Detection et Estimation

TP:
Detection

Master SISEA

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Mauricio Caceres

Pierre-Samuel Garreau-Hamard

Enseignant : Di Ge

Chapter 1

Introduction

1.1 Objectif

1.2 Write the log likelihoood funciton $L(A, \phi)$ for a given observation vector

We have the signal

$$x(n) = Acos(2\pi f_0 n + \phi) + w(n) \tag{1.1}$$

The non deterministic part of the signal is the noise. A White Gaussian noise with variance σ^2, wich is common in many fields. So the log-likelihood function is evaluated in function of this noise with the next expression.

$$w(n) = x(n) - Acos(2\pi f_0 n + \phi) \tag{1.2}$$

The WGN distributions are iid so

$$P(w(n)|A, \theta) = \prod_{n=0}^{N-1} P_w(w(n)) \tag{1.3}$$

So the log-likelihood function $L(A, \phi)$ is

$$L(A, \phi) = logP(w(n)|A, \theta) \tag{1.4}$$

$$L(A, \phi) = log \prod_{n=0}^{N-1} P_w(w(n)) \tag{1.5}$$

$$L(A, \phi) = \sum_{n=0}^{N-1} logP_w(w(n)) \tag{1.6}$$

$$L(A,\phi) = \sum_{n=0}^{N-1} log(\frac{1}{\sigma\sqrt{2\pi}} \exp[-1/2(\frac{x(n) - Acos(2\pi f_0 n + \phi) - \mu}{\sigma})^2] \tag{1.7}$$

$$L(A,\phi) = \sum_{n=0}^{N-1} B + ([-1/2(\frac{x(n) - Acos(2\pi f_0 n + \phi) - \mu}{\sigma})^2]a \tag{1.8}$$

1.3 Show that the maximun likelihood estimators are the solution of the following equations

$$\frac{\partial L(A,\phi)}{\partial A} = \sum_{n=0}^{N-1} (\frac{x(n) - Acos(2\pi f_0 n + \phi) - \mu}{\sigma})(\frac{cos(2\pi f_0 n + \phi}{\sigma}) \tag{1.9}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = \sum_{n=0}^{N-1} -(\frac{x(n) - Acos(2\pi f_0 n + \phi) - \mu}{\sigma})(\frac{Asin(2\pi f_0 n + \phi}{\sigma}) \tag{1.10}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} -(\frac{Ax(n)sin(2\pi f_0 n + \phi)}{\sigma^2} - \frac{A^2 cos(2\pi f_0 n + \phi) sin(2\pi f_0 n + \phi)}{\sigma^2}) = 0 \tag{1.11}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} -(\frac{Ax(n)sin(2\pi f_0 n + \phi)}{\sigma^2} - \frac{A^2 cos(2\pi f_0 n + \pi) sin(2\pi f_0 n + \phi)}{\sigma^2}) = 0 \tag{1.12}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \frac{A}{\sigma^2} \sum_{n=0}^{N-1} -(\frac{Ax(n)sin(2\pi f_0 n + \phi) - A^2 cos(2\phi f_0 n + \pi) sin(2\phi f_0 n + \phi)}{\sigma^2}) = 0 \tag{1.13}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} (x(n)sin(2\pi f_0 n + \phi) - Acos(2\pi f_0 n + \pi)) sin(2\phi f_0 n + \phi)) = 0 \tag{1.14}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} (x(n)sin(2\pi f_0 n + \phi) - A \sum_{n=0}^{N-1} cos(2\pi f_0 n + \phi) sin(2\pi f_0 n + \phi)) = 0 \tag{1.15}$$

Using the entity

$$\sum_{n=0}^{N-1} cos(2\pi f_0 n + \phi) sin(2\pi f_0 n + \phi)) \approx 0 \tag{1.16}$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} (x(n) sin(2\pi f_0 n + \phi) = 0 \tag{1.17}$$

Developing the sum of angles

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} x(n)(sin(2\pi f_0 n + \phi)cos(2\pi f_0 n + \phi) + cos(2\pi f_0 n + \phi)sin(2\pi f_0 n + \phi)) = 0$$

$$\frac{\partial L(A,\phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} x(n)(sin(2\pi f_0 n + \phi)cos(\phi) = -\sum_{n=0}^{N-1} x(n)cos(2\pi f_0 n + \phi)sin(\phi)) \tag{1.18}$$

$$\frac{\sum_{n=0}^{N-1} x(n) sin(2\pi f_0 n)}{\sum_{n=0}^{N-1} x(n) cos(2\pi f_0 n)} = \frac{sin(\phi)}{cos(\phi)} = tg(\phi) \tag{1.19}$$

