Robust Mathematical Formulation of Agent-Based Computational Economic Market Models

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Abstract

In science and especially in the economic literature, agent-based modeling has become a widely used modeling approach. These models are often formulated as a large system of difference equations. In this study, we discuss two aspects of numerical modeling for two agent-based computational economic market models: the Levy-Levy-Solomon model and the Franke-Westerhoff model. We derive time-continuous formulations of both models and, for the Levy-Levy-Solomon model, we discuss the impact of the time-scaling on the model behavior. For the Franke-Westerhoff model, we proof that a constraint required in the original model is not necessary for stability of the time-continuous model. It is shown that a semi-implicit discretization of the time-continuous system preserves this unconditional stability. In addition, this semi-implicit discretization can be computed at cost comparable to the original model.

Keywords: agent-based models, Monte Carlo simulations, time scaling, continuous formulation, continuous limit, numerical solver

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1 Introduction

Agent-based computational economic market (ABCEM) models have become a popular modeling tool in economic research. They can be regarded as large dynamical systems formulated as difference equations. The model by Stigler [43] may be seen as the first ABCEM model but generally the model by Kim and Markowitz [31, 41] is referred to as the first modern ABCEM model. ABCEM models are a notable class in the research field econophysics. The general goal of ABCEM models is to reproduce persistent statistical features present in financial data all over the world which are known as stylized facts. Possible research questions include: evaluate the kind of stylized facts microscopic agent dynamics create; and estimate the impact of regulations on a financial market. Thanks to Monte Carlo simulations, it is possible to study the time evolution of statistical quantities such as the wealth distribution or the stock return distribution. The importance of agent-based modeling in economics has been emphasized by several authors [20, 28, 44].

The drawback of this procedure is that these empirical results solely rely on computational experiments. In addition, the sensitivity of many ABCEM models w.r.t. their parameters, observed in a tendency towards blow ups for particular choices of model parameters, motivated this work. The deeper study of these phenomena has revealed that this behavior often originates from the time stepping scheme of difference equations which can be viewed as explicit Euler discretizations with a normalized time step of one. It is a well known property of the explicit Euler method to perform poorly in the case of stiff differential equations. For this reason, we believe that a time continuous formulation of difference equations helps understanding several numerical issues. Albeit, we have to recognize that starting from a difference equation, the continuous formulation, respectively the continuum limit, is not uniquely defined. To our knowledge, this approach is the first work within the ABCEM community addressing this important issue.

For these reasons, the goal of this paper is to study the continuous formulation of ABCEM models and to present strategies for a robust mathematical formulation of ABCEM models. Clearly these aspects apply to a wider class of agent-based models. In this work we analyze the following issues:

1. **Continuous formulation and limit**: We discuss the connections of difference equations and differential equations. In particular, we emphasize that the continuum formulation of a difference equation is not unique. In addition, we claim that a continuous models, such as stochastic and ordinary differential equations, have several advantages in comparison to difference equations.

2. **Numerical discretization and solver**: Numerical discretization include time discretization schemes of stochastic and ordinary equations as well as discretizations of differential occuring within formulas. Numerical solvers include numerical solvers for non-linear equations. We highlight that the choice of numerical discretizations and solvers is paramount, e.g. for the solution of stiff ordinary differential equations either explicit methods with sufficiently small time steps or implicit methods are required.

The choice to formulate an ABCEM model continuously or as a difference equation is a questions of modeling. Difference equations are able to model a time evolution of quantities with a fixed time lag. In contrast, a time continuous model, e.g. an ordinary differential equation (ODE) or stochastic differential equation (SDE), may be discretized using various time steps and is thus suitable to model the same quantity resolving different time scales. Any difference equations can be interpreted as a scaled Euler-type discretization of a time continuous differential equation. Note that starting from a difference equation, the time continuous formulation is not unique. Furthermore, the advantage of a time continuous formulation is that it may be used to explain instabilities on the level of the difference equations and thus to guide the choice of appropriate discretization schemes.

Finally, we emphasize that a time continuous dynamical system may be translated to mesoscopic descriptions modeled using partial differential equations (PDEs) [32, 40]. This limit process leading from microscopic dynamics to a mesoscopic description is at the heart of kinetic theory which has been successfully applied to several ABCEM models in the past [1, 2, 38, 46, 48]. Thus, one may see this work as a first step from ABCEM models, mostly formulated as difference equations, to financial market models in the physical or mathematical literature, modeled as PDEs.

In this study we exemplarily discuss the continuous formulation and limit using the example of the Levy-Levy-Solomon model [33] and Franke-Westerhoff model [22]. The Levy-Levy-Solomon model is one of the most influential ABCEM models and an early example of an ABCEM model in general [41]. The Levy-Levy-Solomon model considers the wealth evolution of agents and the stock price evolution. Furthermore, each agent has to decide in each time step on the optimal asset allocation between the stocks and the asset class bonds. The stock price is fixed in each time step by the clearance mechanism that perfectly matches supply and demand and can be consequently seen as a rational market. The authors claimed that their model is able to reproduce several stylized facts from financial markets such as fat-tails in asset returns. It has been documented in [52] that the stock price returns are normally distributed and that the model exhibits finite-size effects. In comparison to the Levy-Levy-Solomon model, the Franke-Westerhoff model is rather recent and has been first introduced in 2009 [22]. The model tracks the time evolution of two agent groups and not of an individual agent as in the Levy-Levy-Solomon model. The stock price is modeled as a stochastic difference equation and is thus modeled as a disequilibrium model. The Franke-Westerhoff model is fully described by a system of three difference equations. It has been documented that the Franke-Westerhoff model is able to reproduce several stylized facts. The reason to choose the Franke-Westerhoff model in this study is on the one hand the simplicity of this model and on the other hand the prototypical nature of this model especially w.r.t. the stock price update mechanism. We have selected the Levy-Levy-Solomon model in this work not only because of the popularity but as well in virtue of the wealth evolution of agents.

The paper is structured as follows. In the next section, we discuss the connection of difference equations and time continuous differential equations in the context of disequilibrium financial market models. Furthermore, we shortly discuss the passage from ABCEM models to partial differential equations. In section three, we present the continuous formulations and continuous limits of the Levy-Levy-Solomon and Franke-Westerhoff model. We specifically discuss the impact of different time scalings in the Levy-Levy-Solomon model. Furthermore, we show that a naive numerical discretization of the time continuous Franke-Westerhoff model leads to blow ups. Therefore, we introduce a semi-implicit discretization of the time continuous Franke-Westerhoff model and show that the qualitative output coincides with the original model. We finish this work with a short conclusion.

2 Mathematical Perspective on ABCEM Models

We introduce the connection of ABCEM models to time continuous dynamical systems. In particular, we state the advantages of time continuous models in comparison to difference equations.

As mentioned before, ABCEM models may be interpreted as discretized dynamical systems. Many models in literature neglect the time dependence respectively normalize the time step to one [22, 27, 29]. Then in fact, many models are rather formulated by difference equations. In this section, we lay out the connection between difference equations and discretized differential equations. Furthermore, we introduce a rather general model for an irrational market which is broadly used in literature. Finally, we give a short outlook on possible numerical discretization strategies and discuss the advantage of discretized differential equations in comparison to difference equations.

