TETRAHEDRA UROP PROJECT 2

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1. Introduction

A tetrahedron is a polyhedron with 4 faces. Suppose T is a tetrahedron with angles $\alpha_{12}, \alpha_{13}, \alpha_{14}, \alpha_{23}, \alpha_{24}, \alpha_{34}$. My team and I were curious about tetrahedra whose angles, when viewed as elements of $\mathbb{R}/2\pi\mathbb{Q}$, span a 5-dimensional $\mathbb{Q}$ vector space. In particular, we wanted to know if one such tetrahedra could have Dehn invariant zero. Showing that no such tetrahedron with this property have Dehn invariant zero would have serious ramifications. For instance, Debrunner showed that if a polyhedron P tiles 3D space then it must have Dehn invariant zero.

2. The Problem

Given a tetrahedron T, enumerate the vertices 1,2,3,4. Denote e_{ij} as the edge between vertices i and j and θ_{ij} as the dihedral angle of e_{ij}. One convention that I follow throughout this paper is listing edges and angles using the ordering $12, 13, 14, 23, 24, 34$. Suppose we're given dihedral angles $\theta_{12}, ..., \theta_{34} \in \mathbb{R}/\mathbb{Q}\pi$ that span a 5-dimensional $\mathbb{Q}$-vector space. For a tetrahedron with such angle to have Dehn invariant 0, the edge lengths $e_{12}, ..., e_{34}$ must be proportional to a tuple of positive integers such that $\sum_{i<j} e_{ij}\theta_{ij} = 0 \mod \mathbb{Q}\pi$. Let $z_{ij} = e^{i\theta_{ij}}$. Let $w_{ij} = e^{i(2\theta_{ij})}$. Then $2\cos\theta_{ij} = z_{ij} + z_{ij}^{-1}$ and $2\cos 2\theta_{ij} = w_{ij} + w_{ij}^{-1}$. By Theorem 1 of Wirth-Dreiding, $\cos\theta_{ij} = \frac{D_{ij}}{\sqrt{D_{ijk}D_{ijl}}} \in \sqrt{\mathbb{Q}}$ and consequently $\cos 2\theta_{ij} = 2\cos^2\theta_{ij} - 1 \in \mathbb{Q}$. Then w_{ij} satisfies the polynomial relation $w_{ij}^2 - c_{ij}w_{ij} + 1 = 0$ where $c_{ij} = 2\cos 2\theta_{ij}$.

3. Numerical Approach

The code I developed as part of this project is distributed across 3 branches of my forked repository of Abdelatiff Chentouf Anas' original work. It includes my p-adic approach to resolving the problem, my numerical approach, and some functions that were generally essential for making computations with tetrahedra.

Suppose $\sum_{i<j} e_{ij}\theta_{ij} = 0 \mod \mathbb{Q}\pi$. Then $\sum e_{ij}\theta_{ij} = q\pi$ for some $q \in \mathbb{Q}$ so $\sum_{i<j} e_{ij}\theta_{ij} i = iq\pi$. Then $\prod z_{ij}^{e_{ij}}$ is a root of unity in a field obtained by adjoining square roots to $\mathbb{Q}$. Now, $Gal(\mathbb{Q}(\zeta_n/\mathbb{Q}) = (\mathbb{Z}/n\mathbb{Z})^\times$ so $\zeta_n \in \mathbb{Q}(\sqrt{\mathbb{Q}})$ if and only if $(\mathbb{Z}/n\mathbb{Z})^\times$ is an elementary 2-group. This forces $(\mathbb{Z}/n\mathbb{Z})^\times = \prod_{p|n}(\mathbb{Z}/p^{e_p}\mathbb{Z})^\times$ where $e_2 \leq 3, e_3 \leq 1$ and $e_p = 0$ for all other primes p. Thus, $n|24$. Thus, $(\prod_{j=1}^6 w_{ij}^{e_{ij}})^{24} = 1$. For a tetrahedra T, we can calculate $W = \prod_{j=1}^6 w_{ij}^{e_{ij}})^{24}$ with a computer and see if it falls within some ϵ of 1. To identify a counterexample, I iterated over randomly generated sextuples, checked if they determined a tetrahedron according to Lemma 4 of Wirth-Dreiding. I had some setbacks when I undertook this approach. For a while, I was applying the conditions for Lemma 4 of Wirth-Dreiding into my computer

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incorrectly. When I did realize that I was doing it wrong, I had to end my AWS EC2 instance because my free plan was running out. It might be worth further exploring this approach.

4. p-adic Approach

If we think about p-adics now, if $\sum_{i<j} e_{ij}\theta_{ij} = 0 \mod \mathbb{Q}\pi$ then $\sum e_{ij}v_p(w_{ij}) = 0$ for every prime p. Now, $c_{ij} \in \mathbb{Q}$ so computing $v_p(c_{ij})$ is easy. Calculating, $v_p(w_{ij})$ is more difficult to calculate since it's in a field extension of $\mathbb{Q}$. We can use the following tool:

$$v_p(w_{ij}) = \begin{cases} 0 & v(c_{ij}) \geq 0 \\ \pm v(c_{ij}) & v(c_{ij}) < 0 \end{cases}$$

If $v_p(w_ij) < 0$, whether $v_p(w_{ij}) = +v(c_{ij})$ or $v_p(w_{ij}) = -v(c_{ij})$ we can't know.

