# On the relationship between the parameters of the distributions of fiber diameters, breaking loads, and fiber strengths

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The development of continuous fiber-reinforced ceramic matrix composites (CFCCs) has been made possible, in part, by the availability of stiff and strong small-diameter ceramic fibers. The principal role of the reinforcing fibers in these materials is to bridge the wake of advancing cracks in the matrix, and ultimately, the tensile strength of CFCCs is controlled by the distribution of tensile strengths of the fibers. In contrast to the design considerations with monolithic structural ceramics where a narrow distribution of strengths is desired, for CFCCs it is desired to use fibers with a wide distribution of strengths since the large dispersion is one of the reasons why CFCCs exhibit a "graceful" mode of failure.

Knowledge of the parameters of the distribution of fiber strengths is of prime importance to predict the mechanical behavior and performance of CFCCs. For example, Curtin has derived expressions that relate the parameters of the distribution of fiber strengths to the ultimate tensile strength of CFCCs [1] while Lara-Curzio has obtained expressions that relate the parameters of the distribution to the life of CFCCs when these are subjected to stress-rupture conditions [2]. The parameters of the distribution of fiber strengths are estimated by determining the strength of several fibers that belong to a subset that is representative of the entire fiber population, while the strength of each fiber (i.e., the ratio of the breaking load and the cross-sectional area of the fiber at the fracture plane) is determined by subjecting it to monotonic tensile loading until failure. For the last twenty-five years, ASTM D3379 "Standard Test Method for Tensile Strength and Young's Modulus for High-Modulus Single-Filament Materials" [3] has been the accepted standard for the determination of fiber strength[1]. According to this document, the tensile strength of a fiber is determined by the ratio of the breaking load and the average of the cross-sectional areas of a sample of fibers that is assumed to be representative of the fiber population. The decision for using the average of the fiber cross-sectional areas for computing each fiber strength is defended by arguing that the measurement of fiber dimensions is a tedious and time-consuming process and that often the fracture surfaces of the fiber cannot be recovered after failure[2] [4]. However, as it is demonstrated by the results presented in this paper, the use of the average of the fiber cross-sectional areas for the calculation of each fiber strength can lead to serious errors in the estimation of the parameters of the distribution of fiber strengths, particularly when the dispersion in fiber diameters is large.

The purpose of this paper is to show, using a Monte Carlo simulation, that it is necessary to measure/determine *each* fiber diameter, and in turn, the cross-sectional area of the fiber when estimating the parameters of the distribution of fiber strengths.

The analysis conducted in this paper is summarized schematically by the diagram in Fig. 1 and it is described in detail as follows. Let us consider a collection of $N$ fibers with circular cross-sectional area and similar length. Let us assume that the diameters of these fibers, $\phi$, are distributed according to a normal distribution (Equation 1) with known mean fiber diameter $\Phi_\mu$ and standard deviation $\Phi_{sd}$ (Fig. 2)

$$f(\phi) = \frac{1}{\Phi_{sd}\sqrt{2\pi}} \exp\left(\frac{-(\phi - \Phi_\mu)^2}{2\Phi_{sd}^2}\right) \tag{1}$$

The cross-sectional areas of the $N$ fibers are obtained as follows. First, $N$ random numbers[3] are generated between 0 and 1 and then $N$ fiber diameters are obtained using those numbers and Equation 1[4]. Then, for each fiber diameter, the corresponding cross-sectional area is calculated using Equation 2. Using arbitrary, but representative, values of $\Phi_\mu = 12\ \mu$m and $\Phi_{sd} = 1\ \mu$m, for the mean and standard deviation of the fiber diameters, respectively, Table I lists the values of the random numbers, fiber diameters and cross-sectional areas.

