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Heterogeneous speculators, endogenous fluctuations and interacting markets: A model of stock prices and exchange rates

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ABSTRACT

We develop a discrete-time model in which the stock markets of two countries are linked via and with the foreign exchange market. The foreign exchange market is characterized by nonlinear interactions between technical and fundamental traders. Such interactions may generate complex dynamics and recurrent switching between "bull" and "bear" market phases via a well-known pitchfork and period-doubling bifurcation path, when technical traders become more aggressive. The two stock markets are populated by fundamentalists, and prices tend to evolve towards stable steady states, driven by linear laws of motion. A connection between such markets is established by allowing investors to trade abroad, and the resulting three-dimensional dynamical system is analyzed. One goal of our paper is to explore potential spill-over effects between foreign exchange and stock markets. A second, related goal is to study how the bifurcation sequence which characterizes the market with heterogeneous speculators is modified in the presence of interactions with other markets.

1. Introduction

The literature about the dynamics of prices in speculative markets, based on the interaction of boundedly rational heterogeneous agents, has become well developed in recent decades. A considerable portion of this literature has focused, in particular, on the dynamics of financial asset prices. Excellent recent surveys include Hommes (2006), LeBaron (2006) and Lux (2008).

Most models include nonlinear elements. Typically, nonlinearity arises from agents' trading rules or demand functions (e.g. Day and Huang, 1990; Chiarella, 1992; Rosser et al., 2003), from evolutionary switching between available strategies, based on certain fitness measures (e.g. Brock and Hommes, 1997, 1998), and from phenomena of contagion and consequent transition of speculators among "optimistic" and "pessimistic" groups (Kirman, 1991; Lux, 1995, 1997). Such nonlinearities are of course the mathematical reason for some typical dynamic outcome of these models, such as long-run price oscillations (often characterized by chaotic behavior) around an unstable "fundamental" steady state, the existence of alternative "nonfundamental" equilibria and the emergence of bubbles and crashes.

Within this literature, a large number of models are based on the so-called chartist–fundamentalist approach. We cite, in particular, Day and Huang (1990), Chiarella (1992), Huang and Day (1993), Brock and Hommes (1998), Lux (1998), Chiarella and He (2001), Chiarella et al. (2002), Hommes et al. (2005), De Grauwe and Grimaldi (2005), Anufriev et al. (2006), Diks and Dindo (2008), Georges (2008), and He and Li (2008). Such models are able to capture—albeit in a stylized

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way—an important determinant of price fluctuations. Chartists use technical trading rules to forecast prices, and in particular believe in the persistence (and thus exploitability) of bullish and bearish market episodes, and formulate their demands accordingly. In contrast, fundamentalists place their orders by assuming that prices will return towards their "fundamental value", thus playing a stabilizing role in the market. Prices are set as a function of aggregated investors' demand by assuming market clearing or, more often, the mediation of market makers. Endogenous price dynamics are thus generated by the interplay between the destabilizing forces of technical trading strategies and the mean reverting forces set in action by fundamental traders.

Generally speaking, one central insight of these models is that price movements are at least partially driven by endogenous laws of motion. A second important result is that some of these models have the potential to replicate a number of important stylized facts of financial markets, such as excess volatility, bubbles and crashes, fat tails for the distribution of the returns and volatility clustering.[1] Finally, these models seem to be useful for testing how certain regulatory measures work. For instance, Wieland and Westerhoff (2005) explore the effectiveness of popular central bank intervention strategies, while He and Westerhoff (2005) discuss how price caps may affect the dynamics of commodity markets. In a recent paper, Bauer et al. (2009) demonstrate that target zones may stabilize financial market.

Such literature generally focuses on the dynamics of a single speculative market, driven by the interplay of boundedly rational heterogeneous investors. In particular, most studies of the behavior of asset prices concentrate on the case of a stylized market with a single risky asset and a riskless asset. Recently, the basic ideas have been extended so as to model the dynamics of a market with multiple risky assets, or more generally to explore the dynamics of interacting speculative markets. For instance, Böhm and Wenzelburger (2005) and Chiarella et al. (2005, 2007) establish dynamic setups where prices and returns of multiple risky assets coevolve over time due to dynamic mean–variance portfolio diversification and updating of heterogeneous beliefs. Westerhoff (2004) and Westerhoff and Dieci (2006) model the interactions between different asset markets with fundamental and technical traders, where the connections arise from traders switching between markets, depending on relative profitability. Another related paper is by Corona et al. (2008), in which a model with interacting stock and foreign exchange markets is studied via numerical simulations and calibrated such that it is able to match some statistical properties of actual financial market dynamics. Brock et al. (2009) modify the stylized model of Brock and Hommes (1998), by including a market for derivative securities, and demonstrate how the latter, by providing hedging opportunities, may destabilize financial markets. Overall, such models show how interactions may destabilize otherwise stable markets and become a further source of nonlinearity and complex price dynamics, depending on the parameters which characterize agents' behavior. Apart from such initial contributions, however, the dynamic analysis of models of interacting markets within the "heterogeneous agent" approach still remains a largely unexplored research area.

This paper develops and explores a further stylized model of interacting markets, populated by boundedly rational heterogeneous investors, namely the case of two *stock markets* denominated in different currencies, which are linked *via* and *with* the related *foreign exchange market*. In our simplified model, which obviously neglects several possible channels of interaction between the stock markets and the foreign exchange market, connections arise simply because the trading decisions of stock market traders who are active abroad are based also on expected exchange rate movements, which are influenced by observed exchange rate behavior; in addition, their orders generate transactions of foreign currency and lead to consequent exchange rate adjustments. A model based on such a plain mechanism of interaction is therefore an ideal setup to address the question of potential spillover effects between different speculative markets. In particular, we investigate whether the existence of such connections may contribute to dampening, or amplifying and spreading the price fluctuations which arise in one of the markets due to the interplay of heterogeneous speculators. To do this we assume that—in the absence of interaction—the dynamics in one of the markets (the foreign exchange market) is governed by a chartist–fundamentalist model similar in structure to that developed by Day and Huang (1990). We select this model because it captures in the simplest possible way the essential features of fundamentalist–chartist interaction and its impact on price dynamics. Despite its stylized nature, the model is characterized by a rich bifurcation route, which has the potential to generate erratic switching between "bull" and "bear" market situations. A link between the three markets is introduced by allowing investors to trade abroad. It turns out that, even in such a simple setup, the role played by market interactions (i.e. whether they are *stabilizing* or rather *destabilizing*) is strongly dependent on some behavioral parameters which govern the intensity of speculative demand. Our findings also allow us to better understand how the well-known bifurcation route described by Day and Huang (1990) is modified in a higher-dimensional model of interconnected markets.

This paper is structured as follows. In Section 2 we describe the details of the model with regard to traders' demand and price adjustment mechanisms. In particular, Sections 2.1 and 2.2 contain our assumptions about the two stock markets, respectively, whereas Section 2.3 focuses on the exchange rate market. Section 3 describes the resulting three-dimensional, nonlinear dynamical system in discrete-time, and derives analytical results about the steady states of the model and their stability. This is initially done in the case of independent markets (Section 3.1) and subsequently for the full system of interacting markets (Section 3.2). Section 4 discusses the conditions under which market interactions have a stabilizing or destabilizing impact on the dynamics. Such a discussion is based on both the steady-state analysis carried out in Section 3 and additional computer simulations. In particular, Section 4 performs a numerical study of the amplitude of price

[1] For an overview of the so-called stylized facts of financial markets see Cont (2001), Lux and Ausloos (2002), Lux (2009), among others.

fluctuations (Section 4.1) and the onset of erratic price behavior (Section 4.2), and provides a brief discussion of robustness of results and the role of certain model assumptions (Section 4.3). Section 5 concludes the paper and suggests possible routes for future research. Appendix A provides a micro-foundation for asset demand functions, whereas Appendices B–D contain the proofs of the propositions and some related discussions.

2. The model

In this section we develop a simple three-dimensional discrete-time dynamic model in which two stock markets (denominated in different currencies) are linked *via* and *with* the foreign exchange market. Let us denote the two stock markets with the superscripts *H*(ome) and *A*(broad). In order to highlight the mechanisms by which endogenous dynamics, generated by the interplay of heterogeneous traders, spreads throughout the system of connected markets, we assume that only (national and foreign) fundamental traders are active in each stock market, with fixed proportions. In contrast, we assume the existence of heterogeneous speculators, fundamental traders (or fundamentalists) and technical traders (or chartists), who explicitly focus on the foreign exchange market. Their proportions are assumed to vary over time, depending on market circumstances: the larger the mispricing in the foreign exchange market, the more agents rely on fundamental analysis. For all types of agents, the "beliefs" about future price movements are updated in each period as a function of observed prices.

We focus on a specific mechanism of interaction between markets. Connections occur in two directions. On the one hand, stock market traders who trade abroad base their demand on expected movements of both stock prices and exchange rates. On the other hand, they generate transactions of foreign currencies and consequent exchange rate changes. For each market, we model the price adjustment process by a log-linear price adjustment function. Such a function, which has often been adopted in the literature (see Beja and Goldman, 1980, Chiarella, 1992, among others) relates the price change to the excess demand.

2.1. The stock market in country H

Let us start with a description of the stock market in country *H*. According to the assumed price adjustment function, the change of the log stock price (P_t^H) from time t to time $t+1$ in country *H* may be expressed as

$$P_{t+1}^H = P_t^H + a^H(D_{F,t}^{HH} + D_{F,t}^{HA}), \tag{1}$$

where a^H is a positive price adjustment parameter and $D_{F,t}^{HH}$, $D_{F,t}^{HA}$ stand for the demand for asset *H* of fundamental traders from countries *H* and *A*, respectively. Note that we are simply assuming proportionality of log-price changes to the current excess demand: prices go up (down) if there is excess demand (supply). This preserves the dimension of the model, thus keeping it analytically tractable. The price setting rule (1) may also be interpreted as the stylized behavior of a market maker, who aggregates agents' demands, clears the market by taking an offsetting long or short position, and then adjusts the price for the next period as a function of observed excess demand. The stylized behavioral rule (1) thus captures the market maker's role[2] of market-clearing and price-setting (see Hommes et al., 2005 for a discussion).

The demand[3] by fundamental traders (or fundamentalists) from country *H* is given as

$$D_{F,t}^{HH} = b^H(F^H - P_t^H), \tag{2}$$

where b^H is a positive reaction parameter and F^H is the log fundamental value of stock *H*. Fundamentalists seek to profit from mean reversion. Hence, their demand is positive when the market is undervalued (and vice versa).

Fundamental traders from abroad may benefit from a price correction in the stock market as well as in the foreign exchange market.[4] The log fundamental value of the exchange rate is denoted by F^S and the log exchange rate by *S*. Their demand may thus be written as

$$D_{F,t}^{HA} = c^H(F^H - P_t^H + F^S - S_t), \tag{3}$$

where $c^H \geq 0$. Suppose, for instance, that both the stock market and the foreign exchange market are undervalued. Then the foreign fundamentalists take a larger long position than the national fundamentalists (assuming equal reaction parameters). Should, however, the foreign exchange market be overvalued, then the foreign fundamentalists become more cautious and may even enter a short position.[5]

[2] Note that Franke (2009) and Zhu et al. (2009) recently proposed interesting frameworks in which market makers actively manage their inventory positions, i.e. their price setting behavior also depends on their own positions in the market.

[3] In our stylized setup the agent's portfolio position in asset *H* is zero when there is no mispricing. Alternatively, we may regard (2) as the deviation of agent's portfolio position from some target value. The same remark holds for demand functions defined below.

[4] Here we define the exchange rate as the price of one unit of currency *H* in terms of currency *A*: an increase in the exchange rate thus means an appreciation of currency *H*.

[5] As a slightly different interpretation of Eq. (3), note that this may be rewritten as: $D_{F,t}^{HA} = c^H(\hat{F}^H - \hat{P}_t^H)$, where $\hat{F}^H := F^H + F^S$, and $\hat{P}_t^H := P_t^H + S_t$ are the log-fundamental value and the log-price of the stock in country *H*, respectively, measured in terms of currency *A*.

