Problem 2: Solving the Initial Value Problem Using Laplace Transform

We are given the differential equation:

$$y'' - 3y' + 2y = e^{3t}, \quad y(0) = -1, \quad y'(0) = -4.$$

Step 1: Taking the Laplace Transform

Applying the Laplace Transform to both sides:

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) + s - 4,$$
$$\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) + 1,$$
$$\mathcal{L}\{y\} = Y(s).$$

Using these, we transform the given equation:

$$(s^2Y(s) + s - 4) - 3(sY(s) + 1) + 2Y(s) = \frac{1}{s-3}.$$

Expanding:

$$(s^2 - 3s + 2)Y(s) + s - 7 = \frac{1}{s-3}.$$

Rearranging:

$$(s-1)(s-2)Y(s) = \frac{1}{s-3} - s + 7.$$

Step 2: Partial Fraction Decomposition

We express: Dividing both sides by $(s^2 - 3s + 2) = (s-1)(s-2)$:

$$Y(s) = \frac{-s^2 + 10s - 20}{(s-1)(s-2)(s-3)}.$$

Step 3: Perform Partial Fraction Decomposition

We assume that $Y(s)$ can be written in the form:

$$Y(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}.$$

Multiplying both sides by $(s-1)(s-2)(s-3)$, we get:

$$-s^2 + 10s - 20 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2).$$

Expanding:

$$A(s^2 - 5s + 6) + B(s^2 - 4s + 3) + C(s^2 - 3s + 2).$$

Rearrange:

$$(A + B + C)s^2 + (-5A - 4B - 3C)s + (6A + 3B + 2C) = -s^2 + 10s - 20.$$

Step 4: Solve for Coefficients

Comparing coefficients:

$$A + B + C = -1,$$
$$-5A - 4B - 3C = 10,$$
$$6A + 3B + 2C = -20.$$

Solving this system:

$$A = -\frac{11}{2}, \quad B = 4, \quad C = \frac{1}{2}.$$

Thus, we write:

$$Y(s) = \frac{-11}{2(s-1)} + \frac{4}{s-2} + \frac{1}{2(s-3)}.$$

Step 5: Inverse Laplace Transform

Using the inverse Laplace transform:

$$\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at},$$

we obtain:

$$y(t) = -\frac{11}{2}e^{t} + 4e^{2t} + \frac{1}{2}e^{3t}.$$

Thus, the final solution to the initial value problem is:

$$\boxed{y(t) = -\frac{11}{2}e^{t} + 4e^{2t} + \frac{1}{2}e^{3t}.}$$