# Polynomial optimization on finite sets.

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# LECTURE 3:
# Symmetry in Sums of squares.

# Outline:

Assume $n$ is an even integer and let $X := \{-1, 1\}^n$. In this lecture we will prove lower bounds on the degree needed to represent a nonnegative quadratic function in $\mathbb{R}[X]$ as a sum-of-squares.

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*The quadratic function*

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It follows that $0 = f_{\min} > f_{(n/2)}$ so the smallest degree where equality holds for every quadratic $f$ is at least $n/2 + 1$.

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- The function $f$ is **invariant under all permutations of the coordinates**.
- The proof [due to G.Blekherman (unpublished)] characterizes all possible **invariant sums-of-squares in** $\mathbb{R}[X]$ of degree $\leq r$.
- The key tool for this characterization will be the relationship between **representation theory** and sums-of-squares [Gatermann-Parrilo (2004)].

# Plan for Lecture 3:

1. A primer on representation theory.
2. Invariant sums-of-squares.
3. Application: Invariant sums-of-squares in the hypercube.

# Part 1:
# A primer on representation theory.

Let $G$ be a finite group

**Definition.**

*A* **representation of** $G$ *is a pair* $(V, \rho)$ *where*

1. $V$ *is a finite-dimensional vector space.*
2. $\rho : G \to GL(V)$ *is a group homomorphism.*

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**Definition.**

*A* **morphism** *between two representations* $(V_1, \rho_1)$ *and* $(V_2, \rho_2)$ *of* $G$ *is a linear map* $T : V_1 \to V_2$ *satisfying*

$$T \circ \rho_1(g) = \rho_2(g) \text{ for all } g \in G.$$

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For $V = \mathbb{R}^2$ define $\rho : S_2 \to GL(V)$ by

$$\rho(id) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \rho(\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

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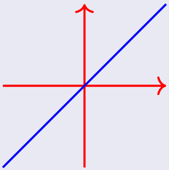
## Remark.

*Representations are the way in which abstract groups become concrete symmetries of a space.*



$$\tau^2 = id \text{ so } \rho(\tau)\rho(\tau) = \rho(id)$$

Let $(V, \rho)$ be a representation of $G$.

**Definition.**

*A vector subspace* $W \subseteq V$ *is a* **stable subspace** *of* $V$ *if*

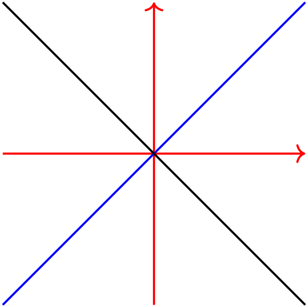
$$\forall w \in W \; \forall g \in G \, (\rho(g)(w) \in W) \, .$$

**Definition.**

*The representation* $(V, \rho)$ *is* **irreducible** *if its only stable subspaces are* $\{0\}$ *and* $V$*.*

# Example

The representation $(V, \rho)$ of $S_2$ has two nontrivial stable subspaces



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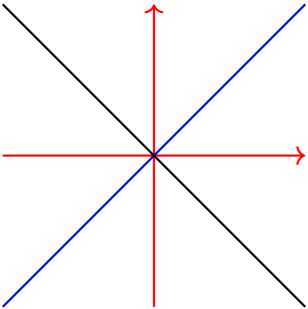

The representation $(V, \rho)$ is NOT irreducible.

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1. Representations appear **everywhere**.For instance if $(V_1, \rho_1)$ and $(V_2, \rho_2)$ are two representations of $G$ then $(V_1 \oplus V_2, \rho_1 \oplus \rho_2)$, $(V_1 \otimes V_2, \rho_1 \otimes \rho_2)$, $(\mathrm{Hom(V_1, V_2)}, \rho_1^* \otimes \rho_2)$ are representations of $G$.
2. There is an effective, complete and rather rigid **classification of the representations of** $G$, summarized in the following two slides...

Let $G$ be a finite group. We have

**Theorem. (Irreducible representations)**

*The following statements hold:*

1. *There exists finitely many non-isomorphic irreducible representations of $G$. We denote them $V^{(1)}, V^{(2)}, \dots, V^{(c)}$ where $c$ is the number of conjugacy classes of $G$.*
2. *The morphisms between irreducible representations satisfy*

$$\mathrm{Hom}_{G}(V^{(i)}, V^{(j)}) \cong \begin{cases} 0, & \text{if } i \neq j \\ \mathbb{C}, & \text{if } i = j. \end{cases}$$

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The irreducible representations serve as building blocks of all other representations...