In genaral one might refer to a market as irrational market if the fixed stock price in each time step does not clear all buy or sell orders. This corresponds to the situation that the aggregated excess demand is non-zero. An early example of an irrational market or disequilibrium model is the Beja and Goldman model [6]. In fact, the Beja and Goldman model can be derived from a rational market model where supply equals demand. Mathematically, such a differential model can be seen as a relaxation of the algebraic demand supply relation. For detailed discussion of the Beja and Goldman model we refer to [6, 47]. In order to be more explicit we give an example of the price adjustment rule present in many models e.g. [3, 9, 17, 35]

$$S(t_{k+1}) = S(t_k) + \Delta t \ ED(t_k). \tag{1}$$

Often the time step Δt is normalized to one such that we are faced with a difference equation. The previous update rule (1) can be interpreted as an explicit Euler discretization of the ODE

$$\frac{d}{dt} S = ED,$$

provided that the aggregated excess demand is deterministic. In the case of the Franke-Westerhoff model [24], the aggregated excess demand is stochastic and thus this pricing mechanism cannot be seen as an approximation of an ODE. The pricing rule of the Franke-Westerhoff model can be interpreted as discretized SDE. For details on this specific case, we refer to appendix A.2. A very general model of an irrational market has been introduced by Trimborn et al. in [47]. It is given by the following SDE

$$dS = F(S, ED) \ dt + G(S, ED) \ dW, \tag{2}$$

with Wiener process W and arbitrary functions F and G. Notice, that (1) is a special case of the model (2). We use the usual notation for Itô stochastic differential equations. Many market mechanism of ABCEM models are special cases of model (2), for example the models presented in [3, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 27, 30, 35, 36, 37, 39, 42, 51]. The simplest discretization of such an SDE (3) is the Euler-Maruyama method. Applying the Euler-Maruyama method to equation (2), we obtain:

$$S(t + \Delta t) = S(t) + \Delta t \ F(S(t), ED(t)) + \sqrt{\Delta t} \ G(S(t), ED(t)) \ \eta, \quad \eta \sim \mathcal{N}(0, 1) \tag{3}$$

for a fixed time $t > 0$ and time step $\Delta t > 0$. In the case of a fully deterministic model, the numerical scheme (3) is identical to the standard Euler method.
From a mathematical perspective, we stress that more sophisticated numerical methods for equation (2) exist, which may improve the quality of approximation remarkably [26, 49]. In particular for the case of stiff SDEs or ODEs, one should use implicit solvers to prevent stability problems.

In the ABCEM literature, most models rely on the explicit Euler (in case of deterministic dynamics) or Euler-Maruyama (in case of stochastic dynamics) discretizations. Often, the numerical approximation is rescaled and fixed such that the time step is set to one. Hence, in ABCEM literature, we are rather faced with difference equations of the following type

$$S_{k+1} = S_k + \bar{F}(S_k, ED_k) + \bar{G}(S_k, ED_k)\ \eta, \tag{4}$$

than with differential equations. The model (4) represents a discretized version of the model (2) with discretizations $\bar{F}, \bar{G}$ of functions F, G. Here $k \in \mathbb{N}$ is the index of the discretized time steps $S_k = S(t + k\ \Delta t)$ for a fixed initial time t and time step $\Delta t > 0$. Finally, we would like to stress that a time continuous model is not only advantageous from a numerical perspective but enables the user to simulate the model on differently coarse time levels by simply adapting the time step in the numerical scheme.

2.1 Connection to Partial Differential Equations

A further advantage of time continuous ODE or SDE models is the possibility to pass to the PDE. Such an description enables the analyses of the the PDE with several mathematical tools. For example, it may be of interest to study the long time dynamics of the system and to derive analytical results such as the steady state distribution. These results may help to understand the model behavior and especially the impact of the agent design on the simulation output.

In the case of a stochastic process, e.g. defined by an SDE, the law or probability density is defined by a PDE. This PDE is usually of Fokker-Planck type. This passage from SDEs to a deterministic probabilistic description is well understood by the so called Feynman-Kac formula [32]. We assume that all needed regularity assumptions of the SDE (2) are satisfied. Then the corresponding Fokker-Planck equation reads

$$\begin{aligned} &\partial_t h(t,s) + \partial_s (F(s,ED)\ h(t,s)) = \frac{1}{2} \partial_{ss} (G(s,ED)^2\ h(t,s)), \\ &h(0,0) = h_0(s). \end{aligned}$$

Thus, $h(t,s),\ s \in \mathbb{R},\ t > 0$ denotes the probability density of the stochastic process defined in (2). The dimension of the phase space of h is directly linked to the dimension of the SDE system (2). Hence, a large dynamical system can be translated into a highly dimensional Fokker-Planck equation. The drawback of a large dimensional PDE are high computational costs and thus for large dimensions there is no computational benefit in comparison to the original model.

Kinetic Theory For agent-based models which consider a large number of agents, one is able to circumvent this problem by using existent concepts from kinetic theory to derive a reduced PDE model out of a large particle dynamics. One method is the mean field limit which corresponds to the situation of an infinite number of agents [25]. This limit can be applied provided that a certain symmetry structure of the agent-based model is given. Alternatively, one may derive Boltzmann type equations out of agent dynamics as shown in [40]. Examples of kinetic models derived from agent-based models are [1, 2, 40, 46, 48].

3 Continuum Limit and Robust Formulation

In this section we discuss the robust formulation of ABCEM models with help of the Levy-Levy-Solomon model and the Franke-Westerhoff model. As pointed out in the introduction are both models structurally very different. The LLS model considers N agents whereas the Franke-Westerhoff model can be regarded as a two agent model.
We introduce possible continuous formulations of the LLS and Franke-Westerhoff model. Furthermore, we show the impact of different continuous formulations on the wealth evolution in the LLS model. In addition, we discuss the impact of different numerical methods on the model behavior of the Franke-Westerhoff model.

All presented results have been generated with the SABCEMM simulator which is freely available on GitHub [45]. For the used pseudo random number generators used in obtaining the results, please confer to table 4.

3.1 Continuum Limit

In this section, we introduce time continuous versions of the LLS and Franke-Westerhoff model. As usual in the ABCEM literature, the models are originally formulated as difference equations. Obviously, such a continuum limit is not uniquely defined and, in the case of the LLS model, we discuss several different time discretizations in the agent dynamic. Thus, we focus on the impact of different time discretizations in the LLS model and derive the continuum limit of the Franke-Weserhoff model which exhibits stability problems in the case of an explicit Euler discretization. A detailed discussion of the continuum version of the LLS and Franke-Westerhoff model can be found in the appendix A.2.

