Quant Finance - Cheat Sheet
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1 Basics

Change of Variables $p(x)dx = g(y)dy \Rightarrow g(y) = \frac{p(x)}{|dy/dx)|}$

1.1 Moments

Variance: $\sigma^2 = E[(X-\mu)^2] = E[X^2] - E[X]^2$

Skewness: $S = E[(X-\mu)^3]/\sigma^3$

Kurtosis: $\kappa = E[(X-\mu)^4]/\sigma^4 - 3$

in General:

$$E[(X-\mu)^p] = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \sigma^p (p-1)!! & \text{if } p \text{ is even.} \end{cases}$$

Covariance: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - \mu_X\mu_Y$

1.2 Distributions

Uniform: $\mu = 1/2, \sigma^2 = 1/12$

Binomial: $p(k;n,p) = \frac{n!}{k!(n-k)!} p^k q^{n-k}$

Normal: $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-(X-\mu)/(2\sigma^2))$

Normal (CDF): $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{t/2} dt = \Phi(\frac{x-\mu}{\sigma})$

LogNormal: $x = log(y)$, then $p(y) = \frac{p(x)}{|dy/dx|} = \frac{1}{y\sqrt{2\pi\sigma^2}} exp(-\frac{(log(y)-\mu)^2)}{2\sigma^2}$

$E[R] = E[e^R - 1] = e^{\mu + \frac{\sigma^2}{2}}$

$Var[R] = E[(R-\mu_R)^2] = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$

Poisson: $p(k;\lambda) = \frac{e^{-\lambda}\lambda^k}{k!}$

$\mu = \lambda, \sigma^2 = \lambda$

1.3 Multiple Random Variables

$S = X_1 + \ldots + X_n$

$E[S] = \sum E[X_i]$

$VAR[S] = \sum_{i,j} w_i w_j E[(X_i - \mu_i)(X_- \mu_j)]$

1.4 Results

Law of Large Numbers: (Binomial example) As n increases, the probability that the mean deviates from np goes to zero.

Central Limit Theorem (CLT) (Binomial example) As n increases for fixed p, the distribution approaches a Gaussian.

Waiting Time : T : Number of tosses until H. $E[T] = 1/p$, p: probability of H

2 Time series Intro

2.1 Definitions

Generalized Random Walk: $r_t = \mu + \sigma z_t$, $r_t = log(P_t/P_{t-1})$, $P_t = P_0 e^{r_0 + \ldots + r_t}$

$MA(q)$: $R_t = c_0 + \sigma_t + \phi_1 z_{t-1} + \ldots + \phi_q z_{t-q}$

$AR(p)$: $R_t = c_0 + c_1 R_{t-1} + \ldots + c_p R_{t-p} + \sigma z_t$

Definitions: Stationary: The joint distribution invariant under time translation $t \mapsto t+s$

Weakly Stationary: First and second moments are invariant ($E[R_t] = E[R_{t+s}]$, $VAR[R_t] = VAR[R_{t+s}]$)

Sample Auto Correlation Function (ACF): Displays the correlation between step t and steps $\{t, t-1, \ldots, p-n\}$

Partial Auto Correlation Function (Partial ACF): Displays the last lag with relevant correlation.

Annualized Statistics: $mean = E[X] \cdot 252$, $std = \sigma \cdot \sqrt{252}$

2.2 Solving $AR(1)$

$R_t = c_0 + c_1 R_{t-1} + \sigma z_t$, $E[R_t] = \frac{c_0}{1-c_1}$, if $\mu = \frac{c_0}{1-c_1}$ and $\lambda = -c_1$, then, $R_t - \mu = -\lambda(R_{t-1} - \mu) + \sigma z_t$

Variance: $\gamma_0 = VAR[R_t] = E[(R_t - \mu)^2] = \lambda^2\gamma_0 + \sigma^2$, then $\gamma_0 = \frac{\sigma^2}{1-\lambda^2}$, $\gamma_k = \lambda^k \gamma_0$

Covariance: $\gamma_k = E[(R_t - \mu)(R_{t-k} - \mu)] = (-\lambda)^k \gamma_0 = \frac{(-\lambda)^k}{1-\lambda^2}\sigma^2$

