Hoy: Regla de la cadena = Método para calcular derivadas de composiciones.

$$\begin{tikzcd}
\vec{a} \in \mathbb{R}^n \arrow[rr, "g"] \arrow[dr] \arrow[rr, phantom, shift right=6ex, "\#"] & & \mathbb{R}^m \ni g(\vec{a}) \arrow[dl, "f"] \\
& \mathbb{R}^p \ni f(g(\vec{a})) &
\end{tikzcd}$$

$h(\vec{x}) = f(g(\vec{x}))$

$\updownarrow$ NOTACIÓN

$h = f \circ g$

Clase anterior:

Si $f: \mathbb{R}^a \to \mathbb{R}^b$ es diferenciable en $\vec{u} \in \mathbb{R}^a$ su derivada es una matriz

$$\underbrace{Df(\vec{u})}_{b \times a} = \begin{array}{c c} & \begin{array}{cccc} x_1 & x_2 & \cdots & x_a \end{array} \\ \begin{array}{c} f_1 \\ \vdots \\ f_b \end{array} & \left[\; \frac{\partial f_i}{\partial x_j}(\vec{u}) \; \right] \end{array}$$

La derivada de f EVALUADA en $\vec{u}$.

Cómo se calcula $Dh(\vec{a})$ en términos de $Df(\cdots)$ y $Dg(\cdots)$?

Teorema [Regla de la cadena] Suponga que

(i) g es diferenciable en $\vec{a}$

(ii) f es diferenciable en $g(\vec{a})$

entonces la composición $h(x) = f(g(x))$ cumple:

(i) h es diferenciable en $\vec{a}$

* (ii) Su derivada en $\vec{a}$ es:

mult de matrices

$$\boxed{\underbrace{Dh(\vec{a})}_{p \times n} = \overbrace{Df(g(\vec{a}))}^{p \times m} \cdot \overbrace{Dg(\vec{a})}^{m \times n}}$$

Obs: Útil porque permite calcular $Dh(\vec{a})$ sin CONOCER $h(x)$ (o incluso sabiendo sólo las derivadas de f y de g en los lugares apropiados)

Obs: Ya conocen esta fórmula en el caso de una sola variable.

Si
$$h(x) = f(g(x))$$
$$h'(x) = f'(g(x))\, g'(x)$$
derivada "interna" es la regla de la cadena
$$h'(a) = f'(g(a)) \cdot g'(a)$$

Ejercicio: [$t(x,y) = (x^2+y^2,\ x^2-y^2)$]
$$s(A,B) = (B^2,\ A-B)$$
Defina
$$[\, r(u,v) := s(t(u,v)) \,]$$
Calcule $Dr(1,2) \overset{?}{=}$

$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "t"] \arrow[d, "h"'] & \mathbb{R}^2 \arrow[dl, "s"] \\ \mathbb{R}^2 & \end{tikzcd}$$

Sol: $Dr(1,2) = \underbrace{Ds(t(1,2))} \cdot \underbrace{Dt(1,2)}$

$t: \mathbb{R}^n \to \mathbb{R}^m$

$$Dt(x,y) = \begin{array}{cc} & \overbrace{\cancel{x} \qquad\quad \cancel{y}}^{n} \\ \left.\begin{matrix}\cancel{x^2+y^2} \\ \cancel{x^2-y^2}\end{matrix}\right\}_{m} & \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix} \end{array}$$

$$Dt(1,2) = \begin{bmatrix} 2 & 4 \\ 2 & -4 \end{bmatrix}.$$

$$s(A,B) = (B^2,\ A-B)$$

$$Ds(A,B) = \begin{array}{c} \\ B^2 \\ A-B \end{array} \begin{array}{cc} A & B \\ \end{array}$$

$$Ds(A,B) = \begin{array}{c|cc} & A & B \\ B^2 & 0 & 2B \\ A-B & 1 & -1 \end{array}$$

$$t(1,2) = (1^2+2^2,\ 1^2-2^2) = (\underset{A}{5}, \underset{B}{-3})$$

$$Ds(t(1,2)) = \begin{pmatrix} 0 & -6 \\ 1 & -1 \end{pmatrix}.$$

$$Dr(1,2) = \begin{pmatrix} 0 & -6 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 2 & -4 \end{pmatrix}$$

$$\boxed{Dr(1,2) = \begin{array}{c|cc} & u & v \\ r_1 \rightarrow & -12 & 24 \\ r_2 \rightarrow & 0 & 8 \end{array}}$$

$$r(u,v) = (r_1(u,v),\ r_2(u,v))$$

$$\frac{\partial r_2}{\partial v}(1,2) = 8$$

Ejercicio: Sea $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ una función escalar

$$\left[h(r,\theta) = f(r\cos\theta,\ r\sin\theta) \right]$$

Calcule $\frac{\partial h}{\partial r}(r,\theta) \stackrel{?}{=}$ y $\frac{\partial h}{\partial \theta}(r,\theta) \stackrel{?}{=}$ $\left(Dh(r,\theta) \stackrel{?}{=} \right)$

en términos de $\frac{\partial f}{\partial x}$ y $\frac{\partial f}{\partial y}$

$$g(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

Sugerencia: $\left[g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \right.$

$(r,\theta) \longmapsto (r\cos\theta,\ r\sin\theta)$

$$\left[h(r,\theta) = f(g(r,\theta)) \right]$$

Por regla de la cadena

$$Dh(r,\theta) = \underbrace{Df(g(r,\theta))}_{?\checkmark} \cdot \underbrace{Dg(r,\theta)}_{?}$$

$$f : \underset{x,y}{\mathbb{R}^2} \longrightarrow \mathbb{R}$$

$f(x,y)$ ✓

$$Df(x,y) = f \begin{array}{cc} x & y \\ \left[\frac{\partial f}{\partial x}(x,y) \right. & \left. \frac{\partial f}{\partial y}(x,y) \right] \end{array}$$

$$Df(g(r,\theta)) = \left[\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta) \quad \frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta) \right]$$

$$g(r,\theta) = (r\cos\theta,\ r\sin\theta)$$

$$Dg(r,\theta) = \begin{array}{c|cc} & r & \theta \\ \hline r\cos\theta & \cos\theta & -r\sin\theta \\ r\sin\theta & \sin\theta & r\cos\theta \end{array}$$

$$Dh(r,\theta) = \left[\frac{\partial f}{\partial x}(\ ) \quad \frac{\partial f}{\partial y}(\ ) \right] \cdot \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

$$\left[\frac{\partial h}{\partial r}(r,\theta),\ \frac{\partial h}{\partial \theta}(r,\theta)\right]$$

$$Dh(r,\theta) = \left[\cos\theta\left(\frac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\right) + \sin\theta\cdot\frac{\partial f}{\partial y}(r\cos\theta, r\sin\theta),\ \underbrace{}\ \right]$$

$$-r\sin\theta\cdot\frac{\partial f}{\partial x}(\ ) + r\cos\theta\cdot\frac{\partial f}{\partial y}(\cdot)$$

Resumiendo:

$$\begin{cases} \dfrac{\partial h}{\partial r}(r,\theta) = \dfrac{\partial f}{\partial x}(r\cos\theta, r\sin\theta)\cdot\cos\theta + \dfrac{\partial f}{\partial y}(r\cos\theta, r\sin\theta)\sin\theta \\[2ex] \dfrac{\partial h}{\partial \theta}(r,\theta) = \dfrac{\partial f}{\partial x}(\ )\cdot(-r\sin\theta) + \dfrac{\partial f}{\partial y}(\cdot)\,[r\cos\theta] \end{cases}$$