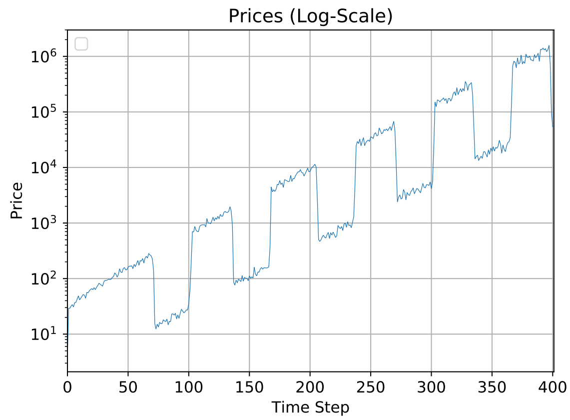
The LLS Model In the following, we introduce a time scale respectively the time step $\Delta t > 0$ in order to perform the continuum limit in a second step. Interpreting the original LLS model as the result of an explicit Euler discretization where Δt is set to 1, a general time-discretized version of the wealth evolution is given by:

$$w(t+\Delta t) = w(t) + \Delta t \left[(1-\gamma(t))\ r + \gamma(t)\ \frac{\frac{S(t+\Delta t)-S(t)}{\Delta t} + D(t)}{S(t)} \right]\ w(t). \tag{5}$$

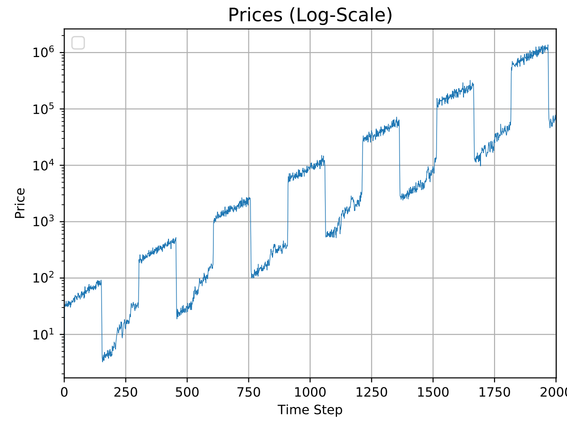
For a proper definition of all parameters and functions we refer to the appendix A.2. Notice that the bond return r and the stock return $\frac{\frac{S(t+\Delta t)-S(t)}{\Delta t}+D(t)}{S(t)}$ are rates and thus scale in time. Equation (5) represents an explicit Euler discretization of the ODE:

$$\frac{d}{dt} w(t) = \left[(1-\gamma(t))\ r + \gamma(t)\ \frac{\frac{d}{dt}S(t) + D(t)}{S(t)} \right]\ w(t)$$

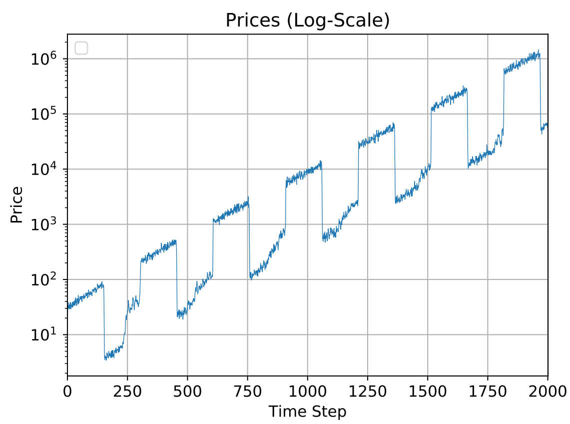
where the time differential $\frac{d}{dt}S(t)$ is approximated by the forward difference quotient $\frac{S(t+\Delta t)-S(t)}{\Delta t}$.
In order to study the time continuous version of the model, we need to properly define the time scaling of the investor. We want to emphasize that several reasonable time scales exist. First, we study the case in which the number of recent timesteps an agent i considers in their



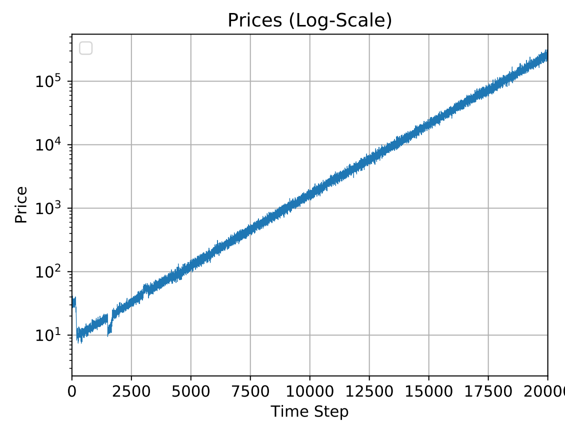
(a) $\Delta t = 0.5$



(b) $\Delta t = 0.1$



(c) $\Delta t = 0.05$



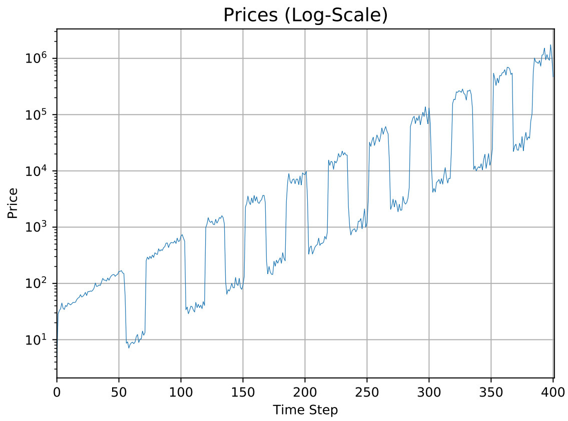
(d) $\Delta t = 0.01$

Figure 1: Simulations of time continuous LLS model with scaled memory variable and different time discretizations. Further parameters as defined in table 1 with $\sigma_\gamma = 0.2$.

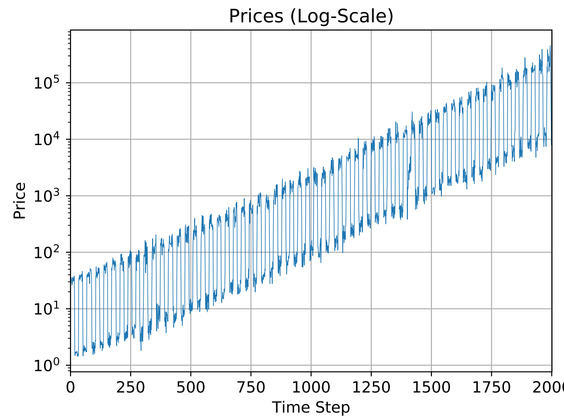
decision, denoted by variable m_i, scales with time, which means: $\bar{m}_i := \lfloor \frac{m_i}{\Delta t} \rfloor$. The results for different time steps using an explicit Euler discretization can be seen in figure 1.

As pointed out in [5], approximately 90% of the optimal investment decisions γ_i in the original model are located at the boundaries of the interval in $[0.01, 0.99]$. Interestingly, in the case of the previously introduced time scaling of the memory variable, this model characteristic changes. For sufficiently small Δt the optimal investment decisions (γ_i) are all located in $(0.01, 0.99)$ and not at the boundaries. This can be explained by the very small optimization horizon and the smoothing effect of a large return history. For $\Delta t = 0.1$ the percentage of extreme decisions reduces to 72% and for $\Delta t = 0.01$ all optimal investment decisions are located in the interior. Note that these statements are based on the average of the results of over 100 runs.

