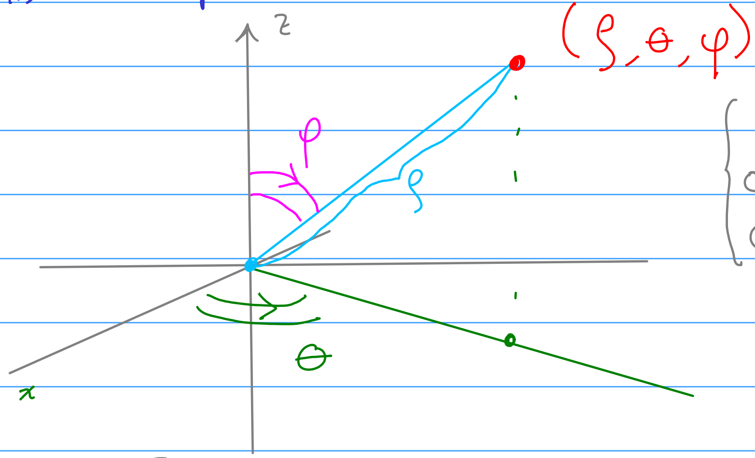
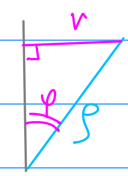
Hoy: Ejemplos de integración en coords esféricas y cilíndricas

1) Coords esféricas



$$\begin{cases} \rho \geq 0 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{cases}$$



$$\begin{cases} x = \rho \sin\varphi \cos\theta \\ y = \rho \sin\varphi \sin\theta \\ z = \rho \cos\varphi \end{cases}$$

$\sin(\varphi) = \frac{r}{\rho} \Leftrightarrow r = \rho \sin\varphi$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \underbrace{\rho^2 \sin\varphi}$$

Factor de ajuste, Jacobiano del cambio de variable

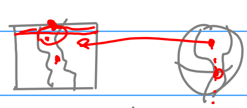
Qué?

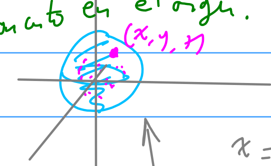
Ejemplo: Calcule $\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dV =$ donde B es la bola de radio 2 centrada en el origen.

en dónde

región sólida

$\{x \in \mathbb{R}^3 : \|x\| \leq 2\}$

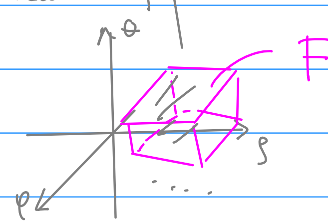




(1) Cómo es B en coordenadas esféricas? φ

$x =$
$y = G(\rho, \theta, \varphi)$
$z =$

$$B = \left\{ (\rho,\theta,\varphi) : \begin{matrix} 0 \leq \rho \leq 2 \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{matrix} \right\}$$



Teo cambio de variable

$$\iiint_B e^{(x^2+y^2+z^2)^{\frac{3}{2}}} \, dV = \int_0^2 \int_0^{2\pi} \int_0^{\pi} \underbrace{e^{(\rho^2)^{\frac{3}{2}}} \rho^2 \sin\varphi} \, d\varphi \, d\theta \, d\rho$$

$$x^2+y^2+z^2 = (\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2 + (\rho\cos\varphi)^2$$

$$= \rho^2\sin^2\varphi\left[\cos^2\theta+\sin^2\theta\right] + \rho^2\cos^2\varphi = \rho^2\sin^2\varphi + \rho^2\cos^2\varphi$$

$$\boxed{\rho^2} = \rho^2(\sin^2\varphi+\cos^2\varphi)$$

$$= \int_0^2\int_0^{2\pi}\underbrace{\int_0^{\pi} e^{\rho^3}\rho^2\sin\varphi\,d\varphi}_{①}\,d\theta\,d\rho =$$

