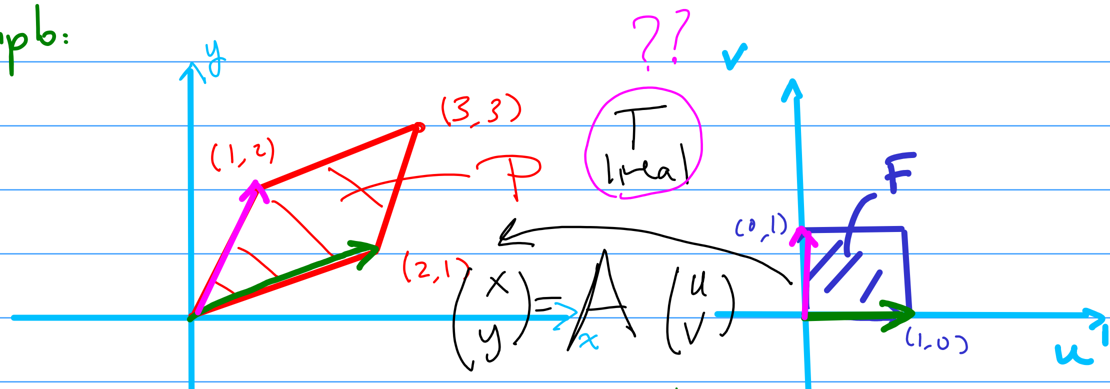
Hoy: Cambio de variable y sistemas de coordenadas en $\mathbb{R}^3$.

Ejemplo:



Use el Teo de cambio de variable para calcular

$$\iint_P f(x,y)\, dA \overset{?}{=}$$

Queremos

$$A = \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{?}$$

que cumpla:

(1) $A \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \boxed{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$

(2) $A \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

Resolviendo los productos encontramos

(1) $\begin{pmatrix} a \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, (2) $\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2u + v \\ u + 2v \end{pmatrix}$$

$$\left\{ \begin{array}{l} x = 2u + v \\ y = u + 2v \end{array} \right. \quad (*)$$

$$\iint_P f(x,y)\,dx\,dy = \left[\int_0^1\int_0^1 f(2u+v,\, u+2v)\ \underbrace{3}_{\text{Jacobiano}}\ du\,dv\right]$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{pmatrix} 2 & 1 \\ 1 & 2\end{pmatrix}\right| = 3$$

$$= \left|\det\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{bmatrix}\right| \quad *$$

Hay dos nuevos sistemas de coordenadas que deben conocer en $\mathbb{R}^3$

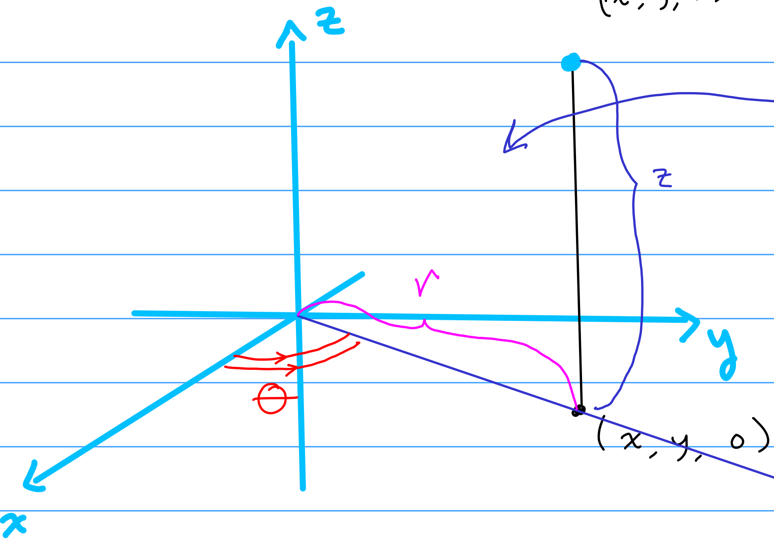
- Coordenadas cilíndricas
- Coordenadas esféricas

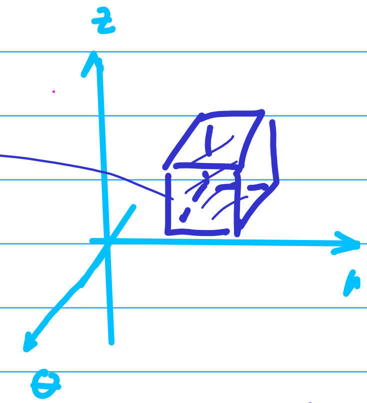
} generalizaciones en $\mathbb{R}^3$ de las coordenadas polares a $\mathbb{R}^2$.

