

# Detection et Estimation

## TP: Detection

### Master SISEA

18 décembre 2016

Mauricio Caceres

Pierre-Samuel Garreau-Hamard

Enseignant : Di Ge



# Chapter 1

## Introduction

### 1.1 Objectif

### 1.2 Write the log likelihood function $L(A, \phi)$ for a given observation vector

We have the signal

$$x(n) = A \cos(2\pi f_0 n + \phi) + w(n) \quad (1.1)$$

The non deterministic part of the signal is the noise. A White Gaussian noise with variance  $\sigma^2$ , which is common in many fields. So the log-likelihood function is evaluated in function of this noise with the next expression.

$$w(n) = x(n) - A \cos(2\pi f_0 n + \phi) \quad (1.2)$$

The WGN distributions are iid so

$$P(w(n)|A, \theta) = \prod_{n=0}^{N-1} P_w(w(n)) \quad (1.3)$$

So the log-likelihood function  $L(A, \phi)$  is

$$L(A, \phi) = \log P(w(n)|A, \theta) \quad (1.4)$$

$$L(A, \phi) = \log \prod_{n=0}^{N-1} P_w(w(n)) \quad (1.5)$$

$$L(A, \phi) = \sum_{n=0}^{N-1} \log P_w(w(n)) \quad (1.6)$$

$$L(A, \phi) = \sum_{n=0}^{N-1} \log\left(\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-1/2\left(\frac{x(n) - A\cos(2\pi f_0 n + \phi) - \mu}{\sigma}\right)^2\right]\right) \quad (1.7)$$

$$L(A, \phi) = \sum_{n=0}^{N-1} B + ([-1/2\left(\frac{x(n) - A\cos(2\pi f_0 n + \phi) - \mu}{\sigma}\right)^2]_a \quad (1.8)$$

**1.3 Show that the maximun likelihood estimators are the solution of the following equations**

$$\frac{\partial L(A, \phi)}{\partial A} = \sum_{n=0}^{N-1} \left(\frac{x(n) - A\cos(2\pi f_0 n + \phi) - \mu}{\sigma}\right) \left(\frac{\cos(2\pi f_0 n + \phi)}{\sigma}\right) \quad (1.9)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = \sum_{n=0}^{N-1} -\left(\frac{x(n) - A\cos(2\pi f_0 n + \phi) - \mu}{\sigma}\right) \left(\frac{A\sin(2\pi f_0 n + \phi)}{\sigma}\right) \quad (1.10)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} -\left(\frac{Ax(n)\sin(2\pi f_0 n + \phi)}{\sigma^2} - \frac{A^2\cos(2\pi f_0 n + \phi)\sin(2\pi f_0 n + \phi)}{\sigma^2}\right) = 0 \quad (1.11)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} -\left(\frac{Ax(n)\sin(2\pi f_0 n + \phi)}{\sigma^2} - \frac{A^2\cos(2\pi f_0 n + \pi)\sin(2\pi f_0 n + \phi)}{\sigma^2}\right) = 0 \quad (1.12)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \frac{A}{\sigma^2} \sum_{n=0}^{N-1} -\left(\frac{Ax(n)\sin(2\pi f_0 n + \phi) - A^2\cos(2\phi f_0 n + \pi)\sin(2\phi f_0 n + \phi)}{\sigma^2}\right) = 0 \quad (1.13)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} (x(n)\sin(2\pi f_0 n + \phi) - A\cos(2\pi f_0 n + \pi))\sin(2\phi f_0 n + \phi) = 0 \quad (1.14)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} (x(n)\sin(2\pi f_0 n + \phi) - A \sum_{n=0}^{N-1} \cos(2\pi f_0 n + \phi)\sin(2\pi f_0 n + \phi)) = 0 \quad (1.15)$$

Using the entity

$$\sum_{n=0}^{N-1} \cos(2\pi f_0 n + \phi) \sin(2\pi f_0 n + \phi) \approx 0 \quad (1.16)$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} (x(n) \sin(2\pi f_0 n + \phi)) = 0 \quad (1.17)$$