2.2. The stock market in country A

Let us now turn to the stock market in country A. We have a set of equations similar to those for stock market H. The log price adjustment is expressed as

$$P^A_{t+1} = P^A_t + a^A(D^{AA}_{F,t} + D^{AH}_{F,t}), \tag{4}$$

where $a^A > 0$. The demand of the fundamentalists from country A investing in stock market A amounts to

$$D^{AA}_{F,t} = b^A(F^A - P^A_t), \tag{5}$$

where $b^A > 0$ and F^A denotes the log-fundamental price of stock market A. The demand of fundamentalists from country H investing in stock market A results in

$$D^{AH}_{F,t} = c^A(F^A - P^A_t + S_t - F^S), \tag{6}$$

where $c^A \geq 0$. Apart from the notation, the only obvious difference to the case described in the previous section is that here agents take the inverse exchange rates into account. The quantity $-S_t = \ln(1/\exp(S_t))$ is the log of the reciprocal value of the exchange rate, and similarly $-F^S$ is the logarithm of the inverse fundamental rate.[6]

A general remark on asset demand functions is in order. In our setup, the diversified portfolio of financial assets of, e.g. fundamentalists from country A, consists of $D^{AA}_{F,t}$ shares of stock A and $D^{HA}_{F,t}$ shares of stock H, given by (5) and (3), respectively, which implies that agents' portfolio share invested in each market does not depend on the mispricing in the other market. Of course, a more general setup should account for possible interdependencies between the two investment decisions. Our simplifying assumption is consistent with a mean–variance setup where agents expect zero correlation between future stock price movements in market A and H, as shown in Appendix A.[7] By neglecting possible correlations in agents' beliefs, we remove direct links between the two stock markets, and focus only on those connections emerging endogenously via the foreign exchange market.

2.3. The foreign exchange market

In the foreign exchange market, the excess demand for currency H results from portfolio positions taken by stock traders who are active abroad and by foreign exchange speculators. The latter group of agents switch between technical and fundamental trading strategies, depending on market conditions. The log exchange rate at time step $t+1$ is determined as

$$S_{t+1} = S_t + d\left[\exp(P^H_t)D^{HA}_{F,t} - \frac{\exp(P^A_t)}{\exp(S_t)}D^{AH}_{F,t} + W_{C,t}D^S_{C,t} + (1 - W_{C,t})D^S_{F,t}\right], \tag{7}$$

where d is a positive price adjustment parameter. According to Eq. (7), the log exchange rate adjustment is proportional to excess demand of currency H, determined by both asset demand of agents who trade abroad and speculative demand in the foreign exchange market. The first two terms in brackets on the right-hand side of Eq. (7) express the demand generated by stock traders. It is important to note that their demand is given in real units. The demand for currency (H or A) of these traders is the product of demand for stock (H or A) times stock prices; in particular, the demand for currency A from traders H investing in stock A, $\exp(P^A_t)D^{AH}_{F,t}$, generates a demand for currency H of the opposite sign, the amount of which is obtained by multiplying the above quantity by the inverse exchange rate.[8] The quantities $D^S_{C,t}$ and $D^S_{F,t}$ denote the demand generated by technical and fundamental foreign exchange speculators, while $W_{C,t}$ and $(1 - W_{C,t})$ denote their market shares, respectively.

As in Day and Huang (1990), we assume that chartist demand may be formalized as

$$D^S_{C,t} = e(S_t - F^S), \tag{8}$$

where $e > 0$. According to Eq. (8), chartists believe in the persistence of a "bull" ("bear") market, and they therefore optimistically hold a long position (pessimistically hold a short position) as long as this is observed. Parameter e governs chartists' confidence in the persistence of deviations from fundamentals, and consequently the "intensity" of their speculative demand. This behavioral parameter will be proven to play a crucial role in the following dynamic analysis. Under the above demand specification, chartists do not take past exchange rate changes into account. Their behavior is based on the simple belief that prices tend to move away from fundamental values. Note that also Brock and Hommes (1998) rely on a model in which chartists behave in a similar manner, and that Boswijk et al. (2007) and Westerhoff and

[6] Eq. (6) admits an alternative interpretation in terms of price of stock A measured by currency H, analogous to that provided for Eq. (3).

[7] Note that demand functions analogous to (5) and (3) are derived from mean–variance utility maximization with multiple risky assets, within the heterogeneous agent portfolio model developed by Chiarella et al. (2005).

[8] Note that this introduces an additional nonlinearity in our model. Dieci and Westerhoff (2009) find that if there are also trend extrapolating chartists in the stock markets (and no speculators in the foreign exchange market) then this "price-quantity" nonlinearity may even be sufficient to create endogenous motion.

Franke (2009) find significant empirical evidence for such trading rules. Moreover, by sketching chartist behavior in the simplest possible way, Eq. (8) avoids the introduction of additional time lags into the model.

By contrast, fundamentalists seek to exploit misalignments and formulate their demand according to

$$D_{F,t}^S = f(F^S - S_t), \tag{9}$$

where $f > 0$.[9] Following He and Westerhoff (2005), we assume that speculators switch between these two trading rules with respect to market circumstances. The proportion of technical traders is defined as

$$W_{C,t} = \frac{1}{1+g(F^S - S_t)^2}, \tag{10}$$

which implies that it decreases as the mispricing in the foreign exchange market increases. The rationale for Eq. (10) is as follows. The more the exchange rate deviates from its fundamental value, the greater the speculators perceive the risk that the bull or bear market might collapse. Hence, fundamental analysis gains in popularity at the expense of technical analysis. Parameter $g > 0$ is a sensitivity parameter. The higher g is, the more sensitive the mass of speculators becomes with regard to a given misalignment. Weighting mechanisms similar to (10), capturing the idea that agents stop using technical trading rules as the mispricing increases, have also been used in Bauer et al. (2009), Gaunersdorfer and Hommes (2007), and De Grauwe et al. (1993).

3. Dynamical system

Eqs. (1), (4), and (7), which model the price adjustments, combined with Eqs. (2), (3), (5), (6), (8)–(10), which fix the excess demand of traders in the three markets, result in a three-dimensional discrete-time dynamical system with the following structure:

$$P_{t+1}^H = G^H(P_t^H, S_t), \quad P_{t+1}^A = G^A(P_t^A, S_t), \quad S_{t+1} = G^S(P_t^H, P_t^A, S_t). \tag{11}$$

Components G^H, G^A and G^S of the map $\mathcal{G}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, which determines the iteration of the system, are expressed, respectively, as (we omit the time index)

$$G^H(P^H, S) = P^H + a^H(D_F^{HH} + D_F^{HA}), \tag{12}$$

$$G^A(P^A, S) = P^A + a^A(D_F^{AA} + D_F^{AH}), \tag{13}$$

$$G^S(P^H, P^A, S) = S + d\left[\exp(P^H)D_F^{HA} - \frac{\exp(P^A)}{\exp(S)}D_F^{AH} + W_C D_C^S + (1 - W_C)D_F^S\right], \tag{14}$$

where

$$D_F^{HH} = b^H(F^H - P^H), \quad D_F^{HA} = c^H(F^H - P^H + F^S - S),$$

$$D_F^{AA} = b^A(F^A - P^A), \quad D_F^{AH} = c^A(F^A - P^A + S - F^S),$$

$$D_C^S = e(S - F^S), \quad D_F^S = f(F^S - S), \quad W_C = \frac{1}{1+g(F^S - S)^2}.$$

Note first that Eqs. (12) and (13), which govern stock price adjustments, are linear in the state variables P^H, P^A, and S, while the exchange rate equation (14) is nonlinear due to both the state-dependent weight W_C and the structure of the demand for currency H from stock market traders. A second important remark concerns the role played by parameters c^H and c^A, which determine the strength of interactions between the markets. In the particular case where $c^H = c^A = 0$, the three equations of the system are decoupled, and each market evolves as an independent one-dimensional system.[10] This "one-dimensional" case will be taken as the starting "reference" case, developed in detail in the next section, before considering the dynamic behavior of the full three-dimensional system in Section 3.2.

3.1. The case of no interactions

In this section we set $c^H = c^A = 0$, which represents the case where no agent trades abroad. The dynamical system takes the following simplified form:

$$P_{t+1}^H = P_t^H + a^H b^H(F^H - P_t^H), \tag{15}$$

[9] Parameters e, f, together with all other parameters governing agents' demand in the stock markets (b^H, c^H, b^A, c^A) may depend, in general, on traders' beliefs about the speed of price correction, the number of traders of each type, their risk aversion, etc. Chiarella et al. (2009) provide an overview of how certain demand functions may be derived from standard utility maximization problems, thus making the role of such parameters explicit.

[10] Note that from the point of view of dynamic analysis, two "intermediate" cases exist, where either $c^H = 0$ or $c^A = 0$. In such cases, two of the three equations evolve as an independent two-dimensional system, which makes the model a bit more tractable analytically than in the full 3D case.

$$P^A_{t+1} = P^A_t + a^A b^A (F^A - P^A_t), \tag{16}$$

$$S_{t+1} = S_t + d\,\frac{(S_t - F^S)[e - fg(F^S - S_t)^2]}{1 + g(F^S - S_t)^2}, \tag{17}$$

i.e. it is described by three independent first-order difference equations, representing the two stock markets and the foreign exchange market, respectively. The stock market equations are linear while the foreign exchange equation is nonlinear (it is of "cubic" type). The steady state[11] properties of the three independent markets are stated in the following

Proposition 1. (a) *The unique steady state in each stock market is represented by the fundamental price, namely*, $\overline{P}^H = F^H$, $\overline{P}^A = F^A$; *stock market steady states are globally asymptotically stable iff* $a^H b^H < 2$, $a^A b^A < 2$, *respectively*.

(b) *The one-dimensional dynamical system* (17), *modelling the exchange rate behavior, always admits three steady states, the fundamental steady state*, $\overline{S} = F^S$, *and two nonfundamental steady states*

$$\overline{S}_l = F^S - \sqrt{\frac{e}{fg}}, \quad \overline{S}_u = F^S + \sqrt{\frac{e}{fg}},$$

located in symmetric positions below and above F^S, *respectively*.

(c) *The fundamental steady state of the foreign exchange market is always unstable. If* $df \le 1$, *the nonfundamental steady states are locally asymptotically stable (LAS), whereas if* $df > 1$ *they are LAS only for* $0 < e < e_{Flip} := f/(df - 1)$, *at which parameter value a period doubling bifurcation occurs*.

Proof. See Appendix B.

According to Proposition 1(a), fundamental stock prices are stable equilibria provided that fundamental traders or prices do not react too strong. This restriction—which ensures "stable" stock markets in the absence of connections with the foreign exchange market—will be assumed in the rest of the paper. The instability of the fundamental equilibrium in the foreign exchange market follows directly from the assumed switching mechanism (10). Near this steady state almost all traders are chartists, and their demand (8) tends to amplify deviations from the fundamental value. However, the same switching rule is also the main reason for the exchange rate dynamics to be bounded, and for two symmetric nonfundamental steady states to exist (Proposition 1(b)). According to Proposition 1(c), such steady states are LAS if the exchange rate is not too responsive to excess demand for currency H (small d). In the opposite case, they undergo a Flip bifurcation when chartists' reaction to misalignments becomes strong enough (large e). Fig. 1 reports bifurcation diagrams associated with the qualitative cases $df \le 1$ (panels a, b) and $df > 1$ (panels c, d), where e is the bifurcation parameter. For each parameter configuration, the diagrams on the same line report the attractors corresponding to two different initial conditions, one above and one below the fundamental value. In the first case (a, b), the attractors are locally stable steady states which do not change qualitatively as e becomes larger, but only increasingly deviate from the fundamental value. In the second case (c, d), both steady states lose stability for $e = e_{Flip}$ and are replaced by stable orbits of period 2, which is then followed by a sequence of period-doubling bifurcations and transition to chaos. This sequence is very similar to that illustrated by Day and Huang (1990) in their well-known one-dimensional stylized model of an asset market with heterogeneous investors and a market maker. The periodic orbits, or chaotic intervals, resulting from this sequence of Flip bifurcations are located either above or below the fundamental steady state, depending on the initial condition. At first, chaotic dynamics take place either in the "bull" or the "bear" market region, which are therefore disjoint *trapping*[12] regions, but at some point, exchange rates start to wander across both regions: the bifurcation diagrams show a drastic enlargement of the chaotic interval, and the dynamics is characterized by intricate fluctuations and erratic switching between bull and bear market episodes (panel e). From an economic point of view, our analysis thus reveals so far that an increase in the aggressiveness of chartists (increase of parameter e) destabilizes the foreign exchange market.

