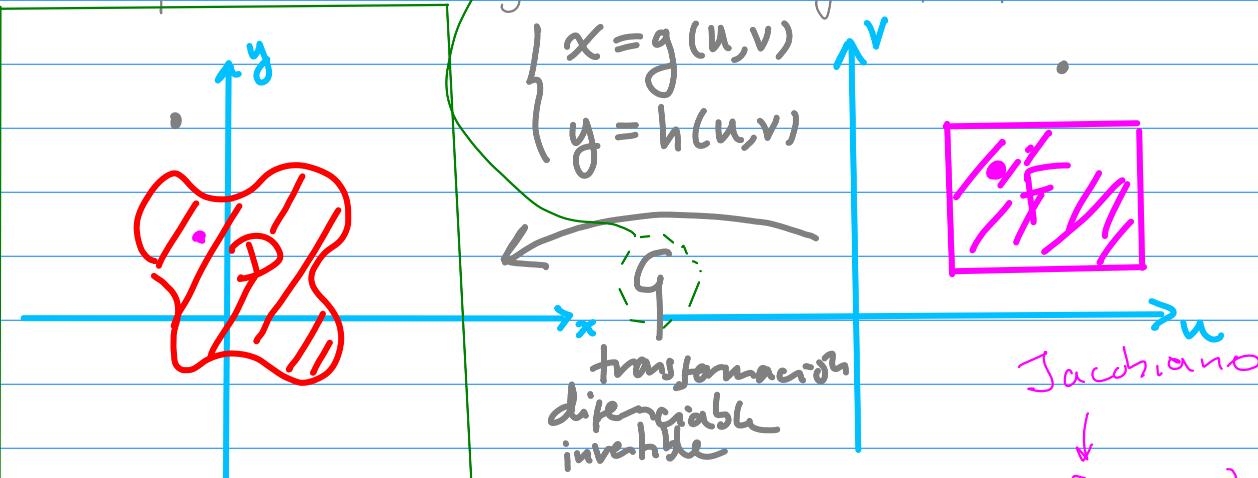
Hoy: ¿Cómo se usan los cambios de variable (más comunes) para calcular integrales dobles y triples?



$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

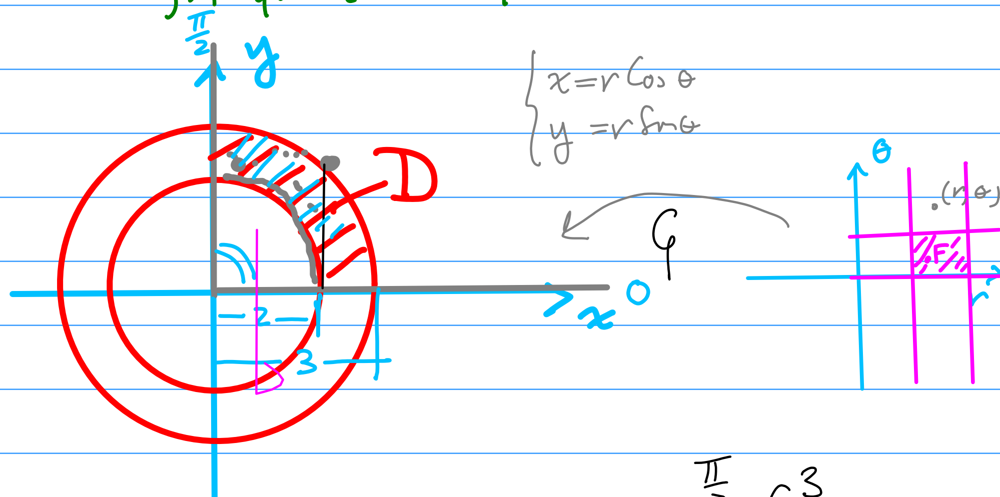
Jacobiano

$$\iint_D p(x,y)\,dA = \iint_F p\big(g(u,v),\, h(u,v)\big)\left[\frac{\partial(x,y)}{\partial(u,v)}\right] du\,dv$$

$$\left|\det\begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}\right|$$

Teorema (del cambio de variable)

Ejemplo: Un plato tiene forma de un cuarto de anillo. Esta encerrado por $x^2+y^2=4$, $x^2+y^2=9$, $x=0$ $y=0$ en el primer cuadrante. La densidad (en kg/m^2) del plato esta dada por $\rho(x,y)=x+y$. Plantee una integral que calcule la masa total.