2.3 MA Example

$r_t = \epsilon_t + a\epsilon_{t-1} + b$, $Cov(r_t, r_{t-k} = E[(\epsilon_t + a\epsilon_{t-1})(\epsilon_{t-k} + a\epsilon_{t-k-1})]) = \{\sigma^2(1+a^2)$ if $k=0, a\sigma^2$ if $k=1, 0$ if $k \geq 2\}$

2.4 Variance Ratio

$VR(q) = \frac{Var(r_t^q)}{qVar(r_t)}$

3 Binomial Tree

$$S_{t+1} = \begin{cases} S_t \cdot u & \text{with prob. } p \\ S_t \cdot d, & \text{with prob} 1-p \end{cases}$$

$P(S_t = S_0 u^k d^{y-k}) = \binom{t}{k} p^k (1-p)^{t-k}$

3.1 Solving Binomial trees

$S_t = S_{t-1} e^{r_t}$, $r_t = a + b \cdot x_t$ $x_t = \begin{cases} 1 & \text{with prob. } p \\ 0, & \text{with prob. } 1-p \end{cases}$, then

$a = \mu - \sigma\sqrt{\frac{p}{1-p}}$, $b = \frac{\sigma}{\sqrt{p(1-p)}}$

$log(u) = \mu + \sigma\sqrt{\frac{1-p}{p}}$, $log(d) = \mu - \sigma\sqrt{\frac{p}{1-p}}$

4 Gambler's Ruin

x : Starting Capital, a : House Capital, Q_x : Probability of ruin.

$Q_x = pQ_{x+1} + qQ_{x-1}$

$$Q_x = \begin{cases} \frac{(q/p)^a - (q/p)^x}{(q/p)^a - 1} & \text{if } p \neq q \\ 1 - \frac{x}{a}, & \text{if } p = q \end{cases}$$

If $a = +\infty$: $Q_x = \begin{cases} 1 & \text{if } p \leq q \\ (q/p)^x, & \text{if } p > q \end{cases}$

Expected Duration:

$$Q_x = \begin{cases} \frac{x}{q-p} - \frac{a}{q-p}\frac{1-(q/p)^x}{1-(q/p)^a} & \text{if } p \neq q \\ x(a-x) & \text{if } p = q \end{cases}$$

5 Mean-Squared Forecast Error (MSFE)

$f_{t,h} = E[X_{t+h} | I_t]$, $f'_{t,1} = E[\Delta X_{t+1} | I_t]$

MSFE: $E[(X_{t+h} - f_{t,h})^2]$, $E[(\Delta X_{t+1} - f'_{t,1})^2]$

6 Ito Calculus - Intro

6.1 Scaling Random Walk

$\Delta t = T/n$, $\lambda = \sqrt{\Delta t} = \sqrt{T/n}$, $\epsilon_t = \lambda z_t$

$B_{\Delta t,T} = \sum_{t=1}^n \epsilon_t = \sqrt{\Delta t}\sum_{t=1}^n z_t$

$E[B_{\Delta t,T}] = 0$,

$Var[B_{\Delta t,T}] = nVar[\epsilon_t] = n\Delta t Var_{z_t} = T$

$\lim_{\Delta t \to 0} B_{\Delta t,T} \sim N(0,T)$

$X(y_1, t_2) = B(t_2) - B(t_1)$, $X \sim N(0, t_2 - t_1)$

$dB_t \sim N(0, dt)$

$$Cov(dB_t, dB_{t'}) = \begin{cases} 0 & \text{if } t \neq t' \\ dt & \text{if } t = t' \end{cases}$$

$$B(T) = B(0) + \underbrace{\int_0^T dB_T}_{\text{Random Variable}}$$

Example: $log(\frac{S_T}{S_0}) = \mu T + \sigma\int dB_T \sim N(\mu T, \sigma^2 T)$

More Generally: $log(\frac{S_{t_2}}{S_{t_1}}) = \int_{t_1}^{t_2} \mu(t)dt + \int_{t_1}^{t_2} \sigma^2 dB_T$