For what concerns the dynamic analysis, the merging of the "bull" and "bear" regions is due to a *homoclinic bifurcation* of the repelling fundamental steady state. Without going into details about such bifurcations, here we simply provide a graphical visualization in panel f, which represents the map (17) for the parameter setting at which the homoclinic bifurcation occurs. This kind of bifurcation is strictly related to the noninvertibility of the map, which is characterized by two *critical points*, one local maximum and one local minimum. This fact enables a repelling steady state to have further "preimages", apart from itself. In the one-dimensional case, such a homoclinic bifurcation occurs at the parameter value for which one of these preimages is a critical point of the map. This is indicated by the arrows in panel f. This bifurcation, together with the symmetry[13] of the 1D map (17) with respect to the fundamental steady state, determines the "merging" of two disjoint trapping intervals into a unique interval (see Dieci et al., 2001; He and Westerhoff, 2005 for the analysis of this type of bifurcation arising from one-dimensional economic examples).

[11] In the following, an overbar denotes steady-state levels for the dynamic variables.

[12] I.e. each region is mapped into itself under iteration of (17).

[13] Note that symmetric points (with respect to F^S) are mapped onto symmetric points under iteration of (17). As a consequence, either an attractor is symmetric with respect to F^S or it admits a symmetric attractor with the same stability properties. Note that such a symmetry property is generally lost when switching to the 3D model of interacting markets, expressed by (12)–(14).

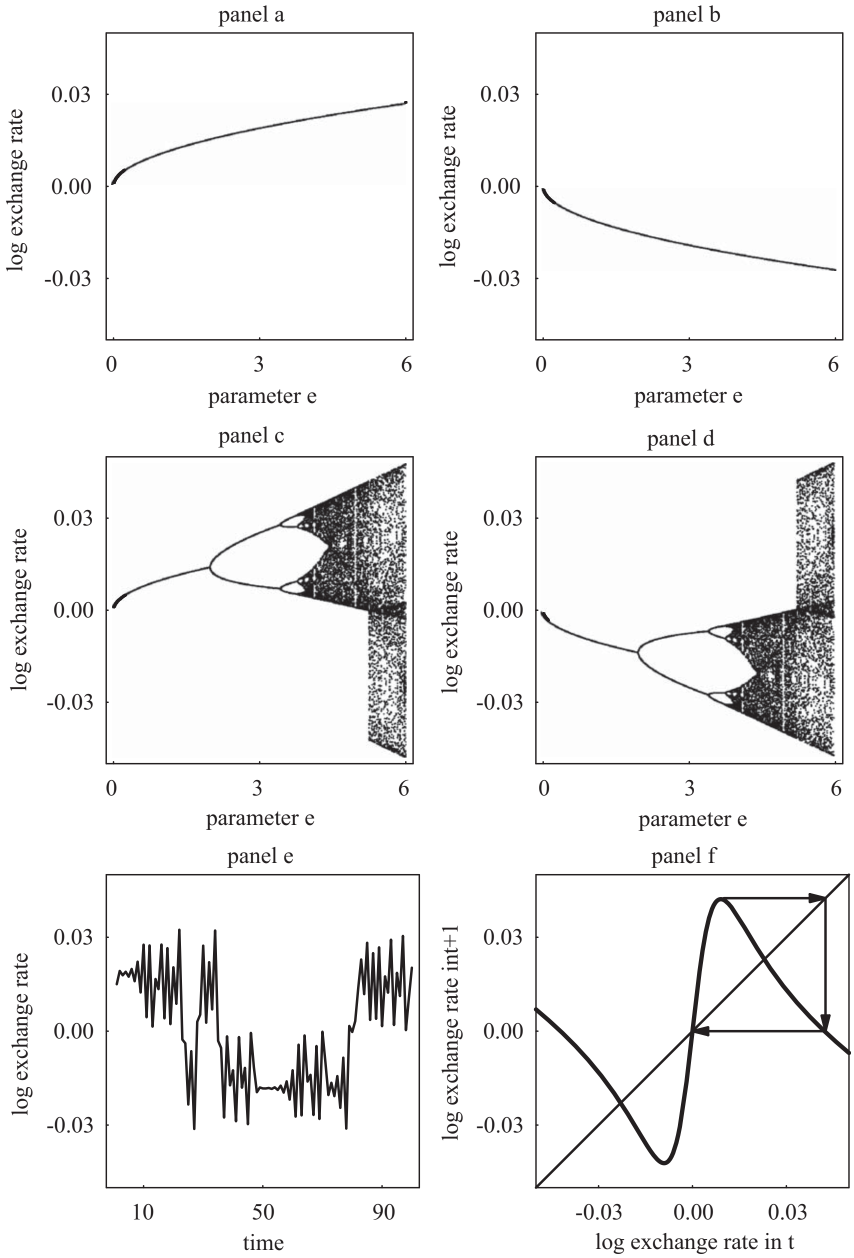

Fig. 1. The case of no market interactions. Panels a and b present bifurcation diagrams for the log exchange rate, for $d=1$, $0<e<6$, $f=0.8$ and two different initial conditions. Panels c and d show the same but now for $d=1.5$ and $f=1$. Panel e depicts the evolution of the log exchange rate in the time domain for $d=1.5$, $e=5.3$, and $f=1$. Panel f shows the 1D map (17) for $d=1.5$, $e\simeq 5.257$ and $f=1$. The gray arrows in this panel indicate that the maximum is in fact a pre-image of the repelling fundamental steady state (and is therefore the minimum). The remaining parameters are $g=10\,000$ and $F^S=0$.

3.2. The case of interacting markets

We now analyze the full system, which is characterized by the existence of stock market traders who trade abroad. This means that at least one of the parameters c^H, c^A is strictly positive. The present section explores in depth the effect of "opening" the economy, in contrast to the reference case of independent markets. In particular the following questions are addressed, which arise quite naturally within this model.

The first question concerns the "destabilizing" or "stabilizing" role played by market interactions. In particular, we try (i) to understand the effect of such interactions on the fundamental equilibrium and its stability in all three markets, (ii) to explore the conditions under which further stable equilibria exist, (iii) to investigate how far these additional equilibria may deviate from the fundamental value, and (iv) to study whether market interactions contribute to an amplification or dampening of price fluctuations around the unstable fundamental steady state, compared with the case of independent markets.

The second issue concerns the persistence of the bifurcation structure described and discussed in the (one-dimensional) case of no interactions. Namely, we try to understand whether such a structure also "survives" in the (three-dimensional) case of interacting markets. Leaving a rigorous analysis to future research, here we simply aim at providing numerical evidence of the existence of a bifurcation sequence similar in quality to that described in the previous section, and in particular of the homoclinic bifurcation that marks the transition to a regime of erratic switching between bull and bear market phases.

The following proposition concerns the steady states of the full model. In order to simplify the notation, we introduce the deviations from fundamentals, $x^H := P^H - F^H$, $x^A := P^A - F^A$, $x := S - F^S$, and express the steady states accordingly. We also define $\Phi^H := \exp(F^H)$, $\Phi^A := \exp(F^A)$, $\Phi^S := \exp(F^S)$.

Proposition 2. (a) *The steady states of the three-dimensional system of interacting markets* (11) *are given by points* $(\overline{x}^H, \overline{x}^A, \overline{x})$ *satisfying*

$$\overline{x}^H = -\frac{c^H}{b^H + c^H}\overline{x}, \quad \overline{x}^A = \frac{c^A}{b^A + c^A}\overline{x}, \tag{18}$$

and such that $\overline{x}$ *solves*

$$x\alpha(x) = x\beta(x), \tag{19}$$

where

$$\alpha(x) = \frac{e - fgx^2}{1 + gx^2}, \quad \beta(x) = \frac{\Phi^H b^H c^H}{b^H + c^H}\exp\left(-\frac{c^H}{b^H + c^H}x\right) + \frac{\Phi^A}{\Phi^S}\frac{b^A c^A}{b^A + c^A}\exp\left(-\frac{b^A}{b^A + c^A}x\right). \tag{20}$$

(b) *A unique (fundamental) steady state exists* ($\overline{x}^H = \overline{x}^A = \overline{x} = 0$) *if the chartist parameter e is sufficiently small, whereas if e is large enough or interactions are sufficiently weak* (*small* c^H, c^A) *two further (nonfundamental) steady states exist.*

Proof. See Appendix C.

As discussed in Appendix C, Fig. 2 represents the possible solutions to equation $\alpha(x) = \beta(x)$. The previously discussed situation of no interactions, $c^H = c^A = 0$, corresponds to the case in which $\beta(x) \equiv 0$, where the two curves intersect at the symmetric points $\mp\sqrt{e/(fg)}$ (panel a). In the case where either c^H or c^A is strictly positive (panels b–d), elementary geometrical considerations suggest that the two curves $\alpha(x)$ and $\beta(x)$ will still intersect each other provided that $\alpha(0) = e$ is sufficiently large with respect to $\beta(0)$, where the latter quantity depends positively on parameters c^H and c^A. In this case further steady states exist, which implies the existence of nonfundamental equilibrium prices in *all* markets, via Eq. (18). This represents a first important effect of the interaction between the three markets. In particular, parameters capturing the strength of interactions, c^H and c^A, have a direct impact on the deviations between nonfundamental and fundamental steady state prices in the two stock markets.

To summarize, by establishing a connection between the markets, we obtain a steady-state structure, which can be regarded as intermediate between those of the original, independent systems. Which of the original situations prevails in the case of interacting markets depends on the parameters of the model. In particular, we focus on the role played by the chartist extrapolation parameter e. For large e, the full system (and therefore also the two stock markets) will display a structure with multiple equilibrium prices, which is "inherited" from the original scenario of the foreign exchange market. In contrast, when e is small, the opposite effect takes place, and the structure with a unique stationary state prevails (also in the foreign exchange market). Moreover, the full 3D model has such a mixed behavior, also with regard to the stability of the steady states. Roughly speaking, for sufficiently small chartist parameter e, the unique steady state of the full system will also be LAS, as occurs in the (independent) stock markets. This will be specified more precisely by the next proposition. Moreover, for large e, two stable nonfundamental steady states will themselves become unstable, and will be replaced by more complex attractors, as in the (independent) foreign exchange market. This will be shown by numerical examples in Section 4. The following proposition concerns the local stability of the fundamental steady state.

Proposition 3. *Assume that the adjustment and reaction parameters in the stock markets satisfy*

$$a^H(b^H + c^H) < 2, \quad a^A(b^A + c^A) < 2. \tag{21}$$

Then if parameters d, e, are sufficiently small, the unique (fundamental) steady state of the dynamical system (11) *is LAS.*

Proof. See Appendix D.