Developing the sum of angles

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} x(n) (\sin(2\pi f_0 n + \phi) \cos(2\pi f_0 n + \phi) + \cos(2\pi f_0 n + \phi) \sin(2\pi f_0 n + \phi)) = 0$$

$$\frac{\partial L(A, \phi)}{\partial \phi} = 0 = \sum_{n=0}^{N-1} x(n) (\sin(2\pi f_0 n + \phi) \cos(\phi) - \cos(2\pi f_0 n + \phi) \sin(\phi)) = - \sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n + \phi) \sin(\phi) \quad (1.18)$$

$$\frac{\sum_{n=0}^{N-1} x(n) \sin(2\pi f_0 n)}{\sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n)} = \frac{\sin(\phi)}{\cos(\phi)} = \tan(\phi) \quad (1.19)$$

$$\hat{\Phi}_{ML} = \arctan\left(-\frac{\sum_{n=0}^{N-1} x(n) \sin(2\pi f_0 n)}{\sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n)}\right) \quad (1.20)$$

Making the derivation to the other parameter

$$\begin{aligned} \frac{\partial^2 L(A, \phi)}{\partial \phi^2} &= \sum_{n=0}^{N-1} -\left(\frac{A \sin(2\pi f_0 n + \phi)}{\sigma}\right) \left(\frac{A \sin(2\pi f_0 n + \phi)}{\sigma}\right) + \\ &\sum_{n=0}^{N-1} -\left(\frac{x(n) - A \cos(2\pi f_0 n + \phi)}{\sigma}\right) \left(\frac{A \cos(2\pi f_0 n + \phi)}{\sigma}\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 L(A, \phi)}{\partial \phi^2} &= \sum_{n=0}^{N-1} -\left(\frac{A^2 \sin^2(2\pi f_0 n + \phi)}{\sigma^2}\right) + \\ &\sum_{n=0}^{N-1} -\left(\frac{x(n) A \cos(2\pi f_0 n + \phi) - A^2 \cos^2(2\pi f_0 n + \phi)}{\sigma^2}\right) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 L(A, \phi)}{\partial \phi^2} &= \sum_{n=0}^{N-1} -\left(\frac{A^2 \sin^2(2\pi f_0 n + \phi)}{\sigma^2}\right) + \\ &\sum_{n=0}^{N-1} \left(\frac{w(n)}{\sigma^2}\right) \end{aligned}$$

## 1.4 To compute the term of Fisher Matrix

$$-\sum_{n=0}^{N-1} E\left[\frac{\partial^2 L(A, \phi)}{\partial \phi^2}\right] = \sum_{n=0}^{N-1} \frac{A^2 \sin^2(2\pi f_0 n + \phi)}{\sigma^2} \quad (1.21)$$

$$-\sum_{n=0}^{N-1} E\left[\frac{\partial^2 L(A, \phi)}{\partial \phi^2}\right] = \frac{A^2 N}{\sigma^2 2} \quad (1.22)$$

$$\begin{aligned} \left[\frac{\partial^2 L(A, \phi)}{\partial \phi^2}\right] &= -\sum_{n=0}^{N-1} \frac{-\cos(2\pi f_0 n + \phi)}{\sigma} \frac{A \sin(2\pi f_0 n + \phi)}{\sigma} \\ &+ \sum_{n=0}^{N-1} -\frac{x(n) - A \cos(2\pi f_0 n + \phi)}{\sigma} \frac{\sin(2\pi f_0 n + \phi)}{\sigma} \end{aligned}$$

$$-\sum_{n=0}^{N-1} E\left[\frac{\partial^2 L(A, \phi)}{\partial \phi^2}\right] = -\sum_{n=0}^{N-1} \frac{-\cos(2\pi f_0 n + \phi)}{\sigma} \frac{A \sin(2\pi f_0 n + \phi)}{\sigma} \quad (1.23)$$

