Question: ¿How many fixed points does a generic morphism $f: \mathbb{P}^n \to \mathbb{P}^n$ have? $f([\vec{x}]) = [\vec{x}]$

Special case: $\mathbb{P}^n \xrightarrow{\varphi} \mathbb{P}^n$ $\quad A \in \mathbb{C}^{(n+1)\times(n+1)}$

$[\vec{x}] = [x_0 : \dots : x_n] \longmapsto [A\vec{x}]$

$[x_0 : x_1]$ $\quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longmapsto [x_0 + 2x_1 : 3x_0 + 4x_1]$

$\mathbb{P}^1$ $\quad \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \longmapsto [x_0 + 2x_1 : x_0 + 2x_1]$

The linear forms given by the components of the map have no common zeroes in $\mathbb{P}^n$ and therefore defines a morphism.

¿How many fixed points?

If $\vec{v}$ is an eigenvector of A $\quad \vec{v} \neq \vec{0}$

$[\vec{v}] \in \mathbb{P}^n$, $\quad A\vec{v} = \lambda \vec{v}$

$[A\vec{v}] = [\lambda \vec{v}] = [\vec{v}]$

If the map is linear the answer should be $\boxed{(n+1)}$ ∞

Idea: Build a space and subvarieties of this space with the property that the number you wish to count corresponds to the number of intersection points among the subvarieties you have built.

We will use the product space $\mathbb{P}^n \times \mathbb{P}^n$ as ambient and will define suitable subvarieties in it.

Given f: $\quad A(\mathbb{P}^n \times \mathbb{P}^n)$

(i) $\mathbb{P}^n \times \mathbb{P}^n \supseteq \Delta = \{([x],[y]) : [x] = [y]\}$

(ii) $\mathbb{P}^n \times \mathbb{P}^n \supseteq \Gamma_{graph} = \{([x],[y]) : [y] = f([x])\}$

$F \underset{s}{\overset{\#}{\rightleftarrows}} \Delta \cap \Gamma = \left\{ ([x],[y]) : \begin{matrix} (1)\ [x]=[y] \\ (2)\ [y]=[f(x)] \end{matrix} \right\} = \{ ([$

$\Delta\cap\Gamma \xrightarrow{\pi_1} \mathbb{P}^n$

$[f(x)] = [x]$

$\pi_1(\Delta\cap\Gamma)$ = Fixed point of the map f

What is the cardinality of this intersection?

When is this intersection proper and reduced

Condition f

$\mathrm{codim}(\Delta) + \mathrm{codim}(\Gamma) \;\; \| \;\; \mathrm{codim}(\Delta\cap\Gamma)$

If $\Delta\cap\Gamma$ is a reduced set of points $[\Delta\cap\Gamma] = \#(\Delta\cap\Gamma)\cdot\{pt\}$

?

$$\underbrace{[\Delta]\cdot[\Gamma]}_{\text{easy}} = [\Delta\cap\Gamma]$$

!

(1) Find $[\Delta]$ and $[\Gamma_f] = [\Gamma]$

(2) Multiply in $A(\mathbb{P}^n\times\mathbb{P}^n)$ and explain why it is correct.

Recall $A(\mathbb{P}^n\times\mathbb{P}^n)$: $\mathbb{P}^n = H_0^0 \supseteq H_0^1 \supseteq H_0^2 \supseteq \dots \supseteq H_0^n = \{pt\}$

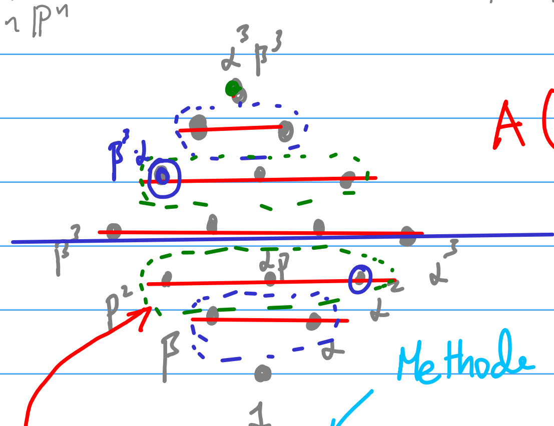
$H_0^i \times H_1^j \subseteq \mathbb{P}^n\times\mathbb{P}^n$

H_0^i: generic subspace of codim i in $\mathbb{P}^n$

H_1^j: generic subspace of codim j

$\alpha^i\beta^j = [H_0^i \times H_1^j] \subseteq A(\mathbb{P}^n\times\mathbb{P}^n)$

$\{\alpha^i\beta^j ;\ 0\le i\le n,\ 0\le j\le n\}$ are a $\mathbb{Z}$-basis for $A(\mathbb{P}^n\times\mathbb{P}^m) = \bigoplus_{j=0}^{n+m} A^j(\mathbb{P}^n\times\mathbb{P}^m)$