$$A = \pi \frac{\phi^2}{4} \tag{2}$$

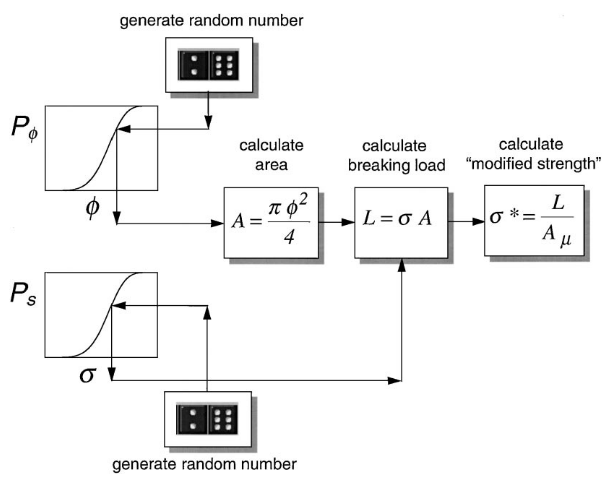


*Figure 1* Schematic of process used in Monte Carlo simulation.

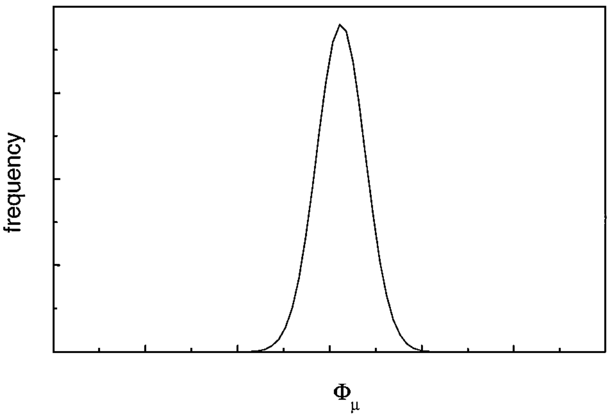


*Figure 2* Normal distribution of fiber diameters.

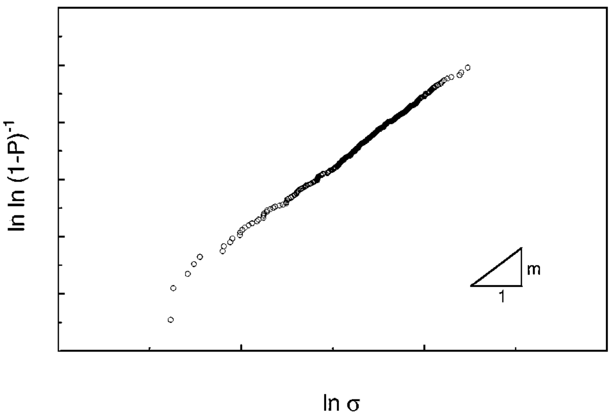


*Figure 3* Distribution of fiber strengths.

Now, let us determine the tensile strength of those $N$ fibers. If the strengths are distributed according to a two-parameter Weibull distribution with known parameters $m$ and $\sigma_{\text{o}}$ (Fig. 3) then the probability that a fiber will fail when subjected to an applied tensile stress, $\sigma$ is going to be:

$$P_{\text{f}} = 1 - \exp\left(-\frac{l}{l_{\text{o}}}\left(-\frac{\sigma}{\sigma_{\text{o}}}\right)^{m}\right) \tag{3}$$

where $l$ is the fiber gauge length and $l_{\text{o}}$ is the characteristic length associated with the characteristic strength $\sigma_{\text{o}}$. By assuming that the gauge length of all the fibers in the population is $l_{\text{o}}^{5}$ then Equation 3 can be re-arranged as follows:

$$\sigma = \sigma_{\text{o}}\left(\ln\frac{1}{1-P_{\text{f}}}\right)^{\frac{1}{m}} \tag{4}$$

Using arbitrary values of $m = 5$ and $\sigma_{\text{o}} = 3.0$ GPa, the tensile strengths of the $N$ fibers were obtained first by generating $N$ random numbers between 0 and 1 and then by using Equation 4 for each one of those numbers. Table II lists the generated random numbers and tensile strengths.

Given the distributions of strengths and fiber cross-sectional areas, the associated distribution of breaking loads for those fibers is determined considering that

$$L_i = A_i\sigma_i \tag{5}$$

where $L_i$ is the breaking load for the $i$-th fiber with strength $\sigma_i$ and cross-sectional area $A_i$. Pairing the data in Tables I and II, the resulting distribution of failure loads is listed in Table III.