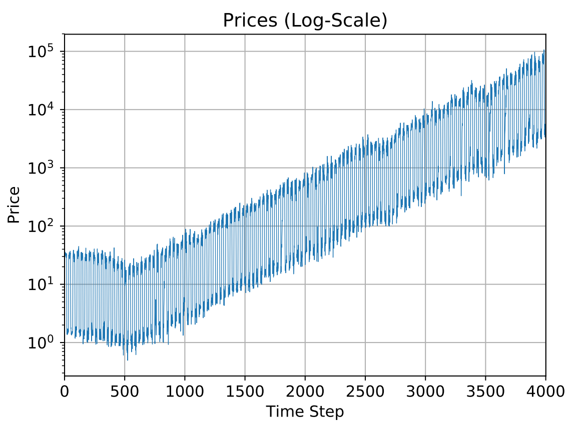
Alternatively, we may assume that the investor's memory does not scale with time, i.e. the number of time steps which corresponds to the agents' memory is always constant. Simulations show that using a non-scaling memory, i.e. with a memory of a fixed number of time steps, oscillating prices for an explicit Euler discretization can be observed for all chosen timesteps (see fig. 2). Averaging over 100 runs also indicates that the percentage of



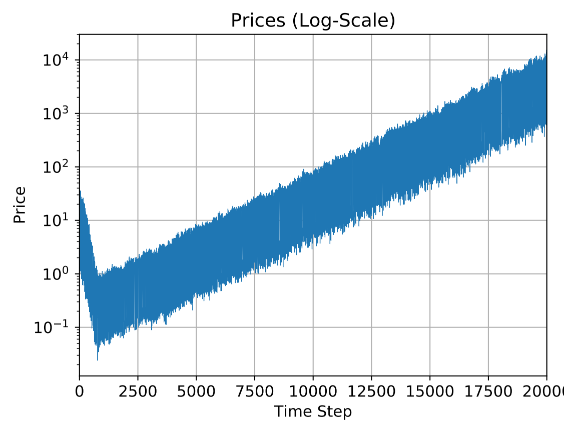
(a) $\Delta t = 0.5$



(b) $\Delta t = 0.1$



(c) $\Delta t = 0.05$



(d) $\Delta t = 0.01$

Figure 2: Simulations of time continuous LLS model with fixed memory variable and different time discretizations. Further parameters as defined in table 1 with $\sigma_\gamma = 0.2$.

extreme decisions remain approximately around 90% for any chosen time discretization. The possibility to study further scales of the LLS model is left for future research.

The Franke-Westerhoff Model Analog to the LLS model, we interpret the Franke-Westerhoff Model as an explicit Euler-type discretization of a a system of ordinary and stochastic differential equations with $\Delta t = 1$. Under this assumption, we introduce the following rescaled version of agents' dynamics in the Franke-Westerhoff Model as a first step towards a time-continuous model ($\Delta t > 0$):

$$\begin{aligned} n^f(t+\Delta t) &= n^f(t) + \Delta t \; n^c(t)\pi^{cf}(a(t)) - \Delta t \; n^f(t)\pi^{fc}(a(t)) \\ n^c(t+\Delta t) &= n^c(t) + \Delta t \; n^f(t)\pi^{fc}(a(t)) - \Delta t \; n^c(t)\pi^{cf}(a(t)) \end{aligned} \tag{6}$$

The corresponding ordinary differential equations (ODE) of the previous dynamics are given by:

$$\begin{aligned} \frac{d}{dt}n^f(t) &= n^c(t)\pi^{cf}(a(t)) - n^f(t)\pi^{fc}(a(t)) \\ \frac{d}{dt}n^c(t) &= n^f(t)\pi^{fc}(a(t)) - n^c(t)\pi^{cf}(a(t)) \end{aligned} \tag{7}$$

As the stock price dynamics include random terms, we interpret the original model as an Euler-Mayurama discretization. Hence, we obtain the following rescaled stock price dynamics:

$$P(t) = P(t-\Delta t) + \mu\ \Delta t\ ED^{FW}(t) + \sqrt{\Delta t}\ \mu\ (\sigma_f + \sigma_c)\ \eta, \quad \eta \sim \mathcal{N}(0,1). \tag{8}$$

For a detailed definition of ED^{FW} we refer to appendix A.2. The stochastic differential equation (SDE) corresponding to equation (8) is given as:

$$dP = \mu\ ED^{FW}(t)\ dt + \mu(\sigma_f + \sigma_c)\ dW \tag{9}$$

The SDE is interpreted in the Itô sense and the usual notation for SDEs is employed. Hence, the time continuous Franke-Westerhoff model reads:

$$\begin{aligned} dP(t) &= \mu\ ED^{FW}(t)\ dt + \mu(\sigma_f + \sigma_c)\ dW \\ \frac{d}{dt}n^f(t) &= n^c(t)\pi^{cf}(a(t)) - n^f(t)\pi^{fc}(a(t)) \\ \frac{d}{dt}n^c(t) &= n^f(t)\pi^{fc}(a(t)) - n^c(t)\pi^{cf}(a(t)) \end{aligned} \tag{10}$$

We derived the ODE-SDE system (10) from the original Franke-Westerhoff Model via the rescaled ODE system (6) and the SDE (9). For a detailed introduction to the Franke-Westerhoff model, we refer to the appendix A.2. In order to clarify the validity of the derived continuum limit, we perform numerical tests.

We first run the model with the parameters defined in [24] and choose the time step Δt to be $\Delta t = 1$. The qualitative results are identical to the original model (see fig. 3). If we change the noise level of the fundamentalist to $\sigma_f = 1.15$, we obtain a blow up of the dynamics (see fig. 4). By blow up we mean that the numerical solution of our equation tends to infinity at finite time. This is an undesirable model characteristic since a minor change in the model parameters has led to an unfeasible model.
This is expected as the only difference is the missing additional constraint (11) for the switching probabilities (cf. Remark 1 in appendix A.2).

One might expect that the large time step $\Delta t = 1$ may be the reason for the numerical instability. The fig. 5 reveals that we still obtain a blow up even for time step $\Delta t = 0.1$.
The reason for the blow up are visible in fig. 6. The values for $n^f(t)$ and $n^c(t)$ leave the interval of $[0,1]$ while preserving the relation $n^f(t) + n^c(t) = 1$. Subsequently this leads to a failure in the price calculation.
It appears Franke and Westerhoff have been aware of this model behavior since they have stated the following additional constraint in [23, 24]

$$\begin{aligned} \pi^{cf}(a(t_{k-1})) &= \min\{1, \nu \exp(a(t_{k-1}))\}, \\ \pi^{fc}(a(t_{k-1}) &= \min\{1, \nu \exp(-a(t_{k-1}))\}, \end{aligned} \tag{11}$$