Proposition 3 proves that when neither the adjustment of the prices in the three markets (parameters a^H, a^A, d) nor the extrapolation of the chartists (parameter e) is too strong, the exchange rate market is *stabilized* by the connections with stable stock markets, such that an unstable fundamental steady state becomes locally (or even globally) asymptotically stable. We provide economic intuition for this result, by considering how exchange rate misalignments affect each component of the excess demand for currency H (in square brackets in Eq. (7)). Assume that the two (isolated) stock

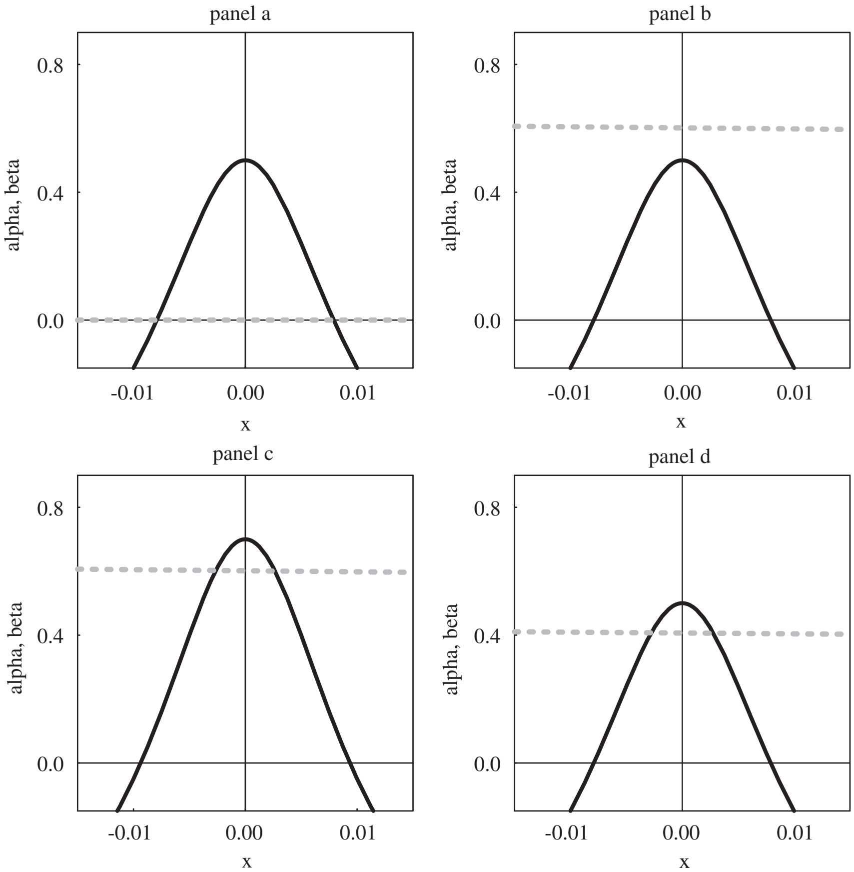

Fig. 2. A characterization of the nonfundamental steady states. In the four panels we plot functions $\alpha(x)$ (black solid line) and $\beta(x)$ (gray dashed line) for different parameter combinations. Panel a: $c^H=0$, $c^A=0$, and $e=0.5$. Panel b: $c^H=0.4$, $c^A=0.4$, and $e=0.5$. Panel c: $c^H=0.4$, $c^A=0.4$, and $e=0.7$. Panel d: $c^H=0.1$, $c^A=0.4$, and $e=0.5$. The remaining parameters are $b^H=1$, $b^A=1.5$, $d=1$, $f=0.8$, $g=10\,000$ and $F^H=F^A=F^S=0$.

markets are stable at their fundamental values. If the (isolated) foreign exchange market is such that $S<F^S$, but close to F^S, then chartists prevail, and their speculative demand $D_C^S<0$ tends to amplify the initial exchange rate deviation. Now open the economy, by allowing fundamental traders to trade in both stock markets. Due to $S<F^S$, investors from country *A* will start buying stock *H*, hoping that the price will be corrected. At the same time investors from country *H* will take negative positions in stock market *A*. Consequently, the two components of demand for currency *H* from stock market investors, $\exp(P^H)D_F^{HA}$ and $-\exp(P^A-S)D_F^{AH}$, will be strictly positive. The impact of such traders thus partly offsets that of foreign exchange speculators, and stabilizes the steady state. This occurs, however, only when stock traders' impact (c^H and c^A) is strong enough and chartist extrapolation (e) is weak. In contrast, for larger e, exchange rate fluctuations break stability in all markets and, in addition, two nonfundamental steady states undergo a sequence of bifurcations similar to that illustrated for the one-dimensional independent foreign exchange market in Figs. 1 c and d. This will be shown in the next section, which contains a broader discussion of the stabilizing/destabilizing impact of interactions.

4. Stabilizing and destabilizing effects of interactions: numerical examples

This section contains numerical examples which illustrate the dynamic behavior of the model and discusses, in particular, the global bifurcations occurring for increasing values of parameter e, which governs the strength of chartists' speculation. First of all it will be shown that in the case of interacting markets large values of e result in a "stronger instability" than in the case of independent markets, and produces an enlargement of the range of fluctuations. This effect is therefore totally different from the stabilizing impact proven analytically (and observed numerically) for small e. Second, it will be shown that, by taking e as a bifurcation parameter, the full three-dimensional model undergoes a sequence of bifurcations which is similar to that observed for the exchange rate market in the one-dimensional case. In particular, we detect also in the 3D case the effects of a *homoclinic bifurcation* of the fundamental steady state, similar to that reported in Section 3.1. Finally, we show that similar results may be obtained in a framework in which a nonlinear, unstable stock market interacts with a linear, stable foreign exchange market. As will become clear, an instability originating from a speculative stock market may as well destabilize a foreign exchange market.

Throughout the examples of this section we use the following common parameter setting: $a^H=1$, $a^A=0.8$, $b^H=1$, $b^A=1.5$, $d=1$, $f=0.8$, $g=10\,000$, $\Phi^H=\Phi^A=\Phi^S=1$ (so that the log fundamental prices F^H, F^A, F^S are all equal to zero); the remaining parameters c^H, c^A, and e may vary across different examples. Note that parameters d, f, g, and F^S are precisely those used in Figs. 1 a, b: this means that in the absence of interactions, the foreign exchange market would be characterized by two coexisting and locally stable nonfundamental steady states for any $e>0$.[14]

4.1. Stability and volatility

Fig. 3 represents bifurcation diagrams for each of the three dynamic variables P^H, P^A, and S, versus the chartist parameter e. In each panel the asymptotic behavior in a case with market interactions (with $c^H=c^A=0.4$) is compared with the corresponding situation with no interactions ($c^H=c^A=0$, gray dashed line). Thus the figure reports the effect of introducing connections of a certain intensity for different values of e. The three panels located on the right are obtained with a different initial condition from those on the left. Note also that parameters a^H, a^A, b^H, b^A, c^H, c^A, d, Φ^H, Φ^A, and Φ^S (i.e. those that play a role for the linearized system around the fundamental steady state), satisfy all of the restrictions we imposed in Appendix D to derive analytical results about local stability.[15] For a range of low values of parameter e, the stabilizing effect of interactions is clear from the bifurcation diagrams, which confirms our local stability results: stability of stock markets remains unaffected, whereas in the foreign exchange market two coexisting LAS nonfundamental steady states (which surround an unstable fundamental equilibrium) are replaced by a unique stable fundamental equilibrium. If we now increase parameter e (in the case $c^H=c^A=0.4$), the latter loses stability for $e=\beta(0)\simeq 0.6015$. The effect of such a bifurcation is a "pitchfork" scenario:[16] for a certain range of e the phase space is thus characterized by the coexistence of two stable equilibria that surround the unstable fundamental steady state, i.e. the two stock markets are destabilized with respect to the situation of no interactions, and steady state prices deviate from fundamentals. Note, however, that in the foreign exchange market the steady state deviation from the fundamental is less pronounced than in the case of independent markets, i.e. a kind of stabilizing effect is still at work here. Larger values of e bring about the sequence of period-doubling bifurcations already reported in the one-dimensional case. Within this range of e we can say that interactions destabilize all three markets. In particular, for $e\simeq 4.856$ the diagram reports a sudden, drastic enlargement of the chaotic region where asymptotic fluctuations are confined. This phenomenon will be further discussed below.

The economic intuition provided by Fig. 3 is therefore that market interactions, combined with strong extrapolation, result in remarkable spillover effects and market volatility, in contrast to the underlying stable markets without interactions.[17]

In Fig. 4, the effect of interactions is analyzed from a slightly different perspective. In the upper group of panels we choose a large value of e ($e=6$). We also set $c^A=0.2$ and increase c^H. Since this represents the parameter that governs the demand for stock in country H by fundamentalists from country A, we are thus increasing the strength of interactions from A to H. By doing this, we notice a transition to increasingly complex dynamics, that is a "destabilizing effect" similar to that already observed in Fig. 3. The results are reported in panels a, b and c, d for log stock price H and the log exchange rate, respectively.[18] By contrast, in the lower panels we select a small value of e ($e=0.5$). We also fix $c^A=0.4$ and increase c^H again. In this case we report the opposite effect, i.e. increasing the strength of interactions stabilizes the system (see panels e, f and g, h for P^H and S, respectively). To summarize, the nature of the impact of market interactions (stabilizing or destabilizing) is determined by the level of a crucial behavioral parameter, strictly related to chartists' confidence in the persistence of deviations from fundamentals. The overall impact is due to a balance between chartist extrapolation in the foreign exchange market and the reaction of the investors from one country to the stock market in the other country.

4.2. Homoclinic bifurcation

The previous subsection has shown how the fully integrated 3D model is able to display the characteristic behavior of each of the starting (independent) markets for different ranges of parameter e. Quite interestingly, as already anticipated by the bifurcation diagrams in Fig. 3, even the homoclinic bifurcation of the steady state reported in the 1D case survives almost identical in the 3D model. Here, two attractors lying in two disjoint regions of the three-dimensional phase-space merge into a unique attractor, thus determining a major qualitative change of the dynamics. The situations before and after the bifurcation are represented in Fig. 5. Panels a, c and e report the projections of the attractors in the plane of the state variables P^H and S. Before the bifurcation, there are two coexisting attractors (panels a, c) and the asymptotic dynamics of the system depends on the initial state. After the bifurcation the two attractors merge into a unique attractor (panel e).

[14] Similar phenomena can easily be detected for a wide region of the parameter space.

[15] One can easily check that conditions (50)–(52) are satisfied for any $e>0$, while condition (49) holds only for $e<e^*\simeq 0.6015$.

[16] Although this is not revealed by the plots in Fig. 3, such a bifurcation occurs via a slightly more complicated mechanism than a pitchfork bifurcation, as further discussed in Appendix C.

[17] This simple observation is particularly meaningful in the light of current world financial turmoil.

[18] Again, left and right panels are characterized by different initial conditions.

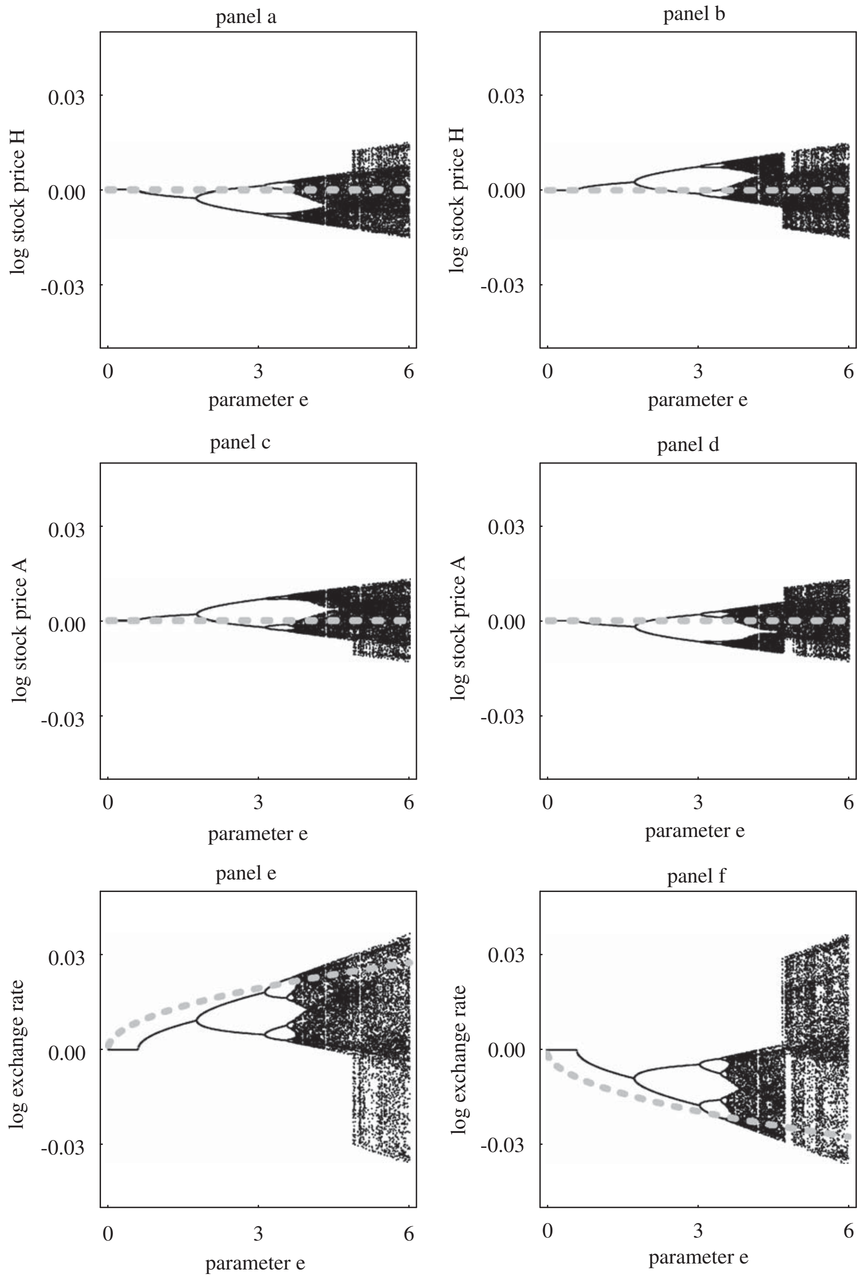

Fig. 3. No market interactions versus market interactions. The six panels present bifurcation diagrams for the log stock price in country H (panels a and b), the log stock price in country A (panels c and d) and the log exchange rate (panels e and f) for two different sets of initial conditions, respectively. The parameters are: $a^H=1$, $a^A=0.8$, $b^H=1$, $b^A=1.5$, $c^H=0.4$, $c^A=0.4$, $d=1$, $0<e<6$, $f=0.8$, $g=10\,000$ and $F^H=F^A=F^S=0$. The superimposed gray dashed lines illustrate the reference case of "no market interactions", i.e. $c^H=0$ and $c^A=0$, for the same values of the other parameters.

