Analysis notes

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Definition 1

A sequence a_n is a map $f : \mathbb{N} - \mathbb{R}$. The sequence converges to a if and only if for all $\forall \varepsilon \exists N s.t. n > N \implies |a_n - a| < \varepsilon$

Theorem 2

If a_n and b_n are sequences that converge to a and b respectively then $a_n b_n$ converges to ab.

Proof. Given ε there exists N_1 such that $n > N_1 \implies |a_n - a| < \frac{\varepsilon}{2(|b|+1)}$. There exists M such that for n bigger than M, $|a_n| - |a| \leq |a - a_n| < 1 \implies |a_n| < |a| + 1$. Given ϵ there exists N_2 such that $n > N_2 \implies |b_n - b| < \frac{\epsilon}{2(|a|+1)}$. $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab| = |a_n(b_n - b) + b(a_n - a)| \leq |a_n||b_n - b| + |b||a_n - a|$. Thus when $n > \max\{N_1, N_2, M\}$, $|a_n||b_n - b| + |b||a_n - a| < (|a| + 1)|b_n - b| + |b||a_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon|b|}{2(|b|+1)} < \epsilon$. $\square$

Proposition 3

If a_n converges to a then $\frac{1}{a_n}$ converges to $1/a$.

Proof. $|\frac{1}{a_n} - \frac{1}{a}| = |\frac{a - a_n}{a_n a}| = \frac{|a - a_n|}{|a_n||a|}$. There exists M such that for n greater than M, $|a| - |a_n| \leq |a_n - a| \leq |a|/2 \implies -|a_n| \leq -|a|/2 \implies |a_n| \geq |a|/2$ and there exists N such that for n greater than N, $|a - a_n| < \varepsilon |a|^2/2$. Thus if $n > \max\{N, M\}$ $\frac{|a - a_n|}{|a_n||a|} \leq \frac{2|a - a_n|}{|a|^2} < \varepsilon$. $\square$

Theorem 4 (Monotone Convergence Theorem)

Any bounded monotone sequence is convergent

Proof. Assume that a_n is monotone increasing. Let $a = \sup a_n$ since a_n is assumed to be bounded. For all ε there exists N such that $a_N > a - \epsilon$. Since the sequence is monotone, $n > N \implies a_n > a - \varepsilon$ $\square$

Theorem 5 (Squeeze theorem)

If $a_n \leq c_n \leq b_n$ for all n and a_n and b_n converge to a then c_n converges to a.

Proof. Given ε there exists N_1 such that $n \geq N_1 \implies |a_n - a| < \varepsilon \implies a - \varepsilon < a_n$. There exists N_2 $s.t.$ $n \geq N \implies |b_n - b| < \epsilon \implies b_n < a + \epsilon$. Let $n \geq \max\{N_1, N_2\}$. Then $a - \varepsilon < a_n \leq c_n \leq b_n < a + \epsilon$. Thus $|c_n - a| < \varepsilon$. $\square$

Definition 6

A sequence a_n is a Cauchy sequence if and only if given ε there exists N such that for $n > N$ and $m > N$ $|a_n - a_m| < \epsilon$.

Theorem 7 (Cauchy's theorem)

Any Cauchy sequence is convergent.

Definition 8

A mapping $T : \mathbb{R} \to \mathbb{R}$ is a contraction mapping if and only if there exists $0 < \gamma < 1$ such that $|T(x) - T(y)| \leq \gamma|x - y|$ for all $x, y \in \mathbb{R}$.

Definition 9

A fixed point of $T : \mathbb{R} \to \mathbb{R}$ is a point x such that $T(x) = x$.

Theorem 10 (Contraction mapping theorem)

Any contraction mapping has a fixed point.

Example 11

The map $T(x) = \frac{1}{2}x+1$ is a contraction mapping since $|T(x)-T(y)| = |(\frac{1}{2}x+1)-(\frac{1}{2}y+1)| = |\frac{1}{2}x-\frac{1}{2}y| = \frac{1}{2}|x-y|$. It has a fixed point by the contraction mapping theorem. $T(x) = \frac{1}{2}x + 1 = x \implies \frac{1}{2}x = 1 \implies x = 2$.

Proof of the contraction mapping theorem. Let T be a contraction mapping with $|T(x) - T(y)| \leq \gamma|x - y|$. Create a sequence inductively by $a_0 = x_0$ where x_0 is some real number and $a_{k+1} = T(a_k) = T^n(x_0)$. $|a_{n+1} - a_n| = |T^{n+1}(x_0) - T^n(x_0)| = |T^n(T(x_0)) - T^n(x_0)| = |T(T^{n-1}(T(x_0))) - T(T^{n-1}(x_0))| \leq \gamma|T^{n-1}(T(x_0)) - T^{n-1}(x_0)| \leq \gamma^n|T(x_0) - x_0|$. For any arbitrary m, n with $m \geq n, |a_m - a_n| \leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + ...|a_{n+1} - a_n| \leq (\gamma^{m-1}+\gamma^{m-2}+...\gamma^n)|T(x_0)-x_0| = \frac{\gamma^n-\gamma^m}{1-\gamma}|T(x_0)-x_0|$. Given ϵ there exists N such that $\frac{\gamma^N}{1-\gamma} < \frac{\epsilon}{|T(x_0)-x_0|}$. If n and m are greater than N then $\frac{\epsilon}{|T(x_0)-x_0|} > \frac{\gamma^N}{1-\gamma} > \frac{\gamma^n}{1-\gamma} > \frac{\gamma^n-\gamma^m}{1-\gamma}$. Thus a_n is Cauchy. $|T(a) - a| = |T(a) - T^{n+1}(x_0) + T^{n+1}(x_0) -$

$T^n(x_0)+T^n(x_0)-a| \leq |T(a)-T^{n+1}(x_0)|+|T^{n+1}(x_0)-T^n(x_0)|+|T^n(x_0)-a| < 2|a-T^n(x_0)|+|T^{n+1}(x_0)-T^n(x_0)| \leq 2|a-T^n(x_0)|+\gamma^n|T(x_0)-x_0|$. Since a_n converges to a, given ϵ there exists N such that $n > N \implies |a-T^n(x_0)| < \epsilon/3$ and since γ^n converges to 0, given ϵ, there exists M such that $n > M \implies \gamma^n|T(x_0)-x_0| < \epsilon/3$. Thus if $n > \max\{N, M\}$, $|T(a)-a| < \epsilon$. Since ϵ can be made arbitrarily small, $T(a) = a$. □

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Fact 12

Cauchy sequences are useful because you can say a sequence converges without identifying the point it converges to.

Proof. Assume first that a_n is convergent. Then given ε there exists N such that $n > N \implies |a_n - L| < \varepsilon/2$. Let $m > N$ then $|a_n - a_m| \leq |a_n - L| + |L - a_m| < \varepsilon$.

Conversely, assume that a_n is a Cauchy sequence. First observe that a Cauchy sequence is bounded. If we set $\varepsilon = 1$ we can find N such tha $n, m \geq N \implies |a_n - a_m| < 1$. In particular, $|a_n - a_N| < 1$ for all but $a_1, ..., a_{N-1}$. Thus a_n is bounded by $\max\{a_1, ..., a_{N-1}, a_N + 1\}$ and $\min\{a_1, ..., a_{N-1}, a_N - 1\}$ which are finite. Define $b_n := \sup_{i \geq n} a_i$. Such a sup exists since a_n is bounded. $b_{n+1} = \sup_{i \geq n+1} a_i \geq \sup_{i \geq n} a_i = b_n$. The b_n sequence is thus monotone so it converges to some number a. Given $\epsilon > 0$ there exists a big enough N such that simultaneously if $n, m \geq N$ then $|a_n - a_m| < \epsilon/3$ and if $n \geq N$ then $|a - b_n| < \epsilon/3$. From the definition of b_n, there exists $i \geq n$ such that $|b_n - a_i| < \varepsilon/3$. Thus $|a - a_n| \leq |a - b_n| + |b_n - a_i| + |a_i - a_n| < \varepsilon$. □

We showed before that a contraction mapping has a fixed pint.

Proposition 13

A contraction mapping has at most one fixed point

Proof. Suppose y_1 and y_2 are fixed points. Since y_1 and y_2 are fixed points, $|T(y_1) - T(y_2)| = |y_1 - y_2|$ and since T is a contraction mapping, $|T(y_1) - T(y_2)| \leq \gamma|y_1 - y_2| < |y_1 - y_2|$. □

Theorem 14 (Bolzano-Weierstrass Theorem)

Any bounded sequence has a convergent subsequence.

Proof. Let p_n be a bounded sequence.

Case 1: For every subsequence q_n of p_n, there exists i such that given N there exists $n \geq N$ such that $q_n > q_i$.

Then there exists i such that given N there exists $n \geq N$ such that $p_i < p_n$. Then let $\alpha_1 = p_i$ and define $q_1 = p_i$ and q_k the $k-1$th term of p_n that is strictly greater than p_i. Now, there exists j such that given N there exists $n \geq N$ such that $q_k < q_n$. Then let $\alpha_2 = q_j$. Continue in this manner to get a strictly increasing subsequence of p_n.

Case 2: There exists a sequence q_n of p_n such that given i there exists N such that $n \geq N \implies q_n \leq q_i$. Let $d_1 = q_1$. There exists i with $q_i \leq q_1$. Let $d_2 = q_i$. Continue in this way to get a decreasing subsequence d_i of p_n.

□

Definition 15

A series is a sequence of the form

$$s_n = \sum_{i=1}^{n} a_i$$

Definition 16 (Geometric series)

Given $0 < \gamma < 1$, let $a_n = \gamma^n$. Then $s_n = \sum_{i=0}^{n} \gamma^i = \frac{1-\gamma^{n+1}}{1-\gamma}$. Then $\lim_{n\to\infty} s_n = \frac{1}{1-\gamma}$. If instead you had started at $i = a$ then you get $\sum_{i=a}^{n} \gamma^i = \frac{\gamma^a}{1-\gamma}$.

Definition 17 (Absolute convergence)

A series $\sum_{i=1}^{n} a_i$ converges absolutely if and only if $\sum_{i=1}^{n} |a_i|$ converges.

Proposition 18

If $\bar{s}_n = \sum_{i=1}^{n} |a_i|$ is convergent then $s_n = \sum_{i=1}^{n} a_i$ is also convergent. The converse is not necessarily true.