$A(\mathbb{P}^3\times\mathbb{P}^3)$ is a Gorenstein ring

$R_d \times R_{n-d} \longrightarrow R_n \to k$

Methode of

points of int

$A\alpha^2 + B\alpha\beta + C\beta^2 = [\ ]$

$\Delta\cdot\alpha\beta^2 = A(\alpha^3\beta^3) = [\ \cap H_0\times H_1^3]$

Let's apply the method of indeterminate coefficients to find $[\Delta]$

$$\mathbb{P}^n \times \mathbb{P}^n \supseteq \Delta = \{([x],[y]) : [x]=[y]\}$$

$$\begin{tikzcd} \Delta \arrow[d, "\pi_1"] \arrow[d, phantom, shift right=4, "\cong"] \\ \mathbb{P}^n \arrow[u, bend left, "s"'] \end{tikzcd} \qquad ([x],[x]) \mapsfrom [x] \qquad \Delta \cong \mathbb{P}^n$$

So Δ is irreducible and of $\dim(\Delta)=n$ so $\operatorname{codim}_{\mathbb{P}^n\times\mathbb{P}^n}(\Delta) = 2n-n = n$

$$[\Delta] = \sum_{j=0}^{n} c_j \alpha^j \beta^{n-j} \qquad c_j ?$$

Fix $j=t$ $\quad c_t \alpha^t \beta^{n-t}$

every other term disappears.

$$[\Delta]\cdot \alpha^{n-t}\beta^{t} = \sum_{j=0}^{n} c_j \alpha^{j+n-t}\beta^{t+n-j} = c_t \alpha^n \beta^n$$

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# of points of int of Δ and $H_0^{n-t} \times H_1^t$

$$\Delta \cap H_0^{n-t} \times H_1^t = \underbrace{H_0^{n-t}}_{L_1 \ldots L_{n-t}} \cap \underbrace{H_1^t}_{A_1 \ldots A_t}$$

$$\Delta \cap A \times B = A \cap B$$

$\Delta = \{([x],[y]) : [x]=[y]\}$

ONE POINT!! in $\mathbb{P}^n$

$\mathbb{C}^{n+1}$ $\{L_1,\ldots,L_{n-t}, A_1 \ldots A_t\}$

$\mathbb{C}^{n+1} \xrightarrow{B} \mathbb{C}^n$

We conclude:

$$[\Delta] = \sum_{k=0}^{n} \alpha^k \beta^{n-k}$$

$$\left[[\Gamma_f] = \sum_{j=0}^{n} e_j^{?} \alpha^j \beta^{n-j} \right]$$

$$\begin{tikzcd} ([x], f([x])) \Gamma_f \arrow[d, "\pi_1"] \\ {[x]} \mathbb{P}^n \arrow[u, bend left, "s"] \end{tikzcd}$$

$$[\Gamma_f] \cdot \underbrace{\alpha^{n-t} \beta^t}_{} = \underset{d^t}{e_t} \alpha^n \beta^n$$

$$[\Gamma_f \cap (H_0^{n-t} \times H_1^t)] \cong H_0^{n-t} \cap f^{-1}(H_1^t)$$

$$= \{([x], f([x]))\} \cap A \times B \quad : [x] \in A,\ f([x]) \in B \qquad A \cap f^{-1}(B)$$

H_0^{n-t}

$$\mathbb{P}^t \xrightarrow{f} \mathbb{P}^n \qquad f^{-1}(H_1^t)$$

$$(d\zeta)^{td} = d^t \{pt\}$$

$f^{-1}(L_1)$
$\vdots$
$f^{-1}(L_t)$

[How many points of intersection do t generic forms of degree d have in $\mathbb{P}^t$] $\rightsquigarrow d^t$

$$\mathbb{P}^n \xrightarrow[d]{f} \mathbb{P}^n \qquad [F_0 : \dots : F_n]$$

Then: These are all.

Fix degree d, choose homogeneous polynomials $F_0, \dots, F_n$ of degree d and require that they have no common zeroes.

$$[\Gamma_f] = \sum_{k=0}^{n} d^t \alpha^t \beta^{n-t}$$

number d

if f has components of degree d.

$$[\Delta]\cdot[\Gamma_f] = \left(\sum_{k=0}^{n} \alpha^k \beta^{n-k}\right)\left(\sum_{j=0}^{n} d^j \alpha^j \beta^{n-j}\right)$$

$$= \underbrace{\left(d^n + d^{n-1} + \dots + d + 1\right)}\alpha^n \beta^n$$

If $d=1$

$$\underbrace{(n+1)}\alpha^n \beta^n$$

$$\left.1 + \dots + 1 + 1\right\} = n+1 \quad \checkmark$$

$\chi(\mathbb{P}^n)$

This is the expected number of fixed points.

When is it the actual number?

Claim: This is correct for "generic" (most) morphisms.

alg. gp.

Theorem 1.7 (Kleiman's transversality thm)

Suppose G acts transitively on a variety X and $A \subseteq X$ is any subvariety

(1) If $B \subseteq X$ is another subvariety $\exists$ $U \subseteq G$ open + dense such that

$$B \cap gA \text{ is gen transverse}$$

(2) If G is affine then $[A] = [gA]$

Example:

$$X = \mathbb{P}^n \times \mathbb{P}^n$$

$$G = PGL(n) \times PGL(n) \rtimes \mathbb{Z}/2\mathbb{Z}$$

G acts transitively on X

$$B = \Delta \qquad A = \Gamma_f$$

$$[\Delta]\cdot[\Gamma_f] \neq [\Delta \cap \Gamma_f]$$

$$[\Delta]\cdot[g\Gamma_f] = [\Delta \cap g\Gamma_f]$$

$$g\Gamma_f = \Gamma_{g^{-1}(f)}$$