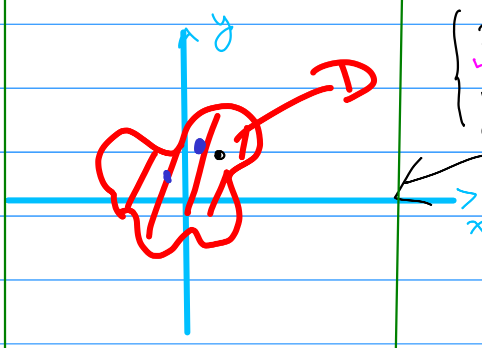
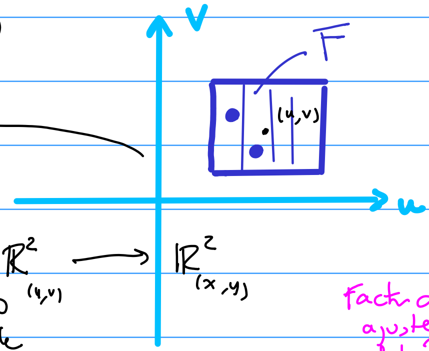
Hoy: Teorema del cambio de variable



$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

T



$T: \mathbb{R}^2_{(u,v)} \longrightarrow \mathbb{R}^2_{(x,y)}$

CAMBIO de variable de (u,v) a (x,y)

Factor de ajuste del área

$$\iint_D p(x,y)\,dA \overset{*}{=} \iint_F p\big(g(u,v),\, h(u,v)\big)\, \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$

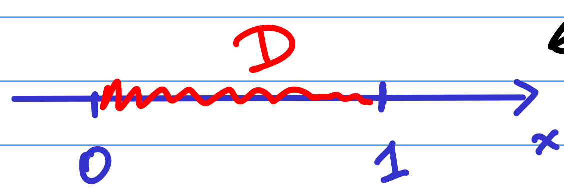
Teorema: Si T es 1-1 y diferenciable entonces la igualdad $*$ es cierta.

Obs: En 1 variable

$$\int_0^1 \sin(e^x)\, e^x\, dx = \int_1^e \sin\left(e^{\ln(u)}\right) e^{\ln(u)} \cdot \frac{1}{u}\, du$$

$x = \ln(u)$

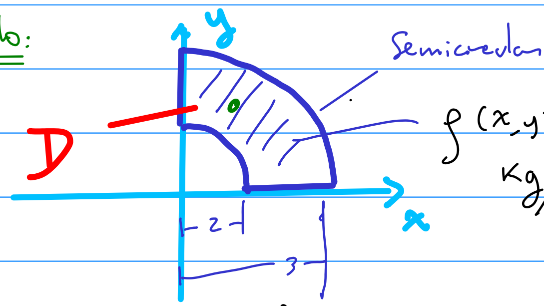
$$\frac{dx}{du} = \frac{1}{u}\, du$$



$$\ln(1001) - \ln(1000) = \frac{1}{1000} \cdot (1001 - 100$$

Obs: En $\mathbb{R}^2$ hay dos cambios de variable principales: polares y lineales

Ejemplo:

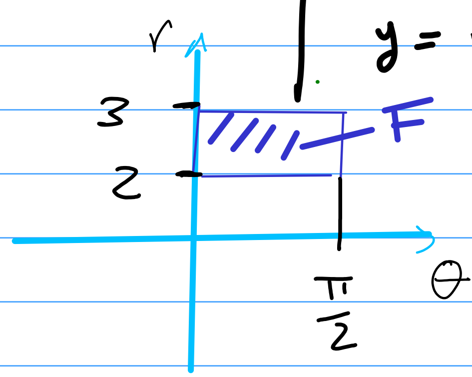


$\rho(x,y) = x^2+y^2+1$ Kg/m^2 (densidad sup.)

Calcule la masa total.

$x = r\cos\theta$

$y = r\sin\theta$



$$\begin{cases} 2 \le r \le 3 \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

Por Teo del cambio de variable

$$r^2\left[\cancel{\cos^2\theta + \sin^2\theta}^{\,1}\right] + 1$$

$$\underbrace{\iint_D \overbrace{(x^2+y^2+1)}^{\text{densidad}}\, dA}_{\text{masa}} = \iint_{F} \underbrace{\left[(r\cos\theta)^2 + (r\sin\theta)^2 + 1\right]}_{r^2+1 \;=\; \dots} \cdot \, dr\, d\theta$$

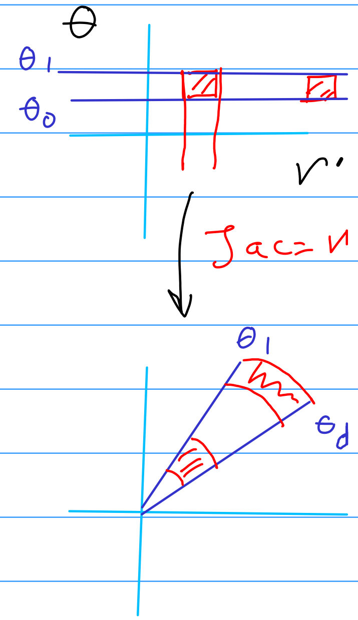
F: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