5. My algorithm

My algorithm exploits the p-adic approach to the problem to find tetrahedra whose angles span a 5-dimensional $\mathbb{Q}$-vector space and have Dehn invariant zero. It does this by randomly generating sextuples, checking if they determine a tetrahedra (using Lemma 4 of Wirth-Dreiding). If it is a tetrahedron, then it calcualtes the denominator of c_{ij} when it is expressed in simplest terms. Recall that

$$c_{ij} = 2\cos 2\theta_j = 4\cos^2\theta_j - 2 = \frac{4D_{ij}^2 - 2D_{ijk}D_{ijl}}{D_{ijk}D_{ijl}}$$

Now, the denominator can be expressed as

$$L_{ij} = \frac{D_{ijk}D_{ijl}}{\gcd(D_{ijk}D_{ijl}, 4D_{ij}^2 - 2D_{ijk}D_{ijl})} = \frac{D_{ijk}D_{ijl}}{\gcd(D_{ijk}D_{ijl}, 4D_{ij}^2)}$$

Then, it gets the prime factors of all these. Then, it finds the valuations of all of these with respect to all the primes. We can arrange these Ls into a list $[L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, L_{34}]$. Taking the valuation with respect to p of these yields

$$v_p(L_{ij}) = \begin{cases} 0 & v_p(c_{ij}) \geq 0 \\ \pm v_p(c_{ij}) & v_p(c_{ij}) < 0 \end{cases}$$

If $\sum_{i<j} \pm e_{ij}v_p(L_{ij}) = 0$ for some combination of pluses and minuses, this is a necessity for the condition to hold. It is not a sufficiency since we don't know which combination of pluses and minuses is right. Here is how I checked if $\sum_{i<j} \pm e_{ij}v_p(L_{ij}) = 0$ computationally:

For every prime p

(1) Identify all L_{ij} such that $v_p(L_{ij}) > 0$. Suppose there are m such L_{ij}s. Insert these valuatioins into an $1 \times m$ matrix VL.
(2) Create an $2^m \times m$ matrix M where the elements in the ith row are the digits of the ith integer (starting from 0) when expressed in base 2.

(3) Perform an elementwise multiplication $M \star VL$ then the matrix multiplication $(M \star VL) \times E$ where

$$E = \begin{pmatrix} e_{12} \\ e_{13} \\ e_{14} \\ e_{23} \\ e_{24} \\ e_{34} \end{pmatrix}$$

(4) Use np.any($(M \star VL) \times E == 0$) to see if any combination of pluses and minuses yielded the correct result.

Not unless np.any($(M \star VL) \times E == 0$) for all primes is the tetrahedron a candidate for having Dehn invariant zero. This seems to never happen.

6. Some Observations and Analysis

I have a GitHub Repo where the code for this project is stored. In here is also a file that's the result of running the p-adic analysis of my code. Here is an example output:

```
[39, 82, 56, 47, 56, 90]
[2314830, 4265533440, 36952226532, 384102810, 3407333959107, 796296384]
193 [0, 0, 0, 1, 1, 1]
2 [1, 13, 2, 1, 0, 6]
3 [1, 3, 2, 7, 6, 2]
5 [1, 1, 0, 1, 0, 0]
7 [1, 1, 0, 1, 0, 0]
73 [1, 0, 1, 0, 1, 0]
13 [0, 0, 2, 1, 3, 1]
19 [0, 1, 1, 0, 0, 1]
151 [1, 0, 1, 0, 1, 0]
29 [0, 1, 1, 0, 0, 1]
primes passed: {3}
```

This is how you interpret the results:

(1) On the first line is randomly generated sextuple of integers that denote the edges of a tetrahedron in the order that we've been using $(e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34})$.
(2) In the second line is the list $[L_{12}, L_{13}, L_{14}, L_{23}, L_{24}, e_{34}]$.
(3) In the lines below that, the number in the first column is a prime that divides some L_{ij}.
(4) The primes that "passed" are those such that there exists a combination of pluses and minuses such that $\sum_{i<j} \pm e_{ij} v_p(L_{ij}) = 0$.

The Erdos-Kac Theorem states that the probability distribution of

$$\frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

where $\omega(n)$ is the number of distinct prime factors of n is the standard normal distribution. With this, we can calculate the average number of distinct prime factors of a number with

9 digits to be 3 with standard deviation approximately 1.6 (need to cite and check if 1.6 is right). L_{ij} is usually around 9 digits and has more than 3 distinct prime factors. For instance, from looking at the table above,

$$384102810 = 193 \times 2 \times 3^7 \times 5 \times 7 \times 13$$

However, the primes that comprise 384102810 tend to be smaller and have a greater power. By contrast 1,000,000,003=23 $\times$ 307 $\times$ 141623. This suggests that the prime factorizations of the L_{ij}s is somewhat anomalous. Further exploring why this is might bear some fruit.

The most common valuation list is [1,1,0,1,0,0] or some tetrahedral rotation of it. Also, you never get [2,0,0,0,0,0,2] or any tetrahedral rotation of this. Which valuation lists are possible? I looked at valuation lists mod 2. I found that only a few valuation lists are possible mod 2. They are

$$[1, 1, 0, 0, 1, 1]$$
$$[1, 1, 0, 1, 0, 0]$$
$$[0, 0, 0, 0, 0, 0]$$

Maybe we can prove the claim by showing that for every tetrahedron, there exists a prime p that yields a valuation list [1,1,0,1,0,0] (or some tetrahedral rotation of it). Note that it's not possible that $\pm e_{12} \pm e_{13} \pm e_{23} = 0$ otherwise this would lead to a degenerate tetrahedron because one of the faces violates the triangle inequality.

In general, my findings support the conjecture that there are no tetrahedra whose angles span a 5-dimensional $\mathbb{Q}$ vector space. The output of my code showed that seldom does a prime pass.