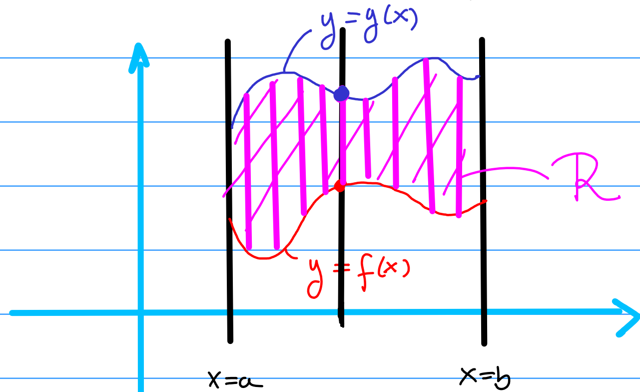
Hoy: Cómo se calculan las integrales dobles? $\iint_R p(x,y)\, dA =$

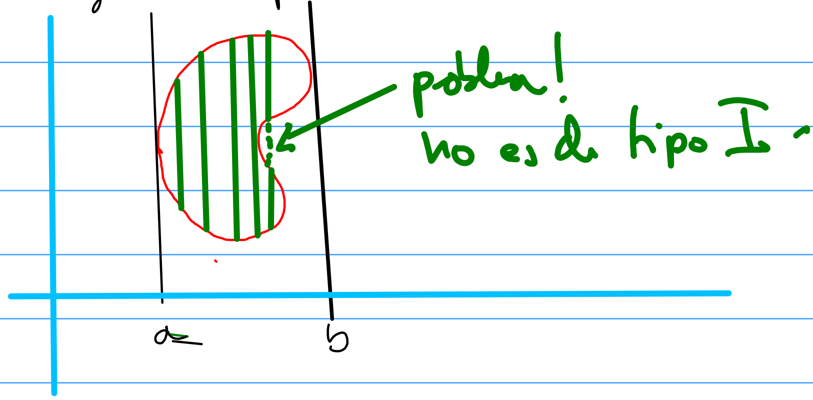
Def:



R es una región tipo I si $\exists$ a, b, $y = f(x)$, $y = g(x)$

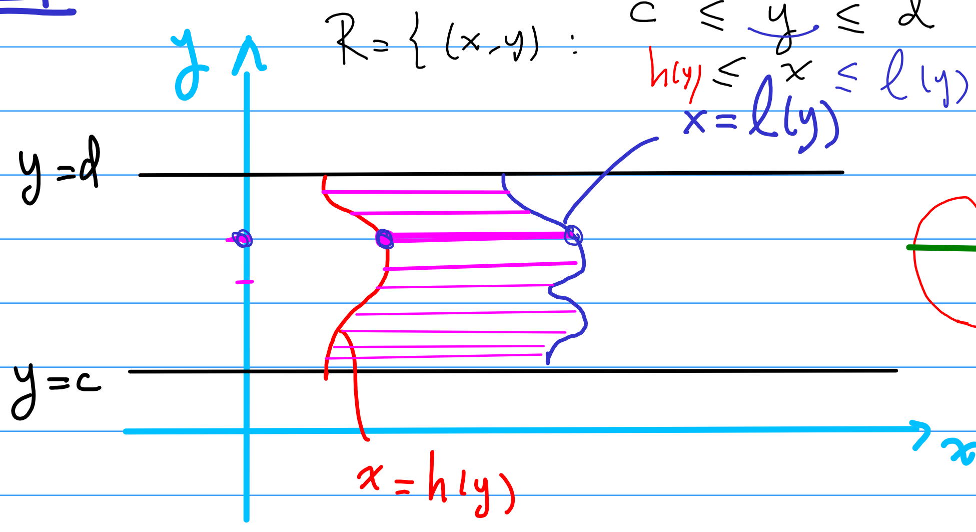
$$R = \left\{ (x,y) : \begin{array}{c} a \leq x \leq b \\ f(x) \leq y \leq g(x) \end{array} \right\}$$

