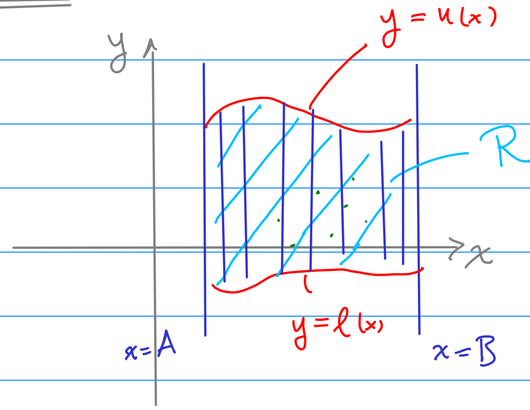
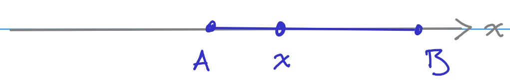
Cálculo de integrales dobles:

Teorema: [Fubini] (Repaso)

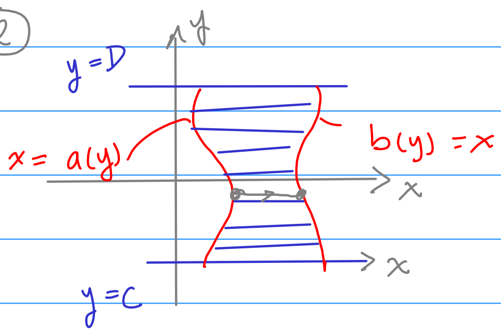
(1)

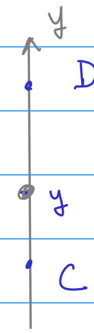




$$\iint_R P(x,y)\,dA = \int_A^B \left[\int_{\ell(x)}^{u(x)} P(x,y)\,dy\right] dx$$

(2)





$$\left[\iint_R P(x,y)\,dA\right] = \int_C^D \left[\int_{a(y)}^{b(y)} P(x,y)\,dx\right] dy$$

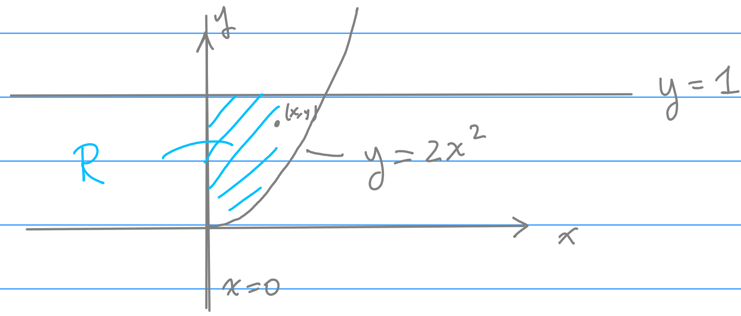
Ejercicio:

Sea R la región del plano encerrada por $y=1$, $y=2x^2$, $x=0$.

(a) Escriba $\iint_R x\cos(y)\,dA$ como integral iterada $dx\,dy$ y $dy\,dx$.

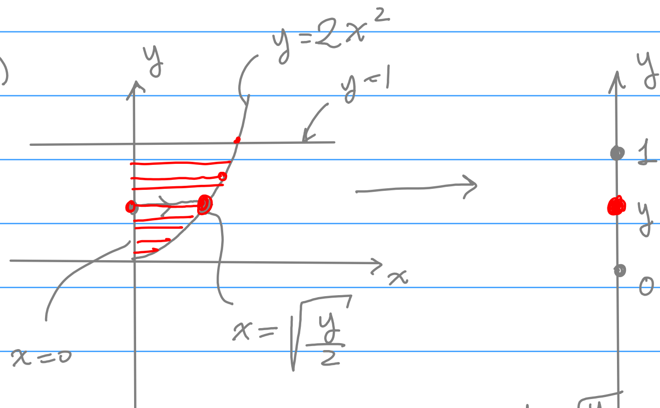
(b) Calcule el valor de la integral.

Sol: (1) Dibujo R – encerrada por $y=1$, $y=2x^2$, $x=0$



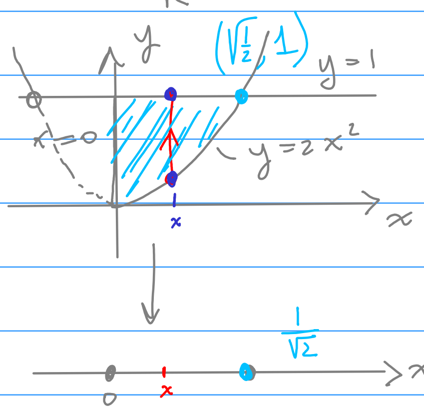
$P(x,y) = x\,\mathrm{Cos}(y)$

(a)



$$\iint_R P(x,y)\,dA = \int_0^1 \left[\int_{x=0}^{x=\sqrt{\frac{y}{2}}} P(x,y)\,dx\right] dy$$

$dx\,dy$



$$\begin{cases} y=1 \\ y = 2x^2 \end{cases}$$

$y=1$

$1 = 2x^2 \Leftrightarrow \frac{1}{2} = x^2$

$x = \pm\sqrt{\frac{1}{2}}$

$$\iint_R P(x,y)\,dA = \int_0^{\frac{1}{\sqrt{2}}} \left[\int_{y=2x^2}^{y=1} P(x,y)\,dy\right] dx$$

