Hoy: (1) Teorema del $\underbrace{\text{cambio de variable}}$

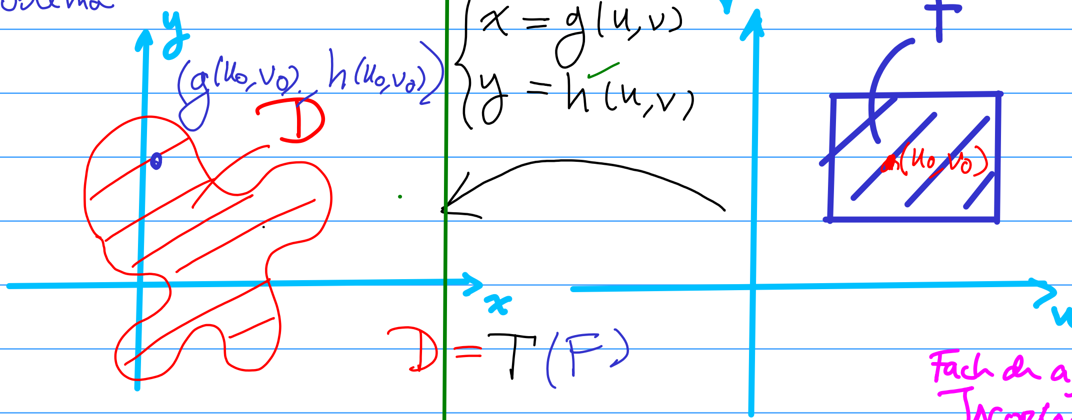
② — "escogencia de coordenadas"

T

"

Problema

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$



$D = T(F)$

Fact de ajuste
JACOBIANO

$$① \quad \iint_D p(x,y)\,dx\,dy \overset{?}{=} \iint_F p\big(g(u,v),\, h(u,v)\big)\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$

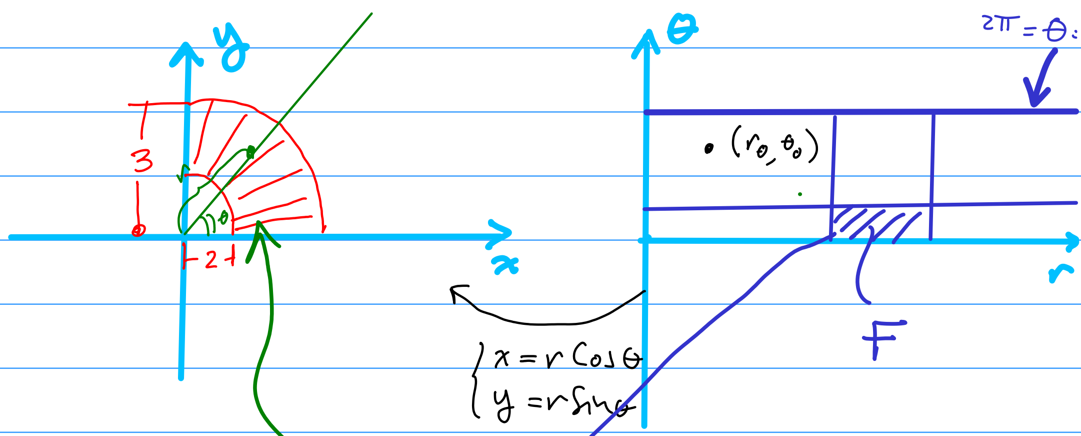
Teorema: [del cambio de variable]

Si $T: \mathbb{R}^2_{(u,v)} \longrightarrow \mathbb{R}^2_{(x,y)}$ es diferenciable y 1-1

y $T(F) = D$, entonces ① es válida.

Ejemplo de transformación: $\mathbb{R}^2_{(x,y)} \xleftarrow{\ T\ } \mathbb{R}^2_{(\theta,r)}$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$ "Coordenadas polares".

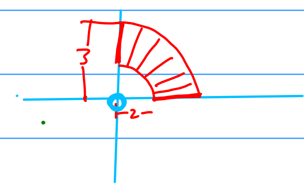


$$\left[\begin{array}{l} 2 \le r \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \end{array}\right]$$

Ejercicio:

Calcule $\displaystyle\iint_D x+y \, dA =$

donde D es



$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left| \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right|$$

Jacobiano cambio a polares

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \left| \det \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \right| =$$

$$\left| r\cos^2\theta - (-r\sin^2\theta) \right| =$$

$$\left| r\left[\cos^2\theta + \sin^2\theta\right] \right| = |r| = r$$

$$\iint_D x+y \, dA = \int_0^{\frac{\pi}{2}} \int_2^3 (r\cos\theta + r\sin\theta)\, r \, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_2^3 r^2 \left[\cos\theta + \sin\theta\right] dr\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\left[\cos\theta + \sin\theta\right] \int_2^3 r^2\, dr \right) d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\cos\theta + \sin\theta\right] \left[\frac{3^3}{3} - \frac{2^3}{3}\right] d\theta$$

