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Abstract

In this paper, we introduce a large system of interacting financial agents in which each agent is faced with the decision of how to allocate his capital between a risky stock or a risk-less bond. The investment decision of investors, derived through an optimization, drives the stock price. The model has been inspired by the econophysical Levy-Levy-Solomon model [26]. The goal of this work is to gain insights into the stock price and wealth distribution. We especially want to discover the causes for the appearance of power-laws in financial data. We follow a kinetic approach similar to [29] and derive the mean field limit of our microscopic agent dynamics. The novelty in our approach is that the financial agents apply model predictive control (MPC) to approximate and solve the optimization of their utility function. Interestingly, the MPC approach gives a mathematical connection between the two opponent economic concepts of modeling financial agents to be rational or boundedly rational. We derive a moment model which is able to replicate the most prominent features of the financial markets: oscillatory price behavior, booms and crashes. Due to our kinetic approach, we can study the wealth and price distribution on a mesoscopic level. The wealth distribution is characterized by a lognormal law. For the stock price distribution, we can either observe a lognormal behavior in the case of long-term investors or a power-law in the case of high-frequency trader. Furthermore, the stock return data exhibits a fat-tail, which is a well known characteristic of real financial data.

Keywords: portfolio optimization, kinetic modeling, model predictive control, stylized facts, stock market, bounded rationality

1 Introduction

The question of allocating capital between a risky and risk-less asset is a well-known issue for private and institutional investors. This research question has a long tradition in economics: for example the famous works of Markowitz [30] or Merton [32, 33].

Another research field which has received a lot of attention in the last decade is the modeling of financial markets. Several financial crashes (Black Monday 1987, Dot-com Bubble 2000,

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Global Financial Crisis 2007) have shown that classical financial market models fail to replicate financial data properly [11, 20]. Since the 1970s, econometricians have detected empirical artifacts in financial data known as *stylized facts*. Stylized facts are universal statistical properties of financial data which can be observed all over the world [12]. The most prominent examples are: booms and crashes of stock prices, the inequality of wealth and fat-tails in the stock return distribution. Several researchers point out that stylized facts play an important role in the creation of financial crisis [27, 41]. For that reason, the discovery of the origin of stylized facts has become a prospering field of economic research. Up to now, this question could only be answered partially and remains widely open [36].

First attempts to discover the origin of stylized facts were made by agent-based financial market models. These models consider a large number of interacting financial agents and share more similarities with particle models in physics than with classical asset-pricing models [28, 41]. These models use tools from statistical physics like Monte Carlo simulations and are part of the new research field econophysics. Major contributions in this field are [26, 28, 39, 13, 22]. These complex systems of interacting agents are not only inspired by physical theories but also by behavioral finance. Thus, the agents are modeled to be boundedly rational in the sense of Simon [40]. These modern market models are capable of reproducing stylized facts. In physics, one might call stylized facts scaling laws [28], which is the motivation to apply tools from physics onto financial models. Numerical experiments of agent-based models indicate that psychological misperceptions of investors can be accounted to be one reason for the appearance of stylized facts [16, 27].

The disadvantage of these particle models is the need to study the complex behavior empirically through computer simulations. In addition, many studies have shown [19, 46, 24, 21] that in several agent-based models stylized facts are caused by finite-size effects of the model and are thus only numerical artifacts. To overcome these problems, it is possible to derive kinetic PDE models out of the microscopic particle models, which give us the possibility to study the appearance of stylized facts analytically. There are several examples of such a kinetic aprroach [29, 6, 15, 14, 18, 31, 5, 43].

The starting point of our work is an agent-based model of financial agents who want to optimize their investment decision. They are faced with the decision of how to allocate their capital in a risky stock or a risk-less bond. To determine the investment strategy, the agents minimize the badness of their portfolio where they estimate the future stock return by a convex combination of a fundamental and chartist return estimate. The stock price is driven by the aggregated demand of financial agents. To fix the stock price, we use a relaxation of Walras equilibrium law [45], utilized in many econophysical models [3, 16, 28]. The microscopic model is inspired by the famous Levy-Levy-Solomon model [25, 26]. Further closely related agent-based models are [10, 8]. One novelty of our model is to apply model predictive control (MPC) to simplify the optimization process and derive the investment decision of agents. This methodology, often applied in the engineering community, has been recently applied to a kinetic opinion formation model [1] but, to our knowledge, never before to a kinetic financial market model. We consider a large system of coupled constrained optimization problems. In order to reduce this system to a set of ordinary differential equations (ODEs), we introduce the game-theoretic concept of Nash equilibria and apply MPC. From the perspective of agent modeling, we first consider rational financial agents and derive through the MPC approach boundedly rational agents. Mathematically, we perform the mean field limit of our microscopic model to derive a mesoscopic description of our dynamics. This means that we look at the limiting case of infinitely many agents and instead of considering each agent individually we can study the dynamics through probability densities. This limit often provides us with Fokker-Planck type equations which enable us to derive analytic solutions and study the long time behavior of our model. This work is closely related to the kinetic financial market model of Maldarella and Pareschi [29]. Besides other approaches, we apply the Boltzmann methodology as performed in [29] and well described in [38] to derive a Fokker-Planck model. We want to point out that, to our knowledge, this is the first model which translates a portfolio model in a kinetic PDE model and consequently considers a wealth and a stock price evolution. We thus are able to analyze the wealth and stock price distribution simultaneously and thus study possible interrelations.

We consider three modeling stages. First, we consider a deterministic model and derive the mean field limit. In addition, we derive a macroscopic moment model which describes the evolution of the stock prices and the average amount of wealth in stocks and bonds. We show that the stock price evolution can create oscillatory solutions, booms and crashes and can thus replicate prominent characteristics of financial markets. Secondly, we add noise to the investment decision of investors and study the outcome on the mesoscopic level. At the final modeling stage, we introduce a new population of financial brokers, equipped with microscopic stock prices. These stock prices are modeled as stochastic differential equations (SDEs). In the mean field limit of infinitely many brokers, we observe a Fokker-Planck equation. This enables us to study the stock price distribution. We distinguish between long-term investors and high-frequency trader. In the case of long-term investors, the stock price distribution is of lognormal type, whereas in the case of high-frequency trader we observe an inverse-gamma distribution. The same distribution has been previously discovered in other financial market models [29, 6, 15]. We want to point out that the inverse-gamma distribution asymptotically satisfies a power-law for large stock prices. In addition, we show numerically that the stock return distributions have a fat-tail. Finally, we want to emphasize that we can observe in all of our models a wealth or portfolio distribution of normal or lognormal type. This is an interesting result, as one might expect to observe a power-law in the portfolio, presumed one has a power-law in the stock return distribution.

The outline of our paper is as follows: in the next section, we first define the microscopic portfolio model. We then apply the MPC approach to simplify the optimization and derive the investment decision of each financial agent. Then, we derive the mean field limit equation in section three and analyze the portfolio distribution. In addition, we derive a moment model which we discuss analytically and numerically. As a next step, we extend our model by adding noise to the investment decision and analyze the resulting PDE-ODE system. In section six, we introduce a population of broker, so that the microscopic stock prices are described by a stochastic process. As it has been done for the previous modeling stages, we perform the mean field limit in order to analyze the stock price distribution. In section seven, we give numerical examples of our model and verify our previous computations. We finish the paper with a short discussion of our results and possible model extensions.

2 Microscopic Model

We consider N financial agents equipped with their personal monetary wealth $w_i \geq 0$. We assume non-negative wealth, and thus do not allow debts. The agents have to allocate their wealth between a risky asset (stock) and a risk-free asset (bond). The wealth in the risky asset is denoted by $x_i \geq 0$ and the wealth in the risk-free asset by $y_i \geq 0$. Thus, the wealth of the i-th agent at time $t > 0$ is given by $w_i(t) = x_i(t) + y_i(t)$.
The time evolution of the risk-free asset is described by a fixed non-negative interest rate $r \geq 0$ and the evolution of the risky asset by the stock return,

$$\frac{\dot{S}(t) + D(t)}{S(t)},$$

where $S(t)$ is the stock price at time t and $D(t) \geq 0$ the dividend. We denote all macroscopic quantities with capital letters. For now, we assume that the stock price and the dividend are given and that the stock price is a differentiable function of time. The agent can shift capital between the two assets. We denote the shift from bonds into stocks by u_i. Thus we have the dynamics

$$\begin{aligned} \dot{x}_i(t) &= \frac{\dot{S}(t) + D(t)}{S(t)}\ x_i(t) + u_i(t) \\ \dot{y}_i(t) &= r\ y_i(t) - u_i(t). \end{aligned}$$

We still need to describe the time evolution of the stock price S. The investment decisions of the agents drive the price through the excess demand

$$ED_N(t) := \frac{1}{N} \sum_{i=1}^{N} u_i(t).$$

The excess demand is positive if the investors buy more stocks than they sell. Thus the **macroscopic stock price evolution** is given by

$$\dot{S}(t) = \kappa\ ED_N(t)\ S(t). \tag{1}$$

where the constant $\kappa > 0$ measures the market depth. This model for the stock price is commonly accepted among economists. The ODE (1) can be interpreted as a relaxation of the well known equilibrium law, supply equals demand, dating back to the economist Walras [45].