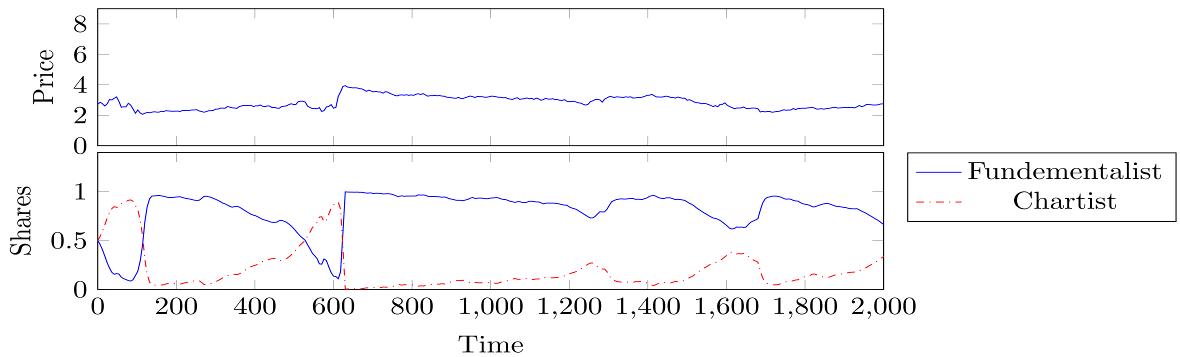


Figure 3: Franke-Westerhoff model with explicit Euler discretization. Parameters as in table table 3.

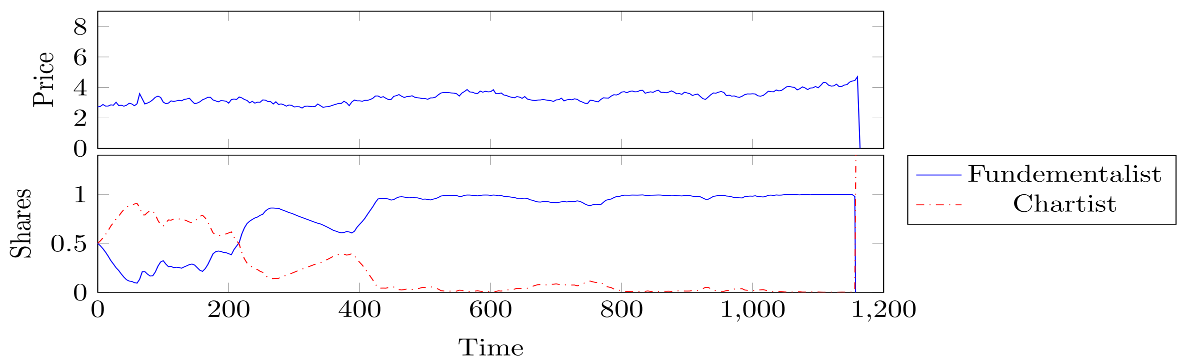


Figure 4: Blow up in the dynamics of the Franke-Westerhoff model with explicit Euler discretization. Parameters as in table table 3 with $\sigma_f = 1.15$.

to their original model [22] introduced in 2009. This additional constraint clearly guarantees the bounds $[0, 1]$ of the fractions of chartists and fundamentalists n^f, n^c. Thus, this additional constraint prevents the dynamics from blowing up. We will show in the next section 3.2 that this constraint can be rendered redundant by applying an improved time discretization.

3.2 Robust Formulation of Franke-Westerhoff Model

In this section we further analyze the stability problem of the Franke-Westerhoff model in the case additional constraint (11) is not enforced. We first show, that the continuous SDE-ODE system (10) is stable without constraint (11). Based on this result, we show that the stability of the continuous system is preserved for all parameter sets when applying an improved semi-implicit time discretization. In the previous section, we have shown numerically that the blow up of the dynamics is caused by the violations of the bounds $[0, 1]$ of the agents' fractions. Proposition 1 shows that these violations are caused by the numerical scheme and are not inherited from the continuous dynamics itself.

Proposition 1. *Any solution of the SDE-ODE system* (10) *remains in the set* $V := \mathbb{R} \times U$ *with* $U := \{(x_1, x_2)^T \in \mathbb{R} | x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$.

For the proof we refer to appendix A.1. Proposition 1 shows that the observed blow ups are introduced by the discretization of the SDE-ODE system (10). In order to avoid the

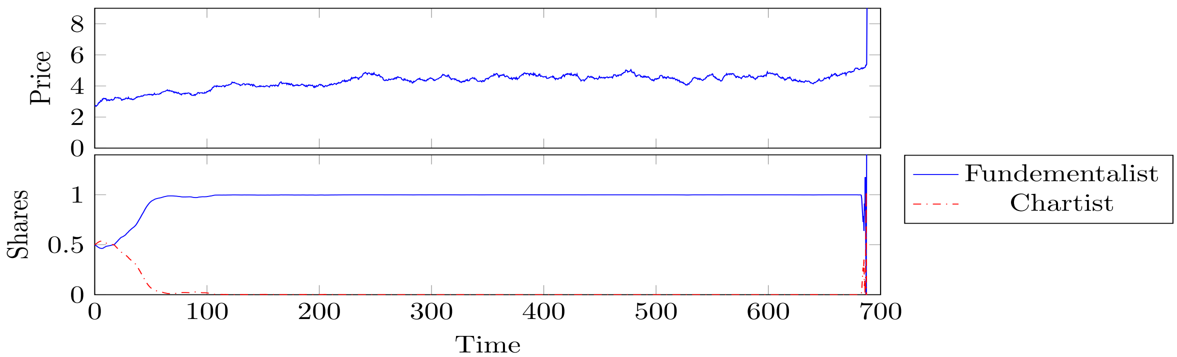


Figure 5: Blow up of the dynamics of the Franke-Westerhoff model with explicit Euler discretization. Parameters as in table table 3 with $\sigma_f = 1.15$ and $\Delta t = 0.1$.

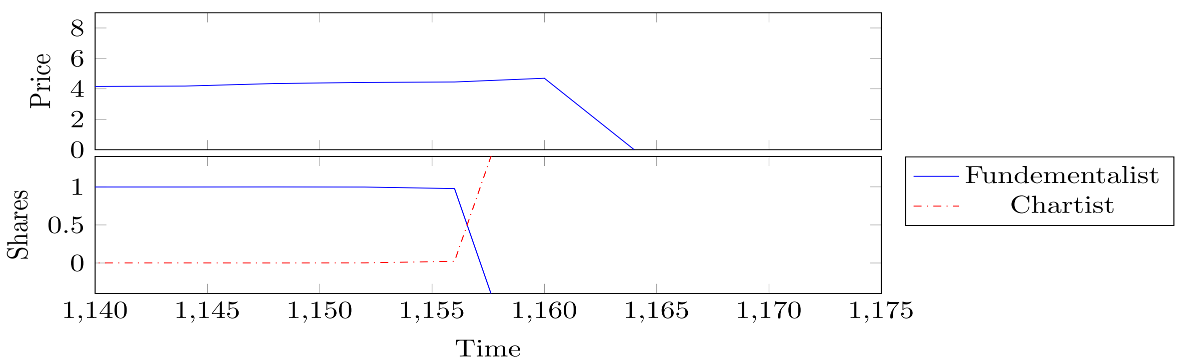


Figure 6: Zoom on the instability of the Franke Westerhoff model with explicit Euler discretization. See fig. 4 for full plot. Parameters as in table table 3 with $\sigma_f = 1.15$.

instabilities we introduce the following semi-implicit Euler discretization:

$$\begin{aligned}
P(t+\Delta t) &= P(t) + \Delta t \; \mu \; ED^{FW}(t) + \sqrt{\Delta t} \; \mu(\sigma_f + \sigma_c) \; \eta, \quad \eta \sim \mathcal{N}(0,1) \\
n^f(t+\Delta t) &= n^f(t) + \Delta t \; [n^c(t+\Delta t)\pi^{cf}(a(t)) - n^f(t+\Delta t)\pi^{fc}(a(t))] \\
n^c(t+\Delta t) &= n^c(t) + \Delta t \; [n^f(t+\Delta t)\pi^{fc}(a(t)) - n^c(t+\Delta t)\pi^{cf}(a(t))]
\end{aligned} \tag{12}$$

In appendix A.2, we show that the semi-implicit scheme (12) can be rewritten in explicit form (see equation (23)). Due to this, the computational cost of the explicit and the semi-implicit schemes are comparable. This semi-implicit scheme (12) preserves the invariant properties of system (10).