7 Ito Process

$dX_t = adt + bdB_t$

$dF = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial X}dX + \frac{b^2}{2}\frac{\partial^2 F}{\partial X^2}dt$

8 Moments of Truth

$E[dB_t] = 0$, $E[(dB_t)^2] = dt$, $E[(dB_t)^3] = 0$, $E[(dB_t)^4] = 3(dt)^2$

So, for Ito's lemma: $E[dx] = adt$, $E[dx^2] = a^2(dt)^2 + b^2dt$, $Var[dx_t^2] = b^2dt$, $Var[dx_t^2] = 2b^4(dt)^2 + O(dt^3)$

8.1 Expectation of Brownian Motion

$E[e^{\alpha Z + \beta}] = e^{\alpha^2/2 + \beta}$

Example: $E[e^B] = e^{t/2}$.

Note that $dX = \mu dt + \sigma dB$, then $X_t - X_0 = \mu t + \sigma\sqrt{t}Z$.

Hence $E[e^X] = E[e^{\mu t + \sigma\sqrt{t}Z}] = e^{t(\sigma^2/2 + \mu)}$

8.2 Ito's Lemma Example

$dS/S = \mu dt + \sigma dB$, then

$dF = d(logS) = [\mu - \sigma^2/2]dt + \sigma dB$

Generalized random Walk

$dX = \mu dt + \sigma dB$

$X_{t_2} - X_{t_1} \sim N(\mu(t_2 - t_1), \sigma^2(t_2 - t_1))$

$dX = a(t)dt + b(t)dB$

$X_{t_2} - X_{t_1} \sim N(\int_{t_1}^{t_2} a(t)dt, \int_{t_1}^{t_2} b(t)^2 dt)$

8.3 Ito Process Examples

Brownian Motion with drift

$dS_T = \mu dt + \sigma dB_t$, $S_T = S_0 + \mu T\sigma(B_T - B_0)$

Geometric Brownian Motion

$dS_T = \mu Sdt + \sigma SdB_t$,

$S_T = S_0 \cdot e^{(\mu - \sigma^2/2)T + \sigma(B_t - B_0)}$

Ornstein-Uhlenbeck (0-U)

$dS_T = \lambda(\mu - S_t -)dt + \sigma dB_t$

Cox-Ingersoll-Ross (CIR)

$d\rho_T = \lambda(\mu - \rho_t -)dt + \sigma\sqrt{\rho}dB_t$

$F = \sqrt{\rho}$, $\frac{\partial F}{\partial \rho} = \frac{1}{2\sqrt{\rho}}$, $\frac{\partial^2 F}{\partial \rho^2} = -\frac{1}{4}\rho^{-3/2}$

$dF = (\frac{4\lambda\mu - \sigma^2}{3F} - \frac{1}{2}\lambda F)dt + \frac{1}{2}\sigma dB_t$

9 Black-Scholes

$\frac{\partial V}{dt} + rS\frac{\partial V}{\partial S} + \frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} - rV = 0$

10 Interest Rates

Short rate

$V = V(t, T, y_t)$, $dy_t = a \cdot dt + b \cdot dB$

10.1 Bond Pricing

$f(t,y) = (\frac{\partial V_i}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_i}{\partial y^2} - yV_i)/\frac{\partial V_i}{\partial y}$

$\frac{\partial V_i}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V_i}{\partial y^2} - yV_i - f(t,y)\frac{\partial V_i}{\partial y} = 0$,

$V_i(T,y) = 1$

Risk Premium

$dV = (\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial y^2})dt + \frac{\partial V}{\partial y}dy =$

$= [(yV + f(t,y)\frac{\partial V}{\partial y}) + a\frac{\partial v}{\partial y}]dt + [b\frac{\partial V}{\partial y}]dB$

Market price of risk: (dt coefficient)

$\frac{dV - yVdt}{b\frac{\partial V}{\partial y}} = [\frac{a+f}{b}]dt + dB$

$\eta = \frac{a+f}{b}$, $f = b\eta - a$

10.2 Interest Rate Models

Ho & Lee: $dy = \psi(t)dt + \sigma dB$

$\psi(t) = -\frac{\partial^2}{\partial t^2}log(V_{mkt}(t_0, t) + \sigma^2(t - t_0)$

