Hoy: Regla de la cadena para calcular derivadas

Teorema: Suponga que $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$, $\mathbb{R}^m \xrightarrow{g} \mathbb{R}^p$ son funciones dadas. Si f es diferenciable en $\vec{a}$ y g es diferenciable en $f(\vec{a})$ entonces

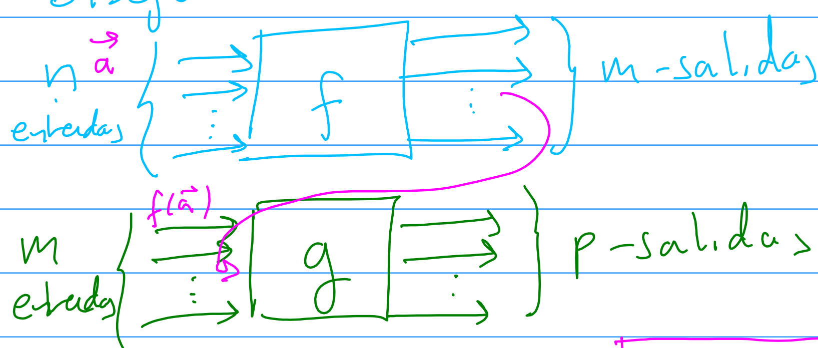
$h = g \circ f$ es diferenciable en $\vec{a}$

y tenemos una manera sencilla de calcular su derivada así:

$$Dh(\vec{a}) = Dg(f(\vec{a})) \cdot Df(\vec{a})$$

Producto de matrices

Dibujo



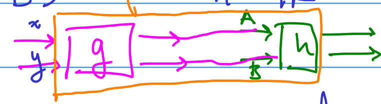
$$h = g \circ f$$

$$\boxed{h(\vec{x}) = g(f(\vec{x}))}$$

$$\boxed{Dh(\vec{a}) = Dg(f(\vec{a})) \cdot Df(\vec{a})}$$

Ejercicio: $g(x,y) = (\overset{A}{x^2+y^2}, \overset{B}{x^2-y^2})$ $\quad g:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$h(A,B) = (B^2, A-B)$ $\quad h:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$



(1) Calcule $T = h\circ g$

(2) Calcule $DT(1,2)$ de dos maneras primero usando (1) y calculando $DT(1,2)$ y luego mediante la regla de la cadena directamente.

(1) $T(x,y) = h(g(x,y)) = \left((x^2-y^2)^2, x^2+y^2-(x^2-y^2)\right)$

$$\boxed{T(x,y) = (\underbrace{x^4+y^4-2x^2y^2}_{T_1}, \underbrace{2y^2}_{T_2})} \leftarrow T(x,y)$$

(2) método 1: a partir de la fórmula de la comp.

$T:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$DT(1,2) = \begin{bmatrix} \frac{\partial T_1}{\partial x}(1,2) & \frac{\partial T_1}{\partial y}(1,2) \\ \frac{\partial T_2}{\partial x}(1,2) & \frac{\partial T_2}{\partial y}(1,2) \end{bmatrix}$$

$\frac{\partial T_1}{\partial x} = 4x^3 - 4xy^2$

$\frac{\partial T_1}{\partial y} = 4y^3 - 4x^2y$

$\frac{\partial T_2}{\partial x} = 0$

$\frac{\partial T_2}{\partial y} = 4y$

R1:

$$DT(1,2) = \begin{bmatrix} -12 & 24 \\ 0 & 8 \end{bmatrix}$$

$4(1)^3 - 4\cdot 1\cdot 2^2 = -12$

$4\cdot 2^3 - 4\cdot 1^2\cdot 2 = 24$ ✓

R2: $g(x,y) = (\overset{g_1}{x^2+y^2}, \overset{g_2}{x^2-y^2})$ $\quad g:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$h(A,B) = (\overset{h_1}{B^2}, \overset{h_2}{A-B})$ $\quad h:\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$T(x,y) = h(g(x,y))$, $\quad DT(1,2)$