Investment strategy Next, we describe how an agent determines his investment strategy. As in classical economic theory, u_i will be a solution of a risk or cost minimization. First, in order to make an investment decision, an agent has to estimate future returns. We take two possible strategies into account, a chartist estimate and a fundamentalist estimate. The estimates need to depend on the current stock price.
Fundamentalists believe in a fundamental value of the stock price denoted by $s^f > 0$ and assume that the stock price will converge in the future to this specific value. The investor therefore estimates the future return of stocks versus the return of bonds as

$$K^f := U_\gamma\left(\omega \frac{s^f - S}{S}\right) - r.$$

Here, U_γ is a value function in the sense of Kahnemann and Tversky [23] which depends on the risk tolerance γ of an investor. A typical example is $U_\gamma(x) = sgn(x)|x|^\gamma$ with $0 < \gamma < 1$ and sign function sgn. The constant $\omega > 0$ measures the expected speed of mean reversion to the fundamental value s^f of fundamental. We want to point out that this stock return estimate is a rate and thus ω needs to scale with time.
Chartists assume that the future stock return is best approximated by the current or past stock return. They estimate the return rate of stocks over bonds by

$$K^c := U_\gamma \left(\frac{\dot{S}/\rho + D}{S} \right) - r.$$

The constant $\rho > 0$ measures the frequency of exchange rates [29]. Both estimates are aggregated into one estimate of stock return over bond return by a convex combination

$$K = \chi\ K^f + (1-\chi)\ K^c.$$

This idea has been previously applied to a kinetic model of opinion formation [2]. The weight χ is determined from an instantaneous comparison as modeled in [28]. We let

$$\chi = W(K^f - K^c),$$

where $W : \mathbb{R} \to [0,1]$ is a continuous function. If for example, $W = \frac{1}{2}\tanh + \frac{1}{2}$, the investor optimistically believes in the higher estimate. Together, if $K > 0$, the investor believes that stocks will perform better and if $K < 0$ that bonds will perform better.

Objective function Next, we can define the minimization problem that determines the agent's actions. We define the "badness" of the portfolio by

$$\Psi_i := \begin{cases} |K|\ \frac{x_i^2}{2}, & K < 0, \\ 0, & K = 0, \\ |K|\ \frac{y_i^2}{2}, & K > 0, \end{cases}$$

which can be conveniently rewritten to $\Psi_i = K \cdot \left(-H(-K)\frac{x_i^2}{2} + H(K)\frac{y_i^2}{2} \right)$, where H is the Heaviside step function, zero at the origin. If stocks are believed to be better ($K > 0$), then being invested in bonds ($y_i > 0$) is bad, and vice versa. The badness is larger, the larger the estimated difference between returns K. The agent tries to minimize the running costs

$$\int_0^T \left(\frac{\mu}{2} u_i(t)^2 + \Psi_i(t) \right) dt.$$

We consider a finite time interval $[0, T]$ and have added a penalty term that punishes transactions. The penalty term is necessary to convexify the problem but is also reasonable, because it describes transaction costs. The transaction costs are modeled to be quadratic which is an often used assumption in portfolio optimization [4, 34].

Hence, in summary, our microscopic model is given by

$$\dot{x}_i(t) = \frac{\dot{S}(t) + D(t)}{S(t)}\; x_i(t) + u_i^*(t) \tag{2a}$$

$$\dot{y}_i(t) = r\; y_i(t) - u_i^*(t) \tag{2b}$$

$$\dot{S}(t) = \kappa\; ED_N(t)\; S(t) \tag{2c}$$

$$u_i^* := \underset{u_i:[0,T]\to\mathbb{R}}{\operatorname{argmax}} \int_0^T \left(\frac{\mu}{2} u_i(t)^2 + \Psi_i(t)\right) dt. \tag{2d}$$

The microscopic model is an optimal control problem. The dynamics are strongly coupled by the stock price in a non-linear fashion. Since all investors want to minimize their individual badness function, one needs to solve the optimal control problem in a game-theoretic context. We choose the concept of Nash equilibria which will be explained in detail in the next section.

3 MPC for microscopic model

In case of many agents, we have a large system of optimization problems (2). Such a system is very expensive to solve. For that reason, we approximate the objective functional (2d) by model-predictive control (MPC). In the MPC framework, one assumes that the investor only optimizes on the time interval $[\bar{t}, \bar{t} + \Delta t]$ for a small $\Delta t > 0$ and fixed $\bar{t}$. One thus assumes that one can approximate the control u on $[0, T]$ by piecewise constant functions on time intervals of length Δt. We can only expect to observe a suboptimal strategy since we perform an approximation of (2d).

We choose the penalty parameter μ in the running costs to be proportional to the time interval so that $\mu = \nu\Delta t$ for some ν. This can be motivated by checking the units of the variables in the cost functional (K is a rate, thus measured in 1/time, Ψ is wealth^2/time, u wealth/time). We see that the penalty parameter μ must be a time unit. Furthermore, we insert the right-hand side of the stock price equation into the stock return. Thus, the constrained optimization problem reads

$$\begin{aligned}
&\int_{\bar{t}}^{\bar{t}+\Delta t} \left(\frac{\nu\Delta t}{2} u_i^2(t) + \Psi_i(t)\right) dt \to \min \\
&\dot{x}_i(t) = \kappa\; ED_N(t)\; x_i(t) + \frac{D(t)}{S(t)}\; x_i(t) + u_i(t), \quad x_i(\bar{t}) = \bar{x}_i, \\
&\dot{y}_i(t) = r y_i(t) - u_i(t), \quad y_i(\bar{t}) = \bar{y}_i, \\
&\dot{S}(t) = \kappa\; ED_N(t)\; S(t), \quad S(\bar{t}) = \bar{S}.
\end{aligned}$$

Game theoretic setting We want to solve our MPC problem in a game theoretic setting. All agents are coupled by the stock price respectively excess demand ED_N. As pointed out previously, it is impossible that all agents act optimal since all agents play a game against each other. In fact, we are faced with a noncooperative differential game. Thus, a reasonable equilibrium concept is needed to solve our optimal control problem. We want to search for Nash equilibria. In this setting, each agent assumes that the strategies of the other players are fixed and optimal. Thus, we get N optimization problems which need to be solved simultaneously. Hence, we have a N-dimensional Lagrangian $L \in \mathbb{R}^N$. The i-th entry L_i corresponds to the i-th player and reads:

$$
\begin{aligned}
L_i(x_i, y_i, S, u_i, \lambda_{x_i}, \lambda_{y_i}, \lambda_S) = &\int\limits_{\bar{t}}^{\bar{t}+\Delta t} \left(\frac{\nu \Delta t}{2} u_i^2(t) + \Psi_i(t) \right) dt \\
&+ \int\limits_{\bar{t}}^{\bar{t}+\Delta t} \dot{\lambda}_{x_i}\ x_i + \lambda_{x_i}\ \kappa\ ED_N\ x_i + \lambda_{x_i}\ \frac{D}{S}\ x_i + \lambda_{x_i}\ u_i\ dt - \lambda_{x_i}\ \bar{x}_i \\
&+ \int\limits_{\bar{t}}^{\bar{t}+\Delta t} \dot{\lambda}_{y_i}\ y_i + \lambda_{y_i}\ r\ y_i - \lambda_{y_i}\ u_i\ dt - \lambda_{y_i}\ \bar{y}_i, \\
&+ \int\limits_{\bar{t}}^{\bar{t}+\Delta t} \dot{\lambda}_S\ S + \lambda_S\ \kappa\ ED_N\ S\ dt - \lambda_S\ \bar{S},
\end{aligned}
$$

with Lagrange multiplier $\lambda_{x_i}, \lambda_{y_i}, \lambda_S$. Notice that the quantities $(x_j^*, y_j^*, u_j^*),\ j = 1, ..., i-1, i+1, ..., N$ are assumed to be optimal in the i-th optimization and therefore only enter as parameters in the i-th Lagrangian L_i. We assume $\lambda_{x_i}(\bar{t}+\Delta t) = \lambda_{y_i}(\bar{t}+\Delta t) = \lambda_S(\bar{t}+\Delta t) = 0$ and thus the optimality conditions are given by