(1) Coordenadas cilíndricas

(r, θ, z)

(x, y, z)

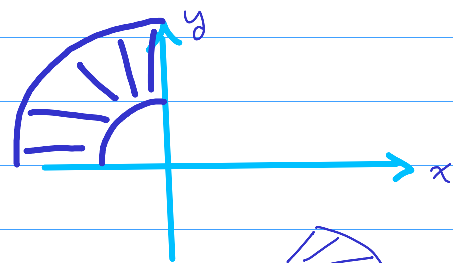


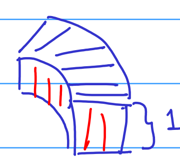


$$\begin{cases} \frac{\pi}{2} \le \theta \le \pi \\ 2 \le r \le 3 \\ 1 \le z \le 2 \end{cases}$$

Cambio de coordenadas

$$\left[\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}\right]$$





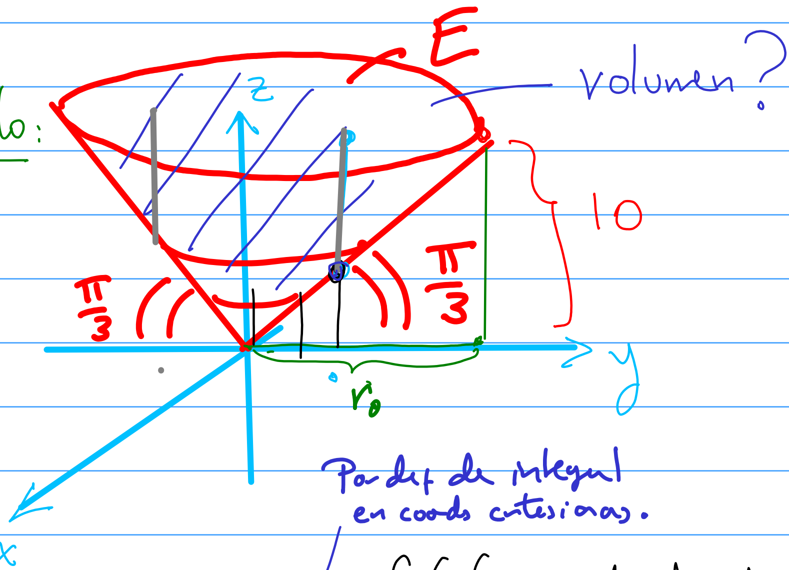
Muy útiles si el problema no cambia al rotar alrededor del eje z (simetría cilíndrica)

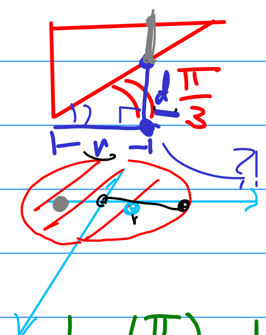
El Jacobiano de coordenadas cilíndricas es: r

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \left|\det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix}\right|$$

$$\left|\det\begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = \left|\det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right| = |r| = r$$

Ejemplo:





$$\tan\left(\frac{\pi}{3}\right) = \frac{10}{r_0}$$

Por def de integral en coords cartesianas.