Por la regla de la cadena

$$DT(1,2) = \underline{Dh(g(1,2))} \cdot \underline{Dg(1,2)}$$

$$Dg(1,2) = \begin{bmatrix} \frac{\partial g_1}{\partial x}(1,2) & \frac{\partial g_1}{\partial y}(1,2) \\ \frac{\partial g_2}{\partial x}(1,2) & \frac{\partial g_2}{\partial y}(1,2) \end{bmatrix} = \begin{bmatrix} 2x & 2y \\ 2x & -2y \end{bmatrix}\Bigg|_{\substack{x=1\\ y=2}}$$

$$= \begin{bmatrix} 2 & 4 \\ 2 & -4 \end{bmatrix} \qquad g(1,2) = (1^2+2^2,\ 1^2-2^2) = (5,-3)$$

Función de A y B

$$h(A,B) = (B^2,\ A-B)$$

$$Dh(g(1,2)) = Dh(5,-3) = \begin{bmatrix} \frac{\partial h_1}{\partial A}(5,-3) & \frac{\partial h_1}{\partial B}(5,-3) \\ \frac{\partial h_2}{\partial A}(5,-3) & \frac{\partial h_2}{\partial B}(5,-3) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -6 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2B \\ 1 & -1 \end{bmatrix}\Bigg|_{\substack{A=5\\ B=-3}}$$

$$\boxed{DT(1,2)} = \begin{bmatrix} 0 & -6 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 \\ 2 & -4 \end{bmatrix} = \begin{bmatrix} -12 & 24 \\ 0 & 8 \end{bmatrix}$$

Obs: La regla de la cadena permite calcular $D(h(g(a,b)))$ SIN calcular $h(g(x,y))$. Gracias a esto nos permitirá entender cómo se relacionan las derivadas entre sí.

$$\left[DT(x,y) = Dh(f(x,y)) \cdot Df(x,y) \right]$$

$g(1,3) = 1^2 - 3^2 = 1 - 9 = -8\,°C$

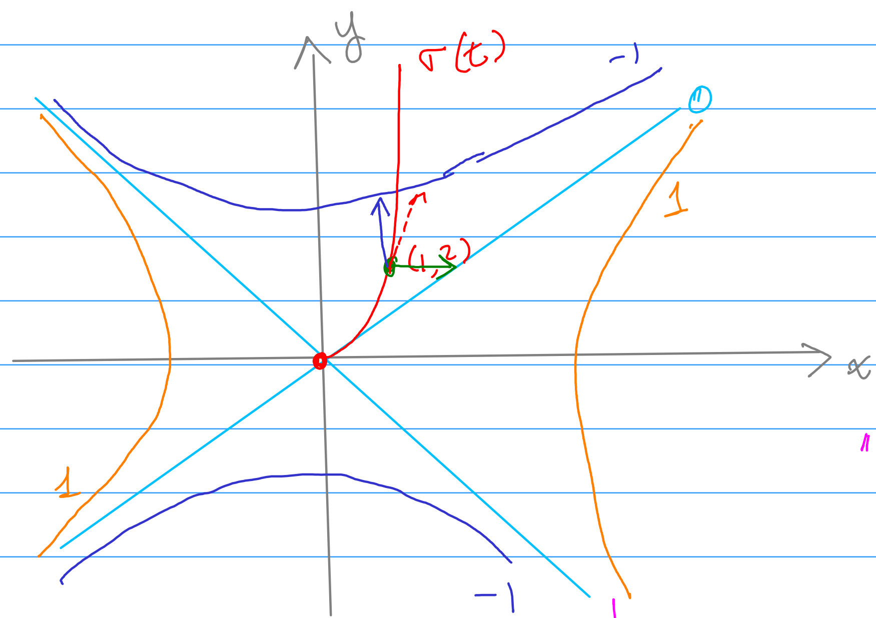
Ejemplo:

$g(x,y) = x^2 - y^2$ = "Temperatura en °C del punto (x,y)"

