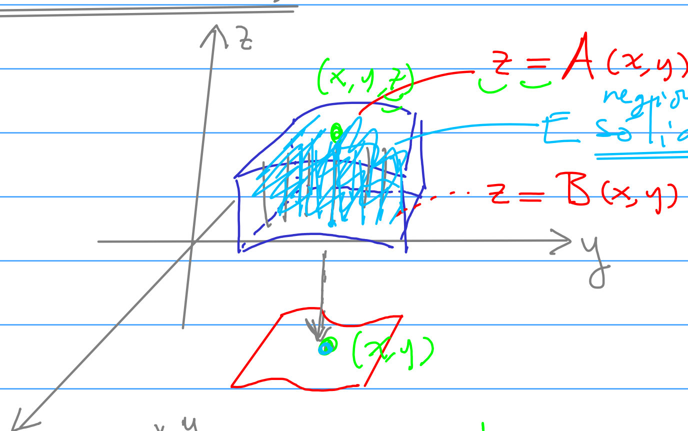
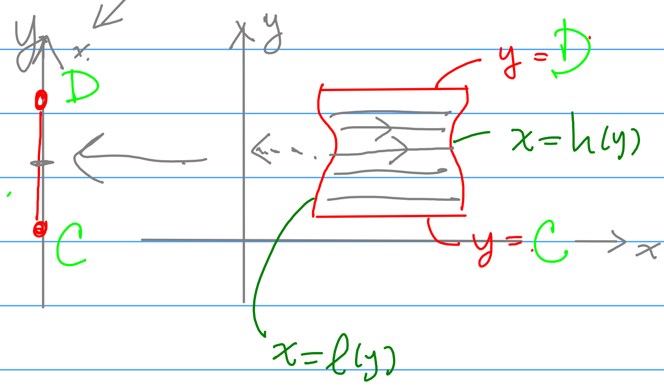
Hoy: (1) Cálculo de integrales triples (Una int triple es un límite de sumas triples . . . difícil de calcular)
(2) Teorema del cambio de variable

Teorema (Fubini)



$$\left(\iiint_E \rho(x,y,z)\,dV\right) \overset{\text{Fubini}}{=} \left\{\int_C^D \left[\int_{\ell(y)}^{h(y)} \left[\int_{B(x,y)}^{A(x,y)} \rho \; dz\right] dx\right] dy\right\}$$



Ejemplo: Sea E la región sólida encerrada por

$z = x^2+y^2+1$ y $z = 5$.

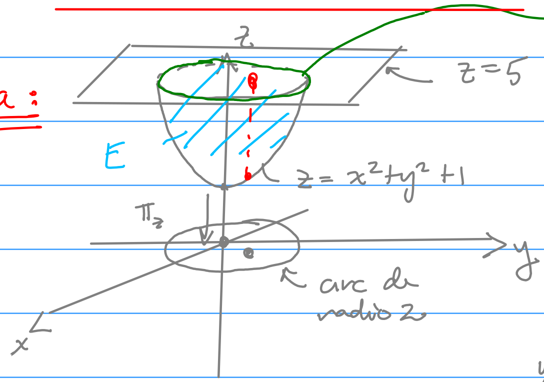
(a) Dibuje E

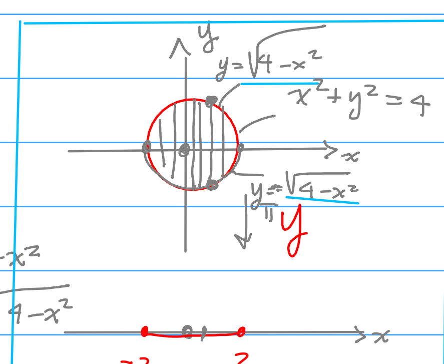
(b) Escriba $\left[\iiint_E 1\, dV\right] = Vol(E)$ como integral iterada en dos órdenes distintos.

Sol a:

$$\begin{cases} z = 5 \\ z = x^2+y^2+1 \end{cases} \iff \begin{cases} z = 5 \\ x^2+y^2 = 4 \end{cases}$$