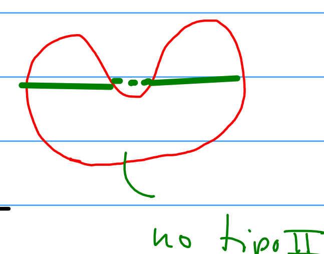
Obs: La región R puede colorearse mediante rectas verticales

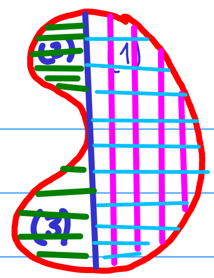


Def: R es del tipo II si

$$R = \left\{ (x,y) : \begin{array}{c} c \leq y \leq d \\ h(y) \leq x \leq \ell(y) \end{array} \right\}$$



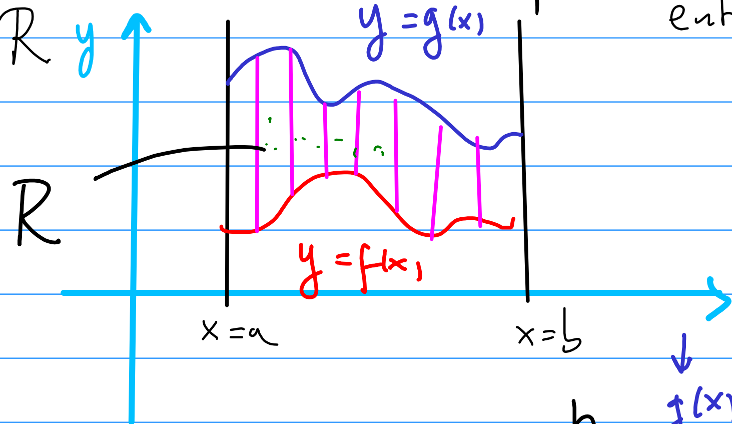




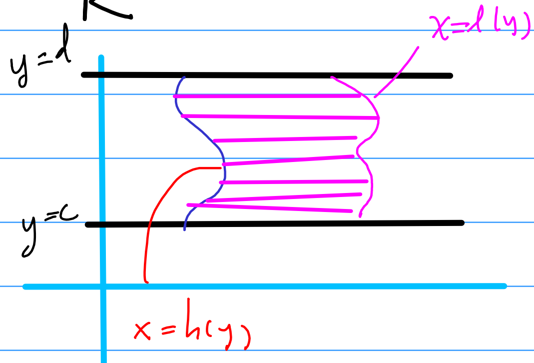
$$\iint_R p(x,y)\,dA = \iint_{①} p(x,y)\,dA + \iint_{②} p(x,y)\,dA + \iint_{③} p(x,y)\,dA$$

Cómo calcular integrales dobles?

Teorema: [Fubini] Si $p(x,y)$ es continua en R y R entonces



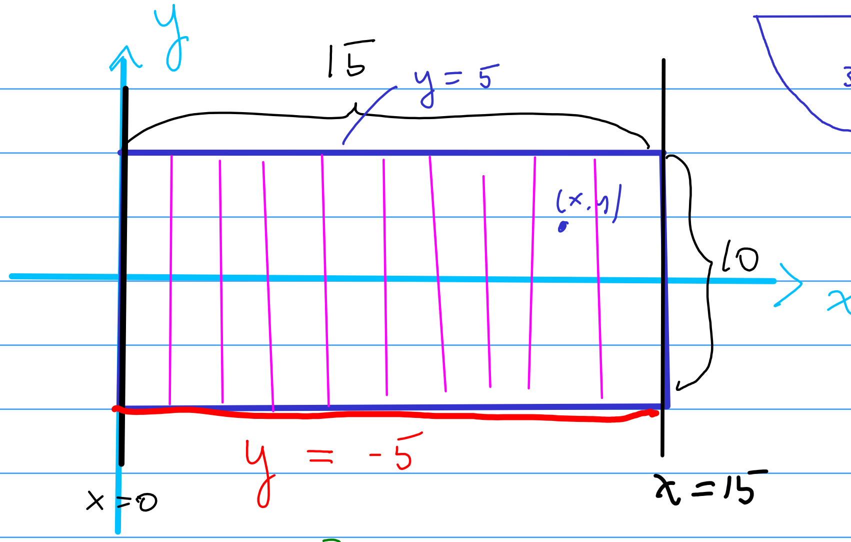
$$\iint_R p(x,y)\,dA = \int_a^b \left(\int_{f(x)}^{g(x)} p(x,y)\,dy \right) dx$$