Sol: $\text{Vol}(E) = \iiint_E 1 \, dx\,dy\,dz$

$$r_0 = \frac{10}{\tan\left(\frac{\pi}{3}\right)}$$

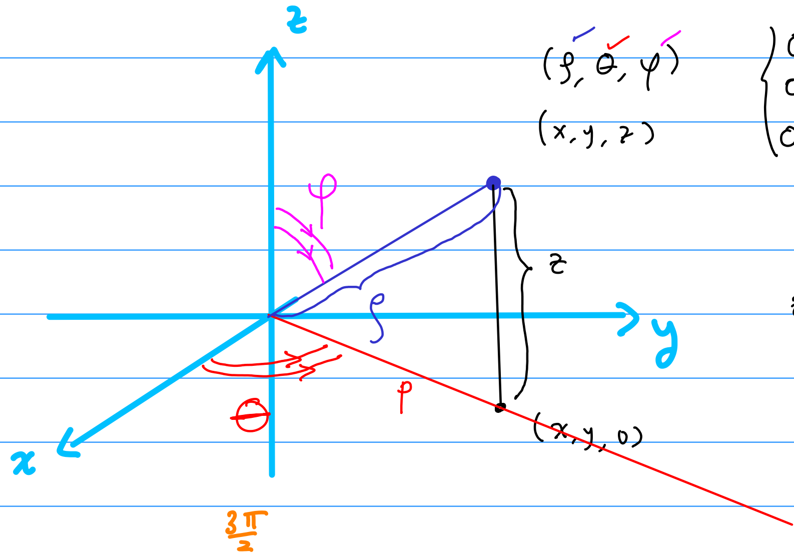
$$\left[r\tan\left(\frac{\pi}{3}\right) = \frac{d}{\cancel{r_0}} \right]$$

Queremos describir E en coordenadas cilíndricas:

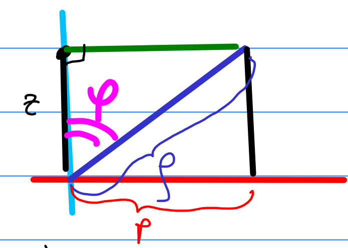
$$E = \left\{(r,\theta,z) : \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le \frac{10}{\tan\left(\frac{\pi}{3}\right)} \\ r\tan\left(\frac{\pi}{3}\right) \le z \le 10 \end{array}\right\} \quad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

$$\mathrm{Vol}(E) = \iiint_E 1\, dx\,dy\,dz = \left[\int_0^{2\pi} \int_0^{\frac{10}{\sqrt{3}}} \int_{r\sqrt{3}}^{10} r \, dz\, dr\, d\theta \right]$$

(2) Coordenadas esféricas



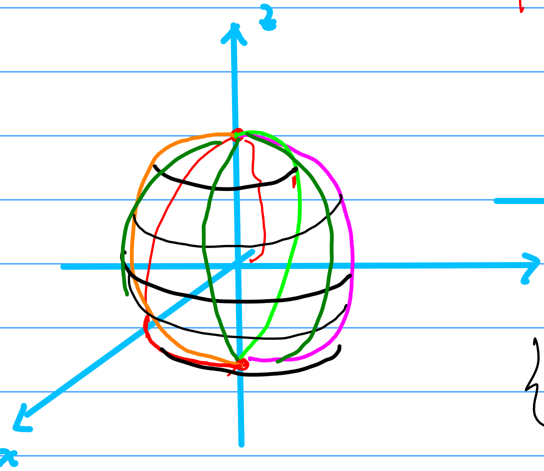
$$\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le \rho \\ 0 \le \varphi \le \pi \end{cases}$$

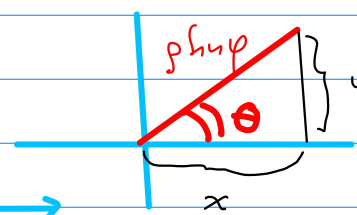


$$\begin{cases} z = \rho \cos(\varphi) \\ p = \rho \sin(\varphi) \end{cases}$$

$$\left[\begin{array}{l} \rho = 1 \\ 0 \le \varphi \le \pi \\ 0 \le \theta \le 2\pi \end{array} \right.$$

esfera de radio 1





$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \end{cases}$$

Se utiliza cuando hay simetría esférica.

Cómo es el cambio de coordenadas?

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos(\varphi) \end{cases}$$

Ejercicio: Demuestre que el Jacobiano de coordenadas esféricas es:

$$\boxed{\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2 \sin(\varphi)}$$

Ejercicio: Calcule el volumen de la bola de radio 1 centrada en (0,0,0).

Sol: E

$$Vol(E) \overset{\text{en cartesianas}}{=} \iiint_E 1\, dx\, dy\, dz =$$

$$E = \left\{ (\rho, \theta, \varphi) : \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \\ 0 \leq \rho \leq 1 \end{array} \right\}$$

Bola unitaria.

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta =$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin\varphi \, \frac{1}{3}\, d\varphi\, d\theta = \frac{2\pi}{3}\int_0^{\pi} \sin\varphi\, d\varphi$$

$$\frac{2\pi}{3}\cdot\left(-\cos(\varphi) \Big|_0^{\pi} \right) = \frac{2\pi}{3}\left(1 - (-1)\right) = \boxed{\frac{4\pi}{3}}$$