circ de radio 2





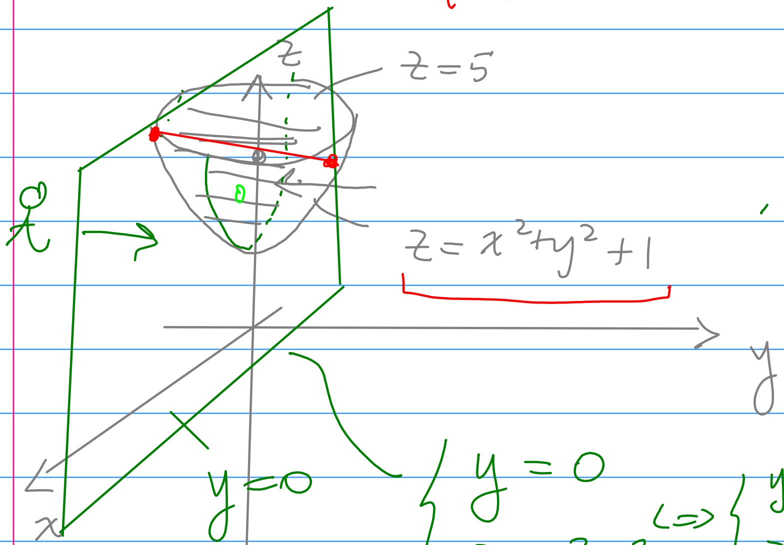
$y^2 = 4 - x^2$

$y = \pm\sqrt{4-x^2}$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2+1}^{5} 1 \, dz\, dy\, dx$$

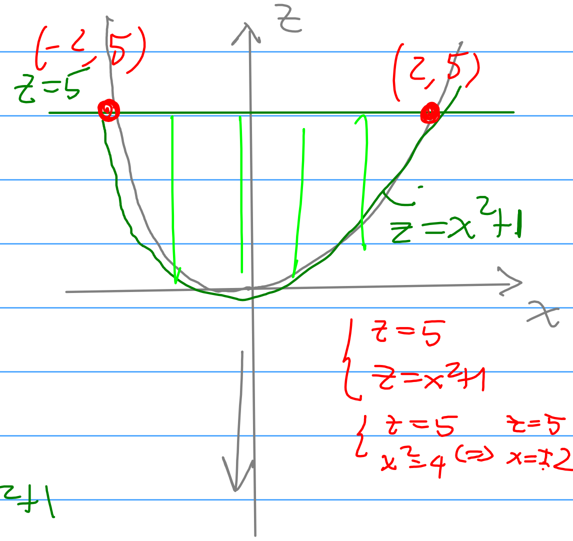
$dz\,dy\,dx$

Cómo quedaría en orden $dy\,dz\,dx$?



$z = x^2+y^2+1$

$$\begin{cases} y = 0 \\ z = x^2+y^2+1 \end{cases} \iff \begin{cases} y = 0 \\ z = x^2+1 \end{cases}$$

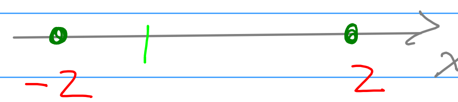


$$\begin{cases} z = 5 \\ z = x^2+1 \end{cases}$$

$$\begin{cases} z = 5 \\ x^2 = 4 \end{cases} \iff \begin{matrix} z = 5 \\ x = \pm 2 \end{matrix}$$

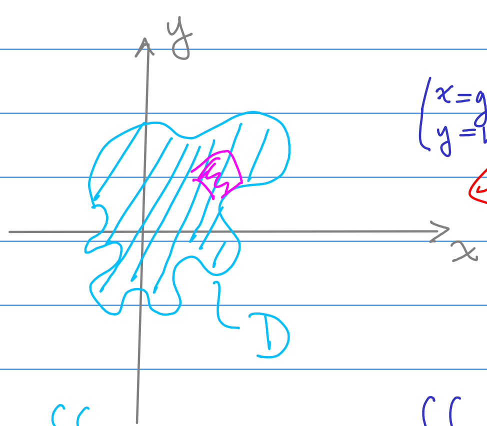
$$z = x^2+y^2+1 \iff y^2 = z - x^2 - 1$$

$$y = \pm\sqrt{z-x^2-1}$$



$$\int_{-2}^{2}\int_{x^2+1}^{5}\int_{-\sqrt{z-x^2-1}}^{\sqrt{z-x^2-1}} 1 \, dy\, dz\, dx$$

1(2) Teorema del cambio de variable:



IDEA: Hay otras coordenadas en la que D es fácil de describir

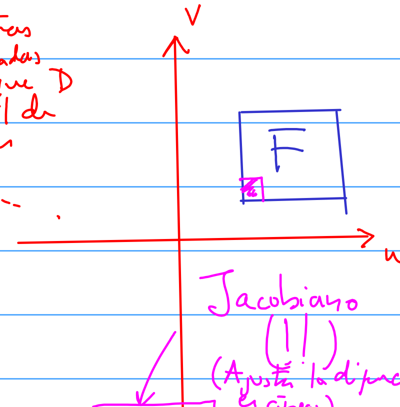
$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