Jacobiano

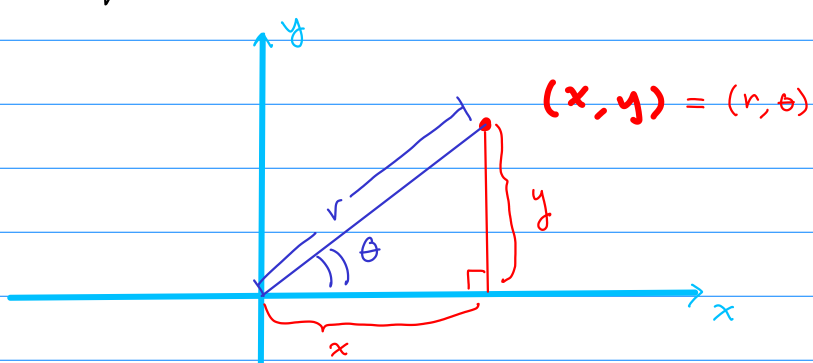
$$M=\iint_D x+y\,dA=\int_0^{\frac{\pi}{2}}\int_2^3 (r\cos\theta+r\sin\theta)\,r\,dr\,d\theta$$

(!!) ← FÁCIL

Idea: Usar polares. (r,θ)

En polares $F=\{(r,\theta):\ 0\le\theta\le\frac{\pi}{2},\ 2\le r\le 3\}$

$D=\varphi(F)$



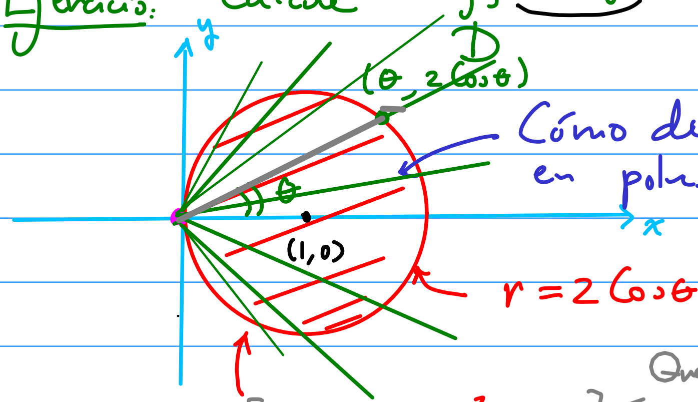
op/hip.

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$

en dónde? ∫∫∫∫ (f) Qué función?

Ejercicio: Calcule $\iint_D \underbrace{x^2+y^2}_{r^2 \cdot r} \, dA$ dónde



Cómo describir esta región en polares?

$r = 2\cos\theta$

$\to \left[(x-1)^2 + y^2 = 1 \right]$ Qué dice en términos de (r,θ)?

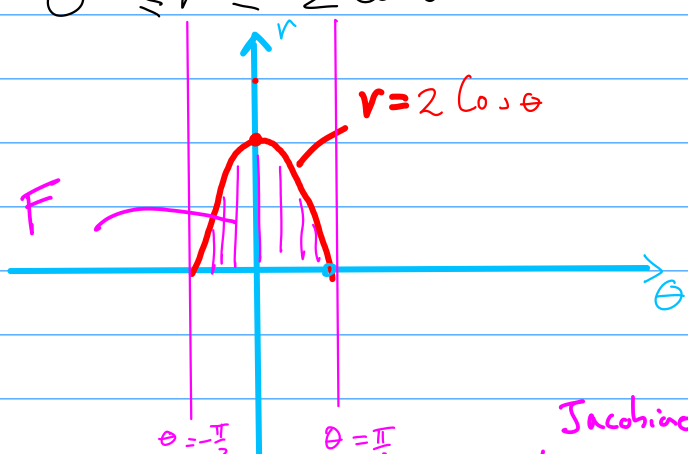
Si $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ la función se vuelve $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$
$= r^2(\cos^2\theta + \sin^2\theta) = r^2$

$(x-1)^2 + y^2 = 1$

$$\begin{cases} (r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1 \\ r^2\cos^2\theta - 2r\cos\theta + \cancel{1} + r^2\sin^2\theta = \cancel{1} \\ r^2\left[\cos^2\theta + \sin^2\theta\right] - 2r\cos\theta = 0 \\ r^2 - 2r\cos\theta = 0 \\ r\left[r - 2\cos\theta\right] = 0 \implies \boxed{r = 2\cos\theta} \end{cases}$$

polares

Concluimos que D está dada por
$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$
$$0 \le r \le 2\cos\theta$$