$$
\begin{aligned}
&\dot{x}(t) = \kappa\ ED_N(t)\ x_i(t) + \frac{D(t)}{S(t)}\ x_i + u_i,\ x_i(\bar{t}) = \bar{x}_i, \\
&\dot{y}_i(t) = r y_i(t) - u_i(t),\ y_i(\bar{t}) = \bar{y}_i, \\
&\dot{S}(t) = \kappa\ ED_N(t)\ S(t),\ S(\bar{t}) = \bar{S}, \\
&\nu\ \Delta t\ u_i(t) = -\lambda_{x_i}(t) - \lambda_{x_i}(t)\ \frac{\kappa}{N}\ x_i(t) + \lambda_{y_i}(t) - \frac{\kappa}{N}\ S(t)\ \lambda_S(t), \\
&\dot{\lambda}_{x_i}(t) = -\kappa\ ED_N(t)\ \lambda_{x_i}(t) - \frac{D(t)}{S(t)}\ \lambda_{x_i}(t) - \partial_{x_i}\Psi_i(t), \\
&\dot{\lambda}_{y_i}(t) = -r\lambda_{y_i}(t) - \partial_{y_i}\Psi_i(t), \\
&\dot{\lambda}_S(t) = \lambda_{x_i}(t)\frac{D(t)}{S^2(t)} x_i - \kappa\ ED_N(t)\ \lambda_S(t) - \partial_S\Psi_i(t).
\end{aligned}
$$

Then we apply a backward Euler discretization to the adjoint equations and get

$$
\begin{aligned}
\lambda_{x_i}(\bar{t}) &= \Delta t\ \partial_{x_i}\Psi_i(\bar{t}+\Delta t), \\
\lambda_{y_i}(\bar{t}) &= \Delta t\ \partial_{y_i}\Psi_i(\bar{t}+\Delta t), \\
\lambda_S(\bar{t}) &= \Delta t\ \partial_S\Psi_i(\bar{t}+\Delta t).
\end{aligned}
$$

Hence, the optimal strategy is given by

$$
u_N^*(x_i, y_i, S) = \begin{cases} \frac{1}{\nu}(K\ y_i - \frac{\kappa}{N}\ S\ (\partial_S K)\ \frac{y_i^2}{2}), & K > 0, \\ 0, & K = 0, \\ \frac{1}{\nu}(K\ x_i + K\ \frac{\kappa}{N}\ x_i^2 + \frac{\kappa}{N} S\ (\partial_S K)\ \frac{x_i^2}{2}), & K < 0. \end{cases}
$$

Feedback controlled model The feedback controlled model reads

$$\dot{x}_i(t) = \kappa \; ED_N(t) \; x_i(t) + \frac{D(t)}{S(t)} \; x_i(t) + u_N^*(t, x_i, y_i, S) \tag{3a}$$

$$\dot{y}_i(t) = r \; y_i(t) - u_N^*(t, x_i, y_i, S) \tag{3b}$$

$$\dot{S}(t) = \kappa \; ED_N(t) \; S(t). \tag{3c}$$

Here, we have inserted the right-hand side of our stock equation (1) into the stock return.

Remark 1. *Alternatively, one might first discretize the system and then optimize. The corresponding optimal control is identical.*

4 Mean field limit of feedback controlled model

In this section, we want to perform the limit of infinitely many agents $N \to \infty$, known as mean field limit. Classical literature on this topic are [7, 17, 35] . The goal is to derive a mesoscopic description of the financial agents instead of considering each agent in the N particle phase space individually. Thus, instead of considering the agents' dynamics in a large dynamical system, we want to describe our dynamics with the help of a density function $f(t, x, y), \; x, y \in \mathbb{R}_{\geq 0}$. The density $f(t, x, y)$ describes the probability that an agent at time t has an amount $x \in \mathbb{R}_{\geq 0}$ of wealth invested in his risky portfolio and $y \in \mathbb{R}_{\geq 0}$ wealth in his risk-free portfolio. The empirical measure is a nice tool to connect the solution of the dynamical system to the mean field limit equation.

Definition 1. *The two-dimensional empirical measure $f^N_{(\boldsymbol{x},\boldsymbol{y})}(x, y)$ for given vectors $\boldsymbol{x} := (x_1, ..., x_N)^T \in \mathbb{R}^N$ and $\boldsymbol{y} := (y_1, ..., y_N)^T \in \mathbb{R}^N$ is defined by:*

$$f^N_{(\boldsymbol{x},\boldsymbol{y})}(x, y) := \frac{1}{N} \sum_{k=1}^{N} \delta(x - x_k) \; \delta(y - y_k).$$

We use the empirical measure to derive the mean field limit equation formally. This is done partially because of the elegance of the method and mainly to clarify the process to the reader. We assume that the microscopic model has a unique solution. Furthermore, we denote the solution of the wealth evolution by $\boldsymbol{x}(t) := (x_1(t), ..., x_N(t))^T \in \mathbb{R}^N$ and

$\boldsymbol{y}(t) := (x_1(t), ..., x_N(t))^T \in \mathbb{R}^N$. We consider a test function $\phi(x, y),\ x, y \in \mathbb{R}_{\geq 0}$ and compute

$$
\begin{aligned}
\frac{d}{dt}\langle f^N_{(\boldsymbol{x(t)},\boldsymbol{y(t)})}(t,x,y), \phi(x,y)\rangle &= \frac{1}{N}\sum_{k=1}^{N}\frac{d}{dt}\phi(x_k(t), y_k(t)) \\
&= \frac{1}{N}\sum_{k=1}^{N}\partial_x\phi(x_k(t), y_k(t))\ \dot{x}_k(t) + \partial_y\phi(x_k(t), y_k(t))\ \dot{y}_k(t) \\
&= \frac{1}{N}\sum_{k=1}^{N}\partial_x\phi(x_k(t), y_k(t))\ \left(\kappa\ ED_N(t)\ x_k(t) + \frac{D(t)}{S(t)}\ x_k(t) + u^*(t, x_k, y_k, S)\right) \\
&\quad + \frac{1}{N}\sum_{k=1}^{N}\partial_y\phi(x_k(t), y_k(t))\ (r\ y_k(t) - u^*(t, x_k, y_k, S)) \\
&= \left\langle f^N_{(\boldsymbol{x(t)},\boldsymbol{y(t)})}(t,x,y),\ \partial_x\phi(x,y)\ \left(\kappa\ ED(t,f,S)\ x + \frac{D(t)}{S(t)}\ x + u^*(t,x,y,S)\right)\right\rangle \\
&\quad + \left\langle f^N_{(\boldsymbol{x(t)},\boldsymbol{y(t)})}(t,x,y),\ \partial_y\phi(x,y)\ \left(r\ y - u^*(t,x,y,S)\right)\right\rangle.
\end{aligned}
$$

Here, $\langle\cdot\rangle$ denotes the integration over x and y. Furthermore, the excess demand ED and optimal control u^* is given by:

$$
\begin{aligned}
ED(t,f,S) &:= \int\int u^*(t,x,y,S)\ f^N_{(\boldsymbol{x(t)},\boldsymbol{y(t)})}(t,x,y)\ dxdy \\
&= \frac{1}{N}\sum_{k=1}^{N} u^*(t, x_k, y_k, S), \\
u^*(t,x,y,S) &:= \begin{cases} \frac{1}{\nu}\ K(t,S)\ x, & K < 0, \\ 0, & K = 0, \\ \frac{1}{\nu}\ K(t,S)\ y, & K > 0, \end{cases} \\
&= \lim_{N\to\infty} u^*_N.
\end{aligned}
$$

Hence, the empirical measure $f^N_{(\boldsymbol{x(t)},\boldsymbol{y(t)})}(t,x,y)$ satisfies the equation

$$
\partial_t f(t,x,y) + \partial_x\left(\left[\kappa\ ED(t,f,S)\ x + \frac{D(t)}{S(t)}\ x + u^*(t,x,y,S)\right] f(t,x,y)\right) \tag{4a}
$$

$$
+\partial_y\left([r\ y + u^*(t,x,y,S)]f(t,x,y)\right) = 0, \tag{4b}
$$

in the weak sense, equipped with the excess demand

$$
ED(t,f,S) = \int\int u^*(x,y,S)\ f(t,x,y)\ dxdy.
$$

We call the PDE (4) the **mean field portfolio equation**. Thus the mean field portfolio stock price evolution is described by the PDE (4) coupled with the macroscopic stock price ODE.

$$
\dot{S}(t) = \kappa\ ED(t,f,S)\ S(t).
$$

Remark 2. *We want to emphasize that the mean field limit optimal control u^* is identical to a best reply strategy of our microscopic model. In a best reply strategy, the optimal control is given by the gradient of the objective function.*