G

$G : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

cambio de coordenadas

$G(F) = D.$



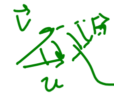
Jacobiano

$(|\ |)$

(Ajusta la diferencia en áreas)

$$\iint_D P(x,y)\, dA = \iint_F P(g(u,v), h(u,v)) \boxed{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}\, du\, dv$$

donde $\frac{\partial(x,y)}{\partial(u,v)} = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix}\right|$



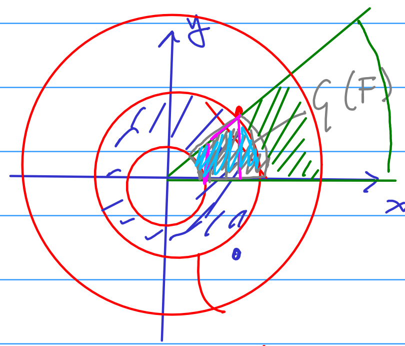
$\left|\det\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}\right|$ = Área paralelogramo

Cambio de coordenadas a polares.

Ejemplo: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$r \geq 0$

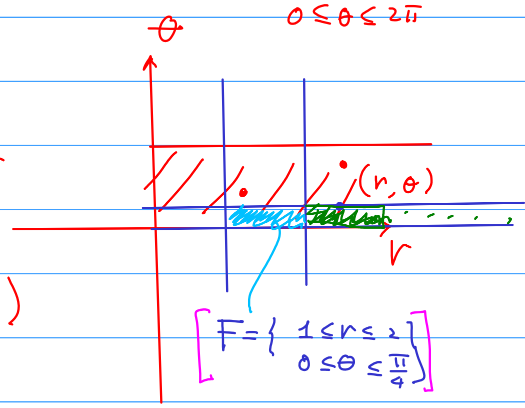
$0 \leq \theta \leq 2\pi$



$G(F)$

1-1

$G(r,\theta)$
$\|$
$(r\cos\theta, r\sin\theta)$



$F = \left\{ \begin{array}{l} 1 \leq r \leq 2 \\ 0 \leq \theta \leq \frac{\pi}{4} \end{array} \right\}$

$G(F')$ = anillo entre 1 y 2.

$F' = \left\{ \begin{array}{l} 1 \leq r \leq 2 \\ \theta\ ? \end{array} \right\}$

Ejercicio: Para $\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$

(a) Calcule $\frac{\partial(x,y)}{\partial(r,\theta)} =$

(b) Encuentre $\iint_D x^2+y^2\, dV$ donde D es el círculo de radio 3 centrado en $(0,0)$.

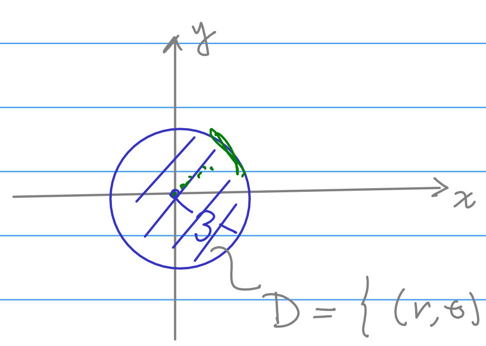
Sol: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left|\det\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix}\right| = \left|\det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}\right|$$

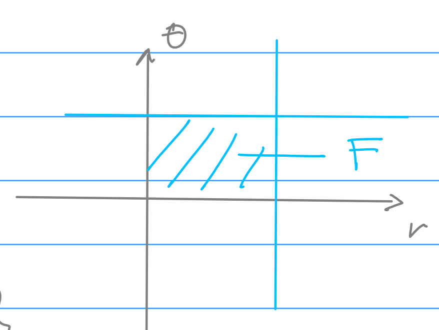
$$= \left| r\cos^2\theta - (-r\sin^2\theta)\right| = \left| r(\cos^2\theta + \sin^2\theta)\right| \underset{\|}{} \; r$$

El Jacobiano al cambiar a polares es r SIEMPRE

(b) Sea D el disco de radio 3 centrado en $(0,0)$



$D = \left\{ (r,\theta) : \begin{array}{l} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{array} \right\}$



F

$$\iint_D \underbrace{x^2+y^2}_{1}\, dx\,dy = \int_0^{2\pi}\int_0^3 r^2\, r\; dr\, d\theta$$

Por Teo cambio de variable a Polares

Jacobiano (!)

$$x^2+y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2\left[\cos^2\theta+\sin^2\theta\right] = r^2$$

$$= \int_0^{2\pi}\left(\int_0^3 r^3\, dr\right) d\theta = \int_0^{2\pi} \frac{3^4}{4}\, d\theta = \boxed{\frac{3^4\, 2\pi}{4}}$$

$$\left[\iint_D 1\, dV = \int_0^{2\pi}\int_0^3 r\, dr\, d\theta = \cancel{2}\pi \frac{3^2}{\cancel{2}}\right]$$