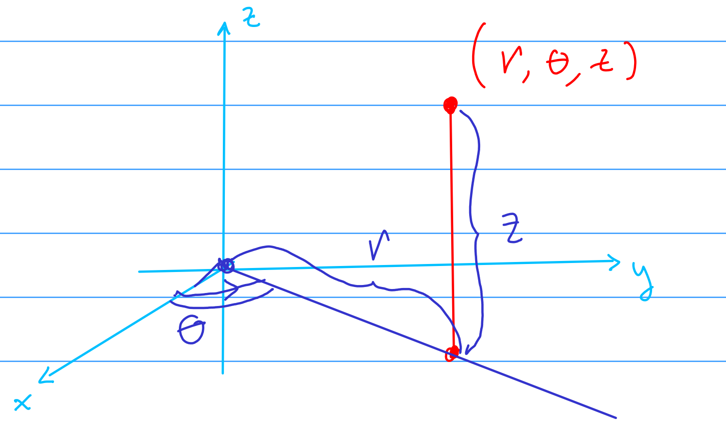
Hoy: ¿Qué cambios de variable hay en $\mathbb{R}^3$
(generalizando las coordenadas polares)?

R: (1) Coordenadas cilíndricas

(2) Coordenadas esféricas

Cilíndricas:



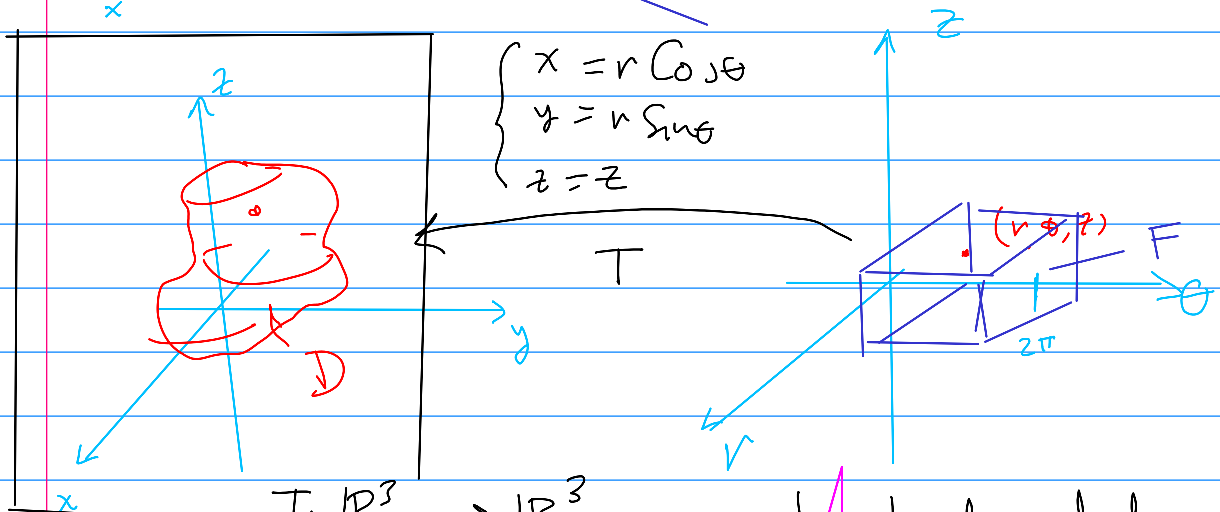
Piso x, y en polares

Altura z igual que antes.

θ en radianes

$$\begin{cases} 0 \leq \theta \leq 2\pi \\ r \geq 0 \\ z \in \mathbb{R} \end{cases}$$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$



$T: \mathbb{R}^3 \to \mathbb{R}^3$, $(r,\theta,z) \mapsto (x,y,z)$ es el cambio de coordenadas

¿Cuál es el factor de ajuste para el cálculo de integrales usando coords. cilíndricas?

$$\iiint_D f(x,y,z)\, dV_{(x,y,z)} = \iiint_F f(r\cos\theta, r\sin\theta, z)\; \boxed{\;}\; dV_{(r,\theta,z)}$$

Jacobiano

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)}$$

Teo cambio de variable

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \left| \det\left(\begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} r \\ \theta \\ z \end{array} & \begin{bmatrix} \frac{\partial x}{\partial r} & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} \end{array} \right) \right|$$

