Geometric Primitives & Transformation

Muhammad Eka Suryana

Universitas Negeri Jakarta

2020

Overview

2D Geometry Primitives

3D Geometry Primitives

2D Transformations

3D Transformation

2D Points

Definition

2D points can be denoted using a pair of values, $\mathbf{x} = (x, y) \in \mathbb{R}^2$

- Alternatively

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Points(2)

- 2D point can also be represented in homogeneous coordinate

Definition

$\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{w}) \in \mathbb{P}^2$

- where vectors differs only by scale are considered equivalent

Definition

$\mathbb{P}^2 = \mathbb{R}^3 - (0, 0, 0)$ is called as 2D *projective space*

2D Points(3)

Definition

Homogeneous points whose last element is $\tilde{w} = 1$, $\tilde{x} = (\tilde{x}, \tilde{y}, 1)$ is called **augmented vector**.

Homogeneous points whose last element is $\tilde{w} = 0$ are called ideal points or points at infinity.

2D Lines

Definition

2D lines is representable in homogeneous coordinate $\tilde{l} = (a, b, c)$ with the corresponding line equation

$$\bar{x}.\tilde{l} = ax + by + c$$

- We can normalize the line equation vector so that $l = (\hat{n}_x, \hat{n}_y, d) = (\hat{n}, d)$ with $||\hat{n}|| = 1$.
- $\hat{x}$ is normal vector perpendicular to the line
- d is its distance to the origin.
- line at infinity $\tilde{l} = (0, 0, 1)$

2D Lines(2)

- $\hat{n}$ can be expressed as function of rotational angle θ

$$\hat{n} = (\hat{n}_x, \hat{n}_y) = (cos(\theta), sin(\theta))$$

The combination of (θ, d) is known as *polar coordinates* and commonly used as Hough Transform line finding algorithm.

2D Lines(3)

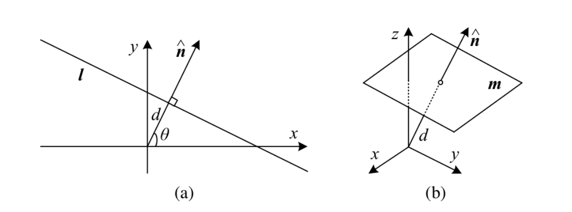


Figure: (a) 2D line equation and (b) 3D plane equation, expressed in terms of the normal $\hat{n}$ and distance to the origin d.

2D Lines(4)

- when using homogeneous coordinates, we can compute the intersection of two lines as

$$\tilde{x} = \tilde{l}_1 \times \tilde{l}_2$$

  where $\tilde{x}$ is cross product operator

- Similarly, line joining two points can be written as

$$\tilde{l} = \tilde{x}_1 \times \tilde{x}_2$$

Cross Product

For 3 dimensional vectors the following rule applied

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Question

What if the vectors high dimensionals?

2D Conics

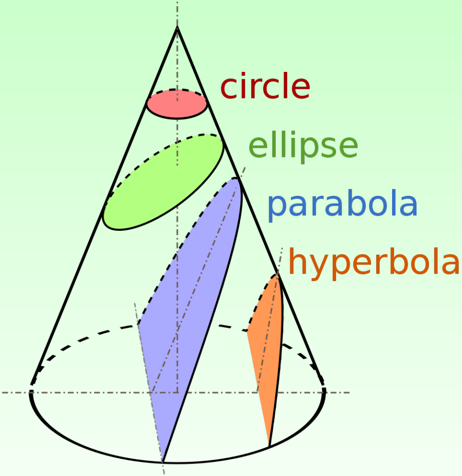


Figure: Different perspectives of conics

2D Conics

Definition

Intersection of a plane and a 3D cone, written as Quadric equation

$$\tilde{x}^T Q \tilde{x} = 0$$

Used extensively for multi-view geometry and camera calibration.

