Hoy: (1) Funciones diferenciables (CRITERIO de diferenciabilidad)

(2) Regla de cadena

Def: Sea $g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ y $\vec{a} \in \mathbb{R}^n$

g es diferenciable en $\vec{a}$ si existe $T \in \mathbb{R}^{m\times n}$ que satisface

$$\lim_{\vec{x}\to\vec{a}} \left[\frac{\| g(\vec{x}) - \overbrace{[g(\vec{a}) + T(\vec{x}-\vec{a})]}^{\ell_a(\vec{x})} \|}{\|\vec{x}-\vec{a}\|} \right] = 0$$

Obs 1: T es única y se llama la DERIVADA DE g en $\vec{a}$ $\quad g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$\begin{array}{cc} & \overbrace{\qquad}^{n} \\ m\Big\{ & \left[Dg(\vec{a}) \right] \end{array} \leftarrow \text{MATRIZ}$$

$$g(x_1,\dots,x_n) = (g_1(x_1,\dots,x_n), g_2(x_1,\dots,x_n), \dots, g_m(x_1,\dots,x_n))$$

$$Dg(\vec{a}) = \begin{array}{cc} & \begin{array}{ccccc} x_1 & x_2 & \dots & j & x_n \end{array} \\ \begin{array}{c} g_1 \\ g_2 \\ \vdots \\ i \\ g_m \end{array} & \left[\begin{array}{ccccc} \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & \\ & & & \frac{\partial g_i}{\partial x_j}(\vec{a}) & \\ & & & & \end{array} \right] \end{array}$$

Derivada parcial i con la x_j

"Deriva contra x_j pensando que todas las demás son constantes y luego evaluamos en $\vec{a}$"

Obs 2: Si $g(x)$ es diferenciable en $\vec{a}$, la función

$$\ell_{\vec{a}}(\vec{x}) := g(\vec{a}) + Dg(\vec{a})(\vec{x}-\vec{a})$$

es una excelente aproximación lineal afín para $g(\vec{x})$ cerca de $\vec{a}$.

Cómo verificar que $g(\vec{x})$ es diferenciable?

Teorema: Sea $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ y $\vec{a} \in \mathbb{R}^n$. m·n funciones

Si TODAS las derivadas parciales de g $\left(\frac{\partial g_i}{\partial x_j}(x_1, \dots, x_n) \right)$ son continuas en $\vec{a}$ entonces g es DIFERENCIABLE en $\vec{a}$.

Ejercicio: * Sea $f(x,y) = (xy, \sin(x+y), y)$ $\quad f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

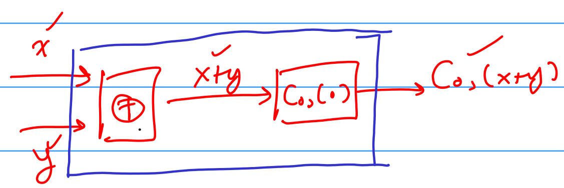
✓(a) Calcule $Df\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

✓(b) Encuentre $\ell_{\left(\frac{\pi}{4}, \frac{\pi}{4}\right)}(x,y)$

→ (c) Demuestre que $f(x,y)$ es diferenciable en todo $\mathbb{R}^2$.

polinomios así que continuas

$$Df(x,y) = \begin{bmatrix} y & x \\ \cos(x+y) & \cos(x+y) \\ 0 & 1 \end{bmatrix}$$



Como todas las derivadas parciales son continuas en $\mathbb{R}^2$ concluimos que $f(x,y)$ es diferenciable en $\mathbb{R}^2$

(a)

$$Df\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \begin{bmatrix} y & x \\ \cos(x+y) & \cos(x+y) \\ 0 & 1 \end{bmatrix} \left(\overset{x}{\frac{\pi}{4}}, \overset{y}{\frac{\pi}{4}} \right)$$

evaluar en.

$$\boxed{Df\left(\frac{\pi}{4}, \frac{\pi}{4}\right) = \begin{bmatrix} \frac{\pi}{4} & \frac{\pi}{4} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\ell : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

funciones lineales afines

$$\text{(b)}\quad \ell_{\left(\frac{\pi}{4},\frac{\pi}{4}\right)}(x,y) = \left(\tilde{\ell}_1,\ \tilde{\ell}_2,\ \tilde{\ell}_3\right)$$

MATRIZ · vector $\quad \vec{x}-\vec{a}$

$$\ell_{\left(\frac{\pi}{4},\frac{\pi}{4}\right)}(x,y) = f\left(\tfrac{\pi}{4},\tfrac{\pi}{4}\right) + Df\left(\tfrac{\pi}{4},\tfrac{\pi}{4}\right) \cdot \overbrace{\left[\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \frac{\pi}{4} \\ \frac{\pi}{4} \end{pmatrix}\right]}^{\vec{x}-\vec{a}}$$

$$f\left(\tfrac{\pi}{4},\tfrac{\pi}{4}\right) = \left(\tfrac{\pi^2}{16},\ \operatorname{sen}\left(\tfrac{\pi}{2}\right),\ \tfrac{\pi}{4}\right) = \left(\tfrac{\pi^2}{16},\ 1,\ \tfrac{\pi}{4}\right)$$

$$\ell = \begin{pmatrix} \frac{\pi^2}{16} \\ 1 \\ \frac{\pi}{4} \end{pmatrix} + \begin{pmatrix} \frac{\pi}{4} & \frac{\pi}{4} \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x - \frac{\pi}{4} \\ y - \frac{\pi}{4} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\pi^2}{16} + \frac{\pi}{4}\left(x-\frac{\pi}{4}\right) + \frac{\pi}{4}\left(y-\frac{\pi}{4}\right) \\ 1 \\ \cancel{\frac{\pi}{4}} + \left(y - \cancel{\frac{\pi}{4}}\right) \end{pmatrix}$$