Obviously, this situation is the higher dimensional equivalent of the merging of two coexisting disjoint intervals in Figs. 1 c, d. Panels b, d and f represent the dynamics of the log-exchange rate in the time domain before and after bifurcation. While initially the dynamics take place in a specified (bull or bear) market region, depending on the initial condition (panels b, d), after bifurcation the dynamics covers both regions, but still switches between the two pre-existing regions at seemingly unpredictable points in time, thus evolving through a series of bubbles and crashes (panel f).

Under the same parameter configuration as Figs. 5e,f, and Fig. 6 (panels a, b, c) represents the time series of all state variables, and shows that the stock prices also jump back and forth between "bull" and "bear" market episodes, triggered by exchange rate fluctuations. Finally, panel d plots in the plane (S_t, S_{t+1}) a trajectory obtained using the same parameters

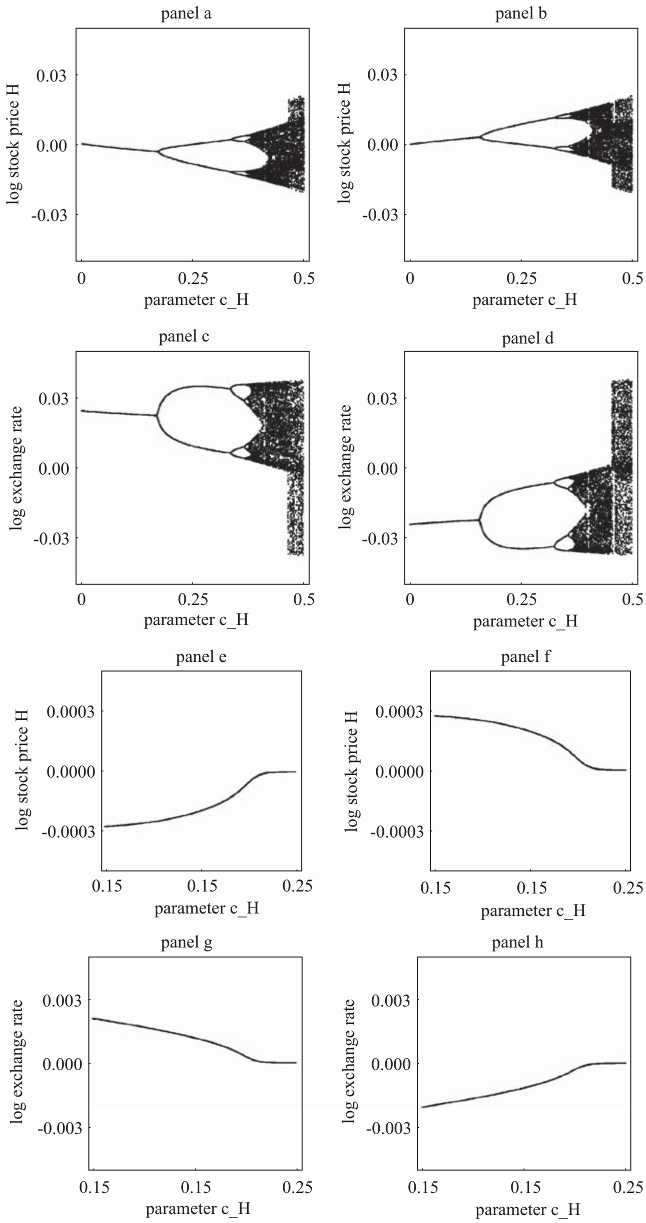

Fig. 4. The destabilizing/stabilizing effect of market interactions. Panels a and b (c and d) show bifurcation diagrams for the log stock price in country H (the log exchange rate) for two different sets of initial conditions. The parameter setting is $a^H=1$, $a^A=0.8$, $b^H=1$, $b^A=1.5$, $0<c^H<0.5$, $c^A=0.2$, $d=1$, $e=6$, $f=0.8$, $g=10\,000$ and $F^H=F^A=F^S=0$. In the bottom four panels we repeat these computations but now use $0.15<c^H<0.25$, $c^A=0.4$ and $e=0.5$.

(and with a large number of iterations). If there are no interactions, such a plot would qualitatively reproduce the graph of the 1D map underlying the dynamical system (17) (as in Fig. 1 f). Since markets interact and there is hence feedback from the stock markets to the foreign exchange market, the plot in panel d does not reduce exactly to such a cubic curve. Put differently, the stock markets create some kind of ‘‘deterministic noise’’ for the exchange rate process. Note, however, that the two pictures in panel d and Fig. 1 f are very similar to each other. Far from being rigorous, we may argue that this feature enables the persistence in the 3D model of the original bifurcation structure, and of the characteristic bull and bear price dynamics.

We do not push ahead with the analysis of such bifurcation mechanisms here, but leave further exploration to future research. As already discussed in Section 3.1, in the one-dimensional system (17) such phenomena are due to a homoclinic bifurcation of the repelling fundamental steady state, strictly related to the fact that the map is noninvertible, with two critical points. While in the 1D case much can be said about this bifurcation on analytical grounds, a similar analysis for the

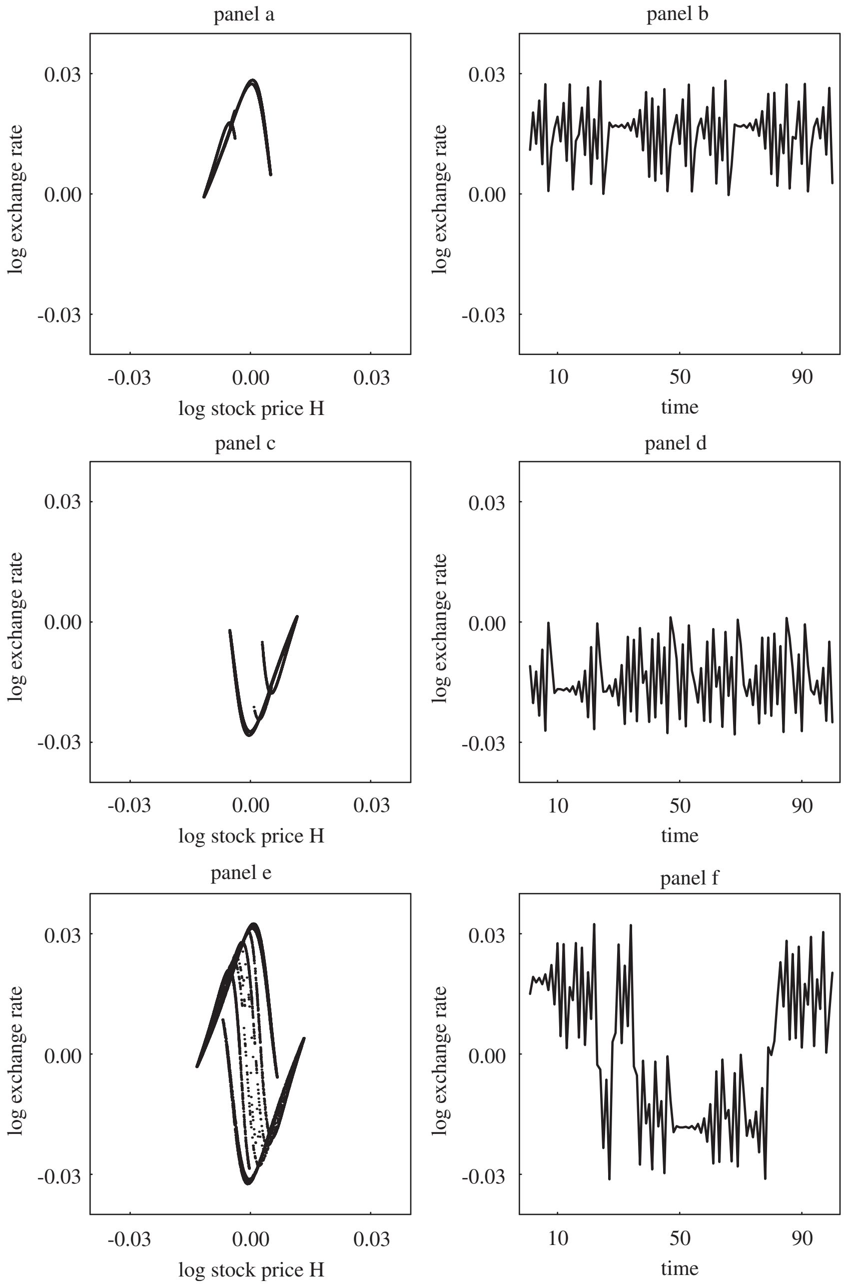

Fig. 5. The emergence of bull and bear market dynamics. In panels a and c the log exchange rate is plotted against the log stock price of country *H* for two different sets of initial conditions. Panels b and d present the corresponding evolution of the log exchange rate in the time domain. The parameter setting is $a^H = 1$, $a^A = 0.8$, $b^H = 1$, $b^A = 1.5$, $c^H = 0.4$, $c^A = 0.4$, $d = 1$, $e = 4.6$, $f = 0.8$, $g = 10\,000$ and $F^H = F^A = F^S = 0$. In panels e and f we do the same but now use one set of initial conditions and assume that $e = 5.3$.

complete model is much harder, due to the higher dimension of the dynamical system, and the fact that the 3D map $\mathcal{G}$ defined by (12)–(14) is no longer symmetric. Some interesting results can be obtained, however, in the intermediate cases where c^H, or c^A, is equal to zero, and thus two of the three state variables evolve as an independent two-dimensional system.[19]

[19] In a related but different 2D model, Tramontana et al. (2009) provide a computer-assisted proof of this homoclinic bifurcation, based on the properties of the so-called *critical curves* of noninvertible maps of the plane (see Mira et al., 1996), which represents the two-dimensional analogue of the critical points for one-dimensional maps.

