Notes for Week 10 Discussion 3D
Winter 2025

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1 Laplace Transform Basics

The Laplace transform of a function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Common Laplace Transforms

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}, \quad \mathcal{L}\{\cos(bt)\} = \frac{s}{s^2+b^2}.$$

2 Shifting Theorem

Time Shifting

If $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a).$$

Frequency Shifting

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s).$$

3 Convolution Theorem

The convolution of two functions $f(t)$ and $g(t)$ is defined as:

$$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\, d\tau.$$

Taking the Laplace transform:

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

4 Heaviside (Unit Step) Function

The Heaviside function $u(t-a)$ is defined as:

$$u(t-a) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$$

Laplace Transform of Heaviside Function

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

$$\mathcal{L}\{f(t)u(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}.$$

5 Summary of Techniques

1. **Use Laplace Transform** to convert differential equations into algebraic equations.
2. **Apply Shifting Theorem** when dealing with exponentials and delays.
3. **Use Convolution Theorem** when solving integral equations.
4. **Heaviside Functions** model piecewise and switch-on behavior.
5. **Partial Fraction Decomposition** helps invert rational Laplace expressions.
6. **Inverse Laplace Transform** retrieves the original time-domain function.