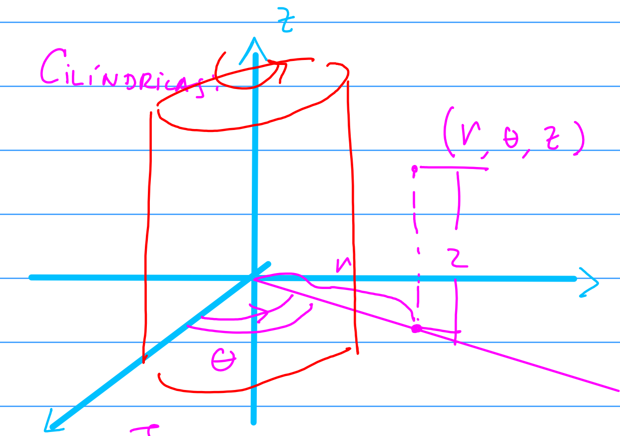
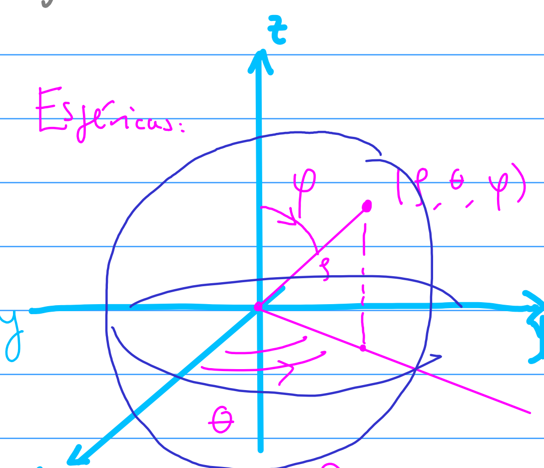
Hoy: Ejemplos del uso de coords esféricas y cilíndricas para calcular integrales



$J = r$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



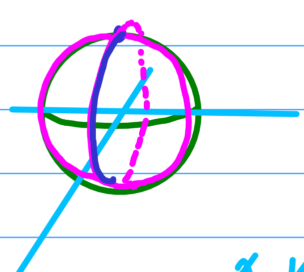
$J = \rho^2 \sin\varphi$

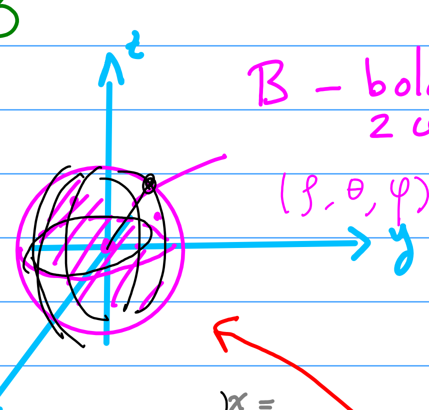
$$(*)\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

Ejercicio: Sea $B = \{(x,y,z) : \|(x,y,z)\| \leq 2\} \subset \mathbb{R}^3$

Calcule $$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV \overset{?}{=}$$

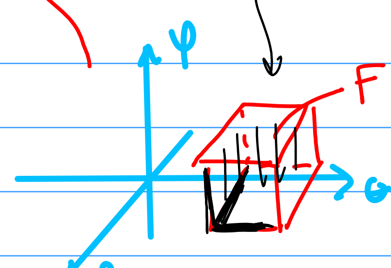
Solución:





B – bola de radio 2 centrada en $(0,0,0)$

$$\boxed{\begin{array}{l} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}}\ (*)$$



$$(*)\begin{cases} x = \\ y = \\ z = \end{cases}$$

Por Teo cambio de variable

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}}\, dV$$

$$\|$$

$\parallel$

$$x^2+y^2+z^2 = (\rho \sin\varphi \cos\theta)^2 + (\rho \sin\varphi \sin\theta)^2 + (\rho\cos\varphi)^2 =$$

$$= \rho^2 \sin^2\varphi \, [\underbrace{\cos^2\theta + \sin^2\theta}_{1}] + \rho^2\cos^2\varphi =$$

$$= \rho^2 (\sin^2\varphi + \cos^2\varphi) = \rho^2$$

Jac. a esféricas

$$= \int_0^{2\pi}\left[\int_0^{\pi}\int_0^{2} e^{\rho^3}\, \boxed{\rho^2 \sin\varphi} \, d\rho\, d\varphi\right] d\theta = \int_0^{2\pi} d\theta \, [\quad]$$

$$= 2\pi \int_0^{\pi} \sin\varphi \Big[\underbrace{\int_0^{2} e^{\rho^3}\rho^2\, d\rho}_{\substack{u=\rho^3 \\ du = 3\rho^2 d\rho}}\Big]\, d\varphi$$