Vasicek: $dy = \alpha(\bar{y} - y)dt + \sigma dB$

Hull & White: $dy = \alpha(\bar{y} - y + \psi(t))dt\sigma dB$

Cox-Ingersoll-Ross:

$dy = \alpha(\bar{y} - y)dt + \sigma\sqrt{y}dB$

10.3 Pure Diffusion not good for rates:

$dy = \alpha(\bar{y} - y)dt + \sigma dB$

$y = e^{-\alpha t}z$, $dy = -\alpha y dt + e^{-\alpha t}dz = \alpha(\bar{y} - y)dt + \sigma dB$

$\Rightarrow dz = e^{\alpha t}[(\alpha\bar{y})dt + \sigma dB]$

$z(t) - z_0 = \bar{y}(e^{\alpha t} - 1) + \sigma\int_0^t e^{\alpha s}dB_s$

$y(t) = y_0 e^{-\alpha t} + \bar{y}(1 - e^{-\alpha t}) + \sigma e^{-\alpha t}\int_0^t e^{\alpha s}dB_s$

$E[y(t)] = y_0 + (\bar{y} - y_0)(1 - e^{-\alpha t})$,

$Var[y(t)] = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t} \to \frac{\sigma^2}{2\alpha})$

10.4 Bond Pricing

$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial y^2} - yV + \alpha(\bar{y} - y)\frac{\partial V}{\partial y} = 0$, Try a solution of the form: $V(t,y) = e^{f(t) - yg(t)}$

11 Solving PDEs

$\frac{\partial p_0}{\partial t} = \frac{\partial^2 p_0}{\partial z^2}$, $p_0 = \frac{1}{\sqrt{2\pi t}}e^{-z^2/2t}$

Gaussian is solution to any derivative with final payoff:

$V(S_t) = \int p(S_T, T; S, t)V(S_T, T)dS_T$

From Black-Scholes to Diffusion:

$V(S,t) = e^{-r(T-t)}U(S,t) \Rightarrow$

$\Rightarrow \frac{\partial U}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 U}{\partial S^2} + rS\frac{\partial U}{\partial S} = 0$

With $\tau \doteq T - t$, $\xi \doteq S$, $S = e^{\xi}$, $\xi \in [-\infty, \infty] \Rightarrow$

$\Rightarrow \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial \xi^2} - (r - \frac{\sigma^2}{2}\frac{\partial U}{\partial \xi}) = 0$

Finally,

$x \doteq \xi(r - \sigma^2/2)\tau = log(S) + (r - \sigma^2/2)\tau \Rightarrow$

$\Rightarrow \frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 U}{\partial x^2} = 0$

A Special Solution:

$U(x,\tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}}e^{-x^2/(2\sigma^2\tau)}$

General Solution: If $p(z, t=0) = f(z)$, then $p(z,t) = \int p(z-w, t)f(w)dw = = \frac{1}{\sqrt{2\pi t}}\int e^{-(z-w)^2/2t}f(w)dw$

Examples:

if $p(z,0) = z^3$, then $p(z,t) = \frac{1}{\sqrt{2\pi t}}\int e^{-(z-w)^2/2t}w^3dw$. With $u = \frac{w-z}{\sqrt{t}}$ and, $du = dw/t$, then $p(z,t) = \frac{1}{\sqrt{2\pi t}}\int e^{-u^2/2}(z + \sqrt{t}u)^3dw = z^3 + zt$, i.e., $\frac{1}{\sqrt{2}}\int e^{-u/2}u^n du = E[(X-\mu)^n]$

12 Continuous Finance

12.1 A few Special Functions

$f_1(S) = max(S - k, 0) = \frac{1}{2}(|S - K| + S - K)$

$\frac{\partial}{\partial S}f_1(S) \doteq \theta(S - K) = \begin{cases} 1 \text{ if } S > K, \\ 0 \text{ if } S \leq K \end{cases}$