TABLE I Distribution of fiber diameters and cross-sectional areas

| $i$ | $P$ | $\phi$ ($\mu$m) | $A$ ($\mu$m)$^2$ |
|---|---|---|---|
| 1 | 0.6741 | 14.26 | 159.62 |
| 2 | 0.0763 | 4.85 | 18.46 |
| 3 | 0.6901 | 14.48 | 164.70 |
| . | 0.1070 | 5.79 | 26.29 |
| . | 0.4083 | 10.84 | 92.30 |
| . | 0.2973 | 9.34 | 68.51 |
| $N$ | 0.9826 | 22.55 | 399.48 |

TABLE II Distribution of fiber strengths

| $i$ | $P$ | $\sigma$ (GPa) |
|---|---|---|
| 1 | 0.5037 | 2.794 |
| 2 | 0.5025 | 2.792 |
| 3 | 0.6647 | 3.054 |
| . | 0.7993 | 3.298 |
| . | 0.4692 | 2.738 |
| . | 0.1136 | 1.965 |
| $N$ | 0.2329 | 2.300 |

Now, if we were to follow ASTM 3379, we would calculate each fiber strength as the ratio of the breaking load and the average of the fiber cross-sectional area, i.e.,

$$\sigma_i^* = \frac{L_i}{A_\mu} \tag{6}$$

where

$$A_\mu = \frac{\pi}{4}\Phi_\mu^2 \tag{7}$$

To differentiate between the actual tensile strength of a fiber and the strength that would be obtained according to Equation 6 hereafter we will refer to the latter, $\sigma^*$, as "modified fiber strength". By applying Equation 6 to the distribution of breaking loads (Table III) the distribution of "modified fiber strengths" is obtained (Table IV). If we assume that the "modified fiber strengths" are Weibull-distributed, then the parameters of the distribution can be estimated as follows. First, the "modified fiber strengths" $\sigma_i^*$ are ranked in ascending order and then probabilities of failure are assigned for each "modified fiber strength" according to the

TABLE III Distribution of breaking loads

| $i$ | $\sigma$ (GPa) | $A$ ($\mu$m)$^2$ | $L$ (N) |
|---|---|---|---|
| 1 | 2.794 | 159.6211 | 0.446 |
| 2 | 2.792 | 18.4623 | 0.052 |
| 3 | 3.054 | 164.7029 | 0.503 |
| . | 3.298 | 26.2914 | 0.087 |
| . | 2.738 | 92.2950 | 0.253 |
| . | 1.965 | 68.5103 | 0.135 |
| $N$ | 2.300 | 399.4849 | 0.919 |

TABLE IV Distribution of "modified strengths"

| $i$ | $L$ (N) | $\sigma^*$ (GPa) |
|---|---|---|
| 1 | 0.446 | 3.943 |
| 2 | 0.052 | 0.456 |
| 3 | 0.503 | 4.447 |
| . | 0.087 | 0.767 |
| . | 0.253 | 2.234 |
| . | 0.135 | 1.190 |
| $N$ | 0.919 | 8.126 |

TABLE V Ranked "modified strengths" and associated probability of failure

| $i$ | $\sigma^*$ | $P$ |
|---|---|---|
| 1 | 0.456 | 0.001 |
| 2 | 0.767 | 0.003 |
| 3 | 1.190 | 0.005 |
| . | 2.234 | 0.007 |
| . | 3.943 | 0.009 |
| . | 4.447 | 0.011 |
| $N$ | 8.126 | 0.999 |

TABLE VI Control set

| | |
|---|---|
| $N$ | 500 |
| $\Phi_\mu$ | 15 $\mu$m |
| $m$ | 5 |
| $\sigma_o$ | 3.0 GPa |

following estimator [5]

$$P_i^* = \frac{i - 0.5}{N_f} \tag{8}$$

The results of this operation are listed in Table V. The parameters $m^*$ and $\sigma_o^*$ of the distribution of "modified fiber strengths" can be estimated by a least-squares linear regression of the data in a plot of ln ln $(1/1 - P^*)$ versus ln $\sigma^*$.

