t+1}) = \rho_m \Delta m_t$ (8)

Using the expectations and forward equation (5):

$E(m_{t+1}) - E(p_{t+1}) = E(y_{t+1}) - \eta E(r_{t+1})$ (9)

Then difference (6) and (5):

$E(m_{t+1}) - m_t - (E(p_{t+1}) - p_t) = E(y_{t+1}) - y_t - (\eta E(r_{t+1}) - \eta r_t)$

$E(\Delta m_{t+1}) - E(\pi_{t+1}) = E(y_{t+1}) - y_t - (\eta E(r_{t+1}) - \eta r_t)$

$E(\pi_{t+1}) = E(\Delta m_{t+1}) - E(\Delta y_{t+1}) + (\eta E(r_{t+1}) - \eta r_t)$ (10)

Then plugging (10) in (6) and using (8):

$y_t - \bar{y}_t = E_t(y_{t+1} - \bar{y}_{t+1}) - \frac{1}{\sigma}[r_t - (\rho_m \Delta m_t - E(\Delta y_{t+1}) + (\eta E(r_{t+1}) - \eta r_t)] - \overline{rr}_t$

Gali 2003 set $\bar{y}_t = 0$, $a_t = 0$, $g_t = 0$, $\rho = 0$, there fore $\overline{rr} = 0$. Then:

$y_t = E_t(y_{t+1}) - \frac{1}{\sigma}[r_t - (\rho_m \Delta m_t - E(\Delta y_{t+1}) + (\eta E(r_{t+1}) - \eta r_t)]$

$y_t = E_t(y_{t+1}) - \frac{1}{\sigma} r_t + \frac{1}{\sigma}\rho_m \Delta m_t - \frac{1}{\sigma} E(\Delta y_{t+1}) + \frac{1}{\sigma}\eta E(r_{t+1}) - \frac{1}{\sigma}\eta r_t$

$y_t = E_t(y_{t+1}) - \left(\frac{1+\eta}{\sigma}\right) r_t + \frac{1}{\sigma}\rho_m \Delta m_t - \frac{1}{\sigma} E(y_{t+1}) + \frac{1}{\sigma} y_t + \frac{1}{\sigma}\eta E_t(r_{t+1})$

$$\left(\frac{1+\eta}{\sigma}\right)r_t = E_t(y_{t+1}) + \frac{1}{\sigma}\rho_m\Delta m_t - \frac{1}{\sigma}E(y_{t+1}) + \frac{1}{\sigma}y_t - y_t + \frac{1}{\sigma}\eta E_t(r_{t+1})$$

$$\left(\frac{1+\eta}{\sigma}\right)r_t = E_t(y_{t+1})\left(1-\frac{1}{\sigma}\right) - y_y\left(1-\frac{1}{\sigma}\right) + \frac{1}{\sigma}\rho_m\Delta m_t + \frac{1}{\sigma}\eta E_t(r_{t+1})$$

$$\left(\frac{1+\eta}{\sigma}\right)r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{\sigma}\right) + \frac{1}{\sigma}\rho_m\Delta m_t + \frac{1}{\sigma}\eta E(r_{t+1})$$

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{\sigma}\right)\left(\frac{\sigma}{1+\eta}\right) + \left(\frac{\sigma}{1+\eta}\right)\frac{1}{\sigma}\rho_m\Delta m_t + \left(\frac{\sigma}{1+\eta}\right)\frac{1}{\sigma}\eta E_t(r_{t+1})$$

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)\rho_m\Delta m_t + \left(\frac{\eta}{1+\eta}\right)E(r_{t+1}) \quad (11)$$

Then iterating forward two times and substituting:

$$E_t(r_{t+1}) = E_t(\Delta y_{t+2})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)E_t(r_{t+2}) \quad (12)$$

$$E_t(r_{t+2}) = E_t(\Delta y_{t+3})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+2}) + \left(\frac{\eta}{1+\eta}\right)E_t(r_{t+3})$$

.

.