Proposition 2. *For all $\Delta t > 0$ and correct initial conditions $(P(t_0), n^f(t_0), n^c(t_0)) \in V$ the numerical solution $(P(t_0 + k\;\Delta t), n^f(t_0 + k\;\Delta t), n^c(t_0 + k\;\Delta t)) \in V,\; k = 0, 1, ..., N$ defined by the scheme in (12) remains in the set V for any number $N \in \mathbb{N}$.*

For the proof we refer to appendix A.1. This shows that an improved numerical discretization can retain the invariance property of the SDE-ODE system for any time step $\Delta t > 0$ and arbitrary choices of constants. This is a huge advantage in comparison of the original model, formulated as a system of difference equations. In particular, the semi-implicit discretization renders the additional constraints (11) as introduced by Franke-Westerhoff in [23] redundant.

We conclude with a numerical example of the semi-implicit discretization (12) showing the effectiveness of the semi-implicit time discretization.

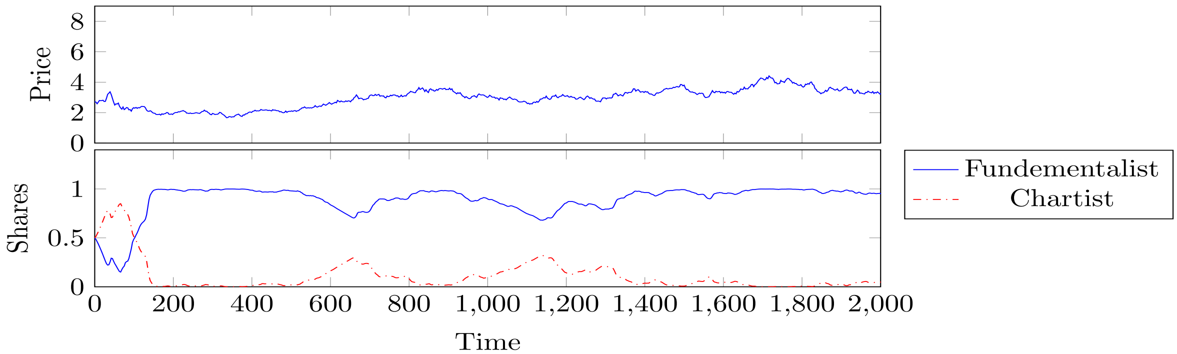


Figure 7: Franke Westerhoff model with semi-implicit discretization. Parameters as in table table 3 with $\sigma_f = 1.15$.

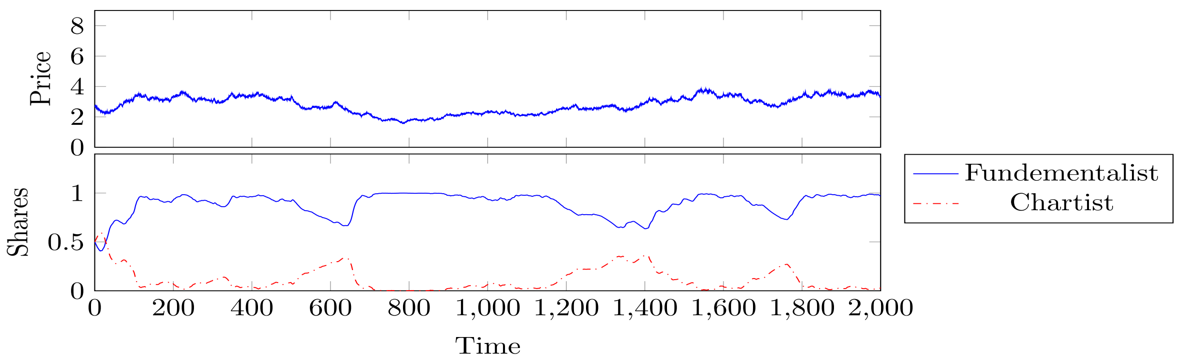


Figure 8: Franke Westerhoff model with semi-implicit discretization. Parameters as in table table 3 with $\sigma_f = 1.15$ and $\Delta t = 0.1$.

4 Conclusion

In this study, we derived the time-continuous formulations of the difference equations used in the LLS and the Franke-Westerhoff model. On the example of the LLS model, we showed that these continuous formulations are not unique. Then, we showed that the numerical instabilities present in the standard Franke-Westerhoff model are not present in the time-continuous SDE-ODE system but stem from the explicit Euler discretization and can be alleviated by applying a proper semi-implicit Euler discretization. In this manuscript, we showed the immanent importance of the proper choice of numerical discretization to the model behavior. As a consequence, we strongly recommend to model ABCEM models on the continuous level as this allows for studying different time scales and the effects of different time discretizations, some of which may be suitable to overcome additional constraints for stability. Furthermore, a continuous formulation allows for the derivation of PDE models which may be simpler to analyze than the original ABCEM models.

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A Appendix

A.1 Analysis

The proof of the Proposition 1 is given by:

Proof. Since we consider a stochastic Itô integral it is clear that P remains in $\mathbb{R}$. It remains to show that $U := \{(x_1, x_2)^T \in \mathbb{R} | x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$ is an invariant set for the equations (7). We define

$$f_1(x_1, x_2) := x_2 \pi^{cf} - x_1 \pi^{fc},$$
$$f_2(x_1, x_2) := x_1 \pi^{fc} - x_2 \pi^{cf},$$

and show the invariance of $U^+ := \{(x_1, x_2)^T \in \mathbb{R} | x_1 \geq 0,\ x_2 \geq 0\}$. We directly obtain that $f_1(0, x_2) = x_2 \pi^{cf} > 0$ and $f_2(x_1, 0) = x_1 \pi^{fc} > 0$ holds and thus U^+ is positive invariant. Secondly, we define $\phi(x_1, x_2) = x_1 + x_2$ and compute the Lie derivative of f along ϕ.

$$\langle \nabla \phi(x_1, x_2), f(x_1, x_2) \rangle = x_2 \pi^{cf} - x_1 \pi^{fc} + x_1 \pi^{fc} - x_2 \pi^{cf} = 0.$$

Hence, we have shown the positive invariance of the set $\{(x_1, x_2)^T \in \mathbb{R} | \phi(x) \leq 1\}$ and consequently U is an invariant set of our ODE system (7).