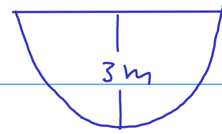


Reduce el problema a calcular varias integrales de una variable cada una.

$$\iint_R p(x,y)\,dA = \int_c^d \left[\int_{h(y)}^{l(y)} p(x,y)\,dx \right] dy$$

Ejemplo:





$$p(x,y) = 3 + \frac{3}{25} y^2$$

$$m^3 \text{ de agua} = \iint_R p(x,y)\,dA \approx 600.0011\ m^3$$

$$\iint_R \left(3 + \frac{3}{25} y^2\right) dA = \int_0^{15} \Bigg[\underbrace{\int_{-5}^{5} \left(3 + \frac{3}{25} y^2\right) dy}_{40} \Bigg] dx$$

$$\int_{-5}^{5} \left(3 + \frac{3}{25} y^2\right) dy = 3y + \frac{\cancel{3}}{25} \frac{y^3}{\cancel{3}} \Bigg|_{y=-5}^{y=5}$$

$$= 3 \cdot 5 + \frac{5^3}{25} - \left[3(-5) + \frac{1}{25}(-5)^3 \right] =$$

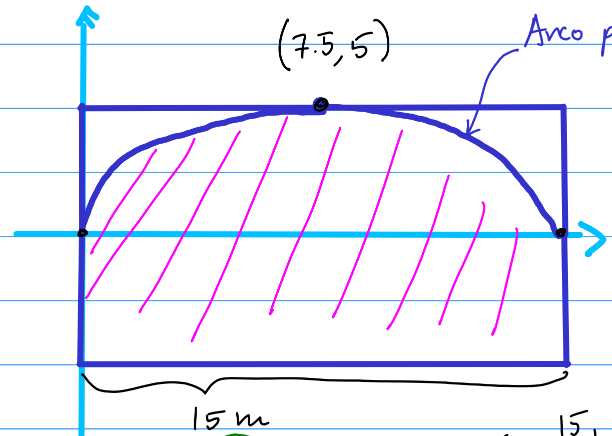
$$= 3 \cdot 10 + 10 = 40$$

$$\int_0^{15} 40\,dx = 40 \int_0^{15} dx = 40 \cdot 15 = \boxed{600\ m^3}$$

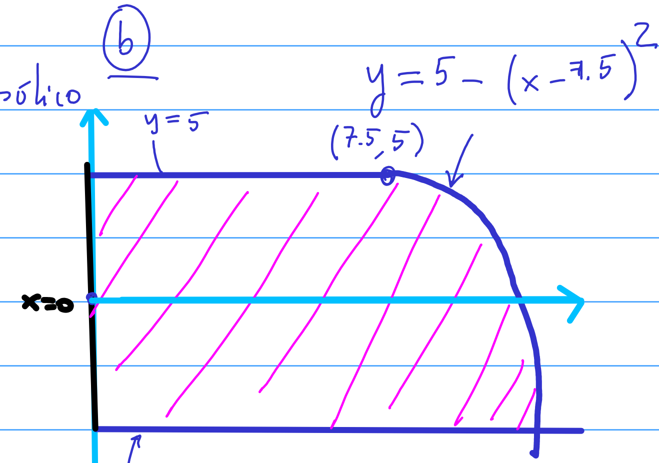
Como tipo II: $\iint_R p(x,y)\,dA = \int_{-5}^{5} \left[\int_0^{15} p(x,y)\, dx \right] dy$ //

Ejemplo 2: Cuánta agua tiene la piscina?