$$\int_0^8 e^u \frac{du}{3} = \frac{e^u}{3}\Big|_0^8 = \frac{e^8-1}{3}$$

$$-\cos(\varphi)\Big|_{\varphi=0}^{\varphi=\pi}$$

$$= 2\pi \int_0^{\pi} \sin\varphi \left[\frac{e^8-1}{3}\right] d\varphi = 2\pi\left[\frac{e^8-1}{3}\right] \int_0^{\pi} \sin\varphi\, d\varphi$$

$$= \boxed{4\pi\left[\frac{e^8-1}{3}\right]}$$

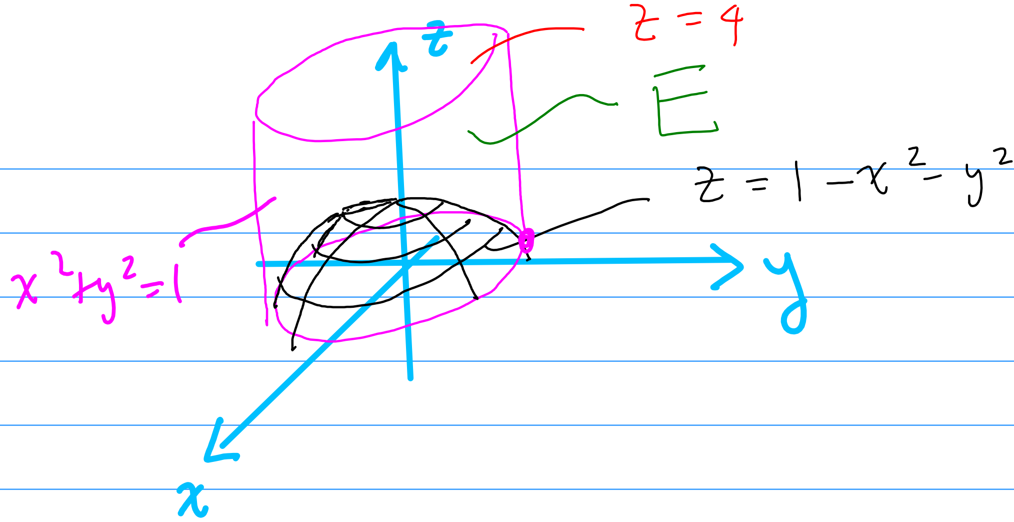
Ejercicio: Sea E el sólido adentro de $x^2+y^2=1$, debajo de $z=4$ y encima de $z=1-x^2-y^2$. Calcule:

(a) $\mathrm{Vol}(E) = \iiint_E 1\, dV$

(b) masa (E) si la densidad es proporcional a la distancia con el eje z y vale $10\ kg/m^3$ en $(1,0,1)$

$\rho(x,y,z) =$ " kg/m^3 del metal del que está hecho (x,y,z) ".

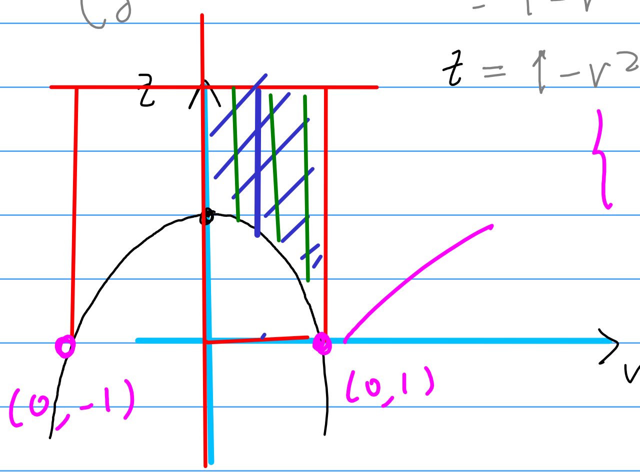
$$\text{masa}(E) = \iiint_E \rho(x,y,z)\, dV$$



$z = 1 - x^2 - y^2$ a cilíndricas:

$$\begin{cases} z = z \\ x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$z = 1 - (r\cos\theta)^2 - (r\sin\theta)^2 =$$
$$= 1 - r^2\left[\cos^2\theta + \sin^2\theta\right]$$