Proof. If $\bar{s}_n = \sum_{i=1}^{n} |a_i|$ is convergent then this is equivalent to saying that $\bar{s}_n$ is a Cauchy sequence. Let $n > m$. Then $|s_n - s_m| \leq |a_{m+1}| + \ldots + |a_n| = |\bar{s}_n - \bar{s}_m|$. Thus if $\bar{s}_n$ is Cauchy (convergent) then s_n is Cauchy (convergent) □

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3.1 Tests for convergence of series

Definition 19 (Comparison Test)

$$\sum_{n=1}^{\infty} a_n \quad \sum_{n=1}^{\infty} b_n$$

Suppose $0 \leq a_n \leq b_n$ for all n If b_n converges then a_n converges.

Proof. Since $a_n \geq 0$, $s_N = \sum_{i=1}^{N} a_i$ is monotone increasing. It is bounded above by $\sum_{n=i}^{\infty} b_i$. By the monotone convergence theorem, s_N converges. □

Definition 20 (Ratio Test)

$$\sum_{n=1}^{\infty} a_n$$

If $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| < 1$ then the sequence converges.

Proof. If $\alpha = \limsup |\frac{a_{n+1}}{a_n}| < 1$, then we can find $\alpha < \beta < 1$ and an integer N such that $|\frac{a_{n+1}}{a_n}| < \beta$ for $n \geq N$. In particular, $|a_{N+1}| < \beta |a_N|, |a_{N+2}| < \beta |a_{N+1}| < \beta^2 |a_n|, |a_{N+p}| < \beta^p |a_N|$. Thus $|a_n| < |a_N| \beta^{-N} \beta^n$ for $n > N$ and the proof the claim follows from the comparison test and absolute convergence test since $\sum \beta^n$ converges. □

Definition 21 (Alternating Series Test)

Suppose $\sum_{n=0}^{\infty} (-1)^n a_n$ where a_n is monotone. Then the series converges if and only if $\lim_{n\to\infty} a_n = 0$.

Example 22

We are going to see if $\sum_{n=0}^{\infty} n 2^{-n}$ converges. Do the ratio test. $\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)2^{-(n+1)}}{n 2^{-n}} = \frac{n+1}{n}\frac{1}{2}$ which converges to $\frac{1}{2}$ which is less than 1.

Example 23

$\sum_{n=0}^{\infty} \frac{2^n}{n!}$. The ratio test says that $\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{(n+1)!}\frac{n!}{2^n} = 2\frac{n!}{(n+1)!} = 2\frac{n!}{(n+1)n!} = \frac{2}{n+1}$ which converges to zero as n goes to infinity.

Example 24

$(n!)^{1/n} = (n(n-1)(n-2)...2\cdot 1)^{1/n}$. Note that $n! \geq n(n-1)(n-2)...\frac{n}{2} \geq (\frac{n}{2})^{\frac{n}{2}}$. Thus $(n!)^{1/n} \geq ((\frac{n}{2})^{\frac{n}{2}})^{\frac{1}{n}} = (\frac{n}{2})^{\frac{1}{2}}$.

Definition 25 (Root Test)

Suppose we have $\sum_{n=0}^{\infty} a_n$. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$ then the sum converges.

Proof. Let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. If $\alpha < 1$, we can choose β such that $\alpha < \beta < 1$ and an integer N such that $n > N \implies \sqrt[n]{|a_n|} < \beta \iff |a_n| < \beta^n$. Since $0 < \beta < 1$, $\sum \beta^n$ converges. Convergence of $\sum a_n$ follows from the comparison test. □

Example 26

Consider

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and we ask ourselves for which x does this converge?

$$\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \frac{x^{n+1}n!}{(n+1)!x^n} = \frac{x}{n+1}$$

. By the ratio test, the sum is convergent for all x.

Example 27

Look at

$$\sum_{n=0}^{\infty} x^n.$$

We already know that this is convergent if and only if $|x| < 1$. The ratio test and root test output identical answers, that $|\frac{x^{n+1}}{x^n}| = |x| < 1$. We can think of 1 as being the "radius of convergence of the sum.

Definition 28 (A continuous function)

A function $f : I \to \mathbb{R}$ (where I is an interval) is continuous at $x_0 \in I$ if and only if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. f is continuous on I if it is continuous at all $x_0 \in I$.

Proposition 29

Let h and f be continuous functions. Then $h(x) + f(x)$ is also a continuous function.

Proof. Given $x_0 \in I$ and $\varepsilon > 0$, pick δ such that simultaneously $|x - x_0| < \delta \implies |f(x) - f(x_0) < \epsilon/2$ and $|x - x_0| < \delta \implies |h(x) - h(x_0)|\varepsilon/2$. Then if $|x - x_0| < \delta$, $|f(x) + h(x) - (f(x_0) + h(x_0))| \leq |f(x) - f(x_0)| + |h(x) - h(x_0)| < \varepsilon$. □

Proposition 30

If f and g are continuous functions then fg is continuous.

Proof. Let $h(x) = f(x)g(x)$. Given ε and x_0 there exists δ such that simultaneously $|f(x) - f(x_0)| < \frac{\varepsilon}{2(|g(x_0)|+1)}$ and $|g(x) - g(x_0)| < \frac{\varepsilon}{2(|f(x_0)|+1)}$ and $|g(x)| - |g(x_0)| \leq |g(x_0) - g(x)| < 1$. Then $|h(x) - h(x_0)| = |f(x)g(x) - f(x_0)g(x_0)| = |f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)| \leq |g(x)||f(x) - f(x_0)| + |f(x_0)||g(x) - g(x_0)| < (|g(x_0)| + 1)|f(x) - f(x_0)| + |f(x_0)||g(x) - g(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon|f(x_0)|}{2(|f(x_0)|+1)} < \varepsilon$. □

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Last time we showed that if f and g are continuous then fg is continuous. A corollary to this is that if f is continuous and it not 0 on an interval then $\frac{1}{f}$ is continuous on that interval.

Example 31

A constant function is continuous.

Example 32

Since we know that constant functions and x are continuous, all polynomials are continuous.

Theorem 33

Let f and g be continuous functions that map $\mathbb{R}$ to $\mathbb{R}$ and assume that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Then $f(x) = g(x)$ for all $x \in \mathbb{R}$. In other words $f|_{\mathbb{Q}}$ uniquely defines f over the reals. In general f restricted to any dense subset of $\mathbb{R}$ uniquely defines f.

Proof. Given ε and x_0, there exists δ such that simultaneously $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon/2$ and $|g(x) - g(x_0)| < \varepsilon/2$. Then $|f(x_0) - g(x_0)| \leq |f(x_0) - f(x)| + |f(x) - g(x)| + |g(x) - g(x_0)|$. For all $\delta > 0$ there exists $x \in \mathbb{Q}$ such that $|x_0 - x| < \delta$. Thus $|f(x_0) - g(x_0)| \leq |f(x_0) - f(x)| + |f(x) - g(x)| + |g(x) - g(x_0)| < \varepsilon/2 + 0 + \varepsilon/2 = \varepsilon$. Since ε can be made arbitrarily small, $f(x_0) = g(x_0)$ for all $x_0 \in \mathbb{R}$. $\square$

Definition 34 (Power Series)

Let's go back to the power series. Let $\{a_n\}$ be a sequence and $\sum_{n=0}^{\infty} a_n x^n$ is a power series.

Example 35

Last time we used the ratio test to show that $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$. This function is also continuous. $E(0) = 1$ and $E(1) = e$

Proposition 36

$E(x + y) = E(x)E(y)$

Corollary 37

Let $n, p, q \in \mathbb{Z}$

- $E(n) = E(\sum^n 1) = E(1)^n = e^n$.
- $E(\frac{1}{n}) = \sqrt[n]{e}$ since $e = E(\frac{1}{n})^n$.
- $E(\frac{p}{q}) = \sqrt[q]{e^p} = e^{\frac{p}{q}}$
- $E(x) = e^x$ since they are the same on the rationals. (we just need to prove that $E(x+y) = E(x)E(y)$ and $E(x)$ is continuous).

Proof of proposition 37. Let $E(x) = \sum_{n=0}^{\infty} \frac{x^i}{i!}$ and $E(y) = \sum_{n=0}^{\infty} \frac{y^i}{i!}$.

Observe that $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.

$\sum_{i+j \leq N} \frac{x^i}{i!}\frac{y^j}{j!} = \sum_{n=0}^{N} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i} = \sum_{n=0}^{N} \frac{(x+y)^n}{n!}$ so $\sum_{n=0}^{N} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$ converges to $E(x+y)$.

Observe that

$$\sum_{i+j \leq N} \frac{x^i y^j}{i!j!} < \sum_{i=1}^{N} \frac{x^i}{i!} \sum_{j=1}^{N} \frac{y^j}{j!} < \sum_{i+j \leq 2N} \frac{x^i y^j}{i!j!}$$

As N goes to infinity, the middle term is $E(x)E(y)$ and the left and right terms converge to $E(x+y)$. By the squeeze theorem $E(x)E(y) = E(x+y)$. □

4.1 Tests of convergence for series

Comparison	Suppose $0 \leq a_n \leq b_n$ for all n. If b_n converges then a_n converges.
Ratio	If $\limsup_{n \to \infty} \lvert\frac{a_{n+1}}{a_n}\rvert < 1$ then the sum converges.
Root	If $\limsup_{n \to \infty} \sqrt[n]{\lvert a_n \rvert} < 1$ then the sum converges.
Alternating Series Test	If $a_{n+1}a_n < 0$, $\lvert a_{n+1}\rvert \leq \lvert a_n \rvert$, and $\lim_{n \to \infty} a_n = 0$ then the sum converges.

Example 38

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is an alternating series. Sign of $a_n = \frac{(-1)^n}{n}$ changes, $|a_{n+1}| \leq |a_n|$, and $|a_n|$ tends to zero.

Definition 39 (Alternating Series Test)

If the following hold:

- $a_{n+1}a_n < 0$
- $|a_{n+1}| \leq |a_n|$
- $\lim_{n \to \infty} a_n = 0$ then the alternating series test implies that this series converges.