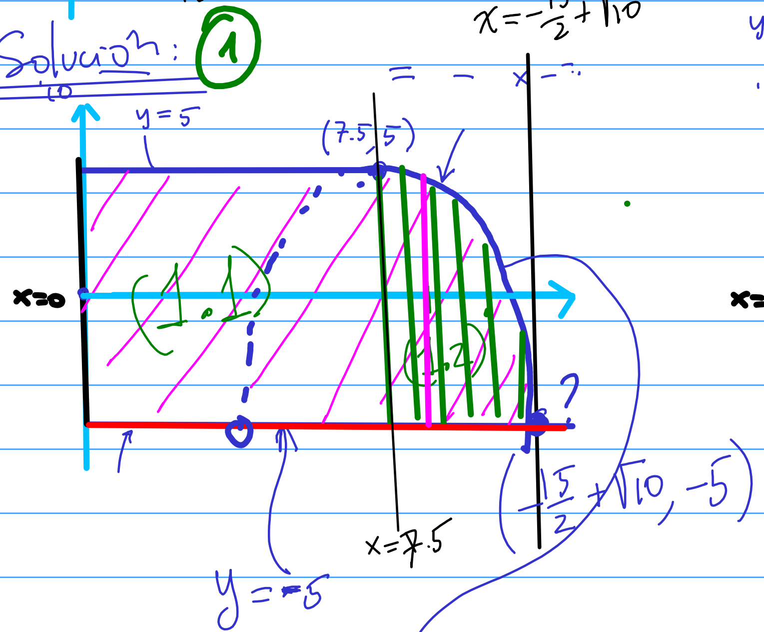
(a)



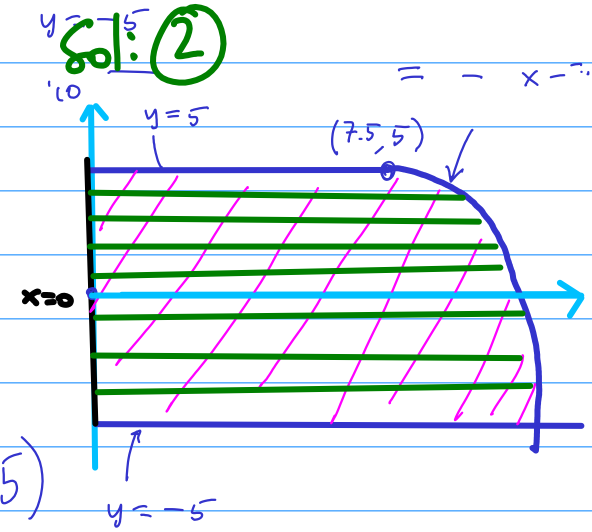
(b)



Solución: 1



Sol: 2



$$y = 5 - \left(x - \frac{15}{2}\right)^2$$

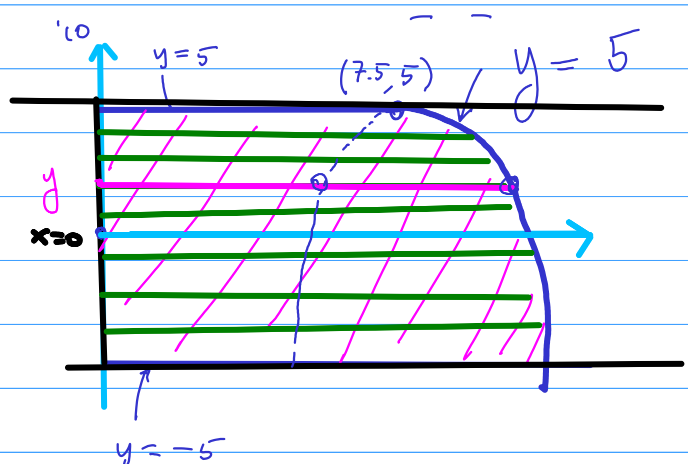
$$\begin{cases} y = 5 - \left(x - \frac{15}{2}\right)^2 \\ y = -5 \end{cases}$$