$\square$

The proof of the Proposition 2 reads:

Proof. Since the Euler-Maruyama method updating rule is simply a sum of real numbers and the real line is a closed set for any stock price, $P \in \mathbb{R}$ holds. The updating rule of the agents' fraction can be rewritten as follows:

$$n^f(t_0 + (k+1)\Delta t) = \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))}$$
$$n^c(t_0 + (k+1)\Delta t) = \frac{n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))}.$$

Then, we can perform a simple induction. We assume that $0 < n^f(t_0 + k\Delta t) < 1,\ 0 < n^c(t_0 + k\Delta t) < 1$ holds and obtain

$$\begin{aligned}
n^f(t_0 + (k+1)\Delta t) &= \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\
&\leq \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t)) + n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\
&= 1, \\
n^c(t_0 + (k+1)\Delta t) &= \frac{n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\
&\leq \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t)) + n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\
&= 1,
\end{aligned}$$

since $\pi^{fc}, \pi^{cf} > 0$ holds by definition and we have used that $n^f + n^c = 1$ holds. The previous inequality shows that $n^f(t_0 + (k+1)\Delta t), n^c(t_0 + (k+1)\Delta t)$ remain in the set U for all $k \in \mathbb{N}$. $\square$

xs

A.2 Models

LLS Model We have implemented the model as defined in [33, 34]. As described in section 3.1 we have added the correct time scale to the model. In order to obtain the original model one needs to set $\Delta t = 1$.

The model considers $N \in \mathbb{N}$ financial agents who can invest $\gamma_i \in [0.01, 0.99],\ i = 1, ..., N$ of their wealth $w_i \in \mathbb{R}_{>0}$ in a stocks and have to invest $1 - \gamma_i$ of their wealth in a safe bond with interest rate $r \in (0, 1)$. The investment propensities γ_i are determined by a utility maximization and the wealth dynamic of each agent at time $t \in [0, \infty)$ is given by

$$\begin{aligned} w_i(t) &= w_i(t - \Delta t) \\ &+ \Delta t \left((1 - \gamma_i(t - \Delta t))\ r\ w_i(t - \Delta t) + \gamma_i(t - \Delta t)\ w_i(t - \Delta t)\ \underbrace{\frac{\frac{S(t) - S(t - \Delta t)}{\Delta t} + D(t)}{S(t)}}_{=:x(S,t,D)} \right). \end{aligned}$$

The dynamics is driven by a multiplicative dividend process. Given by:

$$D(t) := (1 + \Delta t\ \tilde{z})\ D(t - \Delta t),$$

where $\tilde{z}$ is a uniformly distributed random variable with support $[z_1, z_2]$. The price is fixed by the so called *market clearance condition*, where $n \in \mathbb{N}$ is the fixed number of stocks and $n_i(t)$ the number of stocks of each agent.

$$n = \sum_{i=1}^{N} n_i(t) = \sum_{k=1}^{N} \frac{\gamma_k(t)\ w_k(t)}{S(t)}. \tag{13}$$

The utility maximization is given by

$$\max_{\gamma_i \in [0.01, 0.99]} E[\log(w(t + \Delta t, \gamma_i, S^h))].$$

with

$$\begin{aligned} E[\log(w(t + \Delta t, \gamma_i, S^h))] = \frac{1}{m_i} \sum_{j=1}^{m_i} U_i \Bigg(&(1 - \gamma_i(t)) w_i(t, S^h)\ (1 + r\Delta t) \\ &+ \gamma_i(t) w_i(t, S^h) \Big(1 + x\big(S, t - j\Delta t, D\big)\ \Delta t\Big) \Bigg). \end{aligned}$$

The constant m_i denotes the number of time steps each agent looks back. Thus, the number of time steps m_i and the length of the time step Δt defines the time period each agent extrapolates the past values. The superscript h indicates, that the stock price is uncertain and needs to be fixed by the market clearance condition. Finally, the computed optimal investment proportion gets blurred by a noise term.

$$\gamma_i(t) = H(\gamma_i^*(t) + \epsilon_i),$$

where ϵ_i is a normally distributed random variable with standard deviation σ_γ. The function H ensures that γ_i remains in the interval $[0.01, 0.99]$. Finally, we have to update the price after the nosing process. Since the investment fraction is constant we are able to compute the stock price explicitly:

$$S(t) = \frac{\frac{1}{n}\sum_{i=1}^{N} \gamma_i(t)\Big(w_i(t-\Delta t) + \Delta t\ w_i(t-\Delta t)\big(\gamma_i(t-\Delta t)\frac{D(t-\Delta t-)S(t-\Delta t)}{\Delta t\ S(t-\Delta t)} + (1-\gamma_i(t-\Delta t))\ r\big)\Big)}{1 - \frac{1}{n}\sum_{i=1}^{N} \frac{\gamma_i(t)\gamma_i(t-\Delta t)w_i(t-\Delta t)}{S(t-\Delta t)}}.$$

Utility maximization Thanks to the simple utility function and linear dynamics we can compute the optimal investment proportion in the cases where the maximum is reached at the boundaries. The first order necessary condition is given by:

$$f(\gamma_i) := \frac{d}{dt}E[\log(w(t+\Delta t, \gamma_i, S^h))] = \frac{1}{m_i}\sum_{j=1}^{m_i} \frac{\Delta t\ \big(x\big(S, t-j\Delta t, D\big) - r\big)}{\Delta t\ \big(x\big(S, t-j\Delta t, D\big) - r\big)\ \gamma_i + 1 + \Delta t\ r}.$$

Thus, for $f(0.01) < 0$ we can conclude that $\gamma_i = 0.01$ holds. In the same manner, we get $\gamma_i = 0.99$, if $f(0.01) > 0$ and $f(0.99) > 0$ holds. Hence, solutions in the interior of $[0.01, 0.99]$ can be only expected in the case: $f(0.01) > 0$ and $f(0.99) < 0$. This coincides with the observations in [41].