2) Regla de la cadena

Repaso Calc Diferencial:

Si $h(x) = f(g(x))$ $\qquad h'(x) = f'(g(x))\, g'(x)$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "g"] \arrow[rr, bend right, "\#", "h"'] & \mathbb{R} \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

<u>Teorema:</u>

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "g"] \arrow[rr, bend right, "\#", "h"'] & \mathbb{R}^m \arrow[r, "f"] & \mathbb{R}^p \end{tikzcd} \quad \text{Sea } h(\vec{x}) = f(g(\vec{x}))$$

Si g es diferenciable en $\vec{a}$ y f es diferenciable en $g(\vec{a})$

entonces: (1) h es diferenciable en $\vec{a}$

y (2)

$$\boxed{Dh(\vec{a}) = \overbrace{Df(g(\vec{a}))}^{p\times m} \cdot \overbrace{Dg(\vec{a})}^{m\times n}}$$

prod de matrices

<u>Obs</u>: Permite calcular $Dh(\vec{a})$ SIN CONOCER h explícitamente.

escalar.

$$\left[h(x,y) = f\big(g_1(x,y),\, g_2(x,y),\, g_3(x,y)\big) \right]$$

Encuentre una fórmula para $\dfrac{\partial h}{\partial x}$ en términos de las derivadas parciales de f y de g.

$$\begin{tikzcd}
\overset{x,y}{\mathbb{R}^2} \arrow[r, "g", "\#"'] \arrow[rr, bend right, "h"'] & \overset{z_1,z_2,z_3}{\mathbb{R}^3} \arrow[r, "f"] & \mathbb{R}
\end{tikzcd}$$

$$Dh(x,y) = Df(g(x,y)) \cdot Dg(x,y)$$

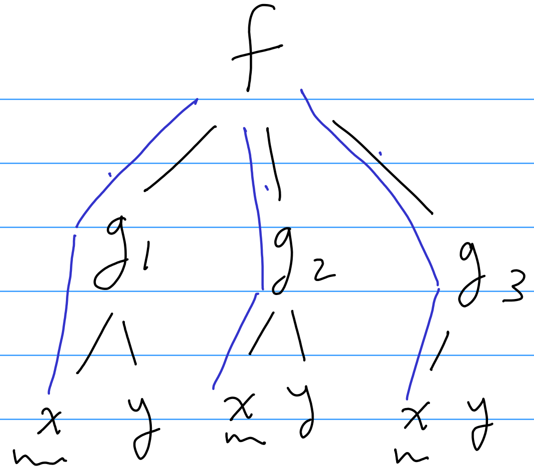
$$\left[\frac{\partial h}{\partial x}(x,y) \quad \frac{\partial h}{\partial y}(x,y) \right] = \begin{array}{c} \begin{array}{ccc} z_1 & z_2 & z_3 \end{array} \\ \left[\frac{\partial f}{\partial z_1}(g(x,y)) \quad \frac{\partial f}{\partial z_2}(g(x,y)) \quad \frac{\partial f}{\partial z_3}(g(x,y)) \right] \end{array} \cdot$$

$$\cdot \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} \frac{\partial g_1}{\partial x}(x,y) & \frac{\partial g_1}{\partial y}(x,y) \\ \frac{\partial g_2}{\partial x}(x,y) & \frac{\partial g_2}{\partial y}(x,y) \\ \frac{\partial g_3}{\partial x}(x,y) & \frac{\partial g_3}{\partial y}(x,y) \end{bmatrix} \end{array}$$

$$\left[\frac{\partial h}{\partial x}(x,y) = \left[\frac{\partial f}{\partial z_1}(g(x,y)) \right] \cdot \frac{\partial g_1}{\partial x}(x,y) + \frac{\partial f}{\partial z_2}(g(x,y)) \cdot \frac{\partial g_2}{\partial x}(x,y) + \frac{\partial f}{\partial z_3}(g(\cancel{x},y)) \cdot \frac{\partial g_3}{\partial x}(x,y) \right]$$

h:

$$\frac{\partial h}{\partial x} =$$



$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial g_1}(g(x,y)) \cdot \frac{\partial g}{\partial x}(x,y) + \frac{\partial f}{\partial g_2}(g(x,y)) \frac{\partial g_2}{\partial x}(x,y) + \frac{\partial f}{\partial g_3}(g(x,y)) \cdot \frac{\partial g_3}{\partial x}$$