4.1 Moment model

The objective of this section is to derive a macroscopic ODE system of our PDE-ODE system of the previous section. Furthermore, we analyze the moment system and verify with the help of simulations that our stylized model can reproduce reasonable financial data. We define the average amount of wealth in stocks X and wealth in bonds Y by

$$X(t) = \int\int x\ f(t,x,y)\ dxdy, \quad Y(t) = \int\int y\ f(t,x,y)\ dxdy.$$

Thus, the excess demand simplifies to

$$\begin{aligned} ED(t,X,Y,S) &= \frac{1}{\nu}\ K(S)\ \int\int (H(-K(t,S))\ x + H(K(t,S))\ y)\ f(t,x,y)\ dxdy \\ &= \begin{cases} \frac{1}{\nu} K(t,S)\ X(t), & K(t,S) < 0, \\ 0, & K(t,S) = 0, \\ \frac{1}{\nu} K(t,S)\ Y(t), & K(t,S) > 0. \end{cases} \end{aligned}$$

where we have used the simple affine-linear form of the control in x and y. The mean field portfolio equation reads

$$\begin{aligned} \partial_t f(t,x,y) + \partial_x \left((\kappa\ ED(t,X,Y,S)\ x + \frac{D(t)}{S(t)} x + u^*(t,x,y,S)\ f(t,x,y) \right) \\ + \partial_y \big((r\ y - u^*(t,x,y,S))\ f(t,x,y) \big) = 0. \end{aligned} \tag{5}$$

We can derive a moment system. We multiply (5) by x, y respectively and integrate over all variables to get the ODE system

$$\frac{d}{dt} X(t) = \kappa\ ED(t,X,Y,S)\ X(t) + \frac{D(t)}{S(t)}\ X(t) + ED(t,X,Y,S) \tag{6a}$$

$$\frac{d}{dt} Y(t) = r\ Y(t) - ED(t,X,Y,S) \tag{6b}$$

$$\frac{d}{dt} S(t) = \kappa\ ED(t,X,Y,S)\ S(t). \tag{6c}$$

Booms, crashes and oscillatory solutions We want to study whether our stock price satisfies the most prominent features of stock markets. These are crashes, booms and oscillatory solutions. Mathematically, a boom or crash is described by exponential growth or decay of the price. In order to study the price evolution, we assume that the weight $W \in [0,1]$ is constant, the exchange rate frequency is $\rho \equiv 1$ and that the value function U_γ is given by the identity. The corresponding equations and solutions can be found in the appendix A.

- Fundamentalists merely ($\chi = 1$) influence the price by their fundamental value s^f. The price is driven to the steady state $S_\infty = \frac{\omega\ s^f}{\omega + r}$ exponentially. Interestingly, the convergence speed depends on the market depth κ, the interest rate r, the expected speed of mean reversion ω and the amount of wealth invested.

- Chartists merely ($\chi = 0$) build their investment decision on the current stock return. The price gets driven exponentially to the equilibrium stock price $S_\infty = \frac{D}{r}$ or away from the equilibrium stock price. This behavior is determined by the average wealth invested in stocks or bonds. In general, we observe exponential growth or decay of the stock price (e.g. $D \equiv 0$). Hence, the chartist behavior can create market booms or crashes. We can thus expect that an interplay of fundamental and chartist strategies leads to oscillatory behavior around the equilibrium prices.

- In our last case, we consider a mix of chartist and fundamental return expectations with a constant weight $\chi \in (0,1)$. In that case, the price converges to the equilibrium price $S_\infty = \frac{\chi\ \omega\ s^f + (1-\chi)\ D}{\chi\ \omega + r}$ which is a combination of the previous equilibrium prices. Thus, the weight χ heavily influences the price dynamic. Furthermore, we can expect to observe oscillatory solutions if we consider a non constant weight $\chi(t,S)$.

Wealth evolution We can analyze the wealth evolution in the same manner as previously the stock price equation. We consider each portfolio separately. The computation can be found in the appendix A, as well.

- We have exponential growth in the stock portfolio, if the wealth gets transferred from bonds to stocks. In the opposite case, the decay of wealth is described by an exponential as well.
- In the bond portfolio, we also observe an exponential increase if the wealth gets shifted into the bond portfolio. If stocks are assumed to perform substantially better ($K(S) > r$)), we have exponential decay in the bond portfolio.

Simulations We want to provide first insights into the portfolio dynamics. The goal is to verify the existence of oscillatory solutions of this simple ODE model. Our previous analysis indicates that there is an interplay between different steady states. We choose the value function U_γ and the weight function W as follows:

$$W(K^f - K^c) := \beta\ \left(\frac{1}{2}\tanh\left(\frac{K^f - K^c}{\alpha}\right) + \frac{1}{2}\right) + (1-\beta)\ \left(\frac{1}{2}\tanh\left(-\frac{K^f - K^c}{\alpha}\right) + \frac{1}{2}\right),$$
$$\alpha > 0,\ \beta \in [0,1],$$
$$U_\gamma(x) := \begin{cases} x^{\gamma+0.05}, & x > 0, \\ -(|x|)^{\gamma-0.05}, & x \le 0, \quad \gamma \in [0.05, 0.95]. \end{cases}$$

The weight function W models the instantaneous comparison of the fundamental and chartist return estimate. The constant $\beta \in [0,1]$ determines if the investor trusts in the higher ($\beta = 1$) or lower estimate ($\beta = 0$) and we thus call this constant the trust coefficient. The constant $\alpha > 0$ simply scales the estimated returns.

The value function U_γ models psychological behavior of an investor towards gains and losses. In order to derive the value function, one needs to measure the attitude of an individual as a deviation from a reference point. We have chosen the reference point to be zero, since $U_\gamma(0) = 0$ holds. In figure 1 we have plotted U_γ and $\bar{U}_\gamma := U_\gamma - 1$. The value function $\bar{U}_\gamma$ is an example of a value function with a negative reference point. Our choice of value function satisfies the usual assumptions: the function is concave for gains and convex for losses, which corresponds to risk aversion and risk seeking behavior of investors. Furthermore, our value function is steeper for losses than for gains, which models the psychological loss aversion of

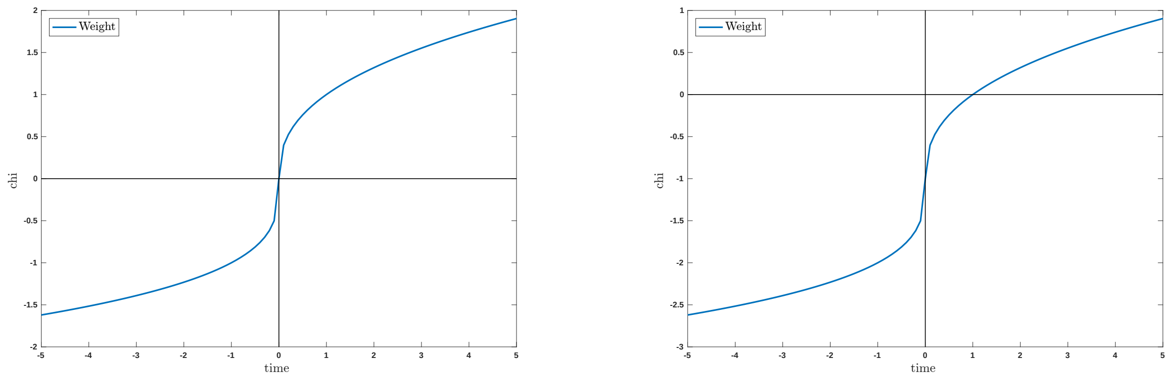


Figure 1: Example of Value functions with different reference points.

financial agents (see figure 1).