① $\int_0^{\pi} e^{\rho^3}\rho^2\sin\varphi\,d\varphi = e^{\rho^3}\rho^2\int_0^{\pi}\sin\varphi\,d\varphi = e^{\rho^3}\rho^2\left(-\cos\varphi\Big|_0^{\pi}\right) = 2e^{\rho^3}\rho^2$

$$= \int_0^2\int_0^{2\pi} 2e^{\rho^3}\rho^2\,d\theta\,d\rho = 4\pi\int_0^2 e^{\rho^3}\rho^2\,d\rho = 4\pi\left(\frac{e^{\rho^3}}{3}\Bigg|_{\rho=0}^{\rho=2}\right)$$

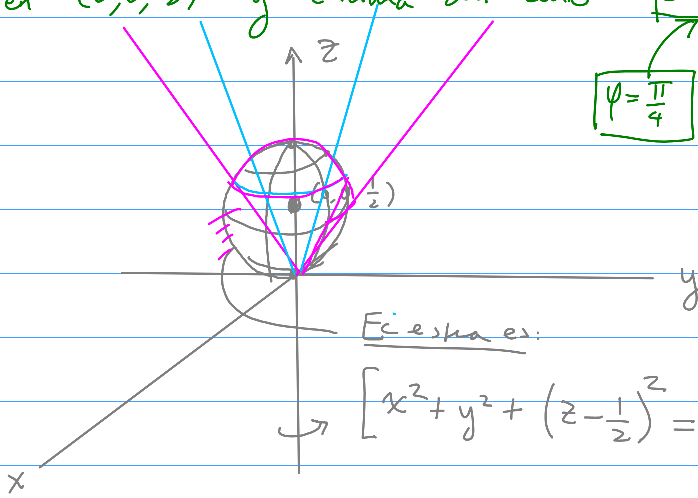
$$= \boxed{\frac{4\pi(e^8-1)}{3}}$$

② $\int_0^{2\pi}\left[2e^{\rho^3}\rho^2\right]d\theta = 2e^{\rho^3}\rho^2\left(\int_0^{2\pi}d\theta\right) = 2e^{\rho^3}\rho^2(2\pi)$

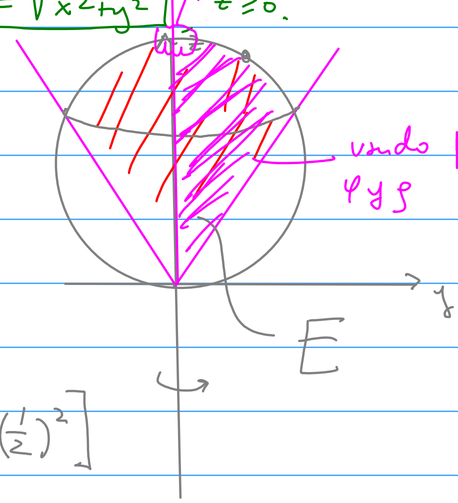
$$\iiint_E 1\,dV =: \mathrm{Vol}(E)$$

Ejemplo 2 *: Calcule el volumen del sólido E que está dentro de la esfera de radio $\frac{1}{2}$ centrada en $(0,0,\frac{1}{2})$ y encima del cono $\boxed{z=\sqrt{x^2+y^2}}$ $\leftrightarrow z\geq 0$.

$\boxed{\varphi=\frac{\pi}{4}}$

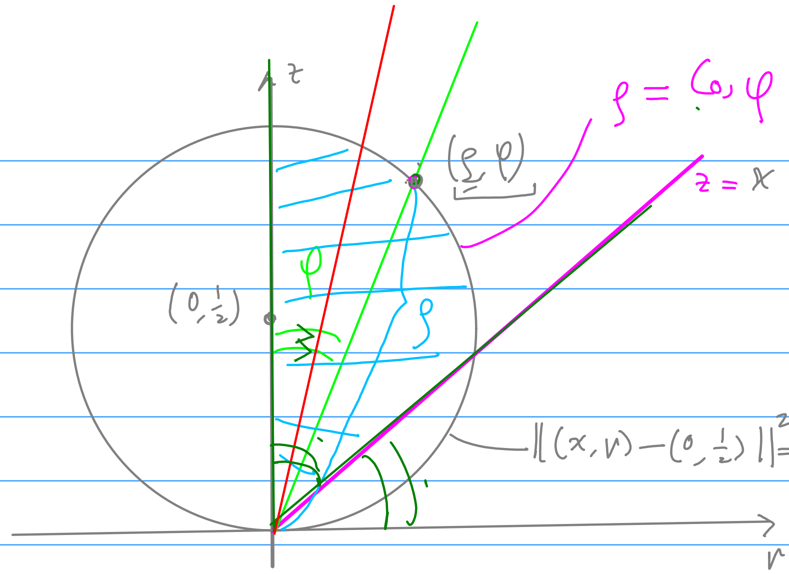


Ec. esfera es: $\left[x^2+y^2+(z-\frac{1}{2})^2=(\frac{1}{2})^2\right]$



Cómo describir en esféricas:

