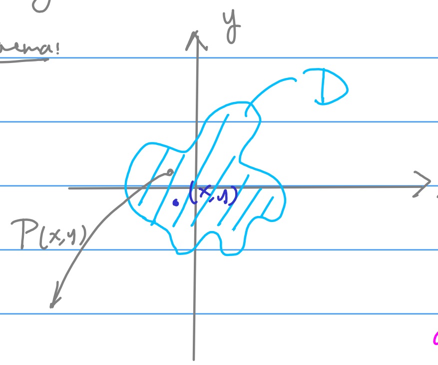
Hoy (¿Cómo se usa el Teorema del cambio de variable?)

Teorema:

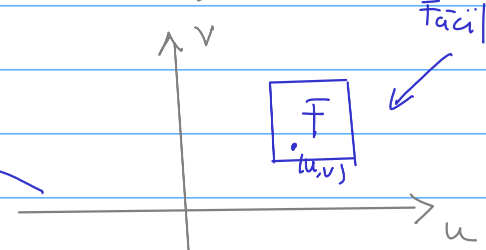


$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

G

Transformación no nec. lineal

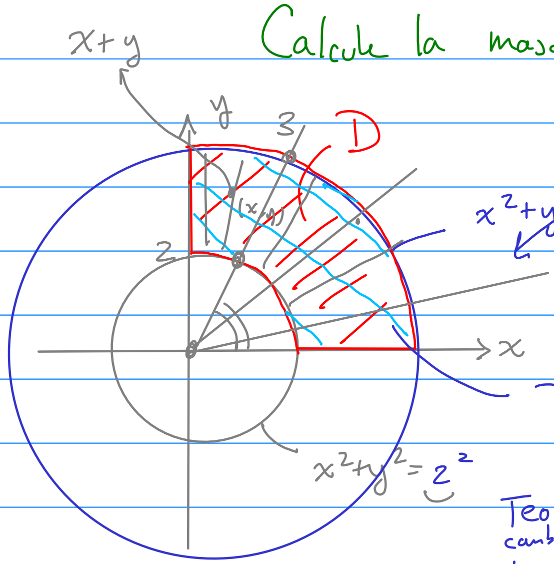
dif y biyectiva



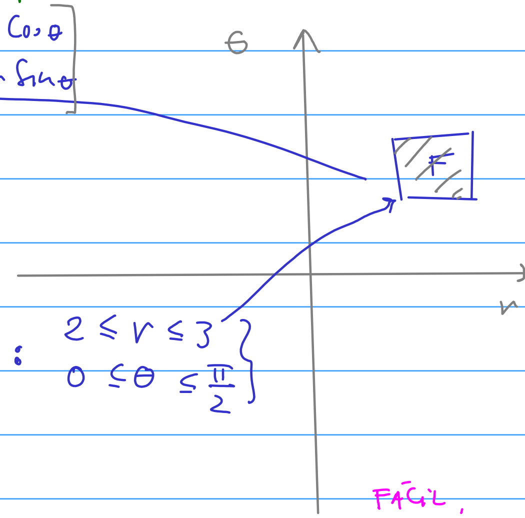
$$\iint_D P(x,y)\,dA = \iint_F P(g(u,v),h(u,v)) \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| du\,dv$$

$$\left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \frac{\partial(x,y)}{\partial(u,v)}$$

Ejercicio: Un plato tiene forma de un cuarto de anillo. Más precisamente el círculo interior es $x^2+y^2=4$, el exterior es $x^2+y^2=9$ y el plato está en el cuadrante $x\geq 0$, $y\geq 0$. La densidad del material en (x,y) es $\rho(x,y) = x+y$ Kg/m^2. Calcule la masa total del plato.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$D = \left\{(r,\theta) : \begin{array}{l} 2 \leq r \leq 3 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{array}\right\}$$

Teo cambio de var.

FÁCIL.

$$M \overset{?}{=} \iint_D \underbrace{\rho(x,y)}_{x+y}\,dA = \int_2^3\int_0^{\frac{\pi}{2}} (r\cos\theta + r\sin\theta)\, r\, d\theta\, dr = \ldots$$