We have solved the moment system with a simple forward Euler discretization. The time step has been chosen sufficiently small to exclude stability problems due to stiffness. We verified the results with the `ode15i` Matlab solver. We have chosen a trust coefficient $\beta = 0.25$ for the

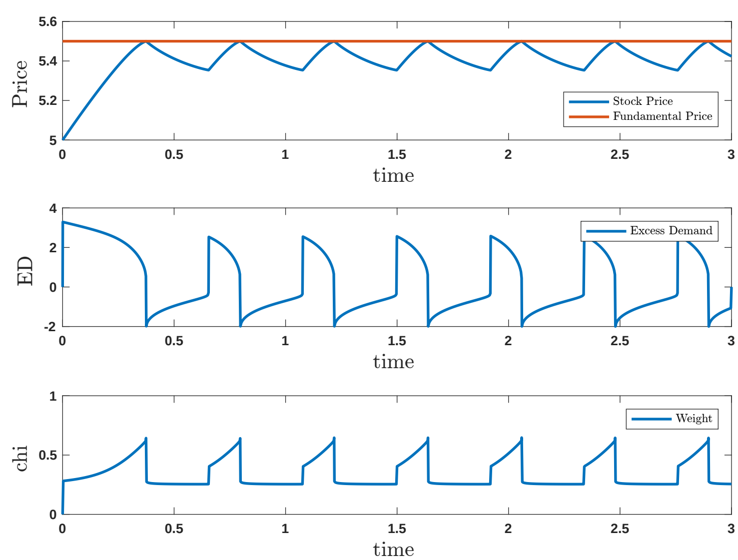


Figure 2: Stock price

simulations in figure 2 and 3. We refer to the appendix B for further settings. The oscillations of the stock price is caused by oscillations in the excess demand. The stock price is always less than or equal to the fundamental price. In addition, the oscillations get translated to the wealth evolution of the portfolios. Increasing wealth in the stock portfolio leads to decreasing

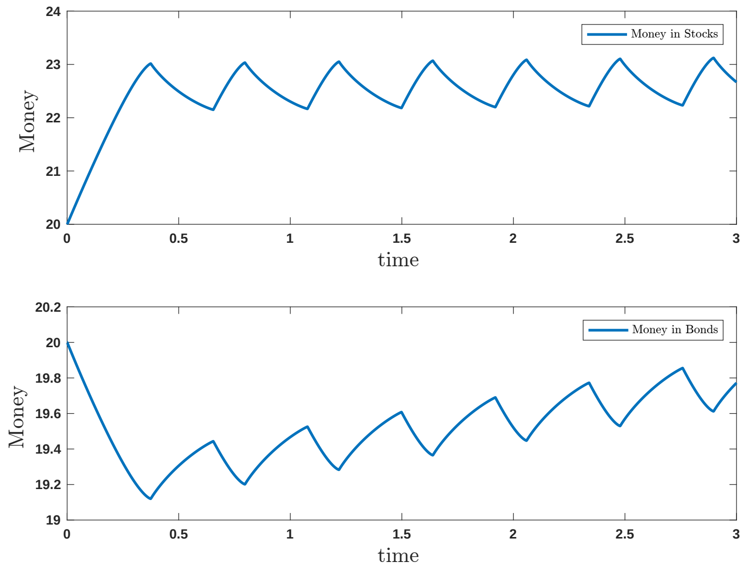


Figure 3: Wealth evolution.

wealth in the bond portfolio. Furthermore, we can observe on average a small positive slope of the wealth invested in bonds (see figure 3). This is caused by the positive interest rate r. In our next simulations (figure 4), we have altered the trust coefficient to study the impact on the price behavior. As figure 4 reveals, the trust coefficient β influences the amplitude

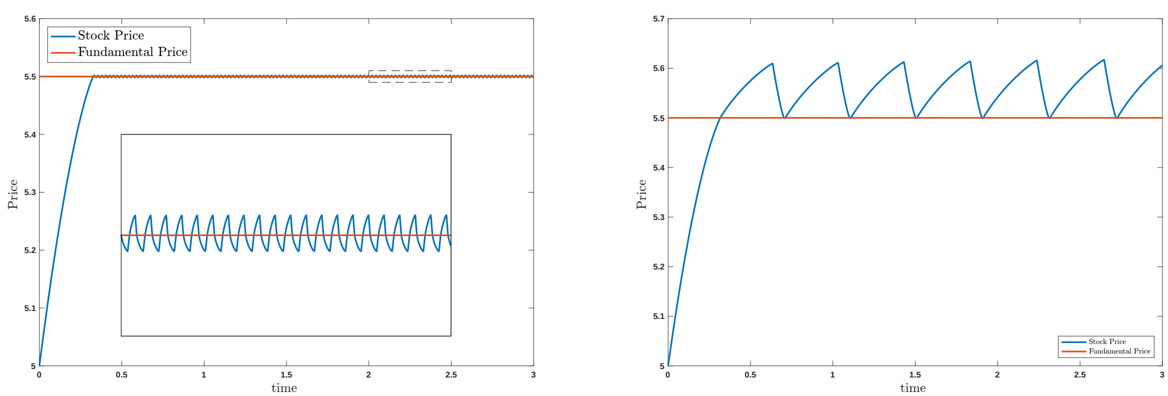


Figure 4: Stock price with trust coefficient $\beta = 0.7$ (left figure) and trust coefficient $\beta = 0.85$ (right figure).

and frequency of our oscillations. In addition, β determines the location of the oscillatory stock price evolution with respect to the fundamental value s^f. A low trust coefficient leads to oscillations located below the fundamental price and a high trust coefficient to oscillations above the fundamental value.

We want to point out that the price behavior is very sensitive with respect to the parameters $\gamma, \kappa, \omega, \alpha$ and β.

Remark 3. *The parameters influence the price dynamics as follows:*

- *A larger risk tolerance γ leads to smaller wave periods and smaller amplitudes. A high risk tolerance heavily changes the price characteristics. We could thus observe convergence of the price to the fundamental value.*
- *The market depth κ influences the amplitude of the oscillations. A bigger κ value leads to a larger amplitude.*
- *The speed of mean reversion ω and the scale parameter α influence the wave period and amplitude. The wave period and amplitude decrease with increasing ω, respectively α.*

Macroscopic steady states In order to obtain steady states, the equations

$$\begin{aligned} 0 &= \kappa\; ED(X,Y,S)X + \frac{D}{S}X + ED(X,Y,S) \\ 0 &= rY - ED(X,Y,S) \\ 0 &= \kappa\; ED(X,Y,S)\; S, \end{aligned}$$

need to be fulfilled. Besides the trivial solution the following steady state configurations are possible.

i) $X = 0, \quad Y = 0, \quad S$ arbitrary

ii) $K(S) = 0, \quad Y = 0, \quad D = 0, \quad X$ arbitrary

iii) $K(S) = 0, \quad r = 0, \quad D = 0, \quad X$ and Y arbitrary

iv) $K(S) > 0, \quad Y = 0, \quad D = 0, \quad X$ arbitrary

v) $K(S) < 0, \quad X = 0, \quad r = 0, \quad Y$ arbitrary

The case $i)$ corresponds to the situation when all investors are bankrupt. In the cases $ii)$ and $iii)$, the investors expect to have no benefit of shifting the capital between both portfolios. This means that the expected return $K(S)$ is zero, which is equivalent to

$$U_\gamma \left(\omega \; \frac{s^f - S}{S} \right) \; \chi + (1 - \chi) \; U_\gamma \,(0) = r.$$

If we choose the value function U_γ to be the identity, we observe

$$S^\infty = \frac{\chi \; \omega \; s^f}{1 + r},$$

as the equilibrium stock price. One might assume that the reference point of the value function is not zero. This means that the financial agent has a fixed bias towards potential gains or losses in his opinion. Mathematically, $U_\gamma(0) \neq 0$ holds and thus the steady state would be shifted by the reference point. Hence, psychological misperceptions of investors lead to changes of the equilibrium price. The case $iv)$ corresponds to the situation that the investor wants to shift wealth from the bond portfolio into the stock portfolio. In fact, no transaction takes place, since there is no wealth left in the bond portfolio. Thus $K(S) > 0$ has to hold, which means

$$U_\gamma \left(\omega \, \frac{s^f - S}{S} \right) \, \chi + (1 - \chi) \, U_\gamma \, (0) > r.$$

In the simple case of the identity function as utility function, we obtain:

$$\frac{\omega \, \chi \, s^f}{r + \omega \, \chi} > S. \tag{7}$$

In this equilibrium case, the amount of transactions have been too low to push the price above a certain threshold defined by inequality (7). The reason for the steady state is the bankruptcy in the bond portfolio. Such a situation does not reflect a usual situation in financial markets. In case v), we face the opposite situation. Here, the investor wants to shift wealth from stocks to bonds although there is no wealth left in the stock portfolio.

4.2 Marginals of mean field portfolio equation

In the previous section, we have analyzed the macroscopic properties of our model. In addition, we want to study the distribution of wealth. The mesoscopic behavior can be studied by the mean field portfolio equation. Compared to models, which only consider ODEs, this is a huge benefit of our kinetic approach. We are interested in discovering the marginal distributions of the mean field portfolio equation

$$\begin{aligned}\partial_t f(t,x,y) + \partial_x ((\kappa \; ED(t,X,Y,S) \; x + \frac{D(t)}{S(t)} x + u^*(t,x,y,S) \; f(t,x,y)) \\ + \partial_y ((r \; y - u^*(t,x,y,S)) \; f(t,x,y)) = 0.\end{aligned}$$

In fact, we can derive equations for the distribution of wealth in stocks and wealth in bonds. The corresponding marginals of f are defined by

$$g(t,x) := \int f(t,x,y) \; dy, \quad h(t,y) := \int f(t,x,y) \; dx.$$

Hence, g is a probability density function of the wealth invested in stocks and h is the probability density function of wealth invested in bonds. We then integrate the mean field portfolio equation over y respectively x to observe equations for g and h. Since the optimal control u^* depends on both microscopic quantities, we cannot expect to get a closed equation for g or h in general.