Let $(W, \rho)$ be any representation of a finite group $G$ and let $\langle , \rangle$ be a $G$-invariant inner product on $W$ ($\langle u, v\rangle = \langle \rho_W(g)u, \rho_W(g)v\rangle$).

**Theorem.**

*There exist mutually orthogonal stable subspaces of $W$:*

$$V_1^{(1)}, \ldots, V_{m_1}^{(1)}, V_1^{(2)}, \ldots, V_{m_2}^{(2)}, \ldots, V_1^{(c)}, \ldots, V_{m_c}^{(c)} \subseteq W$$

*Such that:*

1. *The restriction of $W$ to $V_j^{(i)}$ is isomorphic to $V^{(i)}$*
2. $$W = \left(\bigoplus_{i_1=1}^{m_1} V_{i_1}^{(1)}\right) \oplus \left(\bigoplus_{i_2=1}^{m_2} V_{i_2}^{(2)}\right) \oplus \cdots \oplus \left(\bigoplus_{i_c=1}^{m_c} V_{i_c}^{(c)}\right)$$
3. *The integers $m_1, \ldots, m_c$ and the* **isotypical components**

$$W_j := \left(\bigoplus_{i_j=1}^{m_j} V_{i_j}^{(j)}\right) \quad \textit{for } j = 1, \ldots, c$$

*are uniquely determined.*

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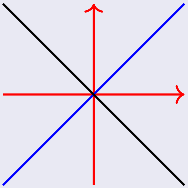

*With respect to bases adapted to the stable subspaces we have*

$$\rho(id) := \begin{array}{c c} & \begin{array}{cc} \color{blue}{triv} & sgn \end{array} \\ \begin{array}{c} \color{blue}{triv} \\ sgn \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad \rho(\tau) := \begin{array}{c c} & \begin{array}{cc} \color{blue}{triv} & sgn \end{array} \\ \begin{array}{c} \color{blue}{triv} \\ sgn \end{array} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

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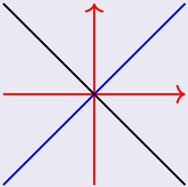

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*Both matrices became* ***simultaneously diagonal****.*

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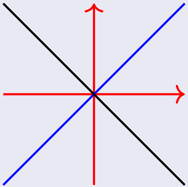

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*Both matrices became* **simultaneously diagonal***. In general the matrices will become* **simultaneously block-diagonal***.*

# Part 2:
# Invariant sums of squares.

Let $X \subseteq \mathbb{R}^n$ be a finite set and let $G$ be a subgroup of permutations of $X$. The ring $\mathbb{R}[X]$ is naturally a representation of $G$.

**Definition.**

*The* **contragradient representation** $(\mathbb{R}[X], \rho^*)$ *is the representation defined for* $f \in \mathbb{R}[X]$ *and* $g \in G$ *by the formula*

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This representation contains an important subring, the **ring of invariants** of $X$,

$$\mathbb{R}[X]^G := \{f \in \mathbb{R}[X] : \forall g \in G\,(\rho^*(g)(f) = f)\}\,.$$

it is the isotypical component of the trivial representation.

Our main objective will be trying to understand the **invariant sums-of-squares** of some degree $r$, that is the sets

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Example:

*Consider the action of $S_2$ in $\mathbb{R}[x, y]$ by coordinate permutations. The polynomial $x^2 + y^2$ is invariant but $x, y$ are not. However*

$$x^2 + y^2 = \left(\frac{x+y}{\sqrt{2}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2$$

*The $s_i$ are not invariant but do live in a fixed isotypical component.*

**Definition.**

*The* **averaging operator** *is the map* $\mathcal{A} : \mathbb{R}[X] \to \mathbb{R}[X]$ *given by*

$$\mathcal{A}(f) := \frac{1}{|G|} \sum_{g \in G} \rho^*(g)(f)$$

The following properties hold:

1. For every $f \in \mathbb{R}[X]$ the image $\mathcal{A}(f) \in \mathbb{R}[X]^G$.
2. $\mathcal{A}(f) = f$ if and only if $f \in \mathbb{R}[X]^G$
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# Main result:

Assume $\mathbb{R}[X]_{\leq r}$ is a stable subspace of $\mathbb{R}[X]$ as a representation of the group $G$.