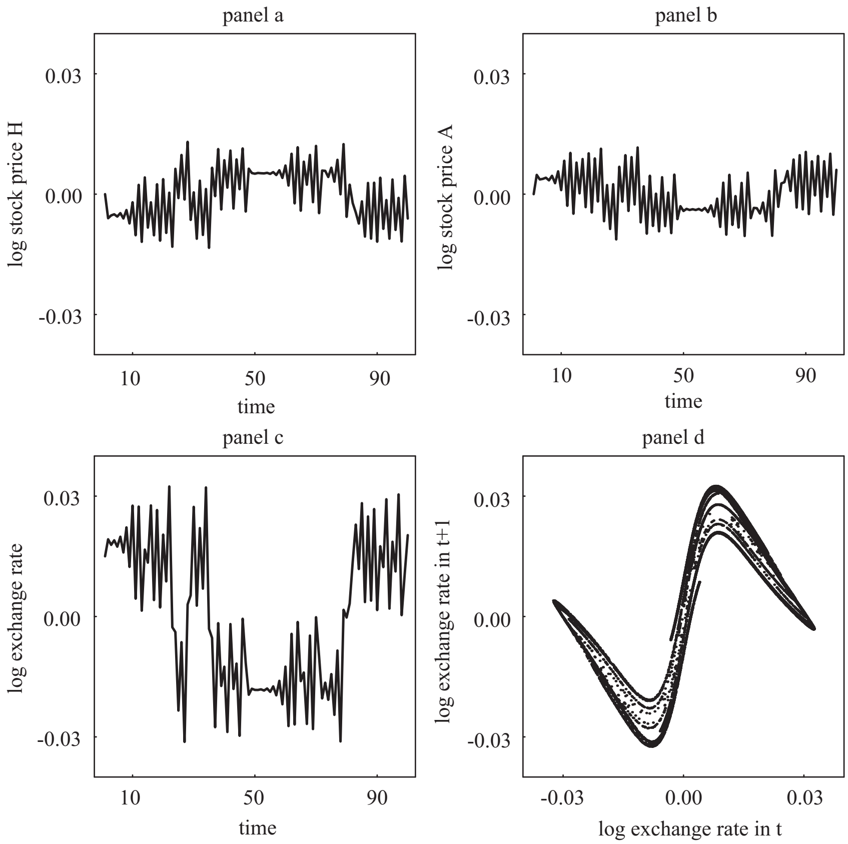

Fig. 6. Bull and bear market dynamics in action. Panels a–c show the evolution of the log stock price of country H, the log stock price of country A and the log exchange rate in the time domain, respectively. In panel d the log exchange rate at time step $t+1$ is plotted against the log exchange rate at time step t. The parameter setting is $a^H=1$, $a^A=0.8$, $b^H=1$, $b^A=1.5$, $c^H=0.4$, $c^A=0.4$, $d=1$, $e=5.3$, $f=0.8$, $g=10\,000$ and $F^H=F^A=F^S=0$.

4.3. "Sources" of instability

A final comment concerns the robustness of our analytical and numerical results under possibly different "sources" of instability. The main message from our dynamic analysis is that market interactions may be stabilizing or destabilizing, depending on the "strength" of speculative behavior of technical traders. We have observed this by starting from a situation with three independent markets, where the fundamental exchange rate equilibrium is unstable, while the fundamental stock market equilibria are stable. In such a situation, once markets are allowed to interact, the complicated dynamics originating in the foreign exchange market are the main driving force to the unstable and complicated price dynamics in the stock markets. Therefore, the current model specification cannot be directly used to explore whether the converse may happen, too, namely, a possible destabilizing effect to an otherwise stable foreign exchange market, due to the instability of stock markets. The phenomena that we have illustrated are indeed fairly general. One can easily set up simple variants[20] of the model, where technical traders are absent from the foreign exchange market, so that the speculative demand component in Eq. (7) reduces to the fundamentalist component $D^S_{F,t}=f(F^S-S_t)$. In the simplest case, a chartist demand component $D^{HH}_{C,t}:=\beta^H(P^H_t-F^H)$ from country H to stock market H is then introduced on the side of the fundamentalist component $D^{HH}_{F,t}$ defined by Eq. (2), together with a weighting mechanism analogous to (10). The resulting dynamical model has, in the case of no interactions ($c^H=c^A=0$), a structure that is formally identical to (15)–(17), where S_t and P^H_t are now driven by a linear and a cubic law of motion, respectively. Next, by introducing interactions, the impact of new chartist parameter β^H on the model dynamics is analogous to that summarized in Fig. 3 for parameter e.

5. Conclusions

Financial markets are characterized by highly volatile prices and repeatedly display severe bubbles and crashes. The chartist–fundamentalist approach offers a number of endogenous explanations for these challenging phenomena, by

[20] The example sketched below is available from the authors upon request.

stressing the interplay between the destabilizing impact of technical trading strategies and the mean reverting price behavior set in action by fundamentalists. The goal of our paper is to analyze the effect of such a basic determinant of price fluctuations within a system of internationally connected financial markets. This is achieved by studying a stylized deterministic dynamic model of two stock markets that are nonlinearly interwoven—by construction—*via* and *with* the foreign exchange market. Such connections are due to the existence of (domestic) stock market traders who trade abroad, based on observed stock price and exchange rate misalignments, and using the foreign exchange market in order to exchange currency. While the stock markets are modelled as simple as possible by means of linear equations, the foreign exchange market "works" in a nonlinear way, in that currency speculators switch between competing linear trading rules. The model results in a three-dimensional discrete-time dynamical system. The focus of our analysis is on how exchange rate movements generated by the interplay of heterogeneous speculators in the foreign exchange market spill over into the stock markets, and affect the overall behavior of the system. In order to get a first insight into this issue we have chosen, as a one-dimensional benchmark, a very simple model of chartist–fundamentalist interaction, similar in structure to the stylized setup of Day and Huang (1990). The main question is therefore how the behavior of stable markets, in which prices are close to their fundamental values, may be affected by connections to an unstable market, characterized by systematic deviations from the fundamental value, the interplay between heterogeneous speculators and, in particular, the destabilizing action of chartists who bet on the persistence of "bull" or "bear" market dynamics. One may wonder which of the original scenarios, that characterize each market "in isolation", prevails once the connections have been introduced, or whether the behavior of the resulting integrated system stays at some intermediate level. We have reported "mixed" results, despite the simplicity of the model. We have shown—by means of both analytical study and numerical experiments—that market interactions may destabilize stock markets, but may also play a stabilizing effect on the foreign exchange market and on the whole system of interacting markets. The nature of the effect is strictly related to the parameters of the model, in particular to parameter e, which governs the speculative demand of chartists. Our findings on the impact of market interactions may be summarized as follows.

- Steady state analysis (Section 3.2) reveals a possible stabilizing effect of market interactions for low values of the price adjustment parameters and of the chartist parameter e. Although one of the markets (the foreign exchange market) has originally an unstable fundamental equilibrium, coexisting with further equilibria, the integrated model displays a unique stable fundamental steady state. Put differently, if the strength of chartist speculation in the foreign exchange market is weak, establishing a connection with stable markets results in the stabilization of the whole system. This stabilizing effect is also confirmed by the numerical experiments in Section 4.1 (in particular the leftmost part of the bifurcation diagrams versus e, in Fig. 3, and the diagrams in Figs. 4 e–h).
- For larger values of e, two further nonfundamental steady states exist, which results in multiple equilibrium prices in each market. In this case, we observe a "mixed" effect on the steady state structure. On the one hand, connections with stable markets can no more bring the whole system back to "fundamentals", and destabilize the stock markets, too, where prices tend to deviate from their fundamental values in the long run. On the other hand, in the foreign exchange market a kind of stabilizing effect is still in action, at least as long as e is not too high, in the sense that steady-state misalignments are less pronounced than in the case of decoupled markets (see the middle part of the bifurcation diagrams in Figs. 3 e,f).
- For large values of e, numerical simulation reveals a destabilizing effect of interactions. In all markets, previously stable (fundamental or nonfundamental) steady states are replaced by periodic motion or complex endogenous dynamics. The possible deviations from the fundamentals are wider than in the case of no interaction. The amplitude of fluctuations increases with e (as shown in the rightmost part of the bifurcation diagrams in Fig. 3) and with the "strength" of interactions (one example is the bifurcation diagrams in Figs. 4 a–d). As long as e remains within a given range, fluctuations stay confined within the "bull" or the "bear" market regions. Afterwards, the interval covered by asymptotic price fluctuations is drastically enlarged, as an effect of a global bifurcation occurring for large e. This brings about a typical alternance of bull and bear market dynamics, with repeating bubbles and crashes around fundamentals in both the foreign exchange and the stock markets, as shown in Section 4.2. Markets that contain neither technical traders nor "behavioral nonlinearities" may then switch between bull and bear episodes, due to quite natural interaction with more speculative markets.

As already pointed out, our choice to make endogenous motion start from the foreign exchange market is purely conventional, and the same results could be obtained as well by assuming that heterogeneous speculators operate in one of the two stock markets. More generally, our model thus captures the way in which a simple mechanism of interaction between stock markets and the foreign exchange market may contribute to spreading or absorbing price fluctuations that originate in one market. It indicates therefore that stock market volatility (exchange rate volatility) may be caused to some extent by exchange rate changes (stock price movements), and suggests how such volatility may be related to the strength of interactions and the intensity of speculative demand. An immediate generalization of our setup would be that of introducing heterogeneous investors in all markets, together with exogenous noise on agents' demand and fundamental prices, in order to conduct a more thorough analysis on "how much" of the price volatility in each market can be ascribed to the link with other speculative markets. A further interesting extension concerns the impact, on stability and bifurcation routes, of a more general switching mechanism between technical and fundamental trading rules, that allows for *asynchronous* updating of beliefs, along the lines of Hommes et al. (2005). Finally, an important development is to explore the impact of regulatory measures, such as financial market liberalizations and central bank interventions, under different assumptions about agents' behavioral parameters. This can be done, for instance, along the lines of Wieland and Westerhoff (2005).

Acknowledgments

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Appendix A. Micro-foundation of demand functions

This appendix shows how the stylized demand functions assumed in the present paper can be justified within a mean–variance framework. Without loss of generality, we consider the portfolio problem faced by the (representative) fundamental investor from country A, who decides how much of his/her wealth to invest in stock A and how much in stock H. We assume, for simplicity, that the residual wealth is held as cash (or, equivalently, invested in a riskless asset with zero interest rate). In the following we denote by (lowercase) p_t^A, p_t^H and s_t the prices of the two assets and the exchange rate, respectively. Agent's wealth (denominated in currency A) evolves from time t to time $t+\Delta t$ according to

$$\Omega_{t+\Delta t}^A = \Omega_t^A + \Omega_t^{AA}\rho_{t+\Delta t}^A + \Omega_t^{HA}\xi_{t+\Delta t}^H$$

where Ω_t^{AA} and Ω_t^{HA} denote the amount of wealth invested in stock A and stock H, respectively, and where

$$\rho_{t+\Delta t}^A := \frac{p_{t+\Delta t}^A}{p_t^A} - 1, \quad \xi_{t+\Delta t}^H := \frac{p_{t+\Delta t}^H s_{t+\Delta t}}{p_t^H s_t} - 1$$

are the returns on stock A and on wealth invested in stock H, respectively (the latter depending also on the exchange rate return $s_{t+\Delta t}/s_t$). In the following we use log-returns, i.e. we set

$$\rho_{t+\Delta t}^A \approx \ln\left(\frac{p_{t+\Delta t}^A}{p_t^A}\right), \quad \xi_{t+\Delta t}^H \approx \ln\left(\frac{p_{t+\Delta t}^H}{p_t^H}\frac{s_{t+\Delta t}}{s_t}\right).$$

Assume the investor has CARA utility of wealth function, $U^A(\Omega_{t+\Delta t}^A) = -\exp(-\theta^A\Omega_{t+\Delta t}^A)$, where θ^A is the constant (absolute) risk aversion coefficient. If future (log-)returns are Gaussian in agent's beliefs, maximization of expected utility of wealth reduces to the following mean–variance problem:

$$\max_{\Omega_t^{AA},\Omega_t^{HA}}\left\{\mathbb{E}_t^A(\Omega_{t+\Delta t}^A) - \frac{\theta^A}{2}\mathbb{V}_t^A(\Omega_{t+\Delta t}^A)\right\}, \tag{22}$$

where $\mathbb{E}_t^A$ and $\mathbb{V}_t^A$ denote expectation and variance (of agent A), conditional upon information available at time t. As a fundamental trader, the agent believes that price A follows a (geometric) mean reverting process of the type:

$$\Delta\ln p_t^A := \ln p_{t+\Delta t}^A - \ln p_t^A = \eta_t^A(\ln\Phi_t^A - \ln p_t^A)\Delta t + \sigma_t^A\sqrt{\Delta t}\tilde{\varepsilon}_t^A,$$

where Φ_t^A is the fundamental price at time t, η_t^A and σ_t^A are time varying (and possibly state-dependent) mean reversion and volatility coefficients, respectively, and $\tilde{\varepsilon}_t^A \sim \mathcal{N}(0,1)$ is an i.i.d. Gaussian noise component. In a similar manner, price H and the exchange rate are assumed to evolve, respectively, according to

$$\Delta\ln p_t^H := \ln p_{t+\Delta t}^H - \ln p_t^H = \eta_t^H(\ln\Phi_t^H - \ln p_t^H)\Delta t + \sigma_t^H\sqrt{\Delta t}\tilde{\varepsilon}_t^H,$$

$$\Delta\ln s_t := \ln s_{t+\Delta t} - \ln s_t = \eta_t^S(\ln\Phi_t^S - \ln s_t)\Delta t + \sigma_t^S\sqrt{\Delta t}\tilde{\varepsilon}_t^S.$$

