

1. A

$$\begin{aligned}
f(\beta) &= (y - X\beta)^T W (y - X\beta) \\
&= \beta^T X^T W X \beta - 2y^T W X \beta + \dots \\
&= \beta^T X^T W X \beta - 2y^T W^T X^T (X^T W X)^{-1} X^T W X \beta + \dots \\
&= (\beta - (X^T W X)^{-1} X^T W y)^T X^T W X (\beta - (X^T W X)^{-1} X^T W y) + (1)
\end{aligned}$$

2. A

$$\begin{aligned}
l(\beta) &= -\log\left(\prod_i \omega_i^{y_i} (1 - \omega_i)^{1-y_i}\right) \\
&= -\left(\sum_i y_i \log \omega_i + (1 - y_i) \log(1 - \omega_i)\right) \quad (2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \omega_i}{\partial \beta_j} &= \frac{x_{ij} \exp(-x_i \beta)}{(1 + \exp(-x_i \beta))^2} = x_{ij} \omega_i (1 - \omega_i) \\
\frac{\partial l}{\partial \beta_j} &= \sum_i \left(\frac{1 - y_i}{1 - \omega_i} - \frac{y_i}{\omega_i}\right) x_{ij} \omega_i (1 - \omega_i) = \sum_i (\omega_i - y_i) x_{ij}
\end{aligned}$$

B See code.

C Write down the Taylor expansion of l :

$$l(\beta) = l(\beta_0) + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j} \beta_j + \frac{1}{2} \sum_{jk} \frac{\partial^2 l(\beta_0)}{\partial \beta_k \partial \beta_j} \beta_j \beta_k + \dots$$

Compute second order derivative:

$$\begin{aligned}
&\frac{\partial^2 l}{\partial \beta_k \partial \beta_j} \\
&= \sum_i \omega_i (1 - \omega_i) x_{ij} x_{ik} \quad (3)
\end{aligned}$$

l can be written as a quadratic form of diagonal matrix W whose diagonal entries are $\omega_i(1 - \omega_i)$

$$\begin{aligned}
l(\beta) &= l(\beta_0) + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j} \Delta \beta_j + \frac{1}{2} \sum_{jk} x_{ij} x_{ik} f_i(\beta_0) \Delta \beta_j \Delta \beta_k + \dots \\
&= \frac{1}{2} \Delta \beta^T X^T W X \Delta \beta + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j} \Delta \beta_j + \dots \\
&= \frac{1}{2} (z - X \Delta \beta)^T W (z - X \Delta \beta) + \dots \quad (4)
\end{aligned}$$

Determine z :

$$\begin{aligned}
 z^T W X &= -\frac{\partial l(\beta_0)}{\partial \beta_i} \\
 \sum_i z_i \omega_i (1 - \omega_i) x_{ij} &= -\sum_i (\omega_i - y_i) x_{ij} \\
 z_i &= \frac{y_i - \omega_i}{\omega_i (1 - \omega_i)}
 \end{aligned} \tag{5}$$