Prob 2

Moon lander

state $[h, v, m]^T$ (altitude, velocity, mass)

Dynamics:-

$$\dot{h}(t) = v(t)$$

$$\dot{v}(t) = -g + \frac{a(t)}{m(t)}$$

$$\dot{m}(t) = -k\, a(t)$$

$a(t) \in (0, 1)$ on or off thrust

k - const. fuel burning rate

Initial state $[h_0, v_0, m_0]^T$, $h(t^*) = 0$, $v(t^*) = 0$

t^* - terminal time

Optimal control policy for minimum fuel consumption.

$$\min_{a(t)} \; P(a) = \int_0^{t^*} a(t)\, dt$$

which at the end give $h(t^*) = 0$, $v(t^*) = 0$

Not dependent on time.

We use PMP for this problem;

$\rightarrow$

$$f = \begin{bmatrix} v \\ -g + \frac{a}{m} \\ -ka \end{bmatrix}$$

$loss = a$

With this we form the Hamiltonian

$$\therefore \quad H = -loss + \lambda^T f$$

$$= -a + \lambda_1 v + \lambda_2 \left(-g + \frac{a}{m}\right) + \lambda_3 (-ka)$$

According to optimal policy we prioritize or focus on a

$\therefore$

$$a^* = \underset{a \in (0,1)}{\operatorname{argmax}} H$$

$$= \underset{a \in [0,1]}{\operatorname{argmax}} -a + \lambda_1 v + \lambda_2 \frac{a}{m} - \lambda_2 g - \lambda_3 ka$$

$$= \operatorname{argmax} \lambda_1 v - \lambda_2 g + \left(-1 + \frac{\lambda_2}{m} - \lambda_3 k\right) a$$

It a bang bang control policy.

let $b = -1 + \frac{\lambda_2}{m} - \lambda_3 K$

$$\therefore \quad a^* = \begin{cases} 0 & b \leq 0 \\ 1 & b > 0 \end{cases}$$

To validate this theory we need to prove b is monotonic

for monotonicity we can always take the derivative of the Hamiltonian.

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \quad \text{for all i.e. } h, v, m$$

i.e.

$$\left(-\frac{\partial H}{\partial h}, -\frac{\partial H}{\partial v}, -\frac{\partial H}{\partial M} \right)^T = \left[0 \quad -\lambda_1 \quad -\left(-\frac{\lambda_2 a}{m^2} \right) \right]^T$$

$\therefore$ We can write.

$$\dot{\lambda}_1 = 0$$
$$\dot{\lambda}_2 = -\lambda_1$$
$$\dot{\lambda}_3 = \frac{\lambda_2 a}{m^2}$$

Now for the control policy we differentiate b with time

$$\frac{db}{dt} = \frac{d}{dt}\left(-1 + \frac{\lambda_2}{m} - \lambda_3 k\right)$$

$$= \frac{\dot{\lambda}_2}{m} - \frac{\lambda_2 \dot{m}}{m^2} - \dot{\lambda}_3 k$$

$$\dot{\lambda}_2 = -\lambda_1 \quad , \quad \dot{\lambda}_3 = -\frac{\lambda_2 a}{m^2} \quad , \quad \dot{m} = -ka$$

$$\therefore \frac{db}{dt} = \frac{-\lambda_1}{m} + \cancel{\frac{\lambda_2 k a}{m^2}} + \cancel{\frac{\lambda_2 a k}{m^2}}$$

$$= \frac{-\lambda_1}{m}.$$

This proves b is monotonous hence the policy applies.

$\therefore$

The policy states:

$$a^* = 0 \rightarrow t \in [0, \tau]$$
$$a^* = 1 \rightarrow t \in [\tau, t^*]$$

This means thrust will be shut off for the initial time to some time period τ and then turned up to maximum after time τ upto the end time t^*

This also seems logical, this way the velocity will also be zero.