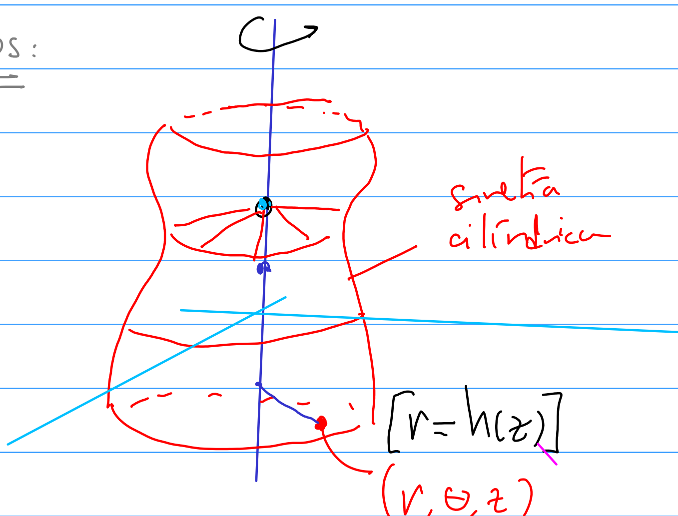
$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

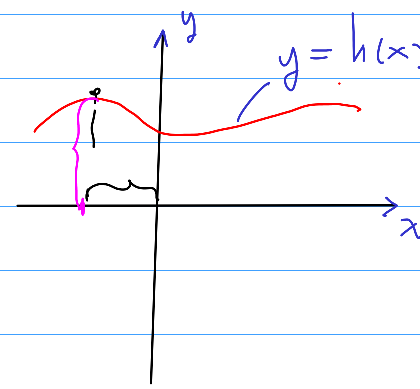
$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \left| \det\left(\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \right|$$

$$= \left| \det\left(\begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \right) \right| = |r| = r$$

porque $r \geq$

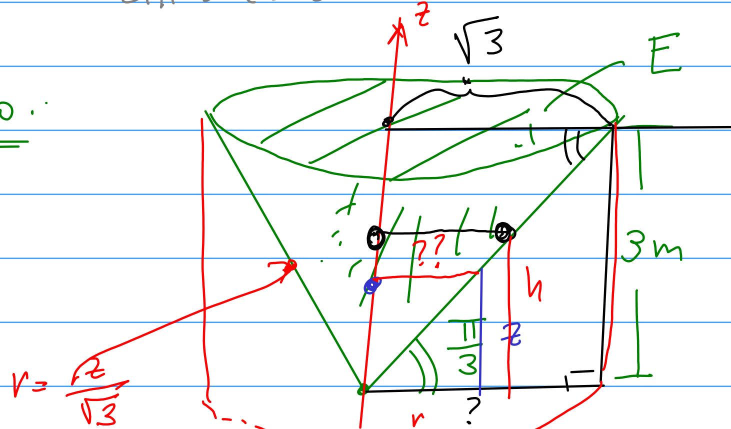
Obs:





Si hay un eje de simetría deberíamos usar cilíndricas alrededor de él.

Ejercicio:



$\mathrm{Vol}(E) = ?$

$$\tan\left(\frac{\pi}{3}\right) = \frac{3}{?}$$

(1) Describimos E en cilíndricas:

$$? = \frac{3}{\tan\left(\frac{\pi}{3}\right)} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

$$E = \left\{ (r,\theta,z) : \begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 3 \\ 0 \leq r \leq \frac{z}{\sqrt{3}} \end{array} \right\} \longleftarrow \ast$$

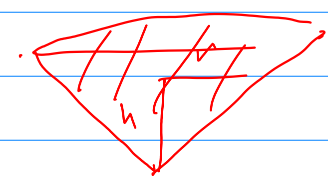
$$\tan\left(\frac{\pi}{3}\right) = \frac{z}{r} \iff r = \frac{z}{\tan\left(\frac{\pi}{3}\right)} = \frac{z}{\sqrt{3}}$$

(2) Calculamos la integral

NOTA: Planteo la integral en coordenadas rectangulares y luego cambio de variable (para no olvidar el Jac).

$$Vol(E) = \iiint_E 1 \, dV \overset{\text{para a cilíndricas}}{=} \left[\iiint_{E \text{ cilíndricas}} 1 \underbrace{\cdot\, r}_{\text{Jacobiano de cilíndricas.}} \, dV_{(r,\theta,z)} \right] =$$