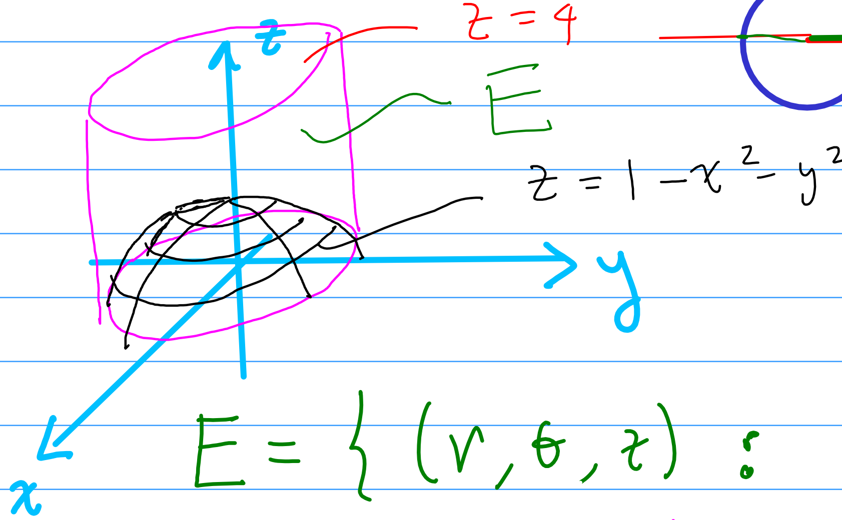


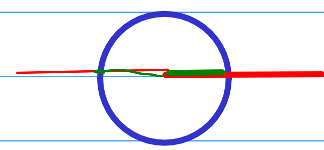
$z = 1 - r^2$

$$\begin{cases} z = 1 - r^2 \\ r^2 = 1 \end{cases} \iff \begin{cases} z = 0 \\ r^2 = 1 \end{cases}$$

$$\begin{cases} r = \pm 1 \\ z = 0 \end{cases}$$

$(0,1)$, $(0,-1)$





$$E = \left\{ (r, \theta, z) : \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 1 - r^2 \le z \le 4 \end{array} \right\}$$

$$E = \left\{ (r,\theta,z) : \begin{array}{c} 0 \le \theta \le 2\pi \\ 0 \le r \le 1 \\ 1-r^2 \le z \le 4 \end{array} \right\}$$

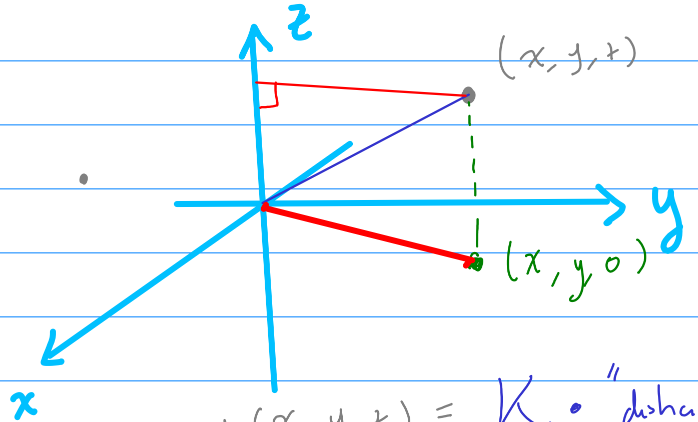
Jacobiano de cilíndricas.

$$\mathrm{Vol}(E) \underset{\text{en cartesianas}}{=} \iiint_E 1 \, dV \underset{\text{Teo cambio varsh}}{=} \left[\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 1 \cdot \boxed{r} \, dz \, dr \, d\theta \right]$$

$$= 2\pi \int_0^1 r \int_{1-r^2}^4 dz \, dr = \int_0^1 r \left[3+r^2\right] dr \cdot 2\pi$$

$$= 2\pi \int_0^1 r\left(3+r^2\right) dr = 2\pi \left[\frac{3}{2} + \frac{1}{4} \right] = \frac{6+1}{2}\pi = \boxed{\frac{7\pi}{2}}$$

(b) La densidad es "proporcional a la distancia con el eje z"



$$\rho(x,y,z) = K \cdot \text{"distancia entre } (x,y,z) \text{ y el eje } z\text{"}$$

$$= K\sqrt{x^2+y^2}$$

?

$$10 = \rho(1,0,1) = K\sqrt{1^2+0^2} = K$$

$$\boxed{\rho(x,y,z) = 10\sqrt{x^2+y^2}} \rightsquigarrow 10r$$

$$\text{masa}(E) \overset{\text{En cartesianas}}{=} \iiint_E p(x,y,z)\, dV$$

$$\overset{\text{Teo cambio var a cilíndricas}}{=} \int_0^{2\pi} \int_0^{1} \int_{z=1-r^2}^{z=4} 10r \cdot r \cdot dz\, dr\, d\theta = \cdots$$

Ejercicio: Calcule el volumen del sólido E dentro de la esfera de radio $\frac{1}{2}$ y centro $(0,0,\frac{1}{2})$ encima del cono $z=\sqrt{x^2+y^2}$, $z \geq 0$