We thus obtain

$$\mathbb{E}_t^A(\Omega_{t+\Delta t}^A) = \Omega_t^A + \Omega_t^{AA}\eta_t^A(\ln\Phi_t^A - \ln p_t^A)\Delta t + \Omega_t^{HA}[\eta_t^H(\ln\Phi_t^H - \ln p_t^H) + \eta_t^S(\ln\Phi_t^S - \ln s_t)]\Delta t. \tag{23}$$

Assuming no correlations, in agent's beliefs, between the random terms $\tilde{\varepsilon}^A$, $\tilde{\varepsilon}^H$, $\tilde{\varepsilon}^S$ yields

$$\mathbb{V}_t^A(\Omega_{t+\Delta t}^A) = (\Omega_t^{AA})^2(\sigma_t^A)^2\Delta t + (\Omega_t^{HA})^2[(\sigma_t^H)^2 + (\sigma_t^S)^2]\Delta t. \tag{24}$$

Plugging (23) and (24) into (22), the following first-order conditions can be computed:

$$\eta_t^A(\ln\Phi_t^A - \ln p_t^A) - \theta^A(\sigma_t^A)^2\Omega_t^{AA} = 0,$$

$$\eta_t^H(\ln\Phi_t^H - \ln p_t^H) + \eta_t^S(\ln\Phi_t^S - \ln s_t) - \theta^A[(\sigma_t^H)^2 + (\sigma_t^S)^2]\Omega_t^{HA} = 0.$$

Note that $\Omega_t^{AA} = D_t^{AA}p_t^A$, $\Omega_t^{HA} = D_t^{HA}p_t^H s_t$, where D_t^{AA} and D_t^{HA} denote demand for asset A and asset H, respectively, expressed in units of stock. This finally yields the optimal demands:

$$D_t^{AA} = \frac{\eta_t^A}{\theta^A(\sigma_t^A)^2 p_t^A}(\ln\Phi_t^A - \ln p_t^A), \tag{25}$$

$$D_t^{HA} = \frac{\eta_t^H}{\theta^A[(\sigma_t^H)^2 + (\sigma_t^S)^2]p_t^H s_t}\left[(\ln\Phi_t^H - \ln p_t^H) + \frac{\eta_t^S}{\eta_t^H}(\ln\Phi_t^S - \ln s_t)\right]. \tag{26}$$

Eqs. (25) and (26) reveal the connections between agents' demands and their risk attitudes and beliefs about the processes of prices and exchange rates. The absence of correlations between different assets, in agents' beliefs, causes the investment decisions of the same speculator operating in two different stock markets to be decoupled. Including possible correlations would obviously result in interdependent investment decisions, jointly impacting on both prices.

From Eqs. (25) and (26), demand for stock A at a given point in time is thus essentially given by the product of a time-varying reaction coefficient by the log-price misalignment in market A, i.e.

$$D_t^{AA} = b_t^A(\ln\Phi_t^A - \ln p_t^A), \quad b_t^A := \frac{\eta_t^A}{\theta^A(\sigma_t^A)^2 p_t^A}. \tag{27}$$

The same holds for the case of the demand for stock H, with the addition of a term with a similar structure, capturing the misalignment on the foreign exchange market, i.e.:

$$D_t^{HA} = c_t^H[(\ln\Phi_t^H - \ln p_t^H) + \gamma_t^H(\ln\Phi_t^S - \ln s_t)], \quad c_t^H := \frac{\eta_t^H}{\theta^A[(\sigma_t^H)^2 + (\sigma_t^S)^2]p_t^H s_t}, \quad \gamma_t^H := \frac{\eta_t^S}{\eta_t^H}. \tag{28}$$

The simplified demand functions used in the paper, (5) and (3), are essentially (27) and (28), respectively, where the fundamental prices and the reaction coefficients b_t^A and c_t^H are assumed to be constant for analytical tractability, and the speed of mean reversion in different markets is the same (i.e. $\gamma_t^H \equiv 1$), in order to reduce the number of parameters.

The optimal portfolio of the trader from country H, investing in stock markets H and A, can be derived in a specular way, starting from wealth denominated in currency H. An even simpler mean–variance setup can be used to obtain optimal demand for currency H from foreign exchange speculators.

We add a few comments on how asset demand impacts on prices in our model. First, we are implicitly assuming that the net supply of each asset is zero. This represents a quite common assumption in such kind of models, and is consistent with our stylized setup, where the "risky assets" H and A pay no dividends, the risk-free rate of return is zero, and therefore agents' steady-state demand is zero, too. Assuming a strictly positive amount of assets H and A would require a more rich setup (where both dividend payments and the risk-free rate are explicitly taken into account) such that the risky assets earn a strictly positive excess return in the "fundamental" steady state solution. Such a "risk premium" would thus drive investors to hold positive amounts of these assets in their steady-state portfolios.[21] Second, it follows from our micro-foundation that Eqs. (1) and (4) relate price changes to agents' (excess) desired *asset holdings*. This simplifying view has often been adopted in the literature, too (see, e.g. Chiarella et al., 2009 and references therein). From a different perspective, one could model price adjustments as related to net *orders*, i.e. *changes* in asset holdings, and rewrite, e.g. Eq. (1) as follows:

$$P_{t+1}^H = P_t^H + a^H(D_{F,t}^{HH} + D_{F,t}^{HA} - Z_{t-1}^H), \tag{29}$$

where $Z_{t-1}^H = D_{F,t-1}^{HH} + D_{F,t-1}^{HA}$. This approach would largely increase the dimension of the dynamical system, unless one strongly simplifies Eq. (29) by simply setting $Z_{t-1}^H = 0$, i.e. by neglecting the effect of agents' accumulated positions.[22] We remark that such modelling issues are strictly connected, in general, to the underlying view about the role and behavior of the market maker, as can be argued from the models recently developed by Franke and Asada (2008) and Zhu et al. (2009).[23]

[21] Note that this more general case can usually be reduced to the case of "zero" supply, via suitable changes of parameters (see, e.g. Hommes, 2006).

[22] We thank one of the referees for suggesting us this alternative interpretation, which may help to reconcile the two approaches based on "orders" and on "positions", respectively.

[23] Although the "order-based" price setting equation (29) may look more realistic than the "position-based" equation (1), it neglects the possible price impact of market maker inventories. Roughly speaking, market maker accumulated inventory is inversely related to agents' aggregate position. Given that market makers seek to limit their inventories, this observation provides an intuitive justification of price adjustment equation (1). See also the conceptual discussion in Franke (2008).

Appendix B. Proof of Proposition 1

The proof of (a) is straightforward.

(b) The one-dimensional nonlinear map (17), which governs the exchange rate dynamics in the case of $c^H = c^A = 0$, can be rewritten as

$$S_{t+1} = S_t + d\left[f(F^S - S_t) + \frac{(e+f)(S_t - F^S)}{1+g(S_t - F^S)^2}\right]. \tag{30}$$

By introducing deviations from the fundamental value, $x := S - F^S$, Eq. (30) takes the form $x_{t+1} = h(x_t)$, where

$$h(x) = x + d\left[-fx + \frac{(e+f)x}{1+gx^2}\right] = (1-df)x + \frac{d(e+f)x}{1+gx^2}. \tag{31}$$

For strictly positive e, f, g, map $h(x)$ admits three fixed points, solutions of $h(x)=x$, given by $\overline{x}=0$, $\overline{x}_l = -\sqrt{e/(fg)}$, $\overline{x}_u = \sqrt{e/(fg)}$. It follows that the dynamical system (30) has three steady states, namely $\overline{S} = F^S$, $\overline{S}_l = F^S - \sqrt{e/(fg)}$, $\overline{S}_u = F^S + \sqrt{e/(fg)}$.

(c) In order to study the stability properties of the steady states, we consider the derivative of map (31)

$$\frac{dh(x)}{dx} = 1 - df + d(e+f)\frac{1-gx^2}{(1+gx^2)^2}. \tag{32}$$

Evaluation of (32) at the fundamental steady state yields

$$\left.\frac{dh}{dx}\right|_{x=0} = 1 + de > 1,$$

which reveals that the fundamental steady state is always unstable. With regard to the nonfundamental steady states $\overline{x}_l = -\sqrt{e/(fg)}$, $\overline{x}_u = \sqrt{e/(fg)}$, note first that $g\overline{x}_l^2 = g\overline{x}_u^2 = e/f$. Therefore we obtain

$$\left.\frac{dh}{dx}\right|_{x=\overline{x}_l} = \left.\frac{dh}{dx}\right|_{x=\overline{x}_u} = 1 - \frac{2def}{e+f} < 1.$$

The nonfundamental steady states are thus LAS if and only if $1 - 2def/(e+f) > -1$, i.e.

$$e(df-1) < f. \tag{33}$$

By taking the chartist parameter e as a bifurcation parameter, the stability condition (33) holds for any $e>0$ if $df \le 1$. In the opposite case, $df > 1$, the stability condition is satisfied only for $e < e_{Flip} := f/(df-1)$, at which value a period doubling bifurcation occurs.

Appendix C. Proof of Proposition 2

(a) The steady states $(\overline{P}^H, \overline{P}^A, \overline{S})$ of the dynamical system (11) are the solutions of the following system of equations

$$\overline{P}^H = G^H(\overline{P}^H, \overline{S}), \quad \overline{P}^A = G^A(\overline{P}^A, \overline{S}), \quad \overline{S} = G^S(\overline{P}^H, \overline{P}^A, \overline{S}), \tag{34}$$

which can be rewritten as follows:

$$(b^H + c^H)(F^H - \overline{P}^H) + c^H(F^S - \overline{S}) = 0, \tag{35}$$

$$(b^A + c^A)(F^A - \overline{P}^A) + c^A(\overline{S} - F^S) = 0, \tag{36}$$

$$c^H(F^H - \overline{P}^H + F^S - \overline{S})\exp(\overline{P}^H) - c^A(F^A - \overline{P}^A + \overline{S} - F^S)\exp(\overline{P}^A - \overline{S}) + \frac{e(\overline{S} - F^S)}{1+g(F^S - \overline{S})^2} + \frac{fg(F^S - \overline{S})^3}{1+g(F^S - \overline{S})^2} = 0. \tag{37}$$

Note that from (35) and (36) it follows, respectively, that

$$\overline{P}^H - F^H = \frac{c^H}{b^H + c^H}(F^S - \overline{S}), \quad \overline{P}^A - F^A = \frac{c^A}{b^A + c^A}(\overline{S} - F^S). \tag{38}$$

The quantities $(F^H - \overline{P}^H + F^S - \overline{S})$ and $(F^A - \overline{P}^A + \overline{S} - F^S)$ in (37) can therefore be rewritten, respectively, as

$$(F^H - \overline{P}^H + F^S - \overline{S}) = -\frac{b^H}{b^H + c^H}(\overline{S} - F^S),$$

$$(F^A - \overline{P}^A + \overline{S} - F^S) = \frac{b^A}{b^A + c^A}(\overline{S} - F^S),$$

while the quantities $\exp(\overline{P}^H)$ and $\exp(\overline{P}^A - \overline{S})$ in the same equation become

$$\exp(\overline{P}^H) = \Phi^H \exp\left[-\frac{c^H}{b^H + c^H}(\overline{S} - F^S)\right],$$

$$\exp(\overline{P}^A - \overline{S}) = \frac{\Phi^A}{\Phi^S}\exp\left[-\frac{b^A}{b^A + c^A}(\overline{S} - F^S)\right],$$

where $\Phi^H := \exp(F^H)$, $\Phi^A := \exp(F^A)$, $\Phi^S := \exp(F^S)$. By introducing deviations from fundamentals, $x^H := P^H - F^H$, $x^A := P^A - F^A$, $x := S - F^S$, Eq. (38) simplifies into (18), while (37) can be rewritten as

$$\frac{e\overline{x} - fg\overline{x}^3}{1 + g\overline{x}^2} - \frac{c^H b^H}{b^H + c^H}\overline{x}\Phi^H \exp\left(-\frac{c^H}{b^H + c^H}\overline{x}\right) - \frac{c^A b^A}{b^A + c^A}\overline{x}\frac{\Phi^A}{\Phi^S}\exp\left(-\frac{b^A}{b^A + c^A}\overline{x}\right) = 0,$$

or $\overline{x}[\alpha(\overline{x}) - \beta(\overline{x})] = 0$, that is Eq. (19), where $\alpha(x)$ and $\beta(x)$ are defined by (20).