Franke-Westerhoff model We present the Franke-Westerhoff model as introduced in [22] and considered with minor modifications in [23, 24]. As described in section 3.1 we have added a time scaling to the model. In order to obtain the original model one needs consider the explicit Euler discretization of the agents' shares and has to set $\Delta t = 1$. The Franke-Westerhoff model considers tow types of agents, chartists and fundamentalists. The demand of each agent reads

$$d^f(t) = \phi(P_f(t) - P(t)) + \epsilon_k^f, \quad \phi \in \mathbb{R}^+, \quad \epsilon_k^f \sim \mathcal{N}(0, \sigma_f^2), \tag{14}$$
$$d^c(t) = \chi(P(t) - P(t-\Delta t)) + \epsilon_k^c, \quad \chi \in \mathbb{R}^+, \quad \epsilon_k^c \sim \mathcal{N}(0, \sigma_c^2), \tag{15}$$

where $P(t)$ denotes the logarithmic market price and $P_f(t)$ denotes the fundamental price. The noise terms ϵ_k^f and ϵ_k^c are normally distributed, with zero mean and different standard deviations σ_c^2 and σ_f^2. The second important features are the fractions of the chartist or fundamental population. In that sense the two agents can be seen as representative agents of a population. The fraction of chartists $n^C(t) \in [0,1]$ and the fraction of fundamentalitst $n^F(t) \in [0,1]$ have to fulfill $n^C(t) + n^F(t) = 1$. Hence, the deterministic excess demand can be defined as:

$$ED^{FW}(t) := \frac{1}{2}(ed^f(t) + ed^c(t)) \tag{16}$$

$$\begin{aligned} ed^f(t) &:= 2\ n^f(t)E[d^f(t)] \\ ed^c(t) &:= 2\ n^c(t)E[d^c(t)]. \end{aligned} \tag{17}$$

Here, E denotes the expected value. The pricing equation is then given by the simple rule

$$P(t) = P(t-\Delta t) + \mu\ \Delta t\ ED^{FW}(t) + \sqrt{\Delta t}\ \mu\ (\sigma_f + \sigma_c)\ \eta, \quad \eta \sim \mathcal{N}(0,1). \tag{18}$$

Finally, we need to specify the switching mechanism. This switching mechanism is known as the transition probability approach (TPA) [35, 50]. We consider the so called switching index $a(t) \in \mathbb{R}$ which describes the attractiveness of the fundamental strategy over the chartist strategy. Thus, a positive $a(t)$ reflects an advantage of the fundamental strategy in comparison to the chartist and if $a(t)$ is negative we have the opposite situation. We define the switching probabilities

$$\pi^{cf}(a(t)) := \nu \exp(a(t)) \tag{19}$$

$$\pi^{fc}(a(t) := \nu \exp(-a(t)). \tag{20}$$

where π^{xy} is the probability that an agent with strategy x switches to strategy y. The flexibility parameter $\nu > 0$ is a scaling factor for $a(t)$.

Remark 1. *A minor modification of the Franke-Westerhoff model introduced in 2011 [23, 24] considers the following switching probabilities*

$$\pi^{cf}(a(t)) := \min(1, \nu \exp(a(t))) \tag{21}$$

$$\pi^{fc}(a(t) := \min(1, \nu \exp(-a(t)))\,. \tag{22}$$

The previous definition ensures that switching probabilities are restricted to the interval $[0, 1] \subset \mathbb{R}$. *In the original model introduced in 2009 [22] there has been no additional limits as introduced in* (21).

Explicit Euler Discretization Then the explicit Euler discretization of the time evolution of chartist and fundamentalist shares as presented in (10) is given by:

$$n^f(t) = n^f(t - \Delta t) + \Delta t n^c(t)\pi^{cf}(a(t - \Delta t)) - \Delta t n^f(t)\pi^{fc}(a(t - \Delta t))$$
$$n^c(t) = n^c(t - \Delta t) + \Delta t n^f(t - \Delta t)\pi^{fc}(a(t - \Delta t)) - \Delta t n^c(t - \Delta t)\pi^{cf}(a(t - \Delta t))$$

Semi-Implicit Euler Discretization Alternatively one may use the semi-implicit scheme introduced in section 3 for the time evolution of chartist and fundamentalist shares.

$$\begin{aligned} n^f(t) &= \frac{n^f(t - \Delta t) + \Delta t \pi^{cf}(a(t - \Delta t))}{1 + \Delta t(\pi^{fc}(a(t - \Delta t)) + \pi^{cf}(a(t - \Delta t)))} \\ n^c(t) &= \frac{n^c(t - \Delta t) + \Delta t \pi^{fc}(a(t - \Delta t))}{1 + \Delta t(\pi^{fc}(a(t - \Delta t)) + \pi^{cf}(a(t - \Delta t)))} \end{aligned} \tag{23}$$

As shown in section 3.1 as well, this numerical approximation stable and conserves the invariance property of the ODEs.

Finally, we have to specify how the switching index $a(t)$ is calculated. The switching index $a(t)$, encodes how favourable a fundamentalist strategy is over a chartist strategy. The switching index is determined linearly out of the the three principles *predisposition, herding and misalignment.*

$$\alpha(t) = \alpha_p + \alpha_h \; (n^f(t) - n^c(t)) + \alpha_m \; (P(t) - P_f)^2,$$

Parameter	Value
N	100
m_i	15
σ_γ	0 or 0.2
r	0.04
$z_1 = z_2$	0.05
Δt	1
time steps	200

(a) Parameters of LLS model.

Variable	Initial Value
μ_h	0.0415
σ_h	0.003
$\gamma(t=0)$	0.4
$w_i(t=0)$	1000
$n_i(t=0)$	100
$S(t=0)$	4
$D(t=0)$	0.2

(b) Initial values of LLS model.

Table 1: Basic setting of the LLS model.

Parameter	Value
N	99
m_i	$10,\ 1 \leqslant i \leqslant 33$ $141,\ 34 \leqslant i \leqslant 66$ $256,\ 67 \leqslant i \leqslant 99$
σ_γ	0.2
r	0.0001
$z_1 = z_2$	0.00015
Δt	1
time steps	$20,000$

(a) Parameters of LLS model.

Variable	Initial Value
μ_h	0.0415
σ_h	0.003
$\gamma_i(t=0)$	0.4
$w_i(t=0)$	1000
$n_i(t=0)$	100
$S(t=0)$	4
$D(t=0)$	0.004

(b) Initial values of LLS model.

Table 2: Setting for the LLS model (3 agent groups).

where $\alpha_p, \alpha_h, \alpha_m > 0$ are weights respectively scaling factors. The sign $\alpha_p \in \mathbb{R}$ determines the predisposition with respect to a fundamental or chartist strategy. For details regarding the modeling we refer to [22].

A.3 Parameter sets

LLS Model The initialization of the stock return is performed by creating an artificial history of stock returns. The artificial history is modeled as a Gaussian random variable with mean μ_h and standard deviation σ_h. Furthermore, we have to point out that the increments of the dividend is deterministic, if $z_1 = z_2$ holds. We used the C++ standard random number generator for all simulations of the LLS model if not otherwise stated.

Franke-Westerhoff Model

Parameter	Value
ϕ	0.18
χ	2.3
α_0	-0.161
α_h	1.3
α_m	12.5
σ_f	0.79
σ_c	1.9
ν	0.05
P_f	1
μ	0.01
Δt	1

(a) Parameters

Variable	Initial Value
$P(t=0)$	1

(b) Initial Values

Table 3: Parameters and initial values for the Franke Westerhoff model.

Simulation	Random Number Generator
Figure 1	C++ MT19937 RNG (64 bit)
Figure 2	C++ MT19937 RNG (64 bit)
Figure 3	IntelMKL MT2203 RNG (64 bit)
Figure 4	IntelMKL MT2203 RNG (64 bit)
Figure 5	IntelMKL MT2203 RNG (64 bit)
Figure 6	IntelMKL MT2203 RNG (64 bit)
Figure 7	IntelMKL MT2203 RNG (64 bit)
Figure 8	IntelMKL MT2203 RNG (64 bit)

Table 4: Random Generators for the Simulations.

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