$\frac{\partial^2}{\partial S^2}f_1(S) \doteq \delta(S - K) = \begin{cases} 0 \text{ if } S \neq K, \\ \infty \text{ if } S = K \end{cases}$

Dirac delta Functional

$\delta(x) = \lim_{t \to 0}\frac{1}{\sqrt{2\pi t e^{-x^2/2t}}} = \begin{cases} 0 \text{ if } x \neq 0, \\ \infty \text{ if } x = 0 \end{cases}$

$\int_{-\infty}^{\infty}\delta(x)dx = 1$

$\int_{-\infty}^{\infty}\delta(x)f(x)dx = f(0)$

$\int_{-\infty}^{\infty}\delta(x-y)f(x)dx = f(y)$

13 Survival Probabilities (no drift)

Probability to arrive without crossing z^*, obeys boundary condition $p_s(z^*, t) = 0$

$$p_s(z,t) = \begin{cases} \frac{(e^{-(z-z_0)^2/2t} - e^{-(z+z_0-2z^*)^2/2t})}{\sqrt{2\pi t}} & z > z^*, \\ 0 \text{ if } z \leq z^* = 0 \end{cases}$$

13.1 Survival example:

z = firm value = $D + E$, z^* = firm debt = D, z_0 = firm current value, $z_0 > z^*$

14 Stock Price Diffusion

If $dS = \mu S dt + \sigma S dB$, then the probability $p(S_T, T; S, t)$ satisfies:

$\frac{\partial p}{\partial t} + \frac{(\sigma S)^2}{2}\frac{\partial^2 p}{\partial S^2} + \mu S \frac{\partial p}{\partial S} = 0$

$E[f(S_T)] = \int p(S_T, T; S, t) f(S_T) dS_T = F(S,t)$, and

$\lim_{t\to T} F(S,t) = \int \delta(S_T - S) f(S_T) dS_T = f(S)$

$V(S,t) = e^{-r(T-t)}F(S,t) = e^{-r(T-t)}E_t[f(S_T)] = e^{-r(T-t)}E[V(S_T,T)]$

15 Black-Scholes Solutions

15.1 Vanilla Call option

$V(S,t) = \int p(S_T,T;S,t)V(S_T,T)dS_T = \frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int e^{-(x-w)^2/(2\sigma^2(T-t))} f(w)dw,$

where $f(w) = g(S') = max(S'-K, 0)$ for a vanilla call option with strike K and expiration T. So,

$V(S,t) =$

$\frac{e^{-r(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}}\int_{w=logK}^{\infty} e^{\frac{-(S-w)^2}{2\sigma^2(T-t)}}(e^w - K)dw =$

$S\Phi(d_+) - Ke^{-r(T-t)}\Phi(d_-)$

$d_\pm \doteq \frac{log(S/Ke^{-r(T-t)})}{\sigma\sqrt{T-t}} \pm \frac{1}{2}\sigma\sqrt{T-t}$

15.2 Binary Call Option

$f(x') = g(S') = \theta(S'-K) = \begin{cases} 1, S' \geq K \\ 0, S' < K \end{cases}$

$V(S,t) = e^{-r(T-t)}\Phi(d_-)$

15.3 Power Option

$S: X_T = S_T^2$, $logX = 2logS$,
$d(logX) = 2d(logS)$,

$S_t = S_0^2 e^{2[(\mu-\sigma^2/2)t + \sigma\sqrt{t}Z]}$

$E^Q[X_T] = S_0^2 e^{2(r-\sigma^2/2)T} e^{2\sigma^2 T}$

$V = S_0^2 e^{rT+\sigma^2 T}$

Using the risk-neutral measure (Q) and the moment-generating function for gaussian:

$Y \sim N(\mu, \sigma^2) \Rightarrow$

$f(\lambda) = E[e^{\lambda Y}] = e^{\lambda\mu + \lambda^2\sigma^2/2}$

15.4 American Perpetual Put

Put option that never expires, so it satisfies:

$\frac{(\sigma S)^2}{2}\frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} - rV = 0$

Trying a power-law solution of the form: $V(S) = S^\alpha$, then $V(S) = cS^{-2r/\sigma 2}$.