(b) Eq. (18) determines equilibrium stock prices as functions of equilibrium exchange rates, where the latter are the solutions of Eq. (19). Eq. (19) always admits the solution $x = 0$, which corresponds to the fundamental steady state $x^H = x^A = x = 0$. Further nonfundamental steady states are related to the possible solutions of $\alpha(x) = \beta(x)$. We discuss the conditions for their existence by means of Fig. 2. Note first that in the case $c^H = c^A = 0$, quantity $\beta(x)$ is identically equal to zero, so that we are back to the case of no interactions where two symmetric exchange rate equilibria exist, $x = \mp\sqrt{e/(fg)}$ (panel a). If either c^H or c^A is strictly positive, then $\beta(x)$ is represented by a (strictly decreasing) negative exponential function,[24] and the graphical visualization of equation $\alpha(x) = \beta(x)$ is as in the panels from b to d. The bell-shaped function $\alpha(x)$ intersects the vertical axis at level $\alpha(0) = e$, while $\beta(x)$ intersects at the level

$$\beta(0) = \frac{\Phi^H b^H c^H}{b^H + c^H} + \frac{\Phi^A}{\Phi^S}\frac{b^A c^A}{b^A + c^A}. \tag{39}$$

Note that the smaller c^H, c^A are, the lower the ordinate $\beta(0)$ of the intersection. Therefore, as long as e is sufficiently large (panel c) or market interactions are sufficiently weak, i.e. c^H or c^A are small enough (panel d), equation $\alpha(x) = \beta(x)$ will still admit two solutions (and the system will then have three steady states) similarly to the case of no interaction. In particular, $e \geq \beta(0)$ represents a sufficient condition for the existence of multiple solutions. By contrast, if e is sufficiently small, there will be no solutions to $\alpha(x) = \beta(x)$ (panel b) and the system will admit a unique steady state.

We add a few comments on the proof of this result. Consider, for instance, the transition from panels b to c, which is obtained by increasing parameter e for fixed values of c^H, c^A. The exact bifurcation value of e at which two new steady states appear can only be computed numerically. Since $\alpha(x)$ is represented by a bell-shaped curve, symmetric with respect to the vertical axis, and $\beta(x)$ is negatively sloped, the tangency between $\alpha(x)$ and $\beta(x)$, and the subsequent crossing, will take place for strictly positive x, and the two new solutions will both be positive, initially. For $e = \beta(0)$, one of the two solutions coincides with $x = 0$, whereas for higher e, the two solutions will have opposite signs. The bifurcation mechanism is in fact that of a *saddle-node* bifurcation, followed by a *transcritical* bifurcation of the fundamental steady state. If $\beta(x)$ is sufficiently flat (as is the case in Fig. 2), the two bifurcations may occur very close to each other. In any case, the ''final'' result is similar to that of a *pitchfork* bifurcation.

Appendix D. Proof of Proposition 3

The Jacobian matrix of the system (11), evaluated at the fundamental steady state $\mathbf{F} := (F^H, F^A, F^S)$, is the following:

$$D\mathcal{G}(\mathbf{F}) = \begin{bmatrix} 1 - a^H(b^H + c^H) & 0 & -a^H c^H \\ 0 & 1 - a^A(b^A + c^A) & a^A c^A \\ -d\Phi^H c^H & d\dfrac{\Phi^A}{\Phi^S}c^A & 1 + d\left[e - \Phi^H c^H - \dfrac{\Phi^A}{\Phi^S}c^A\right] \end{bmatrix}. \tag{40}$$

For notational purposes, it is convenient to define the following aggregate quantities:

$$q^H := 1 - a^H(b^H + c^H), \quad q^A := 1 - a^A(b^A + c^A), \tag{41}$$

$$u^H := a^H(c^H)^2 d\Phi^H, \quad u^A := a^A(c^A)^2 d\frac{\Phi^A}{\Phi^S}, \tag{42}$$

$$k = 1 - d\left[\Phi^H c^H + \frac{\Phi^A}{\Phi^S}c^A\right]. \tag{43}$$

[24] The graph of $\beta(x)$, however, looks approximately flat at the scale of Fig. 2.

With some algebra, the characteristic polynomial of $D\mathcal{G}(\mathbf{F})$ can be rewritten as

$$\mathcal{P}(\lambda) = \lambda^3 + m_1\lambda^2 + m_2\lambda + m_3, \tag{44}$$

where

$$m_1 = -(q^H + q^A + k + de),$$

$$m_2 = (q^H + q^A)(k + de) + q^H q^A - (u^H + u^A),$$

$$m_3 = q^H u^A + q^A u^H - q^H q^A(k + de).$$

Note that from our assumption (21) and from the fact that a^H, b^H, a^A, b^A, are strictly positive and c^H, c^A are nonnegative, it follows that $|q^H| < 1$, $|q^A| < 1$, which implies $|q^H q^A| < 1$, $|q^H + q^A| < 2$. The following set of inequalities imposed on the coefficients of the characteristic polynomial (44) provide a necessary and sufficient condition for all the eigenvalues of (40) to be of modulus smaller than unity (Farebrother, 1973), which implies a locally asymptotically stable steady state:

$$1 + m_1 + m_2 + m_3 > 0, \tag{45}$$

$$1 - m_1 + m_2 - m_3 > 0, \tag{46}$$

$$1 - m_2 + m_3(m_1 - m_3) > 0, \tag{47}$$

$$m_2 < 3. \tag{48}$$

Conditions (45)–(48) can be rewritten in terms of the parameters of the model, respectively

$$(1 - q^H)(1 - q^A) - u^H(1 - q^A) - u^A(1 - q^H) - (1 - q^H)(1 - q^A)(k + de) > 0, \tag{49}$$

$$(1 + q^H)(1 + q^A) - u^H(1 + q^A) - u^A(1 + q^H) + (1 + q^H)(1 + q^A)(k + de) > 0, \tag{50}$$

$$\begin{aligned} &1 - q^H q^A + u^H + u^A \\ &\quad - (q^H u^A + q^A u^H)^2 - (q^H u^A + q^A u^H)(q^H + q^A) \\ &\quad - [(q^H u^A + q^A u^H)(1 - 2q^H q^A) + (q^H + q^A)(1 - q^H q^A)](k + de) + q^H q^A(1 - q^H q^A)(k + de)^2 > 0, \end{aligned} \tag{51}$$

$$q^H q^A - (u^H + u^A) + (q^H + q^A)(k + de) < 3. \tag{52}$$

We now discuss conditions (49)–(52) and show, in particular, that they are simultaneously satisfied under condition (21), provided that parameters d, e are not too large. Note that quantities u^H, u^A, and k depend on d, according to (42) and (43), respectively. In the following we assume that d is as small as necessary, so that in particular quantity k, defined in (43), is strictly positive. In general we will regard the left-hand side of each of conditions (49)–(52) as a function of parameter $e > 0$ for sufficiently small values of d and fixed values of the other parameters (satisfying all of the assumed restrictions).

Condition (49) can be rewritten as

$$(1 - q^H)(1 - q^A)de < (1 - q^H)(1 - q^A)(1 - k) - u^H(1 - q^A) - u^A(1 - q^H).$$

In the above equation, the term on the right-hand side is strictly positive for any $d > 0$. To prove this, simply note that

$$0 \le \frac{a^H c^H}{1 - q^H} = \frac{a^H c^H}{a^H(b^H + c^H)} < 1, \quad 0 \le \frac{a^A c^A}{1 - q^A} = \frac{a^A c^A}{a^A(b^A + c^A)} < 1$$

and therefore

$$1 - k = d\left[\Phi^H c^H + \frac{\Phi^A}{\Phi^S} c^A\right] > d\left[\Phi^H c^H \frac{a^H c^H}{1 - q^H} + \frac{\Phi^A}{\Phi^S} c^A \frac{a^A c^A}{1 - q^A}\right] = \frac{u^H}{1 - q^H} + \frac{u^A}{1 - q^A}.$$

Moreover, since the quantity $(1 - q^H)(1 - q^A)$ is strictly positive, too, it follows that condition (49) is satisfied for sufficiently small e, namely

$$e < \frac{1}{d}\left[1 - k - \frac{u^H}{1 - q^H} - \frac{u^A}{1 - q^A}\right] = \Phi^H \frac{b^H c^H}{b^H + c^H} + \frac{\Phi^A}{\Phi^S}\frac{b^A c^A}{b^A + c^A} := e^*. \tag{53}$$

Condition (50) can be rewritten as

$$(1 + q^H)(1 + q^A)(1 + k) - u^H(1 + q^A) - u^A(1 + q^H) + (1 + q^H)(1 + q^A)de > 0.$$

Since quantities $1 + q^H = 2 - a^H(b^H + c^H)$ and $1 + q^A = 2 - a^A(b^A + c^A)$ are strictly positive under our assumptions, the coefficient of the term containing parameter e is strictly positive, too. The sum of the remaining terms on the left-hand side is strictly positive if

$$1+k-\left(\frac{u^H}{1+q^H}+\frac{u^A}{1+q^A}\right)>0,$$

which results in

$$d<2\left[\Phi^H c^H\frac{2-a^Hb^H}{2-a^H(b^H+c^H)}+\frac{\Phi^A}{\Phi^S}c^A\frac{2-a^Ab^A}{2-a^A(b^A+c^A)}\right]^{-1}:=d^*.$$

Under such a restriction on d, condition (50) is satisfied for any $e>0$.

Condition (51) is much more complicated to deal with, but it can be noted that for $d,e\to 0$ (in which case $k\to 1$, $u^H,u^A\to 0$), the left-hand side becomes

$$(1-q^Hq^A)(1-q^H)(1-q^A),$$

which is strictly positive under our assumptions (21). By continuity arguments, (51) will then be satisfied for sufficiently small values of d and e.

Condition (52) is obviously satisfied when $q^H+q^A\leq 0$, i.e. $a^H(b^H+c^H)+a^A(b^A+c^A)\geq 2$. Assume now $q^H+q^A>0$, in which case the left-hand side of (52) strictly increases with e. Since our assumptions imply $|q^Hq^A|<1$, $0<q^H+q^A<2$, $0<k<1$, we obtain

$$q^Hq^A-(u^H+u^A)+(q^H+q^A)k<3.$$

It follows that when $q^H+q^A>0$, Eq. (52) is satisfied for sufficiently small e, namely for

$$e<\frac{1}{d}\left[\frac{3+(u^H+u^A)-q^Hq^A}{q^H+q^A}-k\right]:=e^{**}. \tag{54}$$

Our assumptions about aggregate parameters q^H, q^A, also imply that threshold e^{**} defined in Eq. (54) is larger than quantity e^* in (53). To see this, note that (since $|q^Hq^A|<1$, $0<q^H+q^A<2$) the following inequalities hold:

$$de^{**}+k=\frac{3+(u^H+u^A)-q^Hq^A}{q^H+q^A}>1>1-\frac{u^H}{1-q^H}-\frac{u^A}{1-q^A}=de^*+k.$$

This implies that condition (54) is fulfilled whenever (53) holds, and therefore condition (52) is redundant.

Our analysis proves that the fundamental steady state is LAS for sufficiently small d and e. It also provides a rough picture of how local bifurcations may occur. Assume that d is as small as necessary (in particular $d<d^*$) and assume also that by increasing parameter e condition (51) is satisfied at least for $0<e\leq e^*$ (this is actually the case of the parameters used for the numerical examples in Fig. 3). In this case, the steady state loses stability for $e=e^*$. Note that bifurcation value e^* is equal to quantity $\beta(0)$ (see Eq. (39) in Appendix C). This means that in this case the loss of stability occurs precisely when one of the two newborn nonfundamental steady states collides with the fundamental steady state. Numerical and graphical analysis suggests that this contact corresponds to a transcritical bifurcation of the fundamental steady state (see also Appendix C).

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