3D Points

Inhomogeneous coordinates

$\mathbf{x} = (x, y, z) \in \mathbb{R}^3$

Homogeneous coordinates

$\tilde{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}) \in \mathbb{P}^3$

Augmented Vector 3D Points

$\bar{\mathbf{x}} = (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{1})$ with $\tilde{x} = \tilde{w}\bar{x}$

3D Planes

Homogeneous Form

$\tilde{m} = (a, b, c, d)$ with a corresponding plane equation

$$\bar{\mathbf{x}}.\tilde{\mathbf{m}} = ax + by + cz + d = 0$$

- Plane equation normalization $\mathbf{m} = (\hat{n}_x, \hat{n}_y, \hat{n}_z, d) = (\hat{\mathbf{n}}, d)$ with $||\hat{n} = 1||$
- $\hat{n}$ is the *normal vector* perpendicular to the plane and d is its distance to the origin.

3D Planes(2)

- the plane at infinity $\tilde{m} = (0, 0, 0, 1)$ cannot be normalized
- $\hat{n}$ can also be expressed in spherical coordinate

function of two angles (θ, ϕ)

$$\hat{n} = (cos(\theta)cos(\phi), sin(\theta)cos(\phi), sin(\phi))$$

3D Lines

- Won't be described in detail here, 3D lines are more complex than 3D plane.
- there are two approaches
  - Intersection of two planes
  - Regression of two points

Regression of two points

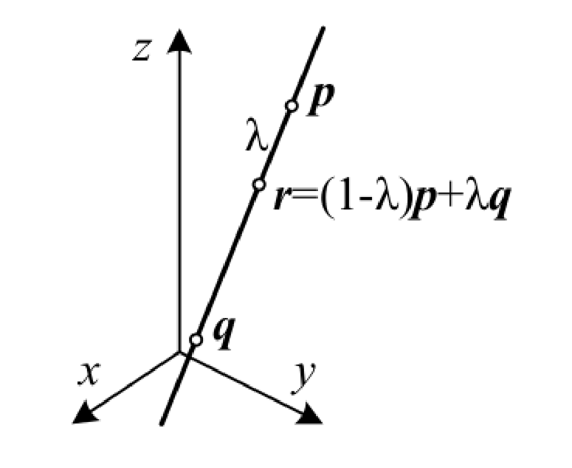


Figure: 3D line equation, $r = (1 - \lambda)p + \lambda q$.

Intersection of two planes

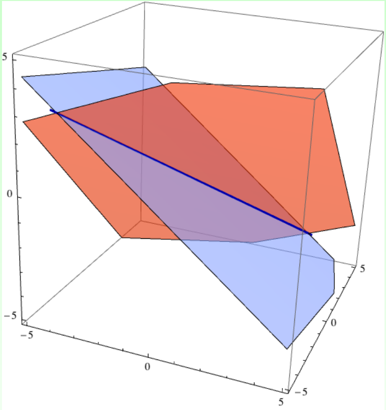


Figure: $\{(x,y,z) \in \mathbb{R}^3 : a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2\}$.

2D Transformation

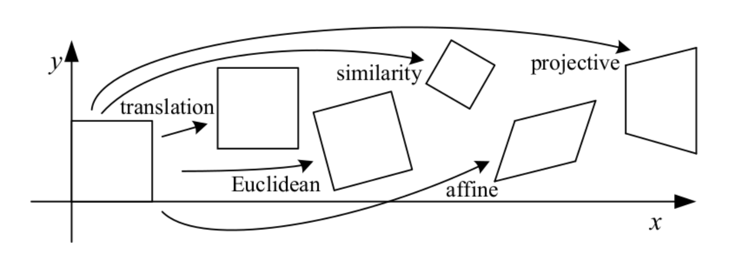


Figure: Basic set of 2D planar transformations

2D Translation

2D translations can be written as $x' = x + t$ or

$$x' = \begin{bmatrix} I & t \end{bmatrix} \bar{x}$$

Where I is 2 $\times$ 2 identity matrix or

$$\bar{x}' = \begin{bmatrix} I & t \\ 0^T & 1 \end{bmatrix} \bar{x}$$

Where 0 is the zero vector

Rotation + translation

This transformation is known as 2D *rigid body motion* or 2D *euclidean transform*, written as $x' = Rx + t$ or

$$\bar{x}' = \begin{bmatrix} R & t \end{bmatrix} \bar{x}$$

with

$$R = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

is an orthonormal rotation matrix $RR^T = I$ and $|R| = 1$.