**Theorem. (Invariant sums-of-squares)**

*If* $\mathbb{R}[X]_{\leq r} = W_1 \oplus \cdots \oplus W_c$ *is the isotypical decompostion of* $\mathbb{R}[X]_{\leq r}$ *then the following equality holds*

$$\Sigma^G_{\leq r} = \mathcal{A}(\Sigma_{W_1}) + \cdots + \mathcal{A}(\Sigma_{W_c})$$

# Proof

Suppose $f = s_1^2 + \cdots + s_N^2$ is an invariant sum of squares.

$$f = \mathcal{A}(f) = \sum_{i=1}^{N} \mathcal{A}(s_i^2)$$

Any summand $s = s_i \in \mathbb{R}[X]_{\leq r}$ has an isotypical decomposition

$$s = w_1 + \cdots + w_c$$

$$\mathcal{A}(s^2) = \mathcal{A}\left(w_1^2 + \cdots + w_c^2 + 2\sum_{i<j} w_i w_j\right)$$

Since the $W_i$ are distinct isotypical components $\mathcal{A}(w_i w_j) = 0$ for $i \neq j$ (**Exercise**) and we conclude

$$\mathcal{A}(s^2) = \mathcal{A}(w_1^2) + \cdots + \mathcal{A}(w_c^2)$$

Collecting similar terms we prove the statement.

# Part 3:

# Invariant sums of squares on the hypercube.

For $n$ even let $X = \{-1,1\}^n$, $G = S_n$ and $\ell(x_1,\ldots,x_n) := \sum_{i=1}^n x_i$

**Theorem. (Blekherman)**

1. *Every function* $f \in \mathbb{R}[X]^G_{\leq n}$ *can be written uniquely as a univariate polynomial in* $\ell$.
2. *If* $f \in \Sigma^G_{\leq n/2}$ *then* $f(\ell) = \sum_{i=0}^{n/2} p_i(\ell)s_i(\ell)$ *where*

$$p_i(\ell) = \prod_{j=1}^{i} \left((2(n+1-j))^2 - \ell^2\right)$$

*and* $s_i$ *is a sum-of-squares of terms of degree* $\leq n/2 - i$ *in* $\ell$.

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As a consequence:

- Every function $f(\ell) \in \Sigma^G_{\leq n/2}$ must be nonnegative in the **real interval** $[-2,2]$ and thus

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*The nonnegative quadratic function*

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**cannot be expressed** *as a sum-of-squares of functions of degree strictly less than* $n/2+1$ *in* $\mathbb{R}[X]$.

# Decomposing the hypercube:

$\mathbb{R}[X]$ where $X = \{-1, 1\}^8$ as $S_8$ representation:

| $d$ | basis | *Isot* | *decomp* | | | |
|---|---|---|---|---|---|---|
| 0 | $1$ | $S^{(8)}$ | | | | |
| 1 | $x_1, \ldots$ | $S^{(7,1)}$ | $S^{(8)}$ | | | |
| 2 | $x_1 x_2, \ldots$ | $S^{(6,2)}$ | $S^{(7,1)}$ | $S^{(8)}$ | | |
| 3 | $x_1 x_2 x_3, \ldots$ | $S^{(5,3)}$ | $S^{(6,2)}$ | $S^{(7,1)}$ | $S^{(8)}$ | |
| 4 | $x_1 x_2 x_3 x_4, \ldots$ | $S^{(4,4)}$ | $S^{(5,3)}$ | $S^{(6,2)}$ | $S^{(7,1)}$ | $S^{(8)}$ |
| 5 | $x_1 x_2 x_3 x_4 x_5, \ldots$ | $S^{(5,3)}$ | $S^{(6,2)}$ | $S^{(7,1)}$ | $S^{(8)}$ | |
| 6 | $x_1 x_2 x_3 x_4 x_5 x_6, \ldots$ | $S^{(6,2)}$ | $S^{(7,1)}$ | $S^{(8)}$ | | |
| 7 | $x_1 x_2 x_3 x_4 x_5 x_6 x_7, \ldots$ | $S^{(7,1)}$ | $S^{(8)}$ | | | |
| 8 | $x_1 \ldots x_8$ | $S^{(8)}$ | | | | |

# Invariant sums-of-squares on the hypercube

Let $\ell(x_1, \ldots, x_n) := \sum x_i$.

**Lemma.**

*The following statements hold:*

1. $\mathbb{R}[C]^G = \mathbb{R}[\ell] / \left( \ell \prod_{j=1}^n ((2j)^2 - \ell^2) \right)$.
2. *The isotypical component of* $\mathbb{R}[C]^G$ *corresponding to the representation* $S^{(n-k,k)}$ *is given by*

$$W_{(n-k,k)} := \left\{ \sum_{j=0}^{n-2k} \ell^j f_j : f_j \in S^{(n-k,k)} \subseteq \mathbb{R}[C]_k \right\}$$

*for* $0 \leq k \leq n/2$.

Our Theorem on invariant sums-of-squares applied to the above isotypical decomposition implies Blekherman's characterization.