Proof. $S_{2n+2} = S_{2n} + a_{2n+1} + a_{2n+2}$ so $S_{2n+2} \leq S_{2n}$ so S_{2n} is monotone decreasing. A similar argument shows S_{2n+1} is monotone increasing. $S_{2n+1} \leq S_{2n}$ and $S_{2n+1} - S_{2n} = a_{2n+1}$ which goes to zero. By the squeeze theorem, the claim follows. □

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Remark 40. *It might be advantageous to fix the proof for the previous problem and take notes on the rearrangement of series.*

Proposition 41

If $f : I \to \mathbb{R}$ is continuous and $x_n \to x$, $x_n, x \in I$, then $f(x_n) \to f(x)$

Proof. Given ε and x there exists δ such that $|x' - x| < \delta \implies |f(x') - f(x)| < \varepsilon$. Given δ there exists N such that $n > N \implies |x_n - x| < \delta$ ao that $|f(x_n) - f(x)| < \varepsilon$. □

Theorem 42 (The Extreme Value Theorem)

If $f : [a, b] \to \mathbb{R}$ is continuous then there exists $M \in I$ such that $f(M) = \sup \operatorname{Im} f$ and there exists m such that $f(m) = \inf \operatorname{Im} f$.

Theorem 43 (Intermediate Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) \leq y \leq f(b)$ then there exists $x \in [a, b]$ such that $f(x) = y$.

Proof of the extreme value theorem. Suppose the function f is not bounded above on the interval $[a, b]$. Then for every natural number n, there exists an $x_n \in [a, b]$ such that $f(x_n) > n$. This defines a sequence $(x_n)_{n \in \mathbb{N}}$. Because $[a, b]$ is bounded, the Bolzano Weierstrass theorem implies that there is a convergent subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of (x_n). Denote its limit by x. As $[a, b]$ is closed, it contains x. Because f is continuous at x, we know that $f(x_{n_k})$ converges to the real number $f(x)$. But $f(x_{n_k}) > n_k \geq k$ for every k, which implies that $f(x_{n_k})$ diverges to ∞, a contradiction. Therefore, f is bounded on $[a, b]$.

$\sup \operatorname{im} f$ exists. It is necessary to find $d \in [a, b]$ such that $M = f(d)$. Let n be a natural number. As M is the least upper bound $M - \frac{1}{n}$ is not an upper bound for f. Therefore, there exists d_n in $[a - b]$ so that $M - \frac{1}{n} < f(d_n)$. This defines a sequence $\{d_n\}$. Since M is an upper bound for f, we have $M - \frac{1}{n} < f(d_n) \leq M$ for all $n>$ Therefore, the sequence $\{f(d_n)\}$ converges to M. By the Bolzano-Weiestrass theorem, there exists a subsequence $\{d_{n_k}\}$, which converges to some d and, as $[a, b]$ is closed d is in $[a, b]$. Since f is continuous at d, the sequence $\{f(d_{n_k})\}$ converges to $f(d)$. But $\{f(d_{n_k})\}$ is a subsequence of $\{f(d_n)\}$ that converges to M so $M = f(d)$. Therefore, f attains its supremum M at d. □

Proof of the intermediate value theorem. First, let $f(a) < u < f(b)$. Let S be the set of all $x \in [a, b]$ such that $f(x) \leq u$. Then S is non-empty since a is an element of S and S is bounded above by b. $c = \sup S$ exists. We claim that $f(c) = u$. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(c)| < \varepsilon$ whenever $|x - c| < \delta$. Thus

$f(x) - \varepsilon < f(c) < f(x) + \varepsilon$ for all $x \in (c - \delta, c + \delta)$. There exists some $a^* \in (c - \delta, c]$ that is contained in S so $f(c) < f(a^*) + \varepsilon \leq u + \varepsilon$. Picking $a^{**} \in (c, c + \delta)$, we knkow that $a^{**} \notin S$ because c is the supremums of S. This means that $f(c) > f(a^{**} - \varepsilon > u - \varepsilon$. Both inequalities mean $u - \varepsilon < f(c) < u + \varepsilon$. Thus $f(c) = u$. □

In both of these, it is crucial that $[a, b]$ be closed sets. The interval $(0, 1]$ doesn't satisfy the extreme value theorem for $f : (0, 1] \to \infty$, $x \mapsto \frac{1}{x}$ since the function is unbounded as you get closer to zero. If you take an unconnected set, then the intermediate value theorem again doesn't work.

Definition 44 (A metric space)

A metric space is a set X endowed with a matric $d : X \times X \to \mathbb{R}$ such that

- $d(x_1, x_2) \geq 0$ with equality if and only if $x_1 = x_2$.
- $d(x_1, x_2) = d(x_2, x_1)$
- $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$

Example 45

- $\mathbb{R}$ endowed with the absolute value metric $d(x_1, x_2) = |x_1 - x^2|$.
- $\mathbb{R}^2$ with $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- $\mathbb{R}^2$ with $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$.
- $C([0, 1], \mathbb{R})$ is the set of continuous functions from $[0, 1]$ to $\mathbb{R}$. For $f, g \in C([0, 1], \mathbb{R})$, $d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|$
- French Railroad metric. The shortest path between two cities via railway. The set here is the earth and the metric is the shortest time it takes to go from one city to another via railway.

6 September 1, 2020

Definition 46 (Convergene in a metric space)

A sequence x_n converges to x if and only if given ε there exists N such that $n \geq N \implies d(x_n, x) < \varepsilon$.

Proposition 47

The limit of a convergent subsequence is unique..

Proof. Suppose x_n converges to x and y. Given ε there exists N such that $n \geq N \implies d(x_n, x) < \varepsilon/2$ and M such that $n \geq M \implies d(x_n, y) < \varepsilon/2$. By the triangle inequality, $d(x, y) < d(x_n, x) + d(x_n, y) < \varepsilon$ so $x = y$. □

Definition 48 (Cauchy Sequence in Metric Space)

Let (X, d) be a metric space and x_n a sequence. x_n is Cauchy if and only if given ε there exists N such that $n, m \geq N \implies d(x_n, x_m) < \varepsilon$

Example 49

If we have, $\mathbb{R}^n$ and x_n a Cauchy Sequence. $|x - y| = d(x, y)$ then we showed that x_n converges

Example 50

In $\mathbb{Q}$ with metric $d(x, y) = |x - y|$, the sequence x_n that tends to $\sqrt{2}$ is Cauchy (since it converges in $\mathbb{R}$) but it does not converge over $\mathbb{Q}$.

Example 51

The sequence $x_n = \frac{1}{n}$ does not converge in $(0, 1)$. The moral of the story is that the cauchy sequence, convergent sequence equivalence is not true for all metric spaces.

Definition 52

A metric space (X, d) is Cauchy complete if and only if every Cauchy sequence is convergent in that metric space.

Example 53 (Set Operations)

$\cap A_\alpha = \{x \in X : x \in A_\alpha \text{ for all } \alpha\}$. $\cup A_\alpha = \{x \in X : \exists \alpha \text{ s.t. } x \in A_\alpha\}$.

Lemma 54

$$X - (\bigcap_\alpha A_\alpha) = \bigcup_\alpha (X - A_\alpha)$$

Definition 55 (Ball)

The ball of radius r with center at x is $B_r(x) = \{y \in X : d(x, y) < r\}$.

Definition 56

In $\mathbb{R}^2$ with the Euclidean metric, the balls are circles without the border.

Definition 57 (Open set)

A subset $A \subseteq X$ is open if for all $x \in A$ there exists $r > 0$ with $B_r(x) \subseteq A$. Observe that the empty set is open by vacuity

Proposition 58

Let A_1 and A_2 be open subsets of a metric space X. The set $A_1 \cup A_2$ is also open. The intersection of A_1 and A_2 is also open. By induction we can then show that the finite union or intersection of open sets is again open.

Proof. Let $x \in A_1 \cap A_2$ then $x \in A_1$ hence $r_1 > 0$ $B_{r_1}(x) \subseteq A_1$ and $x \in A_2$ hence $r_2 > 0 B_{r_2}(x) \subseteq A_2$. Let $r = \min\{r_1, r_2\} > 0$. Thus $B_r(x) \subseteq A_1 \cap A_2$. $\square$

Example 59

The intersection of infinitely many open sets is not necessarily open. For example $\cap_n(-\frac{1}{n}, \frac{1}{n}) = \{0\}$. The min that we used to prove that the intersection of open sets is open is what screws things up.

Proof That the Union of Infinitely Many Open Sets is Open. $x \in \cup_\alpha A_\alpha$ then there exists α such that $x \in A_\alpha$. Since A_α is open, there exists a $r > 0$ with $B_r(x) \subseteq A_\alpha \subseteq \cup_\alpha A_\alpha$. $\square$

Definition 60 (Closed Set)

A set A is closed in a metric space X if and only if $X - A$ is open.

7 October 6, 2020

Proposition 61

The infinite intersection of closed sets is again closed since $X - \cap_\alpha A_\alpha = \cup_\alpha(X - \alpha_\alpha)$ and the infinite union of open sets is open. Again, the infinite union of closed sets is not necessarily closed. For example

$$\bigcup_{0 \leq a < 1} [0, a] = [0, 1)$$

Proposition 62

If a set A is closed then for a sequence $\{x_n\}$ in A, the limit of the sequence as n approaches infinity is contained in A.

Proof. If not, there exists $x \in X - A$. $X - A$ is an open set so there exists $r > 0$ such that $x \in B_r(x) \subset X - A$. On the other hand, $x_n \to x$ so $d(x, x_n) \to 0$. Therefore, for N sufficiently large, $x_n \in B_r(x)$ but this is a contradiction since $x_n \in A$. $\square$

Proposition 63

The converse of Proposition 62 is also true

Proof. Let $A \subset X$. We know that if $x_n \in A$ and $x_n \to x$, then $x \in A$. Our goal is to show that A is closed. If $X - A$ is not open, then there exists $x \in X - A$ such that for all $r > 0$, $B_r(x) \cap A \neq \emptyset$. But $x_n \in B_{\frac{1}{n}} \cap A \neq \emptyset$ and $d(x, x_n) < \frac{1}{n} \implies x_n \to x$. Since x_n converges to x, this forces x to be in A, a contradiction. $\square$

Definition 64

A **cover** of a set A is a collection of sets $\{B_\alpha\}$ (open, closed, or neither) such that $A \subset \cup_\alpha B_\alpha$

Definition 65

A set $A \subset X$ is said to be compact if any cover of X by open subsets has a finite subcover

Example 66

Consider the set $(-1, 1)$. $\cup_n(-1 + \frac{1}{n}, 1 - \frac{1}{n}) = (-1, 1)$. The union of these sets is $(-1, 1)$ yet no finite union of the above union contains $(-1, 1)$.

Example 67

Any set of the form $[a, b] \subset \mathbb{R}$ is compact.

Example 68

In $\mathbb{R}^2$, under the Euclidean metric, the set $\overline{B_r(x)} = \{y \in \mathbb{R} : d(x, y) \leq r\}$ is compact.

