Quiz 1-4

This file contains all the questions from quiz 1 to quiz 4.

1. (2pt) Let R and S be two rings. Is $R \times S$ a Ring? Explain why or why not.
2. Let $(R, +, *)$ be a ring. Is $(R, +, \Delta)$ a ring? Let $a\Delta b = b * a$ for all $a, b \in R$.
3. Let $(R, +, *)$ be a ring. Is $(R, +, \Delta)$ a ring? Let $a\Delta b = -a * b$ for all $a, b \in R$.
4. (2pt) Let R be a ring. Show that $(r \cdot 0 = 0)$ for all $r \in R$.
5. Let R be a ring. Let a and b be two elements of R. Show that $a(-b) = -(ab)$ for all $a, b \in R$.
6. (2pt) Let R be a ring. Let a and b be two elements of R. Show that $(-a)(-b) = ab$ for all $a, b \in R$.
7. Is $\mathbb{Z}_{18}$ a field? Explain why or why not.
8. Let a, b, and n be elements in $\mathbb{N}$. What is the number of solutions to the congruence $ax \equiv b \pmod{n}$? Justify your answer.
9. Show that the cancellation law holds for elements that are not zero divisors.
10. Let $G_n := \{r \in \mathbb{Z}_n \mid r \text{ has a multiplicative inverse}\}$.
    - Find G_{12}.
    - Are G_n fields in general?
11. Let D be an integral domain and let $F = \mathrm{Frac}(D)$ be the field of quotients obtained from the relation
    $$(a, b) \sim (a', b') \iff ab' = a'b \qquad (a, b), (a', b') \in D \times (D \setminus \{0\}).$$
    Show that addition in F is well defined; that is, prove
    $$(a, b) \sim (a', b'),\ (c, d) \sim (c', d') \implies [(a, b)] + [(c, d)] = [(a', b')] + [(c', d')].$$
12. Working in the polynomial ring $\mathbf{Z}_7[t]$, let
    $$f(t) = 3t^3 + t^2 + 5t + 6, \qquad g(t) = 4t^4 + 3t + 1.$$
    Without expanding by the full distributive law, compute the coefficient of t^3 in the product $f(t)g(t)$.
13. Let R be a **commutative** ring with identity 1_R and let
    $$S = R \setminus \{0_R\}.$$
    On the set $R \times S$ define a relation For $(u, v), (x, y) \in R \times S$, define
    $$(u, v) \sim (x, y) \text{ in } R \times S \iff uy = vx \text{ in } R.$$
    Show: If $\sim$ is an equivalence relation, then R is an integral domain.

    Hint:

(a) Because $\sim$ is an equivalence relation, it is, in particular, *transitive*.

(b) Assume, for the sake of contradiction, that R contains a non-zero zero divisor. Choose $a, b \in R$ with $a \neq 0$, $b \neq 0$, and $ab = 0$. Examine the three elements

$$(a, 1), \qquad (0, b), \qquad (0, 1) \in R \times S,$$

and show that transitivity fails.