We verify the expresion

$$\hat{\Phi}_{ML} = arctg(-\frac{\sum_{n=0}^{N-1} x(n) sin(2\pi f_0 n)}{\sum_{n=0}^{N-1} x(n) cos(2\pi f_0 n)}) \tag{1.20}$$

1.4 Give the condition under which $\hat{\Phi}_M L \approx \phi$ different from $E|\hat{\Phi}_M L| \approx \phi$

It's clear that it's not the same when $E|\hat{\Phi}_M L| \approx \phi$ where the mean of the distribution of values of the estimator is close to the true value of ϕ to the next expresion $\hat{\Phi}_M L \approx \phi$ that means that the values of the estimator are always close to the true value of ϕ (the distribution is more strait around the true value)

Using

$$sen(\alpha)cos(\alpha) = 1/2 sin(2\alpha)$$
$$cos^2(\alpha) = \frac{1 + cos(2\alpha)}{2}$$
$$sin^2(\alpha) = \frac{1 - cos(2\alpha)}{2}$$

In the entities of the copy, we obtain

$$\sum_{n=0}^{N-1} cos(4\pi f_0 n) \approx 0 \tag{1.21}$$

$$\sum_{n=0}^{N-1} sin(4\pi f_0 n) \approx 0 \tag{1.22}$$

And the last entity

$$\sum_{n=0}^{N-1} (e^{j4\pi f_0 n}) \approx 0 \tag{1.23}$$

$f_0 t(0, 1/2)$ ne sont pas des valeurs a prendre a cause du theoreme de sannon
Donc il faut pas que fo = 1/2 et 0
Il faut excluire les axes $\phi = 4\pi fo$
$\phi = 2\pi$

1.5 Verify that the maximun-likehood estimator of the amplitude is unbiased if $\phi = \hat{\Phi}_{ML}$

The estimator is unbiased if $E[\hat{A}] = A$ thus

$$E[\frac{2}{N} \sum_{n=0}^{N-1} x(n) cos(2\pi f_0 n + \hat{\Phi}_{ML})] \tag{1.24}$$

with $\phi = \hat{\Phi}_{ML}$

$$\frac{2}{N} E[\sum_{n=0}^{N-1} A cos^2(2\pi f_0 n + \phi)] + E[w(n)] cos(2\pi f_0 n + \phi) = A \tag{1.25}$$

And $E[w(n) = 0$

$$\frac{2}{N} E[\sum_{n=0}^{N-1} A cos^2(2\pi f_0 n] = A \tag{1.26}$$

$$\frac{2}{N} A \frac{N}{2} = A \tag{1.27}$$

And we verify that the estimator is unbiased

$$E[\hat{A}] = A \tag{1.28}$$

1.6 Verify thath thw maximun-likelihood estimator of the amplitude is unbiase if $\phi = \hat{\Phi}_{ML}$

Condition pour $\hat{\Phi}_{ML} = \Phi$

$$tan(\hat{\Phi}_{ML}) = tan(\Phi) \tag{1.29}$$

$$\frac{\sum_{n=0}^{N-1} x(n)sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} x(n)cos(2\pi f_0 n + \phi)} =$$
$$\frac{\sum_{n=0}^{N-1} Acos(2\pi f_0 n + \phi) + w(n)sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} Acos(2\pi f_0 n + \phi) + w(n)cos(2\pi f_0 n + \phi)}$$

$$\frac{\sum_{n=0}^{N-1} x(n)sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} x(n)cos(2\pi f_0 n + \phi)} =$$
$$\frac{\sum_{n=0}^{N-1} Acos(2\pi f_0 n + \phi) + w(n)sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} Acos(2\pi f_0 n + \phi) + w(n)cos(2\pi f_0 n + \phi)}$$

$$\frac{AN/2sin(\phi)\sum_{n=0}^{N-1} w(n)sin(2\pi f_0 n)}{AN/2sin(\phi)\sum_{n=0}^{N-1} w(n)cos(2\pi f_0 n)} \approx tan(\Phi)$$

This term will be next to $tan(\Phi)$ when

$$\sum_{n=0}^{N-1} |w(n)sin(2\pi f_0 n)| << |AN/2sin(\phi)| \tag{1.30}$$

and when we have

$$\sum_{n=0}^{N-1} |w(n)cos(2\pi f_0 n)| << |AN/2cos(\phi)| \tag{1.31}$$

The noise have a normal distribution so the sum of noise have the next distribution

$$\sum_{n=0}^{N-1} |w(n)sin(2\pi f_0 n)| \sim N(0, N\sigma^2/2) \tag{1.32}$$

We take a probabilité of 3σ

$$3\sqrt{\sigma^2 N/2} = 3\sqrt{N/2}\sigma << AN/2sin(\phi)$$
$$3 << \frac{A\sqrt{N}}{\sigma} sin(\phi)$$

For larges values of N $\hat{\Phi}_M L \approx \phi$

We cans observe that the parameters A, σ and N can modified the estimator properties.