Proposition 69

The set $\overline{B_r(x)} = \{y \in X : d(x, y) \leq r\}$

Proof. Let $X - \overline{B_r(x)}$. Then $d = d(z, x) > r$. Let $s = d - r$. $B_s(z) \cap \overline{B_r(x)} = \emptyset$. Assume not. There exists $u \in B_s(z) \cap \overline{B_r(x)}$ so $d(z, u) < s$ and $d(x, u) \leq r$. By the triangle inequality, $d(x, z) \leq d(x, u) + d(u, z) < r + s = d$ $\square$

Definition 70

A set $A \subset X$ is said to be bounded if there exists $0 < r$ and $x \in X$ such that $A \subset B_r(x)$ for some x.

Lemma 71

If K is compact, then K is closed.

Proof. Pick a point $p \notin K$. If $q \in K$, let V_q and W_q be open balls around p and q of radius $\frac{1}{2}d(p,q)$. Observe that if $x \in W_q$ then $d(p,q) \leq d(q,x) + d(x,p) < \frac{1}{2}d(p,q) + d(x,p)$ so $d(x,p) > \frac{1}{2}d(p,q)$ that is, all the points in this ball are at least $\frac{1}{2}d(p,q)$ from p. By compactness, a finite number of these balls, $W_{q1}, ..., W_{qN}$ cover K. Look at the corresponding balls $V_{q1}, ..., V_{qN}$. They are all centered at p. The smallest (their intersection) is a neighborhood of p that contains no point of K. This shows that K^c is open. □

Observe that if X is an compact set, then $X \subset \cup_{x \in X} B_\varepsilon(x)$. But then there exists a finite set S such that $A \subset \cup_{x \in S} B_\varepsilon(x)$

8 October 8, 2020

Last time we looked at the definition of a compact space.

Theorem 72

Let (X,d) be a compact set and x_n a sequence. There exists a subsequence x_{n_k} such that x_{n_k} is a Cauchy sequence.

Proof. Cover X by balls or radius 1. Then $X \subset \cup_{x \in X} B_1(x)$. Compactness tells us that finitely many are needed to cover X. Call these balls $B_1(y_i)$. There exists a ball $B_1(y)$ with infinitely many x_n contained in it. X can be covered by finitely many $B_{\frac{1}{2}}(z_i)$ so so can $B_1(y)$ and so $B_1(y) = \cup_i (B_1(y) \cap B_{\frac{1}{2}}(z_i))$. At least one of $B_1(y) \cap B_{\frac{1}{2}}(z)$ must have infinitely many points of x_n. Continue in this way with $B_{\frac{1}{4}}, B_{\frac{1}{8}}, B_{\frac{1}{16}}, B_{\frac{1}{32}}, ...$. Our subsequence is going to be the x_ns contained in all the balls. Given $\varepsilon > 0$, choose N such that $2^{-N+2} < \varepsilon$. Then $d(x_n, x_m) < \varepsilon$

□

Lemma 73

Let (X,d) be compact and A_α a family of closed subsets. Then $\cap_\alpha A_\alpha = \emptyset \implies$ there exists finitely many $A_1 \cap ... \cap A_m = \emptyset$.

Proof. Define $O_\alpha = X - A_\alpha$ so O_α is open. The union of $\cup O_\alpha = X - \cap_\alpha A_\alpha = X$. By compactness, finitely many O_α cover X so $X = O_1 \cup ... \cup O_m = X - \cap_i A_i$. □

Theorem 74

Let (X, d) be a compact metric space. If x_n is a sequence in X, then there is a convergent subsequence.

Proof. Let (F_n) be a decreasing sequence of closed nonempty subsets of X and let $G_n = X - F_n$. If $\cup_{n=1}^{\infty} G_n = X$ then $\{G_n : n \in \mathbb{N}\}$ is an open cover of X so it has a finite subcover $\{G_{n_k} : k = 1, 2, ...K\}$ since X is compact. Let

$$N = \max\{n_k : k = 1, 2, ..., K\}$$

Then $\cup_{n=1}^{N} G_n = X$ so

$$F_n = \bigcap_{n=1}^{N} F_n = X - \bigcup_{n=1}^{N} G_n = \emptyset$$

contrary to our assumption that every F_n is nonempty. It follows that $\cup_{n=1}^{\infty} G_n \neq X$ and then

$$\bigcap_{n=1}^{\infty} F_n = X - \bigcup_{n=1}^{\infty} G_n \neq \emptyset$$

meaning that X has the finite intersection property for closed sets, so X is sequentially compact. The result follows from the following lemma. $\square$

Definition 75

A metric space has the finite intersection property for closed sets if every decreasing sequence of closed, nonempty sets has nonempty intersection.

Lemma 76

If a metric space has the finite intersection property for closed sets then it is sequentially compact.

Proof. Suppose that X has the finite intersection property. Let (x_n) be a sequence in X and define

$$F_n = \overline{T_n}, \quad T_n = \{x_k : k > n\}$$

Then (F_n) is a decreasing sequence of non-empty closed sets so there exists

$$x \in \bigcap_{n=1}^{\infty} F_n$$

Choose a subsequence (x_{n_k}) of (x_n) as follows. For $k = 1$, there exists $x_{n_1} \in T_1$ such that $d(x_{n_1}, x) < 1$, since $x \in F_1$ and and T_1 is dense in F_1. Similarly, since $x \in F_{n_1}$ and T_{n_1} is dense in F_{n_1}, there exists $x_{n_2} \in T_{n_1}$ with $n_2 > n_1$ such that $d(x_{n_2}, x) < \frac{1}{2}$. Continuing in this way (or by induction), given x_{n_k} we choose $x_{n_{k+1}} \in T_{n_k}$ where $n_{k+1} > n_k$ such that $d(x_{n_{k+1}}, x) < \frac{1}{k+1}$. Then $x_{n_k} \to x$ as $k \to \infty$ so X is sequentially compact. $\square$

Lemma 77

Let (X, d) be a metric space and $A \subset X$ a closed subset. Then A is compact.

Proof. Let $A \subset \cup_\alpha O_\alpha$ for O_α open. Then $X = (\cup_\alpha O_\alpha) \cup (X - A)$. Finitely many of these sets cover X so finitely many of $\cup_\alpha O_\alpha$ cover A. $\square$

Theorem 78

Let $[a, b] \subset \mathbb{R}$ be compact with $a \leq x \leq b$. Then the set of $[a_1, b_1] \times [a_2, b_2] \subset \mathbb{R}^2$ is compact.

Corollary 79

Let $A \subset \mathbb{R}^n$ be a set that's closed and bounded. Then A is compact.

Definition 80 (Continuous Functions Revised)

Let (X, d_X) and (Y, d_Y) be metric spaces. Then $f : X \to Y$ is continuous at x_0 if for all ε there exists $\delta > 0$ such that $d(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon$.

We will talk about everything below and including theorem 80 in next class.

Theorem 81

$\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Theorem 82

If $\sum a_n$ is a series of complex numbers that converges absolutely, then every rearrangement of $\sum a_n$ converges and then all ocnverge to the same sum. Conversely, if the series converges but not absolutely, then given $-\infty \leq \alpha \leq \beta \leq \infty$ there exists a rearrangement of the series such that

$$\liminf_{n \to \infty} \sum a_n = \alpha \quad \text{and} \quad \limsup_{n \to \infty} \sum a_n = \beta$$

9 October 15, 2020

Lemma 83

$[a_1, b_1] \times [a_2, b_2]$ is compact.

Lemma 84

A closed and bounded set A of $\mathbb{R}^n$ with the Euclidean metric is compact.

Proof. Let $A \subset \cup O_\alpha$ with O_α open. Then $X = A \cup (X - A) \subset \cup O_\alpha \cup (X - A)$ Since X is comapact, $X \subset O_1 \cup ... \cup O_n \cup (X - A)$ so $A \subset O_1 \cup ... \cup O_n$. $\square$

Theorem 85

Consider $\mathbb{R}^n$ with the Euclidean metric. A subset $A \subset \mathbb{R}^n$ is compact if and only if it's closed and bounded.

Proof. If A is bounded, $A \subset [-R, R] \times ... \times [-R, R]$ for R sufficiently large. This is a compact set. If A is also closed then it's compact from the previous lemma. $\square$

Definition 86 (Continuity for arbitrary metric spaces)

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$. We say f is continuous at x_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_X(x_0, x) < \delta \implies d_Y(f(x_0), f(x))$.

Lemma 87

The composition of two continuous functions is continuous.

Proof. Given $\varepsilon > 0$ there exists $\delta > 0$ such that $d_Y(f(x_0), f(x)) < \delta \implies d_Z(g(f(x_0), g(f(x))) < \varepsilon$. There exists γ such that $d_X(x, x_0) < \gamma \implies d_Y(f(x), f(x_0)) < \delta$. $\square$

Definition 88 (The Derivative)

Let $f : I \to \mathbb{R}$, $x_0 \in I$ we say that f is differentiable with derivative a if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies \left| \frac{f(x) - f(x_0)}{x - x_0} - a \right| < \varepsilon$$

Example 89

Suppose $f(x) = c$ for all x. Then $\frac{f(x)-f(x_0)}{x-x_0} = 0$ so the derivative of this function is zero.
Let $f(x) = x$. Then the derivative is

$$\lim_{x \longrightarrow x_0} \frac{x - x_0}{x - x_0} = 1$$

Lemma 90

Let f and g be differentiable functions. Then $(f + g)' = f' + g'$, $(fg)' = f'g + g'f$, $(f(g(x))' = f'(g(x))g'(x)$.

Example 91

All polynomials are differentiable. Look at $x^n = x \cdot x \cdot \ldots \cdot x$. Using the Leibniz rule gives $(x^n)' = nx^{n-1}$.

Example 92 (Example of Continuity)

We want to show $f(x) = \frac{1}{x}$ is a continuous function. We show that if $x_0 \in (0, \infty)$ then f is continuous at x_0. Let $\varepsilon > 0$. We want to find $\delta > 0$ such that $|x - x_0| < \delta \implies |\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$, or equivalently $|x - x_0| < \varepsilon x x_0$. There is no $\delta > 0$ for which $|x - x_0| < \delta \implies |x - x_0| < \varepsilon x x_0$ for all $x \in (0, \infty)$. We can remove this problem by requiring x to stay away from zero. For example, let $|x - x_0| < \frac{1}{2}x_0$. Then $\frac{1}{2}x_0 < x$ and $\frac{1}{2}\varepsilon x_0^2 < \varepsilon x x_0$. These inequalities suggest taking

$$\delta = \min\left(\frac{1}{2}x_0, \frac{1}{2}x_0^2\varepsilon\right).$$

For this δ, if $|x - x_0| < \delta$ then $\frac{1}{x} - \frac{1}{x_0}| = \frac{|x-x_0|}{|xx_0|} < \frac{\frac{1}{2}x_0^2\varepsilon}{\frac{1}{2}x_0^2} = \varepsilon$.