.

$$E(r_{t+n}) = E_t(\Delta y_{t+n})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)\rho_m E_t(\Delta m_{t+n}) + \left(\frac{\eta}{1+\eta}\right)E(r_{t+n+1})$$

Substituting 12 into 11, we get:

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_t) + \left(\frac{\eta}{1+\eta}\right)\left[E_t(\Delta y_{t+2})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)E_t(r_{t+2})\right]$$

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_t) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{\sigma-1}{1+\eta}\right)E_t(\Delta y_{t+2}) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)^2 E_t(r_{t+2})$$

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_t) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{\sigma-1}{1+\eta}\right)E_t(\Delta y_{t+2}) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)^2\left(E_t(\Delta y_{t+3})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t\rho_m\Delta m_{t+2} + \left(\frac{\eta}{1+\eta}\right)E_t(r_{t+3})\right)$$

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_t) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{\sigma-1}{1+\eta}\right)E_t(\Delta y_{t+2}) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)^2\left(\frac{\sigma-1}{1+\eta}\right)E_t(\Delta y_{t+3}) + \left(\frac{\eta}{1+\eta}\right)^2\left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+2}) + \left(\frac{\eta}{1+\eta}\right)^3 E_t(r_{t+3})$$

$$r_t = E_t(\Delta y_{t+1})\left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_t) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{\sigma-1}{1+\eta}\right)E_t(\Delta y_{t+2}) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)^2\left(\frac{\sigma-1}{1+\eta}\right)E_t(\Delta y_{t+3}) + \left(\frac{\eta}{1+\eta}\right)^2\left(\frac{1}{1+\eta}\right)E_t(\rho_m\Delta m_{t+2}) +$$

$\left(\frac{\eta}{1+\eta}\right)^3 \Bigg(E_t(\Delta y_{t+1}) \left(\frac{\sigma-1}{1+\eta}\right) + \left(\frac{1}{1+\eta}\right) E_t(\rho_m \Delta m_t) + \left(\frac{\eta}{1+\eta}\right)\left(\frac{\sigma-1}{1+\eta}\right) E_t(\Delta y_{t+2}) +$

$\left(\frac{\eta}{1+\eta}\right)\left(\frac{1}{1+\eta}\right) E_t(\rho_m \Delta m_{t+1}) + \left(\frac{\eta}{1+\eta}\right)^2 \left(\frac{\sigma-1}{1+\eta}\right) E_t(\Delta y_{t+3}) + \left(\frac{\eta}{1+\eta}\right)^2 \left(\frac{1}{1+\eta}\right) E_t(\rho_m \Delta m_{t+2}) +$

$\left(\frac{\eta}{1+\eta}\right)^3 E_t(r_{t+3}) \Bigg)$

.

.

.

If $n \to \infty$ and $|E_t r_{t+k+1}| < \infty$, then $\lim_{k\to\infty} \left(\frac{\eta}{1+\eta}\right)^{k+1} E_t(r_{t+k+1})$=0. Therefore the resulting expression is:

$r_t = \left(\frac{\sigma-1}{1+\eta}\right) \sum_{k=1}^{\infty} \left(\left(\frac{\eta}{1+\eta}\right)^{k-1} E_t[\Delta y_{t+k}] \right) + \left(\frac{1}{1+\eta}\right) \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta}\right)^k E_t(\rho_m \Delta m_{t+1+k})$

Using equation (8) for the first iteration of growth rate of the money supply:

$$r_t = \left(\frac{\sigma-1}{1+\eta}\right) \sum_{k=1}^{\infty} \left(\left(\frac{\eta}{1+\eta}\right)^{k-1} E_t[\Delta y_{t+k}] \right) + \left(\frac{\Delta m_t}{1+\eta}\right) \sum_{i=0}^{\infty} \left(\frac{\eta \rho_m}{1+\eta}\right)^k$$

$$r_t = \left(\frac{\sigma-1}{1+\eta}\right) \sum_{k=1}^{\infty} \left(\left(\frac{\eta}{1+\eta}\right)^{k-1} E_t[\Delta y_{t+k}] \right) + \Delta m_t \left(\frac{\eta \rho_m}{1+\eta}\right) \left(1 + \left(\frac{\eta \rho_m}{1+\eta}\right) + \left(\frac{\eta \rho_m}{1+\eta}\right)^2 \dots \right)$$