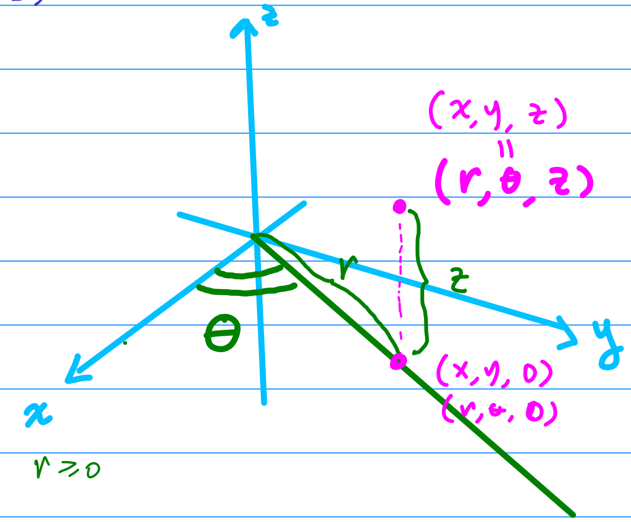
Jacobiano

$$\iint_D x^2+y^2 \, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r \; dr\, d\theta \; (\vdots)$$

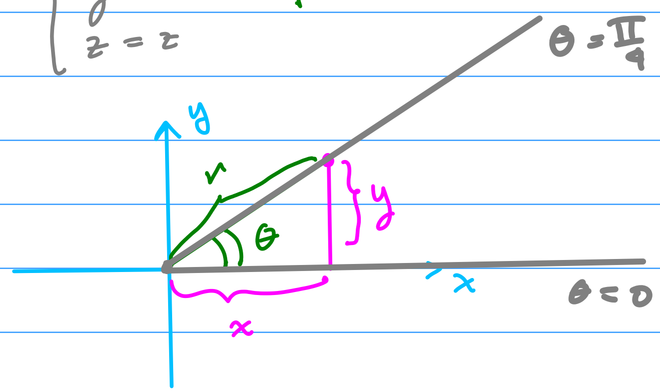
(2) Sistemas de coordenadas en 3D

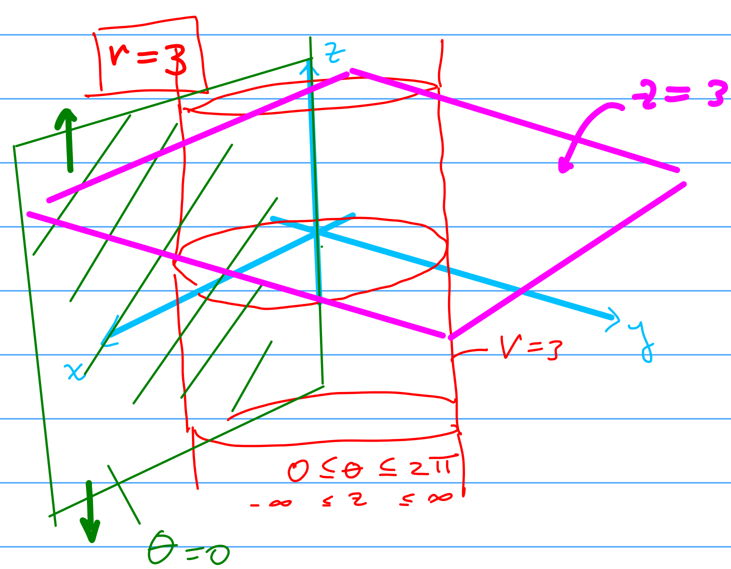
Hay dos generalizaciones distintas de coords polares