If $\hat{S}$ is price when exercised,

$V(\hat{S}) = K - \hat{S} \Rightarrow V(S) = (K-\hat{S})(\frac{S}{\hat{S}})^{-2r/\sigma^2}$

$\frac{\partial V}{\partial \hat{S}}|_{S=\hat{S}} = 0 \Rightarrow \hat{S} = \frac{K}{1+\sigma^2/2r}$

$V(S) = \frac{K\sigma^2/2r}{1+\sigma^2/2r}(\frac{S}{K}(1+\sigma^2/2r))^{-2r/\sigma 2}$

16 The Greeks

Delta

$\Delta \doteq \partial V/\partial S = \begin{cases}\Phi(d_+), \text{ if call} \\ \Phi(-d_+) = \Delta_{\text{call}} - 1 \text{ if put}\end{cases}$

Gamma

$\Gamma \doteq \partial^2 V/\partial S^2 = \frac{\Phi'(d_+)}{\sigma S\sqrt{T-t}}$

Vega

$\nu \doteq \partial V/\partial\sigma = \Phi'(d_+)S\sqrt{T-t}$

17 Martingales

Martingale: An Ito process is martingale iff it has zero drift
(i.e., $dX_t = a\cdot dt + b\cdot dB_t \Rightarrow a = 0$)
$F = e^{-rt}X$ is martingale iff $a = r$.

18 Risk-neutral Pricing

$E_t^Q[\frac{dS_t}{S_t}] = r\cdot dt$

Measure Q:

$\frac{dS_t}{S_t} = rdt + (\mu - r)dt + \sigma dB = rdt + \sigma dB^Q$,

where $dB^Q = \frac{\mu - r}{\sigma}dt + dB$. Then new differential is martingale: $E_t^Q[dB^Q] = 0$,

$Var[dB^Q] = dt$

19 Monte Carlo Pricing

$V_t = E_t^Q[\frac{V_T}{\beta_T/\beta_t}]\begin{cases} E[\cdot] \text{ Sum over paths, equal weights} \\ Q \text{ Use } r \text{ in evolution} \\ V_T \text{ Terminal value path} \\ \beta_T/\beta_t \text{ Discounting } e^{\int_t^T r(s)ds}\end{cases}$

20 Ito Processes in higher Dimensions

$dX_i = a_i(t, X_i, \ldots)dt + b_i(t, X_1, X_2, \ldots)dB_i$

$dF = \frac{\partial F}{\partial t}dt + \sum\frac{\partial F}{\partial X_i}dX_i + \frac{1}{2}\sum\rho_{ij}b_ib_j\frac{\partial^2 F}{\partial X_i\partial X_j}dt$

Rule of thumb:

$(dB_i)^2 \to dt$, $(dB_i)(dB_j) \to \rho_{ij}dt$

$(dX_i)^2 \to b_i^2 dt$, $(dX_i)(dX_j) \to \rho_{ij}b_ib_jdt$

20.1 Examples

1. $F = X_1X_2$, $dF/F = dX_1/X_1 + dX_2/X_2 + (dX_1/X_1)(dX_2/X_2)$

2. $dX_i/X_i = \mu_i dt + \sigma_i dB_i$, $dF/F = (\mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2)dt + \sigma_1 dB_1 + \sigma_2 dB_2$

21 LinAlgebra Of Assets

Let there be n securities and s states. The **payoff matrix** is: $A: \mathbb{R}^n \to \mathbb{R}^s$
$(cols \to rows)$

(Ex:) $A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$

Security $x = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ has payoffs 3, 2, and 1 for states s_1, s_2 and s_3

The Payoff Matrix, if A is **non-singular**, there is a unique solution to $s - A^{-1}b$. Requires: A invertible, Columns independent, securities l.i.

A complete market is one in which every payoff can be generated by some portfolio (from a basis assets), i.e., $rank(A) = s$.