The effect of the dispersion in the distribution of fiber diameters on the parameters of the distribution of "modified fiber strengths" was investigated. The numerical values in Table VI were used as control set, and simulations were conducted for values of the standard deviations of fiber diameters of 1, 2, 3, 4 and 5 $\mu$m. Fig. 4 shows the distribution of fiber diameters for the various values of the standard deviation.

In each case, the procedure outlined in the previous section was repeated ten times and the estimated parameters of the distribution of "modified fiber strengths" were computed for each case.

Fig. 5 is a plot of ln ln $(1/(1 - P^*))$ versus ln $\sigma^*$ for the various values of $\Phi_{sd}$, along with the original distribution of fiber strengths. Note that as the standard deviation in the distribution of fiber diameters increases, the slope of the curves decreases, i.e., the spread of the distribution of "modified fiber strengths" becomes larger than that of the original strength distribution. Fig. 6 is a plot of the ratio of the parameters of the distribution of "modified fiber strengths" and the parameters of the original strength distribution, as a function of the standard deviation of the distribution of fiber diameters. The symbols in Fig. 6 represent the average of ten simulations while the error bars represent the standard deviation of those ten trials. As it was indicated, the Weibull modulus of the distribution of "modified fiber strengths" was found to decrease with $\Phi_{sd}$ while the characteristic "modified strength" increased with $\Phi_{sd}$. Note that for the numerical values used in this exercise, which can be considered to be representative of most ceramic fibers, the Weibull modulus can be underestimated by as much as a factor of four, although the characteristic strength is overestimated by only 25%.

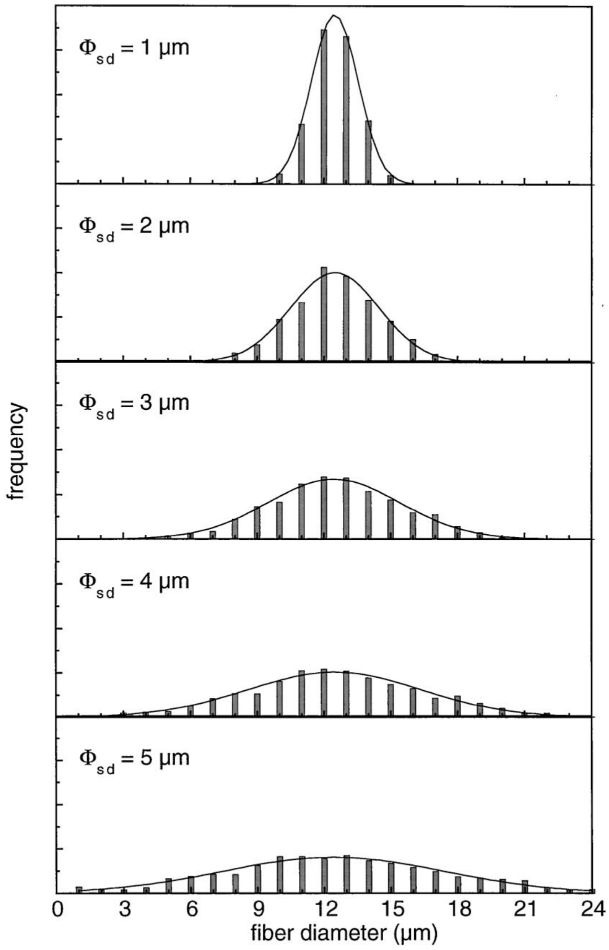


*Figure 4* Normal distribution of fiber diameters for various values of the standard deviation of fiber diameters.