$[0\leq\theta\leq 2\pi]$ — giro sobre valores de θ.



$\|(x,r)-(0,\frac{1}{2})\|^2 = \left(\frac{1}{2}\right)^2 \Leftrightarrow$ $\boxed{r^2 + (z-\frac{1}{2})^2 = \frac{1}{4}}$

$r^2 = x^2+y^2 = \rho^2 \sin^2\varphi$

$x^2+y^2+(z-\frac{1}{2})^2 = \frac{1}{4}$ en esféricas!

$$\left[\rho^2 \sin^2\varphi + \left(\rho\cos\varphi - \frac{1}{2}\right)^2 = \frac{1}{4}\right]$$

$$\rho^2(\sin^2\varphi + \cos^2\varphi) - \rho\cos\varphi + \cancel{\tfrac{1}{4}} = \cancel{\tfrac{1}{4}}$$

$$\rho^2 - \rho\cos\varphi = 0 \Leftrightarrow \rho(\rho - \cos\varphi) = 0$$

$\Leftrightarrow \boxed{\rho = \cos\varphi}$ describe la esfera en coords esféricas.

$$E = \left\{ (\rho,\theta,\varphi) : \begin{array}{l} 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \\ 0 \le \rho \le \cos\varphi \end{array} \right\}$$

$z = \sqrt{x^2+y^2} = \sqrt{(\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2}$

$z = \rho\sin\varphi$ calculando en el caso

$\rho\cos\varphi = \rho\sin\varphi \underset{\rho\neq 0}{\Leftrightarrow} \cos\varphi = \sin\varphi \Leftrightarrow \boxed{\tan\varphi = 1} \Leftrightarrow \boxed{\varphi = \frac{\pi}{4}}$

$$\mathrm{Vol}(E) = \underbrace{\iiint}_{E} 1\, dV \overset{\text{Teo cambio variable (a esféricas)}}{=} \left[\int_0^{2\pi}\int_0^{\frac{\pi}{4}} \underbrace{\int_0^{\cos\varphi} 1\,\rho^2 \sin\varphi \, d\rho}_{(1)} \, d\varphi \, d\theta \right]$$

SIEMPRE empiece en coords cartesianas

$$(*) \quad \int_0^{\cos\varphi} \rho^2 \sin\varphi \, d\rho = \sin\varphi \int_0^{\cos\varphi} \rho^2 \, d\rho = \sin\varphi \left(\frac{\rho^3}{3} \Bigg|_{\rho=0}^{\rho=\cos\varphi} \right)$$

$$= \frac{\cos^3\varphi \sin\varphi}{3}$$

$$\frac{1}{3} \int_0^{\frac{\pi}{4}} \cos^3\varphi \sin\varphi \, d\varphi = \frac{1}{3} \left(- \frac{\cos^4\varphi}{4} \Bigg|_{\varphi=0}^{\varphi=\frac{\pi}{4}} \right)$$

$$= \frac{1}{3} \left(\frac{-\left(\frac{\sqrt{2}}{2}\right)^4}{4} + \frac{1}{4} \right)$$

$$= \frac{1}{3} \left(\frac{1}{4} - \frac{1}{16} \right)$$

$$\int_0^{2\pi} \frac{1}{3} \left(\frac{1}{4} - \frac{1}{16} \right) d\theta = \boxed{\frac{2\pi}{3} \left(\frac{1}{4} - \frac{1}{16} \right)}$$

Vol de la esfera $\frac{4}{3}\pi\left(\frac{1}{2}\right)^3 = \frac{2\pi}{3}\left(\frac{1}{4}\right)$