Using the simplification in equation 1.16

$$-\sum_{n=0}^{N-1} E\left[\frac{\partial^2 L(A, \phi)}{\partial \phi^2}\right] = 0 \quad (1.24)$$

## 1.5 Verify that the maximum-likelihood estimator of the amplitude is unbiased if $\phi = \hat{\Phi}_{ML}$

Condition pour  $\hat{\Phi}_{ML} = \Phi$

$$\tan(\hat{\Phi}_{ML}) = \tan(\Phi) \quad (1.25)$$

$$\begin{aligned} &\frac{\sum_{n=0}^{N-1} x(n) \sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n + \phi)} = \\ &\frac{\sum_{n=0}^{N-1} A \cos(2\pi f_0 n + \phi) + w(n) \sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} A \cos(2\pi f_0 n + \phi) + w(n) \cos(2\pi f_0 n + \phi)} \end{aligned}$$

$$\begin{aligned} &\frac{\sum_{n=0}^{N-1} x(n) \sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} x(n) \cos(2\pi f_0 n + \phi)} = \\ &\frac{\sum_{n=0}^{N-1} A \cos(2\pi f_0 n + \phi) + w(n) \sin(2\pi f_0 n + \phi)}{\sum_{n=0}^{N-1} A \cos(2\pi f_0 n + \phi) + w(n) \cos(2\pi f_0 n + \phi)} \end{aligned}$$

$$\frac{AN/2\sin(\phi) \sum_{n=0}^{N-1} w(n)\sin(2\pi f_0 n)}{AN/2\sin(\phi) \sum_{n=0}^{N-1} w(n)\cos(2\pi f_0 n)} \approx \tan(\Phi)$$

This term will be next to  $\tan(\Phi)$  when

$$\sum_{n=0}^{N-1} |w(n)\sin(2\pi f_0 n)| \ll |AN/2\sin(\phi)| \quad (1.26)$$

and when we have

$$\sum_{n=0}^{N-1} |w(n)\cos(2\pi f_0 n)| \ll |AN/2\cos(\phi)| \quad (1.27)$$

The noise have a normal distribution so the sum of noise have the next distribution

$$\sum_{n=0}^{N-1} |w(n)\sin(2\pi f_0 n)| \sim N(0, N\sigma^2/2) \quad (1.28)$$

We take a probabilité of  $3\sigma$

$$3\sqrt{\sigma^2 N/2} = 3\sqrt{N/2}\sigma \ll AN/2\sin(\phi)$$

$$3 \ll \frac{A\sqrt{N}}{\sigma}\sin(\phi)$$

For larges values of N  $\hat{\Phi}_M L \approx \phi$

We cans observe that the parameters A,  $\sigma$  and N can modified the estimator properties.

The ratio  $\frac{A^2}{\sigma^2}$  represent the RSB

## 1.6 Give the condition under which $\hat{\Phi}_M L \approx \phi$ different from $E|\hat{\Phi}_M L| \approx \phi$

Using

$$\sin(\alpha)\cos(\alpha) = 1/2\sin(2\alpha)$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2}$$

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2}$$

In the entities of the copy, we obtain

$$\sum_{n=0}^{N-1} \cos(4\pi f_0 n) \approx 0 \quad (1.29)$$

$$\sum_{n=0}^{N-1} \sin(4\pi f_0 n) \approx 0 \quad (1.30)$$

And the last entity

$$\sum_{n=0}^{N-1} (e^{j4\pi f_0 n}) \approx 0 \quad (1.31)$$

$f_0 t(0, 1/2)$  ne sont pas des valeurs a prendre a cause du theoreme de sannou

Donc il faut pas que  $f_0 = 1/2$  et 0

Il faut exclure les axes  $\phi = 4\pi f_0$

$\phi = 2\pi$

## Chapter 2

# Implémentation en MATLAB®

2.1 Estimation de l'information mutuelle

2.2 Conclusion