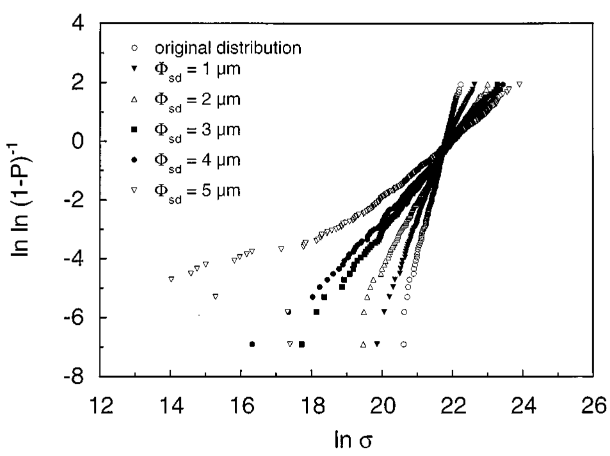


*Figure 5* Distribution of "modified fiber strengths" for various values of the standard deviation of fiber diameters.

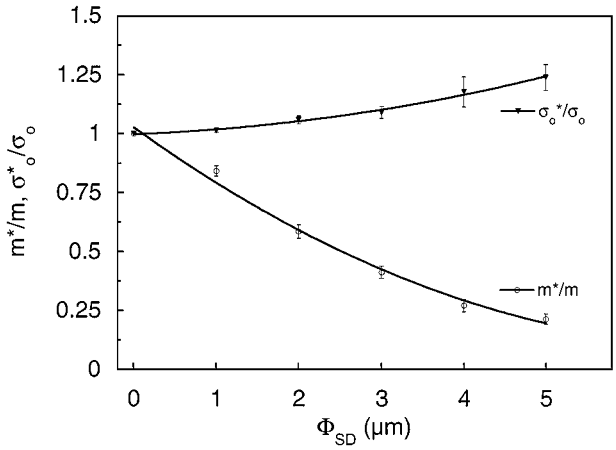


*Figure 6* Ratio of the parameters of the distribution of "modified fiber strengths" to those of the original distribution of fiber strengths.

The results of this exercise can be rationalized as follows. When the breaking loads are divided by the average of the fiber cross-sectional areas the intrinsic relationship between breaking loads and fiber size is lost. In general, large-diameter fibers are more likely to fail with large rather than with small loads. However, when a large load is divided by a cross-sectional area that is smaller than its corresponding cross-sectional area, it results in a "modified fiber strength" that is much larger than the actual strength. Similarly when a small load is divided by a cross-sectional area that is larger than its corresponding cross-sectional area, it results in a "modified fiber strength" that is much smaller than the actual strength. The overall effect of these operations is to produce a new distribution ("modified fiber strengths") that is much wider than the original strength distribution. Furthermore, this effect is magnified when the spread in the distribution of fiber diameters increases, since for the same fiber strength distribution it results in a wider distribution of breaking loads and hence in a wider distribution of "modified fiber strengths".

Note that by considering fibers of similar length but different diameter, the volume of each fiber is different and therefore should be accounted for in the determination of the strength. However calculations have shown that accounting for this correction has a small effect and does not alter the main findings of this exercise [6].

A Monte Carlo simulation was conducted to determine the error in the determination of Weibull parameters when fiber strengths are calculated as the ratio of the experimentally-determined breaking loads and the average of the fiber cross-sectional areas ("modified fiber strengths"). It was found that the parameters of the distribution of "modified fiber strengths" diverge from the actual parameters with increasing dispersion in the distribution of fiber diameters: specifically the Weibull modulus decreases while the "characteristic strength" increases with increasing dispersion in fiber diameters. For the case of typical ceramic fibers it is shown that the Weibull modulus can be underestimated by as much as a factor of four while the characteristic strength is overestimated by 25%. The main conclusion of this exercise is that to obtain an accurate estimate of the parameters of the distribution of fiber strengths *it is necessary* to determine each fiber diameter, and cross-sectional area, contrary to what is prescribed by existing test standards.

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## Notes
1. The jurisdiction of this document was transferred from committee ASTM D30 to committee ASTM C28 where it was decided to withdraw D3379 in favor of a new standard that would address among other things the issues discussed in this paper.
2. Most ceramic fibers shatter upon failure.
3. Using function **RAND()** in the computer application Microsoft Excel 98.
4. Using function **NORMINV** (**probability, mean, standard_deviation**) in the computer application Microsoft Excel 98.
5. Volume effects on strength were not considered at this time.

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6. E. LARA-CURZIO, unpublished results.

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