10 October 27, 2020

Lemma 93

If $f(x)$ is differentiable then it is continuous.

Proof. If $\left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \varepsilon$ then $|f(x) - f(x_0) - f'(x_0)(x - x_0)| < \varepsilon|x - x_0|$. Thus $|f(x) - f(x_0)| < (|f'(x_0)| + 1)|x - x_0|$ when $|x - x_0| < \mu$. To show that f is continuous $\square$

Proposition 94

The derivative of the sum of two functions is the sum of the derivatives of the functions

Proposition 95

$(fg)' = f'g + fg'$

Proof. $\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{g(x)(f(x)-f(x_0))}{x-x_0} + f(x_0)\frac{g(x)-g(x_0)}{x-x_0}$ $\square$

Do chain and quotient rules yourself. We did Rolle's Theorem. Mean Value Theorem. Do the product rule too.

Theorem 96 (Mean Value Theorem)

If f is a continuous real function on $[a, b]$ which is differentiable in (a, b) then there is a point $x \in (a, b)$ at which $f(b) - f(a) = (b - a)f'(x)$.

The proof of this will follow as the corollary the the following lemma:

Lemma 97
If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b) then there is a point $x \in (a, b)$ at which $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$. Note that differentiability is not required at the endpoits.

Proof. Put $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t)$. Then h is continuous on $[a, b]$, h is differentiable on (a, b) and $h(a) = f(b)g(a) - f(a)g(b) = h(b)$. To prove this theorem, we have to show that $h'(x) = 0$ for some $x \in (a, b)$. If h is constant, this holds for every $x \in (a, b)$. If $h(t) > h(a)$ for some $t \in (a, b)$, let x be a point on $[a, b]$ at which h attains its maximum. Thus $h'(x) = 0$. If $h(t) < h(a)$ for some $t \in (a, b)$ the same argument applies of we choose for x a point on $[a, b]$ where h attains its minimum. □

11 October 29, 2020

Last time we showed what it means to be differentiable for a function. If $f(x)$ is differentiable on (a, b) then it is continuous on (a, b). We showed Rolle's Theorem. If $f(a) = f(b)$ for some $a \neq b$ then there exists $a < c < b$ such that $f'(c) = 0$. We showed the Mean Value Theorem. If $f : [a, b] \to \mathbb{R}$ is differentiable, $\frac{f(b)-f(a)}{b-a} = f'(c)$ for some $c \in (a, b)$. Note that if $f(a) = f(b)$ then MVT is equivalent to Rolle's Theorem. Today, we'll talk about the L'Hospital's Law and the Taylor Expansion.

Proposition 98
Let g, f be differential functions on $[a, b]$. Then $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$.

Proof. By the Mean Value Theorem, $\frac{f(b)-f(a)}{b-a} = f'(c)$ and $\frac{g(b)-g(a)}{b-a} = g'(\tilde{c})$. Then $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(\tilde{c})}$. We want to show that $c = \tilde{c}$. Look at $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$. Then $h(a) = (f(b) - f(a))g(a) - (g(b) - g(a))f(a) = f(b)g(a) - g(b)f(a)$. And $h(b) = g(a)f(b) - g(b)f(a)$ so $h(a) = h(b)$. Then there exists c such that $h'(c) = 0$ so $0 = h'(c) = (f(b) - f(a))g'(c) - (g(b) - g(a))f'(c) \implies \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$. We assumed that f and g are differentiable and that f' and g' are continuous. □

11.1 Taylor Expansion

Let $f : [a, b] \to \mathbb{R}$ where f is differentiable and f' is differentiable.

Definition 99 (Taylor Polynomial)
Suppose f is a real function on $[a, b]$ that exists for every $t \in (a, b)$. Let α and β be distinct points of $[a, b]$ and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists $\alpha < x < \beta$ such that $f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n$.

Proof. For $n = 1$, this is just the mean value theorem. In general, the theorem shows that f can be approximated by a polynomial of degree $n - 1$ and that we can estimate the error if we know the bounds on $|f^{(n)}(x)|$.

Let M be the number defined by $f(\beta) = P(\beta) + M(\beta + \alpha)^n$ and put $g(t) = f(t) - P(t) - M(t - \alpha)^n$, $a \leq t \leq b$. We have to show that $n!M = f^{(n)}(x)$ for some x between α and β. Then $g^{(n)} = f^{(n)} - n!M$. Hence the proof will be complete if we can show that $g^{(n)}(x) = 0$ for some x between α and β. Since $P^{(k)}(\alpha) = f^{(k)}(\alpha)$ for $k = 0, ..., n - 1$ we have $g(\alpha) = g'(\alpha) = ... = g^{(n-1)}(\alpha) = 0$. Our choice of M shows that $g(\beta) = 0$ so that $g'(x_1) = 0$ for some x_1 between α and β by the mean value theorem. Sine $g'(\alpha) = 0$, we conclude similarly that $g''(x_2) = 0$ for some x_2 between α and x_1. After n steps, we arrive at the conclusion that $g^{(n)}(x_n) = 0$ for some x_n between α and x_{n-1}. $\square$

Remark 100. *Review what was gone over on October 27 and 29 again. To be honest, just go over the entirety of Chapter 5 from Rudin.*

12 November 3, 2020

A function f is not always equal to its infinite Taylor expansion. It depends on whether the remainder term goes to zero.

Example 101

Let $f(x) = e^x$. Then $f'(x) = f(x)$ and $f(0) = 1$. The Taylor series is $\sum_{i=0}^{\infty} \frac{x^i}{i!}$. The remainder term for $f(x)$ where $a = 0$. $\frac{e^c}{k!}x^k = R_k(x)$ where $0 < c < x$. Then $|R_k(x)| \leq \frac{e^x x^k}{k!} \to 0$ as $k \to \infty$. Not every function has a Taylor series expansion. The functions that are equal to their Taylor series are called **analytic**

Example 102

A function that is not equal to its Taylor series expansion. Let $f(x) = \begin{cases} 0 & x \leq 0 \\ e^{\frac{1}{x^2}} & x > 0 \end{cases}$ This function is infinitely differentiable everywhere except at zero and we can prove with a bit more work that $f^{(k)}(0) = 0$ for all k. The Taylor series gives 0. This happens because the remainder term doesn't go to zero.

Definition 103 (Riemann Integral)

Let $f : [a, b] \to \mathbb{R}$ and $f(x) > 0$. What is the area under the curve?

Suppose we take a rectangle.

Definition 104 (A Partition)

A partition on $[a, b]$ is a set of numbers with $a = x_0 < x_1 < ... < x_{n-1} < x_n = b$.

Define $L_p = \sum m_i(x_{i+1} - x_i)$ and $U_p = \sum M_i(x_{i+1} - x_i)$ and $\int_a^b f = \sup_P L_p = \inf_P U_p$ where the sup *is taken over all partitions. We say a function f is integrable if and only if $\sup_P L_p = \inf_P U_p$.*

Which functions are integrable? All continuous functions are integrable. We prove this next time.

13 November 5, 2020

We went over some stuff about uniform continuity.

Definition 105

A function f is uniformly continuous on an interval I if given $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in I$, $|x - x'| < \delta \implies |f(x) - f(x')| < \varepsilon$.

Theorem 106

Let $f : [a, b] \to \mathbb{R}$. If f is constant, then f is uniformly continuous.

Proof. We argue by contradiction. If not, then there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there exists x_δ, y_δ such that $|x_\delta - y_\delta| < \delta$ and $|f(x_\delta) - f(y_\delta)| \geq \varepsilon$. Let $\delta = 2^{-n}$, $|x_n - y_n| < 2^{-n}$ so $|f(x_n) - f(y_n)| \geq \varepsilon$. But that interval is compact. There exists a subsequence x_{n_k} that converges to x and y_{n_k} converges to y. Thus $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(y)$. There's a bit more to this proof that I didn't get. $\square$

Next we want to use this to show that any continuous function is integrable.

Theorem 107

$f : [a, b] \to \mathbb{R}$ constant. f is integrable.

Proof. Fix $\varepsilon_0 = \frac{\varepsilon}{b-a}$. Use that f is equicontinuous to get a $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon_0$. Let P be a partition. $f : [x_i, x_{i+1}] \to \mathbb{R}$.

$$\inf_{[x, x_{i+1}]} f > \sup_{[x, x_{i+1}]} f - \varepsilon_0.$$

This comes from the fact that the difference of the output of any two inputs that are δ way from each other is less than ε.

$$U(f, P) = \sum \sup_{[x_i, x_{i+1}]} f(x_{i+1} - x_i) < \sum \inf{[x_i, x_{i+1}]} f(x_{i+1} - x_i)$$

f is continuous $\implies$ f is uniformly continuous $\implies$ f is integrable. $\square$

Some more things: $\int_a^b (f+g)dx = \int_a^b f dx = \int_a^b g dx$. *If* $f \geq g$ *for all* x *then* $\int_a^b f \geq \int_a^b g$.

Theorem 108 (Fundamental Theorem of Calculus)

Suppose that f is continuous and $F(x) = \int_a^x f$. Then $F' = f$.

Proof. $\frac{F(x) - F(x_0)}{x - x_0} \to f(x_0)$ as $x \to x_0$.

$$F(x) = \int_a^x f = \int_a^{x_0} f + \int_{x_0}^x f.$$

$f : [a, x_0] \to \mathbb{R}$, $f : [x_0, x] \to \mathbb{R}$. Thus

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{\int_{x_0}^{x} f}{x - x_0}.$$

$\inf_{[x_0,x]} f \leq \frac{F(x)-F(x_0)}{x-x_0} \leq \sup_{[x_0,x]} f$. As $x \to x_0$, $\inf_{[x_0,x]} f \to f(x_0)$ and $\sup_{[x_0,x]} f \to f(x_0)$ so the desired claim follow. □

Theorem 109 (Second Part of the Fundamental Theorem of Calculus)
$F : [a, b] \to \mathbb{R}$. F is differentiable and $F' = f$ is integrable. $F(b) - F(a) = \int_a^b f$

Proof. Let $x_0 < x_1 < ... < x_{n-1}$ be a partition of $[a, b]$. Then $F(b) - F(a) = \sum_{i=0}^{n-1} F(x_{i+1}) - F(x_i)$. This proof isn't finished. □

14 November 10, 2020

We showed last time that continuous functions are integrable. This is also in Rudin. We also showed some things that facilitate calculating the integral. For example, $\sum_a^b (f + g)dx = \int_a^b f dx + \int_a^b g dx$. *Also,* $\int_a^b (cf)dx = c \int_a^b f dx$.