Nevertheless, in the special case $K < 0$, the control u^* only depends on x and the time evolution of g reads

$$\partial_t g(t,x) + \partial_x \left(\left[\frac{K(S(t))}{\nu} \, (\kappa \; X(t) + 1) + \frac{D(t)}{S(t)} \right] x \; g(t,x) \right) = 0.$$

One solution of the equation is given by:

$$g(t,x) = \frac{c}{\sqrt{\pi} x} \; \exp \left\{ - \left(\log(x) - \int_0^t \frac{K(S(\tau))}{\nu} \, (\kappa \; X(\tau) + 1) + \frac{D(\tau)}{S(\tau)} \; d\tau \right)^2 \right\}, \; c > 0,$$

Notice that g is the distribution function of a lognormal law.

We get a closed equation for h, in the case $K > 0$, in the same way. The solution h is of lognormal type as well. We refer to the appendix C for a detailed discussion.

5 Feedback controlled model with noise

We have seen that our model can reproduce the most prominent features of stock price data, namely oscillatory prices, which replicate booms and crashes. Furthermore, we have shown that the distribution of wealth in bonds and stocks can be represented in special cases by lognormal distributions.

So far, we have considered a fully deterministic model. This does not seem to capture all characteristics of financial markets completely. It is generally accepted that stock prices are unpredictable and e.g. news and political decisions influence the behavior of market participants in an uncertain fashion. For that reason, we want to add randomness to our deterministic model. The optimal control of the i-th agent was given by

$$u^\star(x_i, y_i, S) = \begin{cases} \frac{1}{\nu}\ K(S)\ x_i, & K < 0, \\ 0, & K = 0, \\ \frac{1}{\nu}\ K(S)\ y_i, & K > 0. \end{cases}$$

Notice that the investment decision of agents only differs through different personal wealth. Thus, the estimate of stock return over bond return was identical for all investors. This assumption seems to be too simple, so each individual should differ in their return estimate. Hence, we add white noise to our returns estimate. Since the return estimate is a rate, the random variable also needs to scale with time. We use symbolic notation of integrals adopted from the common notation of SDEs to define the integrated noisy optimal control

$$u^\star_{\eta_i}(x_i, y_i, S)\ dt = \begin{cases} \frac{1}{\nu} K(S)\ dt + \frac{1}{\nu} x_i\ dW_i & K < 0, \\ 0, & K = 0, \\ \frac{1}{\nu} K(S)\ dt + \frac{1}{\nu}\ y_i\ dW_i, & K > 0. \end{cases}$$

Here, dW_i denotes the stochastic Itô integral and thus the feedback controlled microscopic system with noise is given by

$$dx_i = (\kappa ED_N\ x_i + u_i^*)\ dt + \frac{1}{\nu}(H(-K)x_i + H(K)y_i)\ dW_i \tag{8a}$$

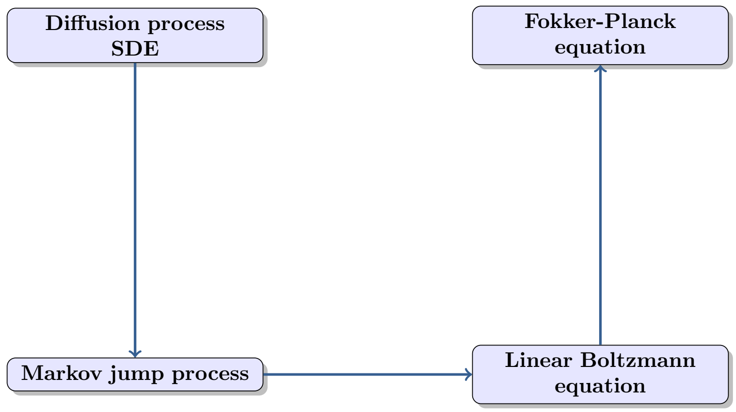
$$dy_i = (r\ y_i - u_i^*)\ dt - \frac{1}{\nu}(H(-K)x_i + H(K)y_i)\ dW_i \tag{8b}$$

$$dS = (\kappa ED_N\ S)\ dt. \tag{8c}$$

5.1 Mean field limit

The goal of this section is to derive a mesoscopic description of our particle dynamics with noise. The classical mean field approaches by Braun, Hepp, Neunzert and Dobrushin [35, 7, 17] do not apply because of the white noise. The only known mean field result in the case of diffusion processes is the convergence of N interacting processes to the kinetic McKean-Vlasov equation [42]. Unfortunately, our model (8) does not satisfy the classical assumptions since our N-particle dynamics are coupled with the macroscopic stock price ODE.

The following modeling approach, well described in [38], is an alternative method to derive the mean field limit of the microscopic model (8) at least formally. The idea is to discretize the diffusion process and interpret it as a Markov jump process. Then, one can derive the corresponding master equation, which can be also interpreted as a linear Boltzmann equation. With the right scaling, known in kinetic theory as grazing limit, one observes in the limit the Fokker-Planck equation. The following diagram illustrates the modeling process.



Boltzmann model As seen before, we consider the probability density $f(t, x, y)$ which describes an investor to have monetary wealth $x \in \mathbb{R}_{\geq 0}$ in his risky portfolio and wealth $y \in \mathbb{R}_{\geq 0}$ in his risk-free portfolio. The portfolio dynamics are characterized by the following linear interactions $(x, y) \mapsto (x', y')$.

$$
\begin{aligned}
x' &= x + a \left(\kappa \; ED(t) + \frac{D(t)}{S(t)} \right) x + a \; u^*_\eta(t, x, y, S), && \text{if } x' > 0, \\
y' &= y + a \; y \; r - a \; u^*_\eta(t, x, y, S), && \text{if } y' > 0,
\end{aligned}
$$

with

$$
\begin{aligned}
u^*_\eta &:= u^*(t, x, y, S) + \frac{1}{\sqrt{a} \; \nu} \; (H(-K) \; x + H(K) \; y) \; \eta, \\
a &:= \Delta t,
\end{aligned}
$$

and a normally distributed random variable η with zero mean and variance one. The time step $\Delta t > 0$ is fixed and originates from the Euler-Maruyama discretization of the SDE. The time evolution of the density function $f(t, x, y)$ is then described by an integro-differential equation of Boltzmann type. In weak form, the equation reads

$$
\frac{d}{dt} \int \phi(x, y) \; f(t, x, y) \; dxdy = (L(f), \phi), \tag{9}
$$

$$
(L(f), \phi) := \left\langle \int \mathcal{K}(x, y, S, ED, D, \eta) \; (\phi(x', y') - \phi(x, y)) \; f(t, x, y) \; dxdy \right\rangle, \tag{10}
$$

with a suitable test function $\phi(x, y)$ and $\langle \cdot \rangle$ denotes the expectation with respect to the random variable $\eta \in \mathbb{R}$. The interaction kernel $\mathcal{K}$ has to ensure that the post interaction portfolio values remain positive:

$$
\mathcal{K}(x, x', y, y', S, ED, D, \eta) := \theta \; \mathbb{1}_{\{x' > 0\}} \mathbb{1}_{\{y' > 0\}} \; \eta,
$$

where $\theta > 0$ is the collision rate and $\mathbb{1}(\cdot)$ the indicator function. The interaction kernel can be simplified if there is no dependence on x and y. This case corresponds to the case of

Maxwellian molecules in the classical Boltzmann equation. This can be achieved by truncating the random variable η in a wary that the post interaction wealth always remains positive. In our case, it is not possible to state explicit bounds for our random variable η since the stock return is not bounded. In fact, for a sufficiently small step size Δt, it is always possible to truncate the random variable in a way that the kernel is independent of x, y. Then, the interaction operator reads:

$$(L(f), \phi) := \left\langle \theta \int \, (\phi(x', y') - \phi(x, y)) \; f(t, x, y) \; dxdy \right\rangle .$$

We can immediately observe that our model conserves the number of agents, which corresponds to the choice $\phi(x, y) = 1$.
We are interested in the asymptotic behavior of the density function f.