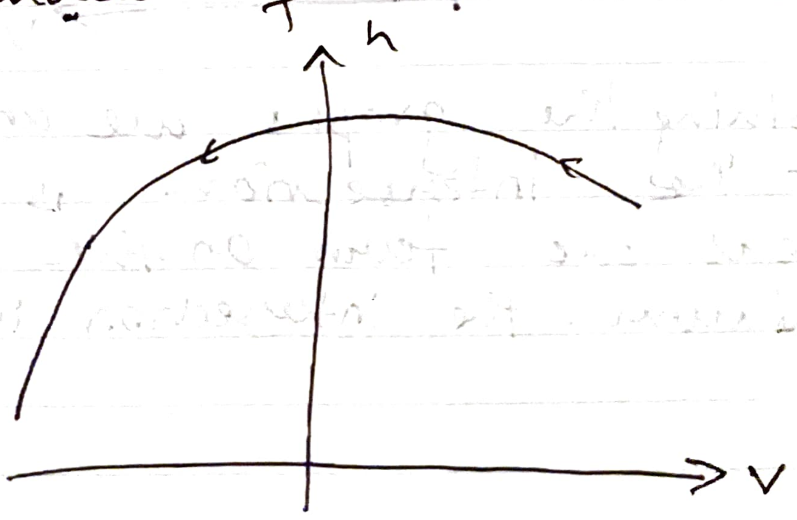
$\therefore$ for $a^* = 0$, @ $t \in (0, \tau]$

This give the dynamics as

$$f = \begin{bmatrix} v \\ -g \\ 0 \end{bmatrix} \qquad \because a = 0$$

$$\therefore \dot{h} = v$$
$$\dot{v} = -g$$
$$\dot{m} = 0$$

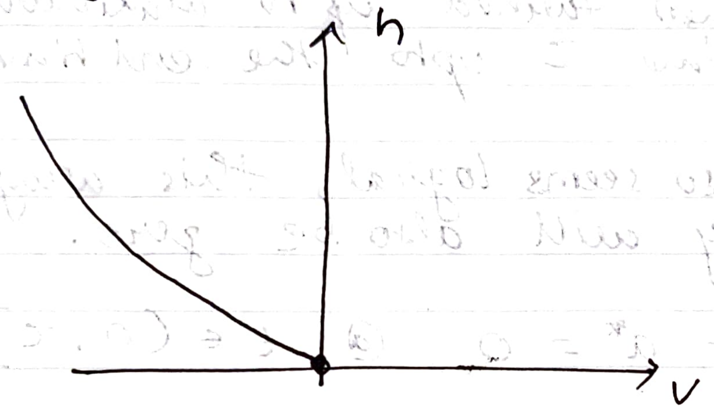
A graphical representation in 2D when its independent of $\dot{m}$.



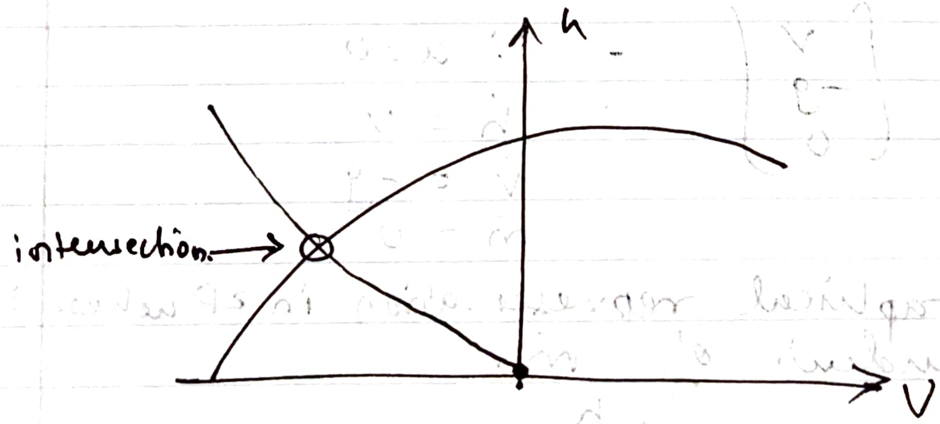
When $a^* = 1$ @ $t \in (\tau, t^*]$

$$\therefore f = \begin{bmatrix} v \\ -g + 1/m \\ -k \end{bmatrix} \therefore \begin{aligned} \dot{h} &= v \\ \dot{v} &= -g + 1/m \\ \dot{m} &= -k \end{aligned}$$

for graphical representation;



Combining the graphs,



Combining the graphs we understand that the intersection is the point where we turn on the thrust to maximum. the intersection is time τ.

There are 3 cases where the lander wont land safely,

1) Weak thrust, heavy lander
2) Not enough fuel.
3) Too near to the surface to do anything.