Jacobino

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left| \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right| = r$$

$$\left| \det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \right| = r\cos^2\theta - (-r\sin^2\theta) = r$$

$$= \iint_F (r^2+1)\cdot r \, dr\, d\theta = \int_2^3 \int_0^{\frac{\pi}{2}} (r^3+r)\, d\theta\, dr$$

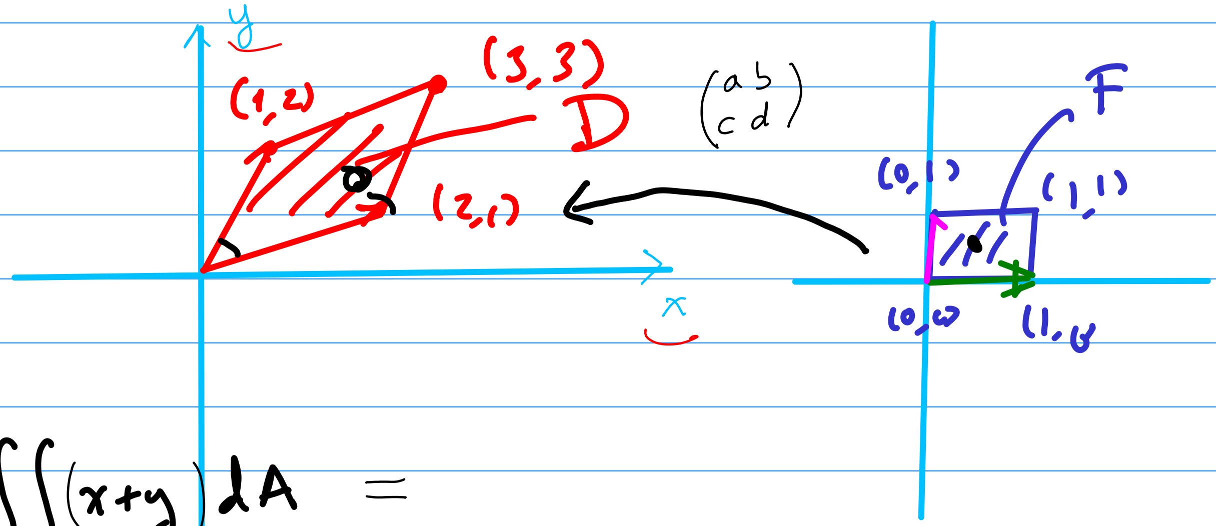


$$= \frac{\pi}{2} \int_2^3 (r^3+r)\, dr =$$

$$= \frac{\pi}{2} \left(\frac{r^4}{4} + \frac{r^2}{2} \Big|_{r=2}^{r=3} \right)$$

$$= \frac{\pi}{2} \left(\frac{3^4}{4} + \frac{3^2}{2} - \frac{2^4}{4} - \frac{2^2}{2} \right)$$

Ejemplo 2: Trasformaciones lineales



$$\iint_D (x+y)\, dA =$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{cases} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \\ \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \end{cases}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Así que proponemos

$$\begin{cases} x = 2u + v \\ y = u + 2v \end{cases}$$

Jacobiano

$$\iint_D (x+y)\, dA = \int_0^1 \int_0^1 (2u+v) + (u+2v) \; 3 \; dv\, du$$

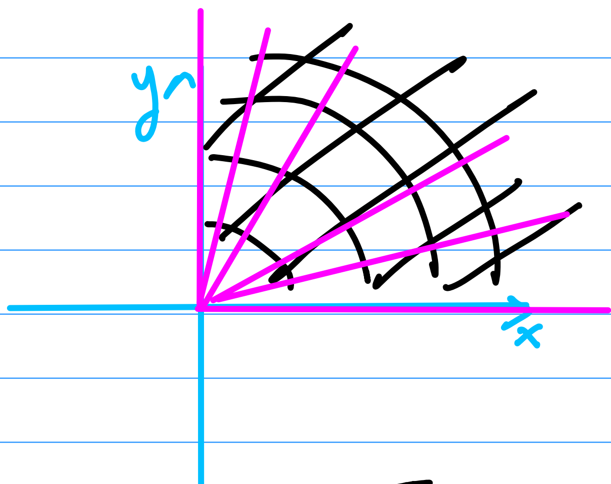
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \right) = 3$$

$$= \int_0^1 \int_0^1 3(u+v) \cdot 3 \, du\, dv =$$

Ejemplo 3: $\int_0^\infty e^{-\frac{x^2}{2}}\,dx \overset{?}{=} I$

$$I^2 = \int_0^\infty e^{-\frac{x^2}{2}}\,dx \cdot \int_0^\infty e^{-\frac{y^2}{2}}\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-\frac{(x^2+y^2)}{2}}\,dx\,dy$$



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$0 \le \theta \le \frac{\pi}{2}$

$$= \int_0^\infty \int_0^{\frac{\pi}{2}} e^{-\frac{r^2}{2}}\, r \; d\theta\, dr$$

$$= \frac{\pi}{2} \int_0^\infty e^{-\frac{r^2}{2}}\, r\, dr = \frac{\pi}{2}$$

$$\left(-e^{-\frac{r^2}{2}} \Big|_{r=0}^{r=\infty} \right)$$

$$I^2 = \frac{\pi}{2} \implies \boxed{I = \sqrt{\frac{\pi}{2}}}$$