The ratio $\frac{A^2}{\sigma^2}$ represent the RSB

1.7 To compute the term of Fisher Matrix

Making the derivation to the other parameter ϕ

$$\frac{\partial^2 L(A,\phi)}{\partial \phi^2} = \sum_{n=0}^{N-1} -(\frac{Asin(2\pi f_0 n+\phi)}{\sigma})(\frac{Asin(2\pi f_0 n+\phi)}{\sigma})+$$
$$\sum_{n=0}^{N-1} -(\frac{x(n)-Acos(2\pi f_0 n+\phi)}{\sigma})(\frac{Acos(2\pi f_0 n+\phi)}{\sigma})$$

$$\frac{\partial^2 L(A,\phi)}{\partial \phi^2} = \sum_{n=0}^{N-1} -(\frac{A^2 sin^2(2\pi f_0 n+\phi)}{\sigma^2})+$$
$$\sum_{n=0}^{N-1} -(\frac{x(n)Acos(2\pi f_0 n+\phi) - A^2 cos^2(2\pi f_0 n+\phi)}{\sigma^2})$$

$$\frac{\partial^2 L(A,\phi)}{\partial \phi^2} = \sum_{n=0}^{N-1} -(\frac{A^2 sin^2(2\pi f_0 n+\phi)}{\sigma^2})+$$
$$\sum_{n=0}^{N-1} (\frac{w(n)}{\sigma^2})$$

$$-\sum_{n=0}^{N-1} E[\frac{\partial^2 L(A,\phi)}{\partial \phi^2}] = \sum_{n=0}^{N-1} \frac{A^2 sin^2(2\pi f_0 n+\phi)}{\sigma^2} \tag{1.33}$$

We obtain one of the terms of the fisher information matrix

$$-\sum_{n=0}^{N-1} E[\frac{\partial^2 L(A,\phi)}{\partial \phi^2}] = \frac{A^2 N}{\sigma^2 2} \tag{1.34}$$

The cross derivation to the other parameter

$$[\frac{\partial^2 L(A,\phi)}{\partial \phi \partial A}] = -\sum_{n=0}^{N-1} \frac{-cos(2\pi f_0 n+\phi)}{\sigma} \frac{Asin(2\pi f_0 n+\phi)}{\sigma}$$
$$+\sum_{n=0}^{N-1} -\frac{x(n)-Acos(2\pi f_0 n+\phi)}{\sigma} \frac{sin(2\pi f_0 n+\phi)}{\sigma} = [\frac{\partial^2 L(A,\phi)}{\partial A \partial \phi}]$$

$$-\sum_{n=0}^{N-1} E[\frac{\partial^2 L(A,\phi)}{\partial\phi\partial A}] = -\sum_{n=0}^{N-1} \frac{-cos(2\pi f_0 n+\phi)}{\sigma}\frac{Asin(2\pi f_0 n+\phi)}{\sigma} \tag{1.35}$$

Using the simplification in equation 1.16 we will obtain a symmetric matrix

$$-\sum_{n=0}^{N-1} E[\frac{\partial^2 L(A,\phi)}{\partial\phi\partial A}] = 0 \tag{1.36}$$

The derivation respect the parameter A

$$\frac{\partial^2 L(A,\phi)}{\partial A^2} = -\sum_{n=0}^{N-1}(\frac{-cos(2\pi f_0 n+\phi)}{\sigma})(\frac{cos(2\pi f_0 n+\phi)}{\sigma}) \tag{1.37}$$

$$\frac{\partial^2 L(A,\phi)}{\partial A^2} = \frac{-1}{\sigma} - \sum_{n=0}^{N-1} cos^2(2\pi f_0 n+\phi) = \frac{-N}{2\sigma} \tag{1.38}$$

The calculation of the expeted value for the fisher information matrix is

$$-\sum_{n=0}^{N-1} E[\frac{\partial^2 L(A,\phi)}{\partial A^2}] = N2\sigma \tag{1.39}$$

The fisher information matrix is

$$J(A,\phi) = \begin{pmatrix} \frac{A^2 N}{\sigma^2 2} & 0 \\ 0 & \frac{N}{2\sigma^2} \end{pmatrix} \tag{1.40}$$

The Cramer-Rao lower bound in the unbiased case is

$$CRLB = J(A,\phi)^{-1} == \begin{pmatrix} \frac{\sigma^2 2}{A^2 N} & 0 \\ 0 & \frac{2\sigma^2}{N} \end{pmatrix} \tag{1.41}$$

Chapter 2

Implementation in Matlab (Scilab)

2.1 Results

2.2 Conclusion