A crucial tool is the fundamental theorem of calculus. If $F : [a, b] \to \mathbb{R}$ *and* F *is and* $F' = f$ *is integrable then* $F(b) - F(a) = \int_a^b f(x)dx$.

The other version of the fundamental theorem of calculus says that if $f : [a, b] \to \mathbb{R}$ *is continuous and* $F(x)$ *is defined by* $\int_a^x f(y)dy = F(x)$ *then* $F'(x) = f(x)$.

Lemma 110
Another neat trick is $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

Proof. P_1 is a partition of $[a, c]$ and P_2 is a partition of $[c, b]$. Let $P = P_1 \cup P_2$ $L(f, P_1) + L(f, P_2) = L(f, P)$. Likewise, $U(f, P) = U(f, P_1) + U(f, P_2)$. If f is integrable on $[a, c]$ and on $[c, b]$ Find a partition P_1 such that $U(f, P_1) < L(f, P_1) + \varepsilon/2$ and a partition P_2 such that $U(f, P_2) < L(f, P_2) + \varepsilon/2$. Add them up and we get $U(f, P) - L(f, P) < \varepsilon$. □

14.1 Arc Length

Suppose we have a curve in the plane. A **curve** *is a map* $\psi : I \to \mathbb{R}^2$ *such that* $t \mapsto (f(t), g(t))$, f *and* g *are continuously differentiable. Some topology shows that since* $\pi_1(\psi)$ *is continuously differentiable and* $\pi_2(\psi)$ *is continuously differentiable then* ψ *is continuously differentiable.*

Definition 111 (Arc Length)
The arc length is defined to be

$$\int_a^b \sqrt{(f')^2 + (g')^2} dx$$

Definition 112 (Improper Integrals)
Let $f : [a, \infty) \to \mathbb{R}$ with f integrable. If $\lim_{x\to\infty} \int_a^x f(y)dy$ exists then we say that $\int_a^\infty f(x)dx$ is an improper integral.

14.2 The functions sine and cosine

Consider the map $s \mapsto (s, \sqrt{1-s^2})$. *Then* $f(s) = s$ *and* $g(s) = \sqrt{1-s^2}$, $g'(s) = \frac{-2s}{2\sqrt{1-s^2}} = -\frac{s}{\sqrt{1-s^2}}$. *Then*

$$\int_{\cos\theta}^{1} \frac{1}{\sqrt{1-s^2}} ds = \theta$$

and

$$\int_0^{\sin\theta} \frac{1}{\sqrt{1-s^2}} ds = \theta.$$

We got these equations by calculating the arc length of a circle $\sqrt{1-x^2} = y$ *between two points.*

15 November 12, 2020

Now, we define the arcsin function which is the inverse function of sin.

$$\arcsin(y) = \int_0^y \frac{1}{\sqrt{1-s^2}} ds$$

$$\arcsin(y) = \frac{\pi}{2} - \int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{1-s^2}} ds$$

Use this formula to investigate what happens when y *goes to* ± 1. *Also use it to show that* arcsin *is continuous on* $[-1, 1]$.

Suppose $f(x) = \sum_{n=0}^\infty a_n x^n$ *is differentiable. What is the derivative of* $f(x)$*? We would think that it's* $f'(x) = \sum_{n=1}^\infty a_n n x^{n-1}$. *How do we prove this?*

Definition 113
Let f_n be a sequence of function. $f_n \to f$ pointwise if for all x_0 in $f_n(x_0) \to f(x_0)$ for all x_0. "Convergence may be at different for each x_0.

Definition 114
A sequence f_n of functions converges uniformly to f if and only if

$$\max_{[a,b]} |f_n - f| \to 0$$

as $n \to \infty$. Equivalently, given $\varepsilon > 0$ there exists N such that $m \geq N \implies |f_n(x) - f(x)| < \varepsilon$

Lemma 115

$f_n : [a,b] \to \mathbb{R}$, $g : [a,b] \to \mathbb{R}$. Assume $f_n \to f$ pointwise and $f_n' \to g$ uniformly. Then if $f'n$ is continuous then g is continuous. Also, we claim that if f is differentiable then $f' = g$.

Proof.

$$\int_a^x f_n'(s)ds + f_n(a) = f_n(x)$$

$$\int_a^x g(s)ds + f(a) = G(x)$$

We claim that $f_n(x) \to G(x)$ uniformly. G is clearly differentiable with $G' = g$ and $f_n \to f$ pointwise. Thus $f = G$. We then want to show that f_n converes to f uniformly.

$$f_n(x) = \int_a^b f_n'(s)ds + f_n(x)$$

$$|f_n(x) - G(x)| = |\int_a^x (f'(s)g(s))ds + f_n(a) - f(a)| \leq \int_a^x |f'(s) - g(s)|ds + (f_n(a) - f(a))$$

$$\leq \int_a^b |f_n'(s) - g(s)|ds + |f_n(a) - f(a)| \leq (b-a)\max_{[a,b]} |f_n' - g|$$

The first term goes to zero since $f_n \to G$ uniformly and $|f_n(a) - g(a)|$ goes to zero as well for big enough N. □

Definition 116 (Weierstrass M-test)

Let $f_n : I \to \mathbb{R}$ be a sequence of functions with $|f_n| \leq M_n$ and $\sum M_n < \infty$. The norm of f_n is defined to be $\max_{x \in I} |f_n(x)|$. Then $\sum_{k=0}^n f_k \to \sum_{k=0}^\infty f_k$ uniformly.

Proof. Let $n > m$. Then $|f_n - f_m| = \sum_{k=m+1}^n f_k$ □

Remark 117. *Just look the Weierstrass M test in the textbook man I don't even know what this man is on about.*

Lemma 118

$\sum_{n=0}^\infty a_n x^n$ and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lambda$. If we let $R = \frac{1}{\lambda}$ We claim that $\sum_{k=0}^n a_k x^k \to \sum_{k=0}^\infty a_k x^k$ uniformly on $[-L, L]$ where $L < R$.

16 November 17, 2020

Uniform convergence is convenient for continuity and integrability. $f_n : [a,b] \to \mathbb{R}$ *is continuous and* $f_n \to f$ *uniformly then* f *is continuous and* $\sum_a^b f_n ds \to \int_a^b f ds$. *Let* $f_n : [a,b] \to \mathbb{R}$ *be continuous and* $f : [a,b] \to \mathbb{R}$ *and* $f_n \to f$ *pointwise. Remember that this means that given* $x_0 \in [a,b]$ *and* ε *there exists* N *such that* $n \geq N \implies |f_n(x_0) - f(x_0)| < \varepsilon$. *Uniform convergence means given* ε *there exists* N *such that* $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$ *for all* x.

We proved that if $f_n' \to g$ *uniformly and* f_n' *is continuous then* f' *is differentiable and* $f' = \lim_{n\to\infty} f_n'$.

Proposition 119

If we have a power series $\sum a_k x^k$ and $\lim_{k\to\infty} |\frac{a_{k+1}}{a_k}| = \lambda$ and $R = \frac{1}{\lambda}$. Then $\sum a_k x^k$ converges uniformly on the domain $[-L, L]$ for all $L < R$ and it converges pointwise on the domain $(-R, R)$.

Note that there exist functions that are infinitely many times differentiable but is not given by a power series.

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

Now let's go back to the contraction mapping theorem. Here is a reminder of the theorem. Let (X, d) *be a metric space and* $T : X \to X$ *a function on the space. It is a contraction map if there exists* $0 < L < 1$ *for all* $x_1, x_2 \in X$ *such that* $d(T(x_1), T(x_2)) < Ld(x_1, x_2)$*. The theorem says that* T *has a unique fixed point (a point* x *such that* $T(x) = x$*). We want to show the existence and uniqueness for a certain Ordinary Differential Equation. An ODE is of the form: Let* $y : [a, b] \to \mathbb{R}$ *be differentiable.*

$$\begin{cases} y'(x) = f(y(x)) \\ y(a) = c \end{cases}$$

The first condition of the ODE implies that $y(x) = \int_a^x f(y(t))dt + c$*.* $C([a, b])$ *is the metric space consisting of continuous functions with*

$$d(h_1, h_2) = \sup_{x\in[a,b]} |h_1(x) - h_2(x)|$$

Define $T(h) = c + \int_a^x f(h(s))ds$*. We want to say that* T *is a contracting map. Once we know this is true we can show there is a unique* h *such that* $h = T(h)$*.*

Proof that T *is a contraction map.*

$$d(T(h_1), T(h_2)) = \sup_{x\in[a,b]} |\int_a^x f(h(s))ds - \int_a^x f(h_2(s)ds|$$

$|f(z_1) - f(z_2)| = |f'(z)(z_1 - z_2)| \leq |z_1 - z_2|$ where $z_1 = h_1(s)$

Then

$$|f(h_1(s)) - f(h_2(s))| \leq L|h_1(s) - h_2(s)|$$

$$\sup_{x\in[a,b]} \int_a^b |f(h_1(s)) - f(h_2(s))|ds$$

We get that the difference between these two functions is bounded by

$$\int_a^b |f(h_1(s)) - f(h_2(s))|ds$$

We have a bound for the integrand though so this is less than or equal to $L\int_a^b |h_1(s) - h_2(s)|ds \leq L(b-a)d(h_1, h_2)$ □

17 November 19, 2020

Last time we showed the existence and uniqueness of ODEs. The key point in the proof was to make the contracting maps theorem and FTC and MVT.

Theorem 120 (Implicit Function Theorem)
Let $F : O \subseteq \mathbb{R}^2 \to \mathbb{R}$ where F_x and F_y are continuous and $F_y(0,0) \neq 0$. Then in a neighborhood of $(0,0)$ y can be as a function $g(x)$ of x and $g'(X) = -\frac{F_x}{F_y}$. The derivatives of F, F_x and F_y, are continuous.

Proof. WLOG, let $F(0,0) = 0$. After looking at a sufficiently small neighborhood of x, $\{(x,y) : F(x,y) = 0\} = \{(x, g(x)\}$.