Note:

Let $r(A) \doteq rank(A)$

$r(A) = r(A^\top) \leq min(n, s)$,

$r(AB) \leq min(r(A), r(B))$,

$r(AA^\top) = r(A)$

22 Market Value and Prices

The vector of prices S acts on portfolio vectors. Compute the market value MV of a portfolio x as:

$MV =$

$\sum S_ix_i = (S_1\ S2\ \ldots)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = S^*x = S[x]$

23 Arbitrage

23.1 Type I Arbitrage

1. Pay nothing now: $V = S^*x = \sum S_ix_i \leq 0$
2. Receive only non-negative later: $Ax \geq 0$, and at least one payoff > 0.

Example:

$A = \begin{pmatrix} 1 & 3 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $x = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

then $S^*x = 0$ but $Ax = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \geq 0$

23.2 Type II Arbitrage

1. There are redundant assets and non-trivial solutions to $Ax = 0$
2. There is arbitrage if the portfolio has a non-zero price $S^*x \neq 0$

Example:

$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$, $S = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 5 \end{pmatrix}$, $x = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$

Then $Ax = 0$ but $S^*x = 4$

24 Arrow-Debreu (AD) Securities

AD-securities $\{e_j\}_{j\leq s}$ is a unitary orthogonal basis of the payoff space (if they exist). An AD e_j security can be replicated by a portfolio x if $Ax = e_j$

The prices of AD securities are called

state prices: $\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_s \end{pmatrix}$

Example:

$A = \begin{pmatrix} 1 & 3 & 1.5 & 2 \\ 1 & 2 & 0.5 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$,

$x_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}$, $x_2 = \begin{pmatrix} -3 \\ 3 \\ -4 \\ 0 \end{pmatrix}$, $x_3 = \begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \end{pmatrix}$

Then, $Ax_1 = e_1$, $Ax_2 = e_2$, $Ax_3 = e_3$

25 Arbitrage Theorem

Consider: a market with n securities, s states and payoff matrix $A: \mathbb{R}^n \to \mathbb{R}^s$. There is no arbitrage if and only if there exists a **strictly positive** state-price vector ψ consistent with the security-price vector, i.e., $\psi \in \mathbb{R}^s, S \in \mathbb{R}^n$ and $S^* = \psi^*A (\Rightarrow S = A^*\psi)$

- if there is at one $\psi > 0$, then no arb.
- if every $\psi < 0$, then there is arb.

$$\begin{tikzcd}
\text{Security Prices } (S) \arrow[d] & \text{State Prices } (\psi) \arrow[l, bend right, "A^*"'] \arrow[d] \\
\text{Portfolios}(x) \arrow[r, bend left, "A"] & \text{Payoffs}(b)
\end{tikzcd}$$

Algebra of Arbitrage

$S[x] = \psi[b] = \psi[Ax]$

$S^*x = \psi^*Ax = (A^*\psi)^*x$

$S = A^*\psi$

26 Arbitrage Pricing Theorem

Given payoff A, prices S, target asset with payoff b:
Find all $\psi > 0$ such that $A^*\psi = S$:

- No solutions $\Rightarrow$ arbitrage
- 1 solution $\Rightarrow$ complete market
- Multiple solutions $\Rightarrow$ incomplete market

27 Optimization

27.1 Taylor's Theorem

$f(\mathbf{x}) = f(\mathbf{x_0}) + (\nabla f)^\top(\mathbf{x}-\mathbf{x_0}) + \frac{1}{2}(\mathbf{x}-\mathbf{x_0})Q(\mathbf{x}-\mathbf{x_0}) + \ldots$

where $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \end{pmatrix}$, and

$Q = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\partial x_2} & \ldots \\ \frac{\partial^2 f}{\partial x_2\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \ldots \\ \vdots & & \ddots \end{pmatrix} = Q^\top$

so that $f(\mathbf{x}) - f(\mathbf{x_0}) \approx \frac{1}{2}(\mathbf{x}-\mathbf{x_0})^\top Q(\mathbf{x}-\mathbf{x_0})$

27.2 Critical points

- if Q's **eigenvalues** are all **positive**, the function is convex up and the critical point is **minimum**.
- if Q's **eigenvalues** are all **negative**, the function is concave and the critical point is **maximum**.
- If Q has both positive and negative eigenvalues, then there are **saddle points**, which are both maximum and minimum, along different directions.
- If any eigenvalues are zero, there are **flat directions**.
- The **eigenvectors** determine the axes of orientation.