Scaled rotation

This transformation is known as *similarity transform* expressed as $x' = sRx + t$ where s is arbitrary scale factor.

$$x' = \begin{bmatrix} sR & t \end{bmatrix} \bar{x} = \begin{bmatrix} a & -b & t_x \\ b & a & t_y \end{bmatrix} \bar{x}$$

This similarity transform preserves angles between lines.

Affine Transform

Generalized form of any transformation operation in 2D space with a transformation matrix

Definition

$x' = A\bar{x}$, where A is arbitrary 2 $\times$ 3 matrix. i.e.

$$x' = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \bar{x}$$

Parallel lines remain parallel under affine transforms.

Projective

Alternatively known as perspective transform or homography, operates on homogeneous coordinates.

$$\tilde{x}' = \tilde{H}\tilde{x}$$

$\tilde{H}$ is homogeneous, two $\tilde{H}$ matrices that differ only by scale are equivalent.

Projective(2)

The resulting homogeneous coordinate $\tilde{x}'$ must be normalized in order to obtain an inhomogeneous result x

$$x' = \frac{h_{00}x + h_{01}y + h_{02}}{h_{20}x + h_{21}y + h_{22}} \tag{1}$$

$$y' = \frac{h_{10}x + h_{11}y + h_{12}}{h_{20}x + h_{21}y + h_{22}} \tag{2}$$

This transformation preserve straight lines.

Co-Vectors

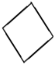
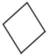
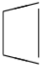
Consider homogeneous equation $\tilde{l} \cdot \tilde{x} = 0$. Substitute $x' = \tilde{H}x$, then

$$\tilde{l}' \cdot \tilde{x}' = \tilde{l}'^T \tilde{H}\tilde{x} = (\tilde{H}^T \tilde{l})^T \tilde{x} = \tilde{l} \cdot \tilde{x} = 0$$

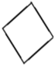
$$\text{i.e. } \tilde{l}' = \tilde{H}^{-T} \tilde{l}$$

- This shows we can do transformation on 2D line instead of doing transformation per points.
- Thus the action of a projective transforms on a co-vector (i.e. 2D Line, 3D normal can be represented by the transposed inverse of the matrix (*adjoint of* $\tilde{H}$).

2D transformations summary

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	2	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	3	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{2\times 3}$	4	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{2\times 3}$	6	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{3\times 3}$	8	straight lines	

3D transformations summary

Transformation	Matrix	# DoF	Preserves	Icon
translation	$\begin{bmatrix} \boldsymbol{I} \mid \boldsymbol{t} \end{bmatrix}_{3\times4}$	3	orientation	
rigid (Euclidean)	$\begin{bmatrix} \boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{3\times4}$	6	lengths	
similarity	$\begin{bmatrix} s\boldsymbol{R} \mid \boldsymbol{t} \end{bmatrix}_{3\times4}$	7	angles	
affine	$\begin{bmatrix} \boldsymbol{A} \end{bmatrix}_{3\times4}$	12	parallelism	
projective	$\begin{bmatrix} \tilde{\boldsymbol{H}} \end{bmatrix}_{4\times4}$	15	straight lines	

3D Rotation

3D rotation is not so straightforward. There are 3 methods to do that:

- Euler angles
- Axis
- Unit Quarternions

Euler Angles

- Product of three separate rotations around 3-axes.
- The results depends on the rotation order.
- It is not always to move smoothly in parameter space (axis), small change in a certain rotation axis impact large change in different axis.
- Bad!

Axis (exponential twist)

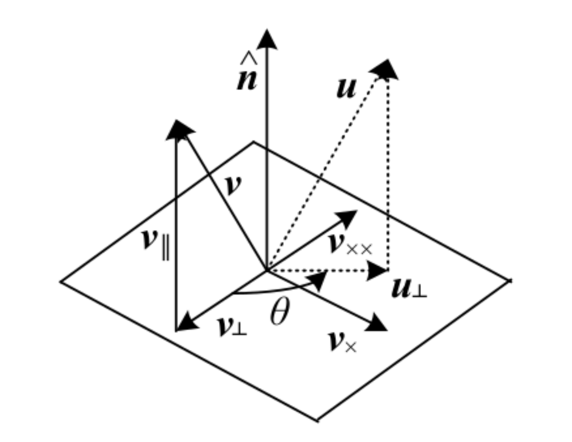


Figure: Rotation around an axis $\hat{n}$ by an angle θ.

Derivation formula on the book!