It's enough to show that if $x_n \to x$ then $g(x_n) \to g(x)$ and $x_n \to x$ nd $g(x_n) \to y$ so $y = g(x)$. Enough to show $F(x_n, g(x_n)) = 0$ and $(x_n, g(x_n)) \to (x,y)$ and $F(x,y) = 0$ so g is continuous. We want to show that g is differentiable. $F(x+h, g(x+h)) - F(x, g(x)) = G(1) - G(0)$. $G(s) = F(x+sh, g(x) + s(g(x+h) - g(x)))$. This is just a function of one variable. $G' = F_x h + F_y(g(x+h) - g(x))$. $0 = F_x h + F_u(g(x+h) - g(x))$
$\frac{g(x+h)-g(x)}{h} = -\frac{F_x(x,g(x))}{F_y(x,g(x))}$ □

18 December 1, 2020

To take a **partial derivative** *with respect to x_1 of a function $f : x_1, x_2, ..., x_n \mapsto f(x_1, x_2, ..., x_n)$, fix $x_2, x_3, ..., x_n$ then take the derivative with respect to x_1.*

Definition 121 (Directional Derivatives)
$h(s) = f(x_0 + su, y_0 + sv)$

Let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ with second derivatives all existing.

19 December 3, 2020

Today is kind of boring.

Theorem 122
If f as above has $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ continuous. Then the order in which we differentiate doesn't matter.

20 Analysis Final Review

Just some stuff to review.

Theorem 123 (Taylor's Theorem)

Suppose we have a function f whose nth derivative exists on $[a, b]$. Suppose we want to get an estimate of f at $x \in [a, b]$ and we know $f^{(k)}(\alpha)$ for all $k \leq n$. Then, there exists $c \in (\alpha, x)$ such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(x-\alpha)^k + \frac{f^{(n)}(c)}{n!}(x-\alpha)^n$$

Comparison	*Suppose $0 \leq a_n \leq b_n$ for all n. If b_n converges then a_n converges.*
Ratio	*If $\limsup_{n\to\infty} \lvert\frac{a_{n+1}}{a_n}\rvert < 1$ then the sum converges.*
Root	*If $\limsup_{n\to\infty} \sqrt[n]{\lvert a_n\rvert} < 1$ then the sum converges.*
Alternating Series Test	*If $a_{n+1}a_n < 0, \lvert a_{n+1}\rvert \leq \lvert a_n\rvert$, and $\lim_{n\to\infty} a_n = 0$ then the sum converges.*

Theorem 124 (Implicit Function Theorem)

Let $F : O \subseteq \mathbb{R}^2 \to \mathbb{R}$ where F_x and F_y are continuous and $F_y(0,0) \neq 0$. Then in a neighborhood U of $(0,0)$, y can be found as a function $g(x)$ of x. What this means is that there exists a function $g : U \to \mathbb{R}$ that satisfies $f(x, g(x)) = 0$. Also, $g'(X) = -\frac{F_x}{F_y}$.

Note that F_x means the partial derivative of F with respect to x and F_y means the partial derivative of F with respect to y. There are also some other bizarre tests for convergence such as

Theorem 125 (Cauchy Condensation Test)

The series $\sum_{n=0}^{\infty} f(n)$ converges if and only if $\sum_{n=0}^{\infty} 2^n f(2^n)$ converges.

Theorem 126

Given $\sum b_n$ and $\sum a_n$. Suppose $\lim_{n\to\infty} \frac{a_n}{b_n} > 0$ then $\sum a_n$ converges if and only if $\sum b_n$ converges.

Proof. Suppose b_n diverges. Let $\lim_{n\to\infty} \frac{a_n}{b_n} = \lambda$. There exists N such that $n \geq N \implies \lvert\frac{a_n}{b_n} - \lambda\rvert < \lambda/2$ or equivalently $\lambda/2 < \frac{a_n}{b_n} < 3\lambda/2$. Thus $b_n\lambda/2 < a_n$. Finally,

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} a_n > \sum_{n=0}^{N-1} a_n + \sum_{n=N}^{\infty} b_n\lambda/2$$

which diverges. The case where b_n converges is again similar. $\square$

Theorem 127

Suppose we have a sequence of functions $f_n : I \to \mathbb{R}$ that satisfy some properties (differentiability, integrability, continuity, etc.) If f_n converges uniformly then

$$\lim_{n\to\infty} \frac{d}{dx} f_n(x) = \frac{d}{dx} \lim_{n\to\infty} f_n(x)$$

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx$$

and $\lim_{n\to\infty} f_n(x)$ is continuous on I.

Some Analysis definitions.

Definition 128

The limit of a_n: a_n converges to L if and only if given $\varepsilon > 0$ there exists N such that $n \geq N \implies |a_n - L| < \varepsilon$

Definition 129

Continuity of f at x_0: f is continuous at x_0 if and only if given $\varepsilon > 0$ there exists δ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. Limits respect continuity. $\lim_{x\to x_0} f(x) = f(x_0)$.

Definition 130

Uniform convergence on interval I The sequence f_n converges to f uniformly if and only given ε there exists N such that $n \geq N \implies |f_n(x) - f(x)| < \varepsilon$ for all $x \in I$.

Some Topology Definitions

Definition 131

A set A is open if and only if for all $x \in A$ there exists $r \in \mathbb{R}$ such that $B_r(x) \subseteq A$. Infinite union and finite intersection.

Definition 132

A set A is closed if and only if $X - A$ is open. Equivalently, A is closed if and only if for all sequences $x_n : \mathbb{N} \to A$, $\lim_{n\to\infty} x_n \in A$. Infinite intersection and finite union.

Definition 133

A set X is compact if and only if for all open covers $X \subseteq \cup_{n=0}^{\infty} A_n$ there exists $\{A_{n_i} : 0 \leq i \leq N\}$ such that $X \subseteq \cup_{i=0}^{N} A_{n_i}$

Some reflections from the homework.

Fact 134

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a good counterexample for some things.

Problem 135

Let $C(I)$ be the set of continuous functions on a closed interval I. Suppose we want to show that $C(I)$ is Cauchy complete with metric

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|$$

Proof. First we show that if f_n converges then it's Cauchy.
Given ε there exists N such that $n, m \geq N \implies d(f_n, f_m) < \varepsilon$. There exists N such that $n, m \geq N \implies d(f_n, f) < \varepsilon/2$ and $d(f_m, f) < \varepsilon/2$. Then $d(f_n, f_m) \leq d(f_n, f) + d(f_m, f) < \varepsilon$. Conversely, suppose for all ε there exists N such that $n, m \geq N \implies |f_n(x) - f_m(x)| < \varepsilon/2$ for all $x \in I$. Then since $f_m(x)$ converges to some function $g(x)$ as $n \to \infty$ since $\mathbb{R}$ is Cauchy complete there exists M such that $m \geq M \implies |g(x) - f_m(x)| < \varepsilon/2$. Thus $|f_n(x) - g(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - g(x)| < \varepsilon$ for all $x \in I$. $\square$

Theorem 136

Suppose we have a series $\sum a_n$ with $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lambda$ and $R = \frac{1}{\lambda}$. Then $\sum a_n$ converges uniformly on $[-L, L]$ where $L < R$ and converges piecewise on $[-R, R]$.

Theorem 137 (The Main Value theorem)

Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Problem 138

Suppose $f(x)$ is differentiable on $(-\infty, 0) \cup (0, \infty)$ and $\lim_{x\to 0} f'(x)$ exists.

Proof. Given ε there exists δ such that $|x| < \delta \implies |f'(x) - L| < \varepsilon$. There exists $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$. Since $|c| < \delta$, $|f'(c) - L| < \varepsilon$ so $|\frac{f(x)-f(0)}{x-0} - L| < \varepsilon$ $\square$

Some stuff with ODEs:

Definition 139

An ODE is of the form

$$\begin{cases} y'(x) = f(y(x)) \\ y(a) = c \end{cases}$$

We can show that every ODE has a solution. Let's first make sure some conditions are met. f must be differentiable and its differentiable must be bounded by L.

Proof. These conditions, coupled with the fundamental theorem of calculus, says that

$$y(x) = \int_a^x f(y(x))dx + c$$

Suppose we're working over the space $I = [a, b]$ with metric

$$d(f, g) = \sup_{x \in I} |f(x) - g(x)|.$$

Define the functional $T : C(I) \to C(I)$ by $T(h(x)) = \int_a^x f(h(s))ds + c$. We now have to show that T is a contraction mapping. This is equivalent to saying that $d(T(\alpha(x)), T(\beta(x))) \le \lambda d(\alpha(x), \beta(x))$ for $\lambda < 1$. A useful assumption to have will be that $L(b - a) < 1$. First, observe that

$$|f(\alpha(x)) - f(\beta(x))| = |f'(c)(\alpha(x) - \beta(x))| \le L|\alpha(x) - \beta(x)|$$

for some $c \in (\alpha(x), \beta(x))$.

$$\begin{aligned}
d(T(\alpha), T(\beta)) &= \sup_{x \in I} |\int_a^x (f(\alpha(s)) - f(\beta(s)))ds| \\
&\le \sup_{x \in I} \int_a^x |f(\alpha(s)) - f(\beta(s))|ds \\
&\le L \sup_{x \in I} \int_a^x |\alpha(s) - \beta(s)|ds \\
&= L \int_a^b |\alpha(s) - \beta(s)|ds \\
&\le L \sup_{s \in I} (b - a)|\alpha(s) - \beta(s)| \\
&= L(b - a)d(\alpha(x), \beta(x))
\end{aligned}$$

□

If instead we were working over $[A, B]$ and $L(B - A) \ge 1$. Then we can cover $[A, B]$ with closed intervals of length $b - a$. We can guarantee that there is a unique solution for the ODE on each of the subintervals. Thus furnishes a function that satisfies the ODE on the entire interval $[A, B]$.

For one of the homework problems, there was the issue that the derivative of f was not bounded above by $L < 1$. We instead showed the uniqueness of a solution and not its existence (though it wouldn't be hard to show). We showed that if $y_1(x_0) = y_2(x_0)$ then $y_1(x) = y_2(x)$ for all $x \in [x_0, x_0 + b]$.