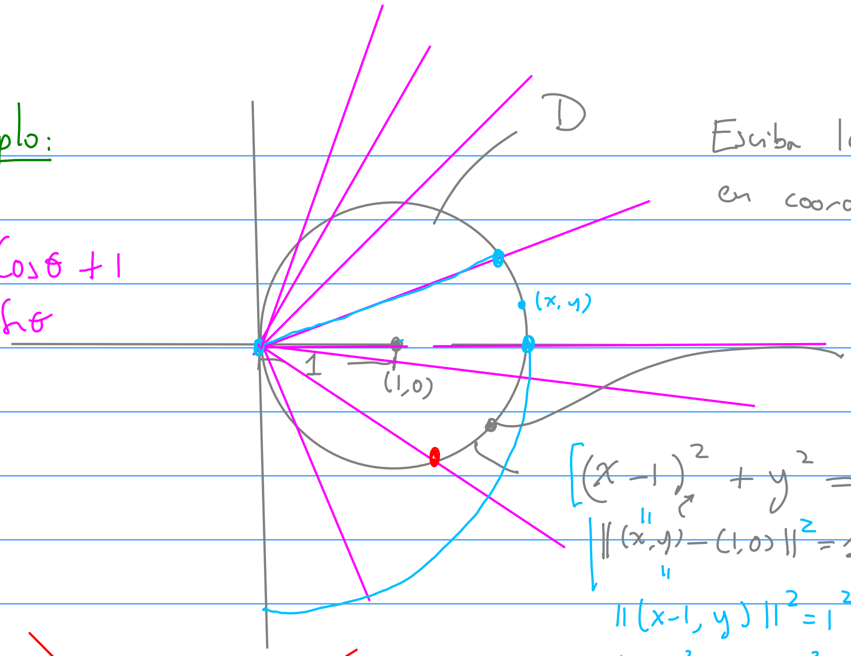
$$\sum\sum \Big[\underbrace{\rho(x,y)\cdot \text{Área}(R)}_{\text{masa}}\Big]$$

Ejemplo:

$$\iint_D x^2+y^2\,dA$$

Escriba la integral sobre D en coordenadas polares.

$$\begin{cases} x = r\cos\theta + 1 \\ y = r\sin\theta \end{cases}$$



$r = 2\cos\theta$

$$[(x-1)^2 + y^2 = 1]$$

$$\|(x,y) - (1,0)\|^2 = 1^2$$

$$\|(x-1, y)\|^2 = 1^2$$

$$(x-1)^2 + y^2 = 1^2$$

Intento 1:

$$\left[\begin{array}{l} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2 \end{array}\right.$$

Sustituimos en la ecuación:

$$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$$

$$r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1}$$

$$r^2(\cos^2\theta + \sin^2\theta) - 2r\cos\theta = 0$$

$$[r^2 - 2r\cos\theta = 0]$$ — despejar r

$$r^2 = 2\cancel{r}\cos\theta$$

$$\boxed{r = 2\cos\theta}$$

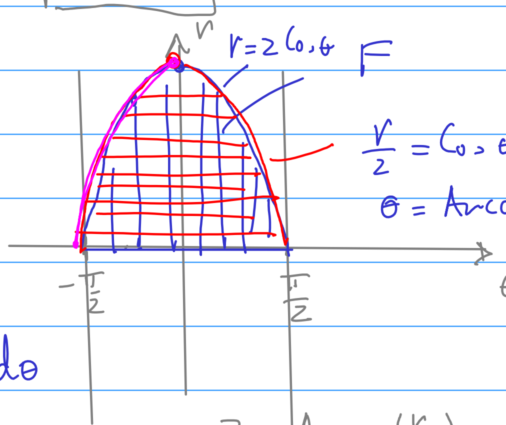
Intento 2:

$$\left[\begin{array}{l} -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \\ 0 \le r \le 2\cos\theta. \end{array}\right.$$

$$M = \iint_D x^2+y^2\,dA =$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} [(r\cos\theta)^2 + (r\sin\theta)^2]\, r\,dr\,d\theta$$

$$= \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_0^{2\cos\theta} r^3\,dr\right) d\theta\right] = \left[\int_0^2 \int_{-\operatorname{Arccos}\left(\frac{r}{2}\right)}^{\operatorname{Arccos}\left(\frac{r}{2}\right)} r^3\,d\theta\,dr\right]$$



$$\frac{r}{2} = \cos\theta$$

$$\theta = \operatorname{Arccos}\left(\frac{r}{2}\right)$$

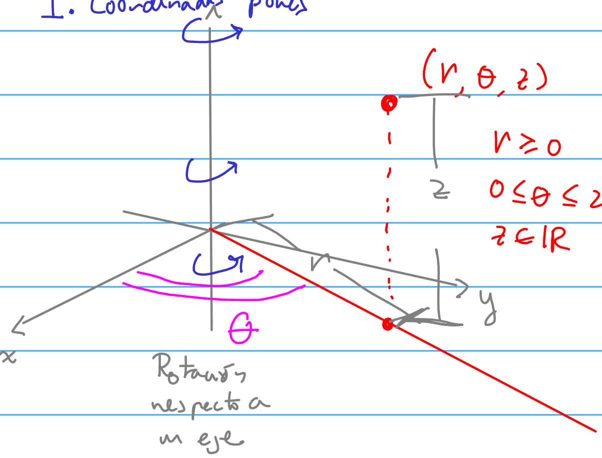
¿Podemos cambiar el orden? Sí, pero con cuidado

(2) Sistemas de coordenadas en 3D:

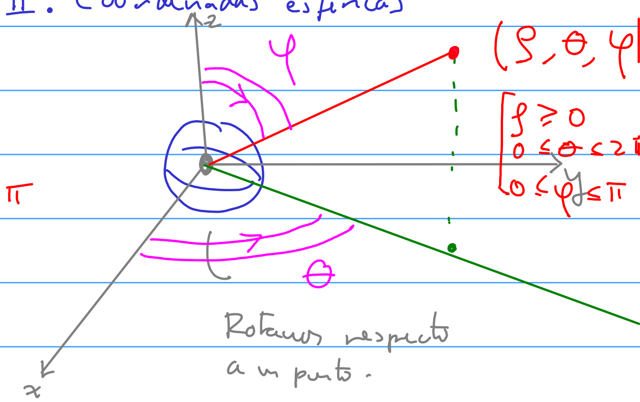
¿Cuál es la versión 3D de coordenadas polares?