$\sigma(t) = (\overset{x}{t}, \overset{y}{2t^2})$ = "Posición de una partícula en el instante t" ← Trayectoria de la partícula

$\boxed{(1)}$ Calcule la composición $h(t) = g(\sigma(t))$ e interpretela físicamente

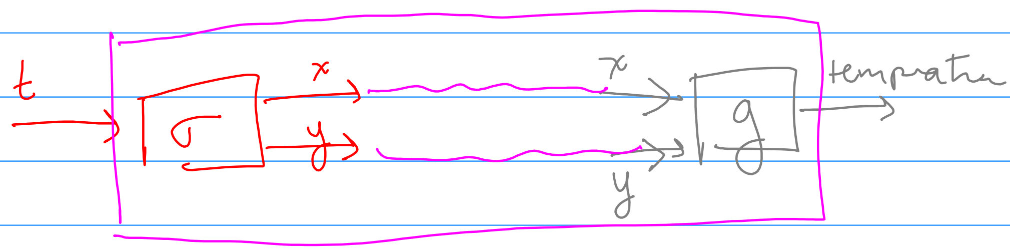
(2) Calcule $Dh(t)$ con los dos métodos en $t = 1$



$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\sigma : \mathbb{R} \longrightarrow \mathbb{R}^2$

$h(t) = g(\sigma(t))$ = "Temperatura de la partícula en el instante t"



$h : \underset{t}{\mathbb{R}} \longrightarrow \mathbb{R}$

$h'(1)$ = "Tasa de cambio de la temperatura de la partícula en el instante $t = 1$" $= Dh(1)$

Sol parte (2):

$g(x,y) = x^2 - y^2$

$\sigma(t) = (\overset{x}{t}, \overset{y}{2t^2})$

$h(t) = g(\sigma(t)) = t^2 - (2t^2)^2 = t^2 - 4t^4$

$Dh(1) = 2t - 16t^3 \big|_{t=1} = 2 - 16 = -14$ °C/sec.

$g(x,y) = x^2 - y^2$

$\sigma(t) = (t, 2t^2)$, $\sigma_1 = t$, $\sigma_2 = 2t^2$

$g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$\sigma : \mathbb{R} \longrightarrow \mathbb{R}^2$

Sol 2, parte ②:

Por regla de la cadena sabemos que $\boxed{h(t) = g(\sigma(t))}$

$$Dh(1) = \underbrace{Dg(\sigma(1))}\cdot\underbrace{D\sigma(1)}$$

$$D\sigma(1) = \begin{bmatrix} \sigma_1'(t) \\ \sigma_2'(t) \end{bmatrix} = \begin{bmatrix} 1 \\ 4t \end{bmatrix}\Bigg|_{t=1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \sigma(1) = (1,2)$$

$$Dg(1,2) = \left[\frac{\partial g}{\partial x}(1,2) \quad \frac{\partial g}{\partial y}(1,2)\right] = \left[2x \quad -2y\right]\Big|_{\substack{x=1\\y=2}} = \left[2 \quad -4\right]$$

$$Dh(1) = \left[2 \quad -4\right]\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 2\cdot 1 + (-4)\cdot 4 = -14 \ {}^{\circ}C/seg.$$

$$Dh(t) = Dg(\sigma(t))\, D\sigma(t)$$

$$= \overbrace{\left[2t \quad -4t^2\right]}^{\shortparallel}\begin{bmatrix} 1 \\ 4t \end{bmatrix}$$

$$= 2t + (-4t^2)(4t)$$

$$= 2t - 16t^3$$

$$Dg(\sigma(t)) = \left[\frac{\partial g}{\partial x}(\sigma(t)),\ \frac{\partial g}{\partial y}(\sigma(t))\right]$$

$$\left[2x \quad -2y\right]\Big|_{\substack{x=t\\y=2t^2}}$$

$$\shortparallel$$

$$\left[2t \quad -4t^2\right]$$