Problem 140

Let $y : [0, 1] \to \mathbb{R}$. Show that the solution to the following ODE is unique.

$$\begin{cases} y'(x) = y^2(X) \\ y(0) = a \end{cases}$$

Proof. Suppose $y_1(x_0) = y_2(x_0)$. Let $M = \sup_{x \in I} |y_1(x) + y_2(x)|$. Then for all $x \in [x_0, x_0 + b]$.

$$\begin{aligned} |y_1(x) - y_2(x)| &= | \int_a^x y_1^2(x) - y_2^2(x) dx| \\ &\leq \int_{x_0}^x |y_1^2(x) - y_2^2(x)| dx \\ &\leq \int_{x_0}^x |y_1(x) + y_2(x)||y_1(x) - y_2(x)| dx \\ &\leq M \int_{x_0}^x |y_1(x) - y_2(x)| dx \\ &\leq Mb \sup_{x \in [x_0, x_0 + b]} |y_1(x) - y_2(x)| \end{aligned}$$

We can set b arbitrarily so let it be $\frac{1}{2M}$. But then this implies that $y_1(x) = y_2(x)$ for all $x \in [x_0, x_0 + b]$. We know from the ODE condition that $y_1(0) = y_2(0)$. But then $y_1(x) = y_2(x)$ for all $x \in [0, b]$. Do this trick again setting $x_0 = b$. We get that $y_1(x) = y_2(x)$ for all $x \in [0, 2b]$. We can do this inductively to get that $y_1(x) = y_2(x)$ for all $x \in [0, 1]$. □

Problem 141

Suppose given ε there exists N such that $n, m \geq N \implies |f_n(x) - f_m(x)| < \varepsilon/2$ for all $x \in I$ and given x and ε there exists M such that $m \geq M \implies |f_m(x) - f(x)| < \varepsilon/2$. Then I claim that for $n \geq N |f_n(x) - f(x)| < \varepsilon$.

Solution to Problem 141. Note that N doesn't depend on x but M does. Then $|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \varepsilon$ for all $n \geq N$ and x. □

Problem 142

Let F be a subset of $\mathbb{R}$ and $E = \{-|x| : x \in F\}$. Express $\inf E$ in terms of $\inf F$.

Solution to Problem 142. I claim that this is $\inf E = \min\{\inf F, -\sup F\}$. Let's say F is both bounded and is nonempty. We know that for $x \in F$, $\inf F \leq x \leq \sup F$. If $x \geq 0, -|x| \geq -\sup F$ and if $x < 0, -|x| \geq \inf F$. Thus $-|x|$ is always bounded below by either $\inf F$ or $-\sup F$. Hence $\inf E \geq \min\{\inf F, -\sup F\}$. Next, $\inf E \leq -|x| \leq x \leq |x| \leq -\inf E$. This shows that $\inf E \leq \inf F$ and $\inf E \leq -\sup F$ so $\inf E \leq \min\{\inf F, -\sup F\}$. Thus $\inf E = \min\{\inf F, -\sup F\}$. □

Problem 143

We know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^k}{k!}.$$

Let's play around with it.

Playing around with e. The **Weierstrass M-Test** says that if there exists M_n such that $M_n \geq |\frac{x^k}{k!}|$ for all $x \in [-L, L]$ and $\sum_{n=0}^{\infty} M_n$ converges then $\sum_{n=0}^{\infty} \frac{x^k}{k!}$ converges uniformly on $[-L, L]$. Let $x \in [-L, L]$. Look at $|\frac{x^k}{k!}| \leq \frac{L^k}{k!}$. We can show using, say, the ratio test, that $\sum_{n=0}^{\infty} \frac{L^k}{k!}$ converges. Then $\sum_{n=0}^{\infty} \frac{x^k}{k!}$ converges uniformly on $[-L, L]$. The weierstrass M-test is actually pretty easy to show.
Now let's use the **Taylor Series Expansion** of e^x. Suppose we know that $\frac{d}{dx} e^x|_{x=0} = 1$. Taylor's theorem says that

$$e^x = \sum_{n=0}^{N-1} \frac{x^k}{k!} + \frac{e^{c_N}}{N!} x^N$$

for some $c_N \in (0, x)$. By the extremal value theorem, f has a maximal value M on $[0, x]$. Now, for a fixed x, the limit as N approaches infinity of $\frac{Mx^N}{N!}$ is 0 so we can be confident that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. □

Problem 144

Let f_n be a sequence of Lipschitz functions each with the same Lipschitz constant L. Show that if $f_n \to f$ pointwise on $[a, b]$ then it converges uniformly.

Proof. It's easy to show that f is Lipschitz with Lipschitz constant L so let's just assume it. Given $\varepsilon > 0$. Create a partition $x_0, x_1, ..., x_m$ of $[a, b]$ such that $x_{i+1} - x_i < \frac{\varepsilon}{3L}$ for all i. For each of these x_i, there exists N_i such that $n \geq N_i \implies |f_n(x_i) - f(x_i)| < \varepsilon/3$. Let $N = \max_i N_i$. Then,

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_n(x_i) + f_n(x_i) - f(x_i) + f(x_i) - f(x)| \\ &\leq |f_n(x_i) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &< L\frac{\varepsilon}{3L} + \frac{\varepsilon}{3} + L\frac{\varepsilon}{3L} \end{aligned}$$

□

Now for the stuff that wasn't in psets/I wasn't paying attention to when it was taught.

Definition 145 (The Directional Derivative)

Let $\vec{u} =< a, b >$ be a unit vector. Then the directional derivative

$$D_u f(x, y) = \lim_{h \to 0} \frac{f(x + ah, y + bh) - f(x, y)}{h}$$

Theorem 146

Let f be continuously differentiable at a point (x_0, y_0). Then

$$\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial x \partial y}$$

Problem 147

Let (X, d) be a compact metric space and A_j a sequence of closed sets. Prove that if

$$X \cap (\bigcap_{n=0}^{\infty} A_j) = \emptyset$$

then

$$X \cap (\bigcap_{n=0}^{N} A_j) = \emptyset$$

for some finite N

Proof. Let $B_j = X - A_j$. B_j are all open sets. Then if $X \cap (\bigcap_{n=0}^{\infty} A_j) = \emptyset$ then

$$\bigcup_{n=0}^{\infty} B_j = X$$

Since X is compact, there are finitely many B_j that cover X. This proves the claim. $\square$

Problem 148

Consider the metric space $(C([0,1]), d)$ where $C([0,1])$ denotes the set of continuous functions on $[0,1]$ and

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

Is the set $S = \{f(x) \in C([0,1]) : f(x) \geq 0 \forall x \in [0,1]\}$ open, closed, or neither?

Proof. S is open if and only if for all $f \in S$, there exists r such that $f \in B_r(f) \subseteq S$. Remember that

$$B_r(f) = \{g \in C([0,1]) : \sup_{x \in [0,1]} |f(x) - g(x)| < r\}$$

Now, it's clear that S can't be open because $f(x) = 0 \in S$ but for all r, $g(x) = -\frac{r}{2} \notin S$ but $|f(x) - g(x)| = \frac{r}{2} < r$. To show it's closed, let's see if the complement T is open.

$$T = \{f \in C([0,1]) : \exists x \in [0,1] s.t. f(x) < 0\}$$

Let $f \in T$ and $r = -\inf_{x \in [0,1]} f(x)$ and $f(x_0) = -r$. Then $r > 0$ and for all $g \in B_{\frac{r}{2}}(f)$, $|g(x_0) - f(x_0)| \leq \sup_{x \in [0,1] | f(x) - g(x)|} < \frac{r}{2}$ so $g(x_0) < -\frac{r}{2} < 0$. Thus S is closed and T is open. $\square$

Definition 149

Given f, and a partition P with $x_0, x_1, \ldots, x_m$, define

$$m_i = \inf_{x \in [x_{i+1}, x_i]} f(x) \quad M_i = \sup_{x \in [x_{i+1}, x_i]} f(x)$$

and

$$L_p = \sum m_i(x_{i+1} - x_i) \quad U_p = \sum M_i(x_{i+1} - x_i)$$

We say f is integrable if and only if

$$\sup_P L_p = \inf_P U_p$$

Observe that integrability is defined on an interval like $[a, b]$. This is unlike continuity or differentiability that are defined at a single point x_0.

Theorem 150 (Calculus)

Here are some theorems related to calculus. Let $f : [a, b] \to \mathbb{R}$ and $F(x) = \int_a^x f(s)ds$.

- Suppose f is continuous. Then f is integrable.
- Suppose f is integrable on $[a, b]$. Then F is continuous on $[a, b]$.
- Suppose f is continuous at x_0. Then F is differentiable at x_0.
- Suppose $F' = f$. Then $\int_a^b f(s)ds = F(b) - F(a)$. We have a verify that F is differentiable and that f is integrable. We get both of these things for free if we just show that f is continuous.

Theorem 151 (L'Hopital's Rule)

Suppose f and g are differentiable on a deleted neighborhood N of a and satisfy the following properties:

- $\lim_{x \to a} f(x) = 0$
- $\lim_{x \to a} f(x) = 0$
- $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists
- For every $x \in N$, $g'(x) \neq 0$.

Proof. We didn't require that f and g be defined at a so let $f(a) = g(a) = 0$ to get a continuous function. (By the generalized mean inequality, there exists $c_x \in (a, x)$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c_x)}{g'(c_x)} \implies \frac{f(x)}{g(x)} = \frac{f'(c_x)}{g'(c_x)}$$

We are justified in expressing this as a fraction because $g(x) \neq 0$ for $x \in N \cap \{x : x > a\}$. If for some $x > a$, we had $g(x) = 0$, Rolle's theorem would furnish a point z such that $g'(z) = 0$, contradicting the fourth constraint. A similar argument shows that $g(x) \neq 0$ for all $x \in N$. Now,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(c_x)}{g'(c_x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$\square$

The case for $\lim_{x \to \infty}$ *is easy if we set* $f(x) = F(-\frac{1}{x})$ *and* $g(x) = G(-\frac{1}{})$.

Theorem 152 (Cauchy's Generalized Mean Theorem)

Suppose f and g are continuous on $[a, b]$ and are differentiable on (a, b). Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Problem 153

Show that $f(x) = \frac{1}{x}$ is continuous on $\mathbb{R} - \{0\}$.

Proof. By the definition of continuity, $f(x)$ is continuous at x_0 if and only if given $\varepsilon > 0$ there exists δ such that $|\frac{1}{x} - \frac{1}{x_0}| < \varepsilon$. Let

$$\delta = \min\{\varepsilon, \frac{1}{2}|x_0|\}$$

Then $|x_0| - |x| \leq |x - x_0| < \frac{1}{2}|x_0| \implies |x| > \frac{1}{2}|x_0|$

$$\begin{aligned} |\frac{1}{x} - \frac{1}{x_0}| &= |\frac{x_0 - x}{xx_0}| \\ &< \frac{\varepsilon}{|x||x_0|} \\ &< \frac{\varepsilon}{\frac{1}{2}|x_0|} \end{aligned}$$

$\square$