$$10 = \left(x + \frac{15}{2}\right)^2 \iff x + \frac{15}{2} = \pm\sqrt{10}$$

$$\boxed{x = -\frac{15}{2} \pm \sqrt{10}}$$

Sol 1:

$$\iint_R p(x,y)\,dA = \underbrace{\int_0^{7.5}\int_{-5}^{5} p(x,y)\; dy\; dx}_{(1.1)} + \int_{7.5}^{-\frac{15}{2}+\sqrt{10}}\left[\int_{-5}^{5-(x-7.5)^2} p(x,y)\; dy\right]dx$$



$$y = 5 - \left(x - \frac{15}{2}\right)^2$$

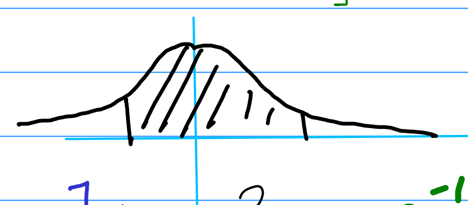
despejamos x:

$$\left(x-\frac{15}{2}\right)^2 = 5-y$$

$$x - \frac{15}{2} = \pm\sqrt{5-y}$$

Tomo la + por geometría del problema

$$\iint_R p(x,y)\,dA = \int_{-5}^{5}\left[\int_0^{\frac{15}{2}+\sqrt{5-y}} p(x,y) \quad dx\right]dy$$

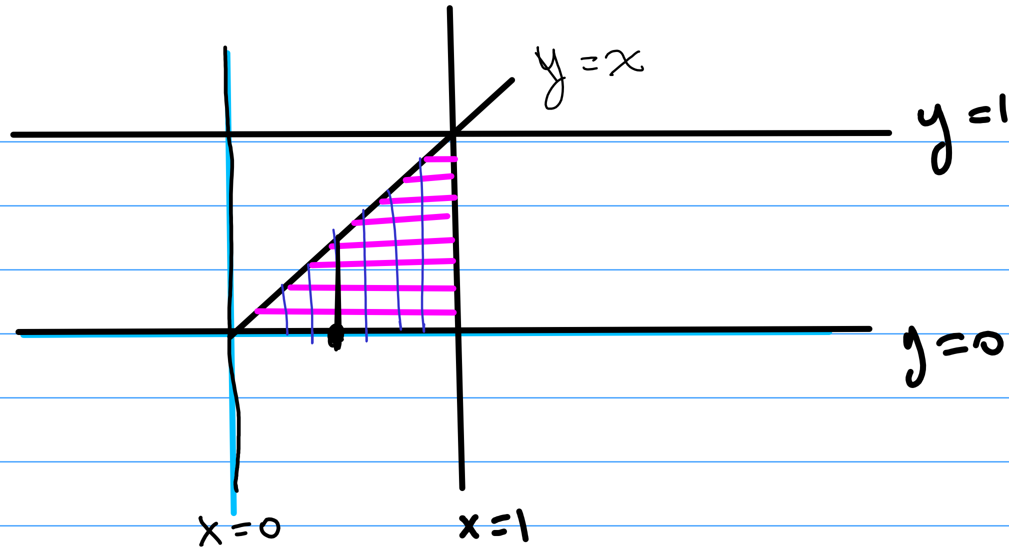


Ejemplo 3:

Calcule

$$\int_{y=0}^{y=1}\left[\int_{x=y}^{x=1} e^{-x^2}\,dx\right]dy \overset{?}{=} \frac{1-e^{-1}}{2}$$

Sugerencia: (1) Dibuje la región
(2) Cambie el orden de integración.



$$\int_0^1 \left[\int_0^x e^{-x^2} \, dy \right] dx$$

$$\int_0^x e^{-x^2} dy = e^{-x^2} y \Big|_{y=0}^{y=x} = x e^{-x^2}$$

$u = x^2$

$du = 2x\,dx$

$$\int_0^1 x e^{-x^2} dx = \int_0^1 e^{-u} \frac{du}{2} =$$

$$-\frac{e^{-u}}{2} \Big|_{u=0}^{u=1} = \boxed{\frac{1-e^{-1}}{2}}$$

✓