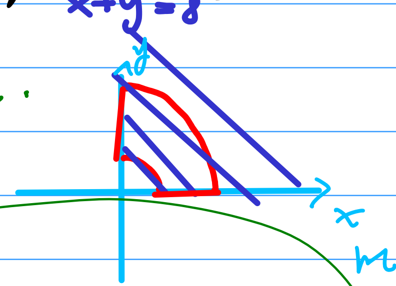
$$= \left[\frac{3^3}{3} - \frac{2^3}{3}\right] \cdot \left(\sin\theta - \cos\theta\right) \Big|_{\theta=0}^{\theta=\frac{\pi}{2}}$$

$$= \left[\frac{3^3}{3} - \frac{2^3}{3}\right] \left(1 - (-1)\right) =$$

$$= \left(\frac{3^3}{3} - \frac{2^3}{3}\right) \cdot 2 = \left(3^2 - \frac{2^3}{3}\right) \cdot 2 = \left(9 - \frac{8}{3}\right) \cdot 2$$

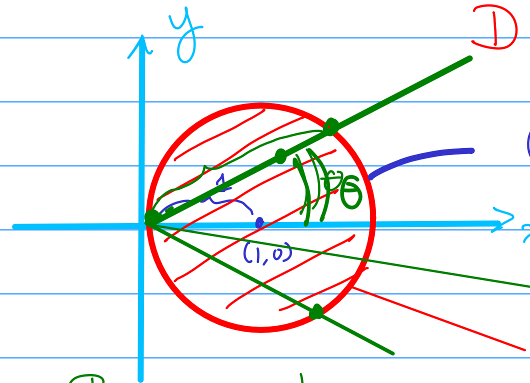
$$= \left[18 - \frac{16}{3}\right] = \iint_D x+y \, dA$$

Respuesta Kg.



"Un plato tiene la forma de la región D y la densidad en Kg/m^2 está dada por $\rho(x,y) = x+y$. Calcule la masa total?

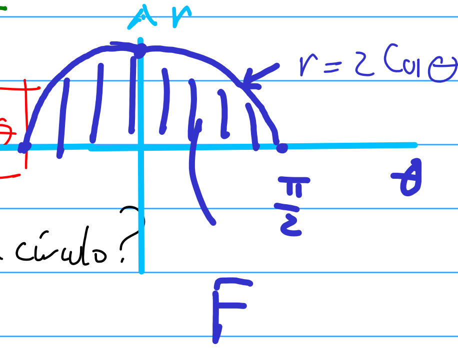
Ejercicio: Calcule $\iint_D x^2+y^2\,dA \overset{?}{=}$



$(x-1)^2+y^2=1$

$$\begin{cases} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta \end{cases}$$

$\boxed{r = 2\cos\theta}$



Sol 1: Pasar a polares

¿Qué relación hay entre (r,θ) en los puntos del círculo?

$$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$$

$$r^2\left[\cos^2\theta + \sin^2\theta\right] - 2r\cos\theta + \cancel{1} = \cancel{1}$$

$$r^2 - 2r\cos\theta = 0$$

$$\left[r(r-2\cos\theta) = 0\right]$$

$$\iint_D x^2+y^2\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r \quad dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{r^4}{4} \Bigg|_{r=0}^{r=2\cos\theta} \right) d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2^4}{4} \cos^4\theta \, d\theta =$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^{4}\theta \, d\theta =$$

$$\begin{cases} 1 = \cos^2\theta + \sin^2\theta \\ \cos(2\theta) = \cos^2\theta - \sin^2\theta \end{cases}$$

$$\left[\frac{1+\cos(2\theta)}{2} = \cos^2\theta\right]$$

$$= 4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta$$

$$= \cancel{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + 2\cos(2\theta) + \cos^2(2\theta)}{\cancel{4}}$$

$$= \pi + 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(2\theta)\, d\theta + \underbrace{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2(2\theta)\, d\theta}_{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1+\cos(4\theta)}{2} d\theta}$$

$$= \pi + \sin(2\theta)\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} + (\quad)$$

$$= \pi + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{\cos(4\theta)}{2}\right) d\theta$$

error
enclose

$$= \pi + \frac{\pi}{2} + \underbrace{\cancel{\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(4\theta)\, d\theta}}_{-\frac{\sin(4\theta)}{4}\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}} \quad 0$$

$$= \boxed{\frac{3\pi}{2}}$$

Sol 2:

$$\begin{cases} x = r\cos\theta + 1 \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{cases}$$