Where

$$\Delta m_t \left(\frac{\eta \rho_m}{1+\eta}\right) \left(1 + \left(\frac{\eta \rho_m}{1+\eta}\right) + \left(\frac{\eta \rho_m}{1+\eta}\right)^2 \dots \right) = \frac{1}{1 - \frac{\eta \rho_m}{1+\eta}} = \frac{1+\eta}{1+\eta(1-\rho_m)}$$

Then

$$r_t = \left(\frac{\sigma-1}{1+\eta}\right) \sum_{k=1}^{\infty} \left(\left(\frac{\eta}{1+\eta}\right)^{k-1} E_t[\Delta y_{t+k}] \right) + \Delta m_t \frac{1+\eta}{1+\eta(1-\rho_m)}$$

To the extent that money growth is positively serially correlated (assume $\boldsymbol{\rho_m = 0.5}$), the nominal rate will necessarily increase in response to a monetary expansion. Under what conditions can the liquidity effect, which would imply negative relationship between nominal interest rate and quantity of money in the economy, be restored?

Using the baseline calibration where the risk-aversion parameter is $\sigma = 1$ and assuming $\rho_m = 0.5$ this will imply that the nominal rate r_t will move in the same direction as an increase in the monetary expansion. Is important to note that this relation is positive, under the long-run neutrality of money, which implies $\lim_{k\to\infty} E_t(y_{t+k+}) \, E_t(y_{t+k+}) = 0$, due whether:

- The output's reversion to its initial level is monotonic
- And/or the semi-elasticity of money demand (η) is large.

In order the liquidity effect to be restored, the risk aversion parameter must be high (σ>1). To the extent that a monetary expansion raises output on impact, the term of the equation 1, which represents the relation with output and the semi-elasticity of money demand, must be negative:

$$\left(\frac{\sigma-1}{1+\eta}\right)\sum_{k=1}^{\infty}\left(\left(\frac{\eta}{1+\eta}\right)^{k-1} E_t[\Delta y_{t+k}]\right)$$

Therefore, the presence of a liquidity effect requires a sufficiently high risk aversion parameter σ (for any ρ_m) or, given σ>1, a sufficiently low money growth autocorrelation ρ_m.

Moreover, we know that the intertemporal elasticity of substitution and the relative risk aversion parameter are inversely related. Hence, in order to generate liquidity effect in this model, the intertemporal elasticity of substitution should be low. Why is the intertemporal elasticity of consumption also important? Lower intertemporal substitution will lead to lower expected increase of consumption, which will reduce the interest rate conditional on the intertemporal substitution being low. This decrease may be higher than the increase induced by the increase in expected inflation. Thus, only if $\varepsilon = \frac{1}{\sigma}$ is low enough it would be possible to have a liquidity effect, which would imply negative relationship between nominal interest rate and quantity of money in the economy.

In addition, lower money income elasticity can lead to the liquidity effect in our model. If money income elasticity is small, then households will not be willing to substitute away from bonds. Therefore, when money supply increases, it is more likely that the interest rate will fall.

Using Dynare to replicate Figure 3 from Gali's paper.

The figure 1 depicts the responses of inflation, output, real and nominal interest-rate given a simple New Keynesian model with sticky prices and an exogenous growth rate of money supply setting as a monetary policy shock. The results show that a 1 percent increase in the annualized rate of money growth implies:

- Inflation will increase by 2.4 percent points, where the slow adjustment implies an increase in real balances.
- Output will increase by more than 1.05 percent point, with the same adjustment of inflation.

- The nominal interest rate will increase by about 0.7 percent points, where the liquidity effects do not have any presence.
- The real interest-rate will decrease persistently (because a higher expected inflation) by (-1.077).

First, a typical monetary shock has strong, and highly persistent, effects on output. On impact, a one percent increase in the money supply raises output by slightly more than 1 percent, while the implied increase in the price level is of about 2.4 percent (annualized). In addition, these effects of money are quite persistent. In particular, the half-life of that output response under the baseline calibration is 3.2 quarters.

Second, a monetary expansion is predicted to raise the nominal rate; in other words, the calibrated model does not predict the existence of a liquidity effect. Still, that feature does not prevent monetary policy from transmitting its effects through an interest rate channel: as shown in the same figure, the real rate declines substantially when the monetary expansion is initiated, remaining below its steady state level for a protracted period. This persistent decline induces the observed expansion in aggregate demand and output.

Figure 1. Dynamic responses to a monetary shock.