Asymptotic limit The goal of the asymptotic procedure is to derive a model of Fokker-Planck type. Thus, the integral operator gets translated into a second order differential operator. The procedure can be described in two steps. First, we perform a second order Taylor expansion of the test function $\phi(x', y')$. Secondly, we rescale characteristic parameters of the model, preserving the main macroscopic properties of the original kinetic equation (9). A closely related approach in kinetic theory is the famous grazing collision limit [44]. We introduce the scaling

$$\theta = \frac{1}{\epsilon}, \quad a = \epsilon,$$

where $\epsilon > 0$ and perform the limit $\epsilon \to 0$. The limit equation is given by the the following Fokker-Planck equation

$$\begin{aligned}
&\partial_t f(t, x, y) + \partial_x ((\kappa \; ED(t, f, S) \; x + u^*(t, x, y, S)) \; f(t, x, y)) \\
&+ \partial_y ((r \; y - u^*(t, x, y, S)) \; f(t, x, y)) + \frac{1}{2 \; \nu^2} \partial^2_{yx} \Big((H(-K)x + H(K)y) \; f(t, x, y) \Big) \\
&= \frac{1}{2 \; \nu^2} \partial^2_x \Big((H(-K)x + H(K)y) \; f(t, x, y) \Big) + \frac{1}{2 \; \nu^2} \partial^2_y \Big((H(-K)x + H(K)y) \; f(t, x, y) \Big),
\end{aligned}$$

coupled with the macroscopic stock price ODE

$$\dot{S}(t) = \kappa \; ED(t, f, S) \; S(t).$$

We call the previously introduced PDE the **diffusive mean field portfolio equation**. In the appendix E, we provide a detailed discussion of the derivation of the diffusive mean field portfolio equation.

5.2 Marginals of diffusive mean field portfolio equation

Again, we are interested in the behavior of the marginal distributions g and h. In the special case $K < 0$, the control u^* only depends on x and the time evolution of g reads.

$$\partial_t g(t, x) + \partial_x \left(\left[\frac{K(S(t))}{\nu} \, (\kappa \; X(t) + 1) + \frac{D(t)}{S(t)} \right] x \; g(t, x) \right) - \partial^2_x \left(\frac{x^2}{2 \; \nu^2} g(t, x) \right) = 0.$$

In order to search for self-similar solutions, we introduce the scaling $\bar{g}(t,\bar{x}) = x\ g(t,x),\ \bar{x} = \log(x)$ and define $b(t) := \frac{K(S(t))}{\nu}\,(\kappa\ X(t)+1) + \frac{D(t)}{S(t)}$. We observe a linear convection-diffusion equation for the evolution of $\bar{g}(t,\bar{x})$

$$\partial_t \bar{g}(t,\bar{x}) + \left(b(t) - \frac{1}{2\ \nu^2}\right)\ \partial_{\bar{x}} g(t,\bar{x}) = \frac{1}{2\ \nu^2}\partial_{\bar{x}}^2 \bar{g}(t,\bar{x}).$$

The solution is given by

$$\bar{g}(t,\bar{x}) = \frac{1}{(2\ \left(\frac{t}{\nu^2}+c\right)\ \pi)^{\frac{1}{2}}} \exp\left\{ -\frac{(\bar{x} + \frac{\frac{t}{\nu^2}+c}{2} - B(t))^2}{2\ (\frac{t}{\nu^2}+c)} \right\},\ c>0,$$

with $B(t) := \int\limits_0^t b(\tau)\ d\tau + \bar{c},\ \bar{c} > 0$. After reverting to the original variables, we get

$$g(t,x) = \frac{1}{x\ (2\ (\frac{t}{\nu^2}+c)\ \pi)^{\frac{1}{2}}} \exp\left\{ -\frac{\left(\log(x) + \frac{\frac{t}{\nu^2}+c}{2} - B(t)\right)^2}{2\ (\frac{t}{\nu^2}+c)} \right\},\ c>0.$$

Thus, the wealth in bonds admits a lognormal asymptotic behavior as well.
Analogously, we obtain a similar equation for h in the case $K > 0$. The solution also satisfies a lognormal law. For details, we refer to the appendix D. At first glance, we did not gain any new information compared to the marginals of the mean field portfolio equation. In both cases, we have observed lognormal behavior. However, this is not true, in the diffusive case, our solution admits a time dependent variance and is not constant in contrast to the deterministic case. In addition, we have observed that adding multiplicative noise does not change the portfolio distribution drastically.

6 Stock price as random process

Until now, the macroscopic stock price evolution has been given by the ODE (1) and was deterministic. We aim to analyze the price behavior in a probabilistic setting and analyze the price distribution. We modify the model by adding a microscopic stochastic model beneath the macroscopic stock price equation (1). To do so, we introduce a new population of $M \in \mathbb{N}$ market makers or brokers. Each broker is equipped with a microscopic stock price $s_j > 0$. The microscopic stock prices are modeled as random processes. The average of broker prices generates the macroscopic stock price S_M.

$$S_M := \frac{1}{M}\sum_{j=1}^{M} s_j.$$

The stochastic nature of microscopic stock prices can be explained by different market accessibility of each broker. Their individual stock price is given by

$$ds_j = \kappa\ ED\ s_j\ dt + s_j\ dW_j, \quad j = 1,...,M, \tag{11}$$

where W_j is a Wiener process and equation (11) has to be interpreted in the Itô sense. Compared to the macroscopic stock price equation (1), there is multiplicative noise added to the price evolution of brokers.

The stock price evolution is coupled with the portfolio evolution in two different ways: First, by the stock return in the stock portfolio and secondly by the investment decision u^*.

$$\begin{aligned}
&\partial_t f + \partial_x((\kappa\ ED\ x + u^*)\ f) + \partial_y((r\ y - u^*)\ f) + \frac{1}{2\ \nu^2}\partial^2_{yx}\Big((-H(-K)x + H(K)y)\ f\Big)\\
&\quad = \frac{1}{2\ \nu^2}\partial^2_x\Big((H(-K)x + H(K)y)\ f\Big) + \frac{1}{2\ \nu^2}\partial^2_y\Big((H(-K)x + H(K)y)\ f\Big),\\
&ds_j = \kappa\ ED\ s_j\ dt + s_j\ dW_j, \quad j = 1, ..., M.
\end{aligned}$$

We need to specify whether the investors' decisions are based on the microscopic or macroscopic stock price. The macroscopic stock price determines the stock return of the agents' portfolio, because this is the global market price. In the case of the investment decision, one can argue that an investor might trade on the microscopic or macroscopic stock price. Arbitrage opportunities are a reason to act on the microscopic scale. In addition, one can argue that the microscopic stock prices have in fact a smaller time scale than the macroscopic stock price since the latter is the average of the former. This leads us to the characterization that investors acting on the micro prices are **high-frequency traders**, whereas agents action on the macro price can be accounted to be **long-term investors**.

Mean field limit As seen before, we want to consider the mean field limit of our microscopic stock price equations. In fact, the microscopic brokers only differ in their initial conditions and multiplicative noise. We have:

$$ds_j(t) = \kappa\ ED(t, f, (\cdot))\ s_j(t)\ dt + s_j(t)\ dW_j, \quad s_j(0) = s^0_j. \tag{12}$$

Thus, there is no coupling between broker and we have a simple setting of McKean-Vlasov type equations. We have written the excess demand as $ED(t, f, (\cdot))$ since we can have $ED(t, f, s_j)$ in the high-frequency case or $ED(t, f, S_M)$ for long-term investors. We assume that the empirical measure $V^N_{\mathbf{s(0)}}(0, s)$, which is defined by the initial conditions of the microscopic system

$$V^N(0, s) := \frac{1}{M}\sum_{k=1}^{M} \delta(s - s^0_k),$$

converges to a distribution function $V(0, s)$. Then, the system (11) converges in expectation to the mean field SDE

$$d\bar{s}_j(t) = \kappa\ ED(t, f, \bar{s}_j)\ \bar{s}_j(t)\ dt + \bar{s}_j(t)\ dW_j, \quad \bar{s}_j(0) = \mathfrak{s}_j, \quad \mathfrak{s}_j \sim V(0, s). \tag{13}$$

Due to the Wiener process, the above set of stochastic processes is independent and in particular identically distributed. We can thus apply the Feynman-Kac formula and the distribution $V(t, s),\ s > 0$ evolves accordingly to

$$\partial_t V(t, s) + \partial_s\left(\kappa\ ED(t, f, (\cdot))\ s\ V(t, s)\right) = \frac{1}{2}\ \partial^2_s(s^2\ V(t, s)). \tag{14}$$

Notice that the macroscopic stock price is the first moment of V.

$$S(t) = \int s\ V(t,s)\ ds.$$

Hence, our **diffusive mean field portfolio stock price** system is given by:

$$\begin{aligned}
&\partial_t f(t,x,y) + \partial_x((\kappa\ ED(t,f,S)\ x + u^*(t,x,y,(\cdot)))\ f(t,x,y,s)) \\
&\qquad - \frac{1}{2\ \nu^2}\partial_x^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) + \frac{1}{2\ \nu^2}\partial_{yx}^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) \\
&\qquad + \partial_y((r\ y - u^*(t,x,y,(\cdot)))\ f(t,x,y)) - \frac{1}{2\ \nu^2}\ \partial_y^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) = 0, \\
&\partial_t V(t,s) + \partial_s\left(\kappa\ ED(t,f,(\cdot))\ s\ V(t,s)\right) = \frac{1}{2}\ \partial_s^2(s^2\ V(t,s)).
\end{aligned}$$

Remember that the influence of the investment decision enters in the stock-price evolution through the excess demand. In the next sections, we want to study the influence of a high-frequency or long-term strategy of investors on the price distribution V.