Hay 2 tipos de "simetrías" "círculos" en $\mathbb{R}^3$

I. Coordenadas polares



II. Coordenadas esféricas



ρ = dist al origen

Polares en (x,y) z como siempre

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

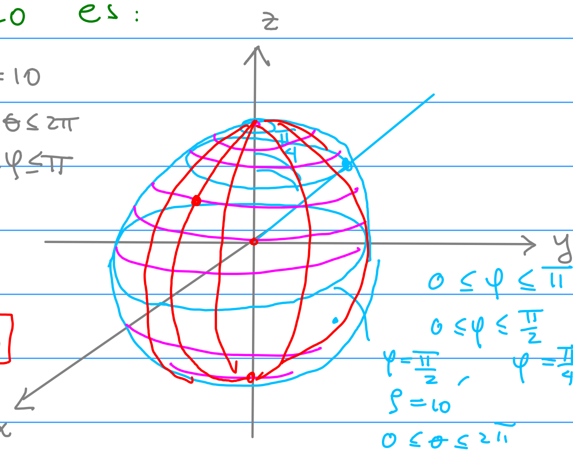
$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

Ejemplo: La esfera de radio 10 es:

Intento 1: ~~$\rho = 10$, $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \cancel{2}\pi$, $\varphi = \frac{\pi}{2}$~~

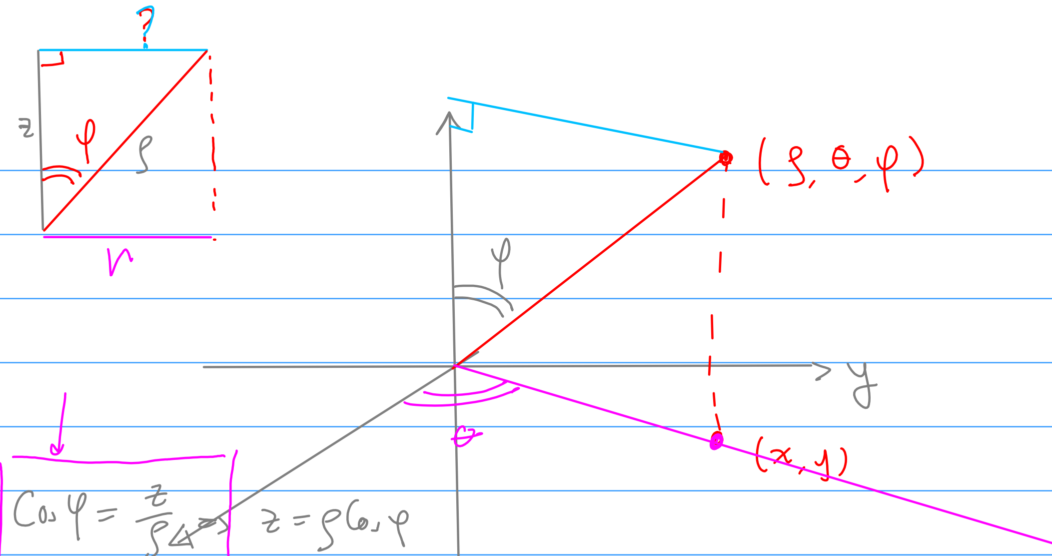
$\rho = 10$
$0 \leq \theta \leq 2\pi$
$0 \leq \varphi \leq \pi$

$\theta = 0$
$\boxed{\rho = 10}$
$0 \leq \varphi \leq \pi$



$\rho = 10$

$$\begin{cases} \varphi = \alpha \\ \theta = \theta_0 \end{cases}$$



$$\cos\varphi = \frac{z}{\rho} \Rightarrow z = \rho\cos\varphi$$

$$\sin\varphi = \frac{?}{\rho} \Rightarrow r = \rho\sin\varphi$$

$$x = r\cos\theta = \rho\sin\varphi\cos\theta$$
$$y = r\sin\theta = \rho\sin\varphi\sin\theta$$

Ejercicio: Demuestre que los jacobianos son:

(1) Cilíndricas r

(2) Esféricas $\rho^2\sin(\varphi)$

$$\left|\det\begin{bmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\varphi} \\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\varphi} \end{bmatrix}\right| = \rho^2\sin\varphi$$