$dy\,dx$

(b)

Sol 1

$dx\,dy$

$$\iint_R x\,\mathrm{Cos}(y)\,dA = \int_0^1 \left[\int_0^{\sqrt{\frac{y}{2}}} x\,\mathrm{Cos}(y)\,dx\right] dy$$

$$\int_0^{\sqrt{\frac{y}{2}}} x\,\mathrm{Cos}(y)\,dx = \mathrm{Cos}(y)\,\frac{x^2}{2}\Bigg|_{x=0}^{x=\sqrt{\frac{y}{2}}} = \underbrace{\mathrm{Cos}(y)\,\frac{y}{4}}\; - 0$$

$$\frac{1}{4}\int_0^1 \mathrm{Cos}(y)\,y\,dy = \begin{cases} u = y & du = dy \\ dv = \mathrm{Cos}(y)\,dy & v = \mathrm{Sin}(y) \end{cases} = \mathrm{Sin}(y)\,y\,\Big|_0^1 - \int \mathrm{Sin}(y)\,dy$$

$$\frac{1}{4}\left[\sin(y)\,y\Big|_0^1 + \cos(y)\Big|_0^1\right] = \frac{1}{4}\left[\sin(1) + \cos(1) - 1\right]$$

Sol 2: [dy dx]

$$\iint_R x\cos(y)\,dA = \int_0^{\frac{1}{\sqrt{2}}}\left[\int_{2x^2}^{1} x\cos(y)\,dy\right]dx$$

$$\int_{2x^2}^{1} x\cos(y)\,dy = x\sin(y)\Big|_{y=2x^2}^{y=1} = x\sin(1) - x\sin(2x^2)$$

$$\int_0^{\frac{1}{\sqrt{2}}}\left[x\sin(1) - x\sin(2x^2)\right]dx = \frac{x^2}{2}\sin(1)\Big|_{x=0}^{x=\frac{1}{\sqrt{2}}}$$

$$+ \frac{\cos(2x^2)}{4}\Big|_{x=0}^{x=\frac{1}{\sqrt{2}}} = \frac{1}{4}\sin(1) + \frac{\cos(1)}{4} - \frac{1}{4}$$

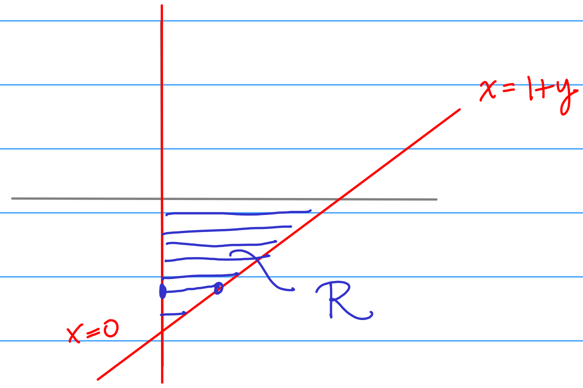
$$= \frac{1}{4}\left[\sin(1) + \cos(1) - 1\right]$$

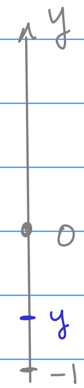
Ejercicio: Dibuje la región sobre la que estamos integando

$$\int_{-1}^{0}\left[\int_{x=0}^{x=1+y} P(x,y)\,dx\right]dy =$$

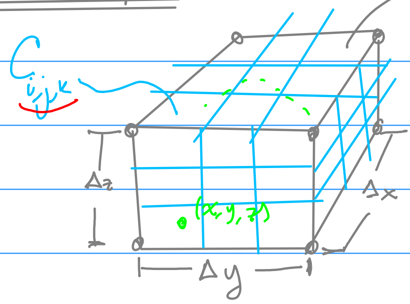
$x = 1+y$

$y = x-1$





Integrales triples:



¿Cuál es la masa total del ladrillo?

(A) El ladrillo tiene densidad c - constante (lingote de oro)

$$M = \boxed{\underset{kg/m^3}{c} \cdot \underset{m}{\Delta x} \cdot \underset{m}{\Delta y} \cdot \underset{m}{\Delta z}} = kg$$

(B) El ladrillo está hecho de bloques de densidad constante ~ δ_{ijk} del bloque C_{ijk}

$$\rho(x,y,z) = 1 + x^2 + y^2$$

$$M = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \delta_{ijk} \cdot \mathrm{Vol}(C_{ijk})$$

(C) El ladrillo tiene una función de densidad cualquiera

Método: (aproximar mediante cubitos cada vez más pequeños)

(1) Partimos cada dirección en N intervalos produciendo N^3 cubitos C_{ijk}, $1 \leq i,j,k \leq N$

(2) En cada cubito escoja un punto $\vec{z}^{\,*}_{ijk}$

Asumimos que el cubo C_{ijk} tiene densidad constante $\rho(\vec{z^*_{ijk}}) = \delta_{ijk}$

(3) Aproximamos la masa total

$$\left[M_N = \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} \rho(\vec{z^*_{ijk}}) \cdot \overbrace{\mathrm{Vol}(C_{ijk})}^{\Delta x \cdot \Delta y \cdot \Delta z} \right] (**)$$

(4) $M = \lim_{N \to \infty} (M_N)$ $\quad \rho : \mathbb{R}^3 \to \mathbb{R}$ — definición de integral

Def: $$\iiint_E [\rho(x,y,z)]\, dV := \lim_{N \to \infty} M_N$$