(2.1) Coords cilíndricas



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$ ✓

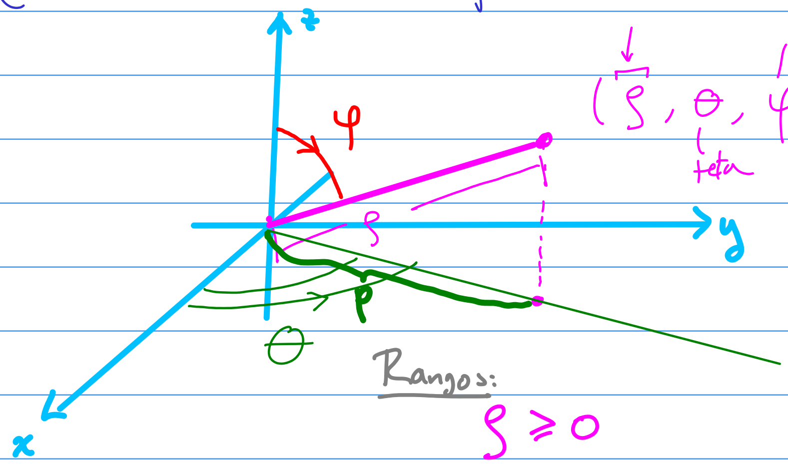




(2.2) Coordenadas esféricas.

rho, fi, teta

$(\rho, \theta, \varphi) = (x, y, z)$



Rangos:

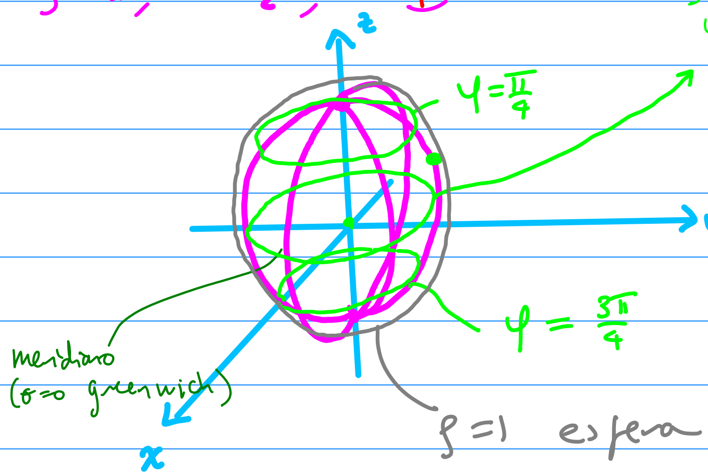
$\rho \geq 0$

$0 \leq \theta \leq 2\pi$

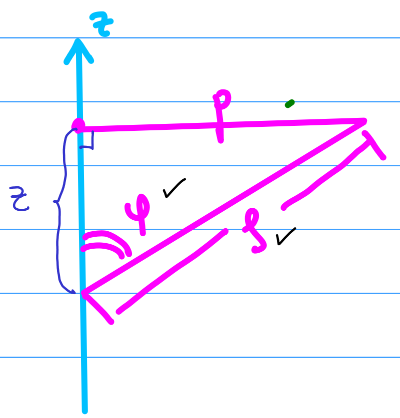
$[0 \leq \varphi \leq \pi]$

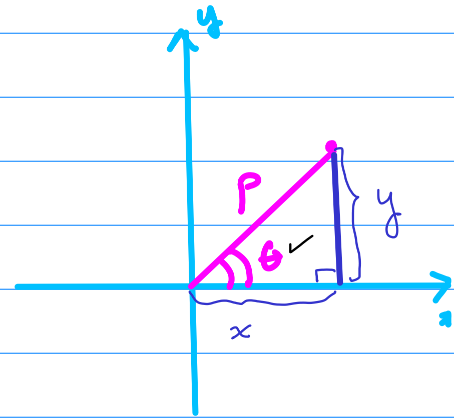
$\rho = 1, \ \theta = \frac{\pi}{2}, \ 0 \leq \varphi \leq \pi$

$\rho = 1, \ 0 \leq \theta \leq 2\pi, \ \varphi = \frac{\pi}{2}$



Cómo son x, y, z dados ρ, θ, φ





$\cos(\varphi) = \frac{z}{\rho} \Rightarrow z = \rho \cos\varphi$

$\sin\varphi = \frac{P}{\rho} \Rightarrow P = \rho \sin\varphi$

$\cos\theta = \frac{x}{P} \Rightarrow x = P\cos\theta$

$x = \rho \sin\varphi \cos\theta$

$\sin\theta = \frac{y}{P} \Rightarrow y = P \sin\theta$

$y = \rho \sin\varphi \sin\theta$

$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

Cambio de variable a coords esféricas.

* **Ejercicio:** Demuestre que

(1) Jacobiano a cilíndricas es r

(2) Jacobiano a esféricas es $\rho^2 \sin\varphi$

$$\left| \det \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{bmatrix} \right| = \rho^2 \sin\varphi$$