28 Lagrange Multipliers

Lagrange function: $L(x,y,\lambda) \doteq h(x,y) - \lambda\cdot(g(x,y) - c)$

Extrema occur when **all partial derivatives** of L vanish.

$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0 \iff \nabla h \propto \nabla g$

Ex: $h(x,y) = x + y$,
$g(x,y) = x^2 + y^2 = r^2$,
$L(x,y,\lambda) = x + y - \lambda(x^2 + y^2 - r^2)$

$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = \frac{\partial L}{\partial \lambda} = 0 \Rightarrow$

$x = 1/(2\lambda), y = x, x = y = \pm r/\sqrt{2}$.

$h_{max} = \sqrt{2}r, h_{min} = -\sqrt{2}r$

29 Portfolio Optimization

Return/Risk: $\mu_P = \mu^\top\boldsymbol{w}$, $\sigma_P^2 = \boldsymbol{w}^\top C\boldsymbol{w}$.

Budget Constraint: $\sum_{i\in P} w_i = 1$

29.1 Minimum-variance portfolio

:

$\mathcal{L}(\boldsymbol{w}, \ell) = \frac{1}{2}\boldsymbol{w}^\top C\boldsymbol{w} + \ell(1 - \iota^\top\boldsymbol{w})$

$\begin{cases} C & \text{covariance matrix} \\ \iota & \text{unit exposure vector}, \iota = (1\ 1\ \ldots) \\ \ell & \text{Lagrange Multiplier} \end{cases}$

Solution satisfies $\frac{\partial\mathcal{L}}{\partial w_i} = \frac{\partial\mathcal{L}}{\partial\ell} = 0$.

1. $\frac{\partial\mathcal{L}}{\partial w_i} = \sum_j C_{ij}w_j - \ell\iota_i = 0$

2. $\boldsymbol{w} = \ell C^{-1}\iota$; 3. $\iota^\top\boldsymbol{w} = \ell(\iota^\top C^{-1}\iota) = 1$

4. Solution is: $\boldsymbol{w}_{min} = \ell C^{-1}\iota = \frac{C^{-1}\iota}{\iota^\top C^{-1}\iota}$

$\sigma_{min}^2 = \ell = 1/(\iota^\top C^{-1}\iota)$

29.2 Risk & Return:

$\mathcal{L}(\boldsymbol{w}, \ell, m) = \frac{1}{2}\boldsymbol{w}^\top C\boldsymbol{w} + \underbrace{\ell(1 - \iota^\top\boldsymbol{w})}_{\text{Budget constraint}} + \underbrace{m(\mu_p - \mu^\top\boldsymbol{w})}_{\text{Return Constraint}}$

-Vary the weights $\frac{\partial\mathcal{L}}{\partial w_i} = \sum_i C_{ij}w_j - \ell\iota_i - m\mu_i = 0$, hence $\boldsymbol{w} = C^{-1}(\ell\iota + m\mu)$.

$\begin{cases} \iota^\top\boldsymbol{w} = \ell(\iota^\top C^{-1}\iota) + m(\mu^\top C^{-1}\iota) = 1 \\ \mu^\top\boldsymbol{w} = \ell(\mu^\top C^{-1}\iota) + m(\mu^\top C^{-1}\mu) = \mu_P \end{cases}$

$\Rightarrow M^{-1}\begin{pmatrix} 1 \\ \mu_P \end{pmatrix} = \begin{pmatrix} \ell \\ m \end{pmatrix}$, $M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$, where

$a \doteq \iota^\top C^{-1}\iota, b \doteq \mu^\top C^{-1}\iota, c \doteq \mu^\top C^{-1}\mu$

$\sigma_P^2 = \boldsymbol{w}^\top C\boldsymbol{w} = (\ell \quad m)M\begin{pmatrix} \ell \\ m \end{pmatrix}$, i.e.,

$\sigma_P^2 = \frac{1}{ac-b^2}(a\mu_P^2 - 2b\mu_P + c)$