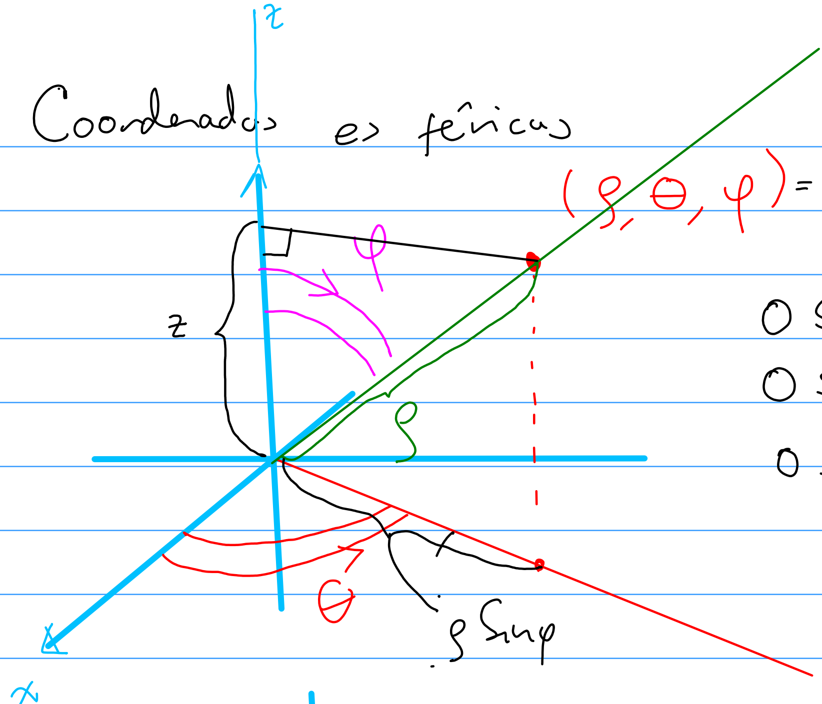
$$\int_0^{2\pi} \int_0^{3} \int_0^{\frac{z}{\sqrt{3}}} r \quad dr\, dz\, d\theta =$$



$$= 2\pi \int_0^3 \left(\frac{r^2}{2} \Big|_{r=0}^{r=\frac{z}{\sqrt{3}}} \right) dz = 2\pi \int_0^3 \frac{z^2}{6} =$$

$$\frac{2\pi}{3 \cdot 6} \left(z^3 \Big|_{z=0}^{z=3} \right) = \frac{2\pi}{3 \cdot 6} 3^3 = \overbrace{\boxed{\frac{1}{3} \pi \cdot 3^2}}^{\frac{1}{3}\pi r^2 h} = 3\pi$$

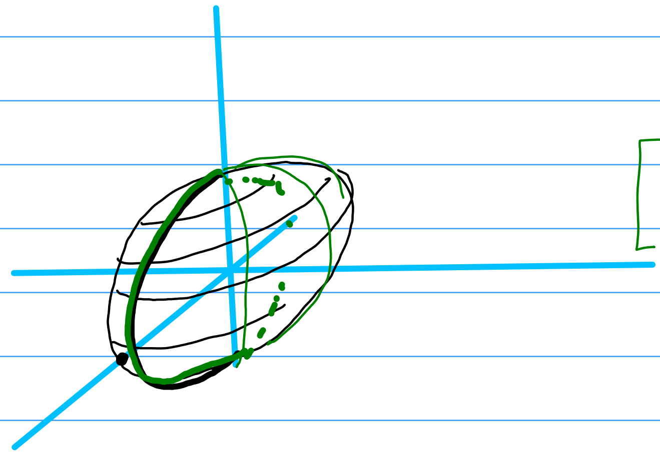
(2) Coordenadas esféricas



$0 \leq \theta \leq 2\pi$

$0 \leq \rho$

$0 \leq \varphi \leq \pi$



$\rho = 1$

$$\left[\begin{array}{l} 0 \leq \theta \leq 2\pi \\ 0 \leq \varphi \leq \pi \end{array}\right.$$

$$\cos\varphi = \frac{z}{\rho} \Rightarrow \begin{cases} z = \rho\cos\varphi \\ x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \end{cases}$$

Ejercicio: Demuestre que:

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \rho^2\sin\varphi$$

Jacobiano de esféricas.