1. A

$$\begin{aligned} f(\beta) &= (y - X\beta)^T W (y - X\beta) \\ &= \beta^T X^T W X \beta - 2y^T W X \beta + \cdots \\ &= \beta^T X^T W X \beta - 2y^T W^T X^T (X^T W X)^{-1} X^T W X \beta + \cdots \\ &= (\beta - (X^T W X)^{-1} X^T W y)^T X^T W X (\beta - (X^T W X)^{-1} X^T W y) + \cdots \end{aligned} \tag{1}$$

2. A

$$\begin{aligned} l(\beta) &= -\log(\prod_i (y_i \omega_i(\beta) + (1 - y_i)(1 - \omega_i(\beta)))) \\ &= -\sum_i \log((2y_i - 1)\omega_i(\beta) + (1 - y_i)) \end{aligned} \tag{2}$$

$$\frac{\partial \omega_i}{\partial \beta_j} = \frac{x_{ij} \exp(-x_i \beta)}{(1 + \exp(-x_i \beta))^2} = x_{ij} \omega_i (1 - \omega_i)$$

$$\frac{\partial l}{\partial \beta_j} = \sum_i \frac{x_{ij} \omega_i (1 - \omega_i)(2y_i - 1)}{(2y_i - 1)\omega_i + (1 - y_i)}$$

B See code.

C Write down the taylor expansion of l:

$$l(\beta) = l(\beta_0) + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j} \beta_j + \frac{1}{2} \sum_{jk} \frac{\partial^2 l(\beta_0)}{\partial \beta_k \partial \beta_j} \beta_j \beta_k + \cdots$$

Compute second order derivative:

$$\begin{aligned} \frac{\partial^2 l}{\partial \beta_k \partial \beta_j} &= \sum_i \frac{x_{ij}(1 - 2\omega_i) x_{ik} \omega_i (1 - \omega_i)((2y_i - 1)\omega_i + (1 - y_i)) - (2y_i - 1)^2 x_{ik} x_{ij} \omega_i^2 (1 - \omega_i)^2}{((2y_i - 1)\omega_i + (1 - y_i))^2} \\ &= x_{ij} x_{ik} w_{ii} \end{aligned} \tag{3}$$

$$w_{ii} = \omega_i (1 - \omega_i) \frac{(1 - 2\omega_i)((2y_i - 1)\omega_i + (1 - y_i)) - (2y_i - 1)^2 \omega_i (1 - \omega_i)}{((2y_i - 1)\omega_i + (1 - y_i))^2}$$

l can be written as a quadratic form of diagonal matrix W

$$\begin{aligned} l(\beta) &= l(\beta_0) + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j} \beta_j + \frac{1}{2} \sum_{jk} x_{ij} x_{ik} f_i(\beta_0) \beta_j \beta_k + \cdots \\ &= \frac{1}{2} \beta^T X^T W X \beta + \sum_j \frac{\partial l(\beta_0)}{\partial \beta_j} \beta_j + \cdots \\ &= (z - X\beta)^T W (z - X\beta) + \cdots \end{aligned} \tag{4}$$

Determine z:

$$
\begin{aligned}
z^T W X &= \frac{\partial l(\beta_0)}{\partial \beta_i} \\
\sum_i z_i w_{ii} x_{ij} &= \sum_i \frac{x_{ij}\omega_i(\beta_0)(1-\omega_i(\beta_0))(2y_i-1)}{(2y_i-1)\omega_i(\beta_0)+(1-y_i)} \\
z_i &= \frac{\omega_i(\beta_0)(1-\omega_i(\beta_0))(2y_i-1)}{((2y_i-1)\omega_i(\beta_0)+(1-y_i))w_{ii}} \qquad (5)
\end{aligned}
$$