6.1 Long-term investors

In the case of long-term investors, the investment decision $u^* = u^*(t,x,y,S)$ depends on the macroscopic stock price S. The stock price equation is given by:

$$\partial_t V + \partial_s\left(\frac{\kappa}{\nu}\ K(S)\ \left[\int\int [H(-K(S))x + H(K(S))y]\ f(t,x,y)\ dxdy\right]\ s\ V\right) = \frac{1}{2}\ \partial_s^2(s^2\ V). \tag{15}$$

We can take the first moment of the previous equation and obtain the macroscopic stock price ODE (1) considered in the previous sections. In addition, we want to point out that in the case of long-term investors, we can derive the same moment system (6), discussed previously.

Asymptotic behavior Due to the fact that the stock price is a stochastic process, we can study the distribution function of our stock price PDE. We define

$$\begin{aligned}
P(t) &:= \int\int [H(-K(S))x + H(K(S))y]\ f(t,x,y)\ dxdy, \\
R(t) &:= \frac{\kappa}{\nu}\ K(S(t))\ P(t),
\end{aligned}$$

and search for self-similar solutions of equation (15). The quantity P is the average amount of wealth in the bond or stock portfolio and R is the average amount of wealth invested in stocks. We consider the scaling $\mathcal{V}(p,t) = s\ V(t,s),\ p = \log(s)$ and $\mathcal{V}$ thus satisfies the following linear convection-diffusion equation

$$\partial_t \mathcal{V}(t,p) + \left(R(t) - \frac{1}{2}\right)\ \partial_p \mathcal{V}(t,p) = \frac{1}{2}\partial_p^2 \mathcal{V}(t,p).$$

The solution of the previous equation is given by

$$\mathcal{V}(t,p) = \frac{1}{\sqrt{2\ \pi}} \exp\left\{-\frac{(p + \frac{t+c_1}{2} - \bar{R}(t))^2}{2}\right\},$$

for a constant $c_1 > 0$ and $\bar{R}(t) := \int\limits_0^t R(\tau)\ d\tau + c_2,\ c_2 > 0$. Hence, by reverting to the original variables, we get

$$V(t,s) = \frac{1}{s\ \sqrt{2\ \pi\ (t+c_1)}} \exp\left\{ -\frac{(\log(s) + \frac{t+c_1}{2} - \bar{R}(t))^2}{2\ (t+c_1)} \right\}.$$

We thus observe lognormal asymptotic behavior of the model.

6.2 High-frequency traders

In the case of high-frequency investors, we have to clarify the dependence of the optimal control u^* on the microscopic stock price s. The investment strategy of **high-frequency fundamentalists** can be translated one to one. We have

$$k^f(t,s) := U_\gamma \left(\omega \frac{s^f(t) - s}{s} \right) - r.$$

The chartist estimated return is more difficult. In fact, the chartists estimate involves a time derivative of the stock price. On the microscopic level, we can insert the right-hand side of the microscopic stock price equation. In addition, we assume that the investor averages over the uncertainty. Thus, for **high-frequency chartists** we get:

$$k^c(t,s) := U_\gamma \left(\frac{\kappa/\rho\ ED(t,f,s) + D(t)}{s} \right) - r,$$

We define the aggregated high-frequency estimate of stock return over bond return by

$$k(t,s) := \chi\ k^f(t,s)\ + (1-\chi)\ k^c(t,s).$$

Hence, the **high-frequency stock price equation** reads

$$\partial_t V(t,s) + \partial_s \Big(\frac{\kappa}{\nu}\ k(t,s)\ \Big[\int \int [H(-k(t,s))\ x + H(k(t,s))\ y]\ f(t,x,y)\ dxdy \Big]\ s\ V(t,s) \Big) \\ = \frac{1}{2}\ \partial_s^2 (s^2\ V(t,s)).$$

Notice that we cannot find a closed equation for the first moment of this equation. In general, it is difficult to solve the high-frequency stock price equation. We want to study admissible states of the high-frequency stock price equation in order to obtain a solution.

In addition, we have to specify the dependence of the diffusive mean field portfolio equation on the microscopic stock price. We want to point out that the diffusive mean field portfolio equation, solely coupled with the high-frequency stock price equation, is not well-defined. This is because of the fact that it is unclear how to interpret the variable s in the optimal control of the diffusive mean field portfolio equation. One solution to this problem is to add the mean field SDE (13) to our model. The diffusive mean field portfolio equation is then coupled with the mean field SDE through the microscopic stock prices $\bar{s}$ in the optimal control. In addition, the diffusive mean field portfolio equation is coupled with the high-frequency stock price equation by the macroscopic stock price S. We get:

$$\begin{aligned}
&\partial_t f(t,x,y) + \partial_x((\kappa\ ED(t,f,S)\ x + u^*(t,x,y,\bar{s}))\ f(t,x,y)) \\
&- \frac{1}{2\ \nu^2}\partial_x^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) + \frac{1}{2\ \nu^2}\partial_{yx}^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) \\
&\quad + \partial_y((r\ y - u^*(t,x,y,\bar{s}))\ f(t,x,y)) - \frac{1}{2\ \nu^2}\ \partial_y^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) = 0, \\
&d\bar{s}(t) = \kappa\ ED(t,f,\bar{s})\ \bar{s}(t)\ dt + \bar{s}(t)\ dW, \\
&\partial_t V(t,s) + \partial_s\left(\frac{\kappa}{\nu}\ k(t,s)\ \left[\int\int [H(-k(t,s))x + H(k(t,s))y]\ f(t,x,y)\ dxdy\right]\ s\ V(t,s)\right) \\
&\quad = \frac{1}{2}\ \partial_s^2(s^2\ V(t,s)).
\end{aligned}$$

Since the solution of our high-frequency stock price equation is the density of the stochastic process $\bar{s}$, we can substitute this PDE by the expected value of the stochastic process $\bar{S}$. The alternative model reads:

$$\begin{aligned}
&\partial_t f(t,x,y) + \partial_x((\kappa\ ED(t,f,S)\ x + u^*(t,x,y,\bar{s}))\ f(t,x,y)) \\
&- \frac{1}{2\ \nu^2}\partial_x^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) + \frac{1}{2\ \nu^2}\partial_{yx}^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) \\
&\quad + \partial_y((r\ y - u^*(t,x,y,\bar{s}))\ f(t,x,y)) - \frac{1}{2\ \nu^2}\ \partial_y^2\Big((H(-K)x + H(K)y)\ f(t,x,y)\Big) = 0, \\
&d\bar{s}(t) = \kappa\ ED(t,f,\bar{s})\ \bar{s}(t)\ dt + \bar{s}(t)\ dW, \\
&S = E[\bar{s}].
\end{aligned}$$

We consider the former model instead of the latter as we can analyze the stock price distribution due to the high-frequency stock price equation.

Steady state We want to show that in special cases the steady state distribution is described by an inverse gamma distribution. We assume that $\chi \equiv 1,\ s^f \equiv c > 0,\ U_\gamma(x) = x$ holds. The stock price equation is then simplified to

$$\begin{aligned}
\partial_t V(t,s) + \partial_s\left(\frac{\kappa}{\nu}\ [\omega\ s^f - s\ (\omega + r)]\ \left[\int\int [H(-k(s))x + H(k(s))y]\ f(t,x,y)\ dxdy\right]\ V(t,s)\right) \\
= \frac{1}{2}\ \partial_s^2(s^2\ V(t,s)).
\end{aligned}$$

Furthermore, we assume that the portfolio distribution f has reached a steady state f_∞. We define

$$\begin{aligned}
P_x^\infty &:= \int\int x\ f_\infty(x,y)\ dxdy > 0, \quad k < 0, \\
P_y^\infty &:= \int\int y\ f_\infty(x,y)\ dxdy > 0, \quad k > 0.
\end{aligned}$$

and assume $P^\infty := P_x^\infty = P_y^\infty$. Hence, the steady state distribution $V_\infty(s)$ satisfies

$$\frac{1}{2}\partial_s^2(s^2\ V_\infty(s)) - \frac{\kappa}{\nu}\ P^\infty\ \partial_s\Big([\omega\ s^f - s\ (\omega + r)]\ V_\infty(s)\Big) = 0. \tag{16}$$

The solution of (16) is given by the inverse gamma distribution

$$V_\infty(s) = C \ \frac{1}{(s)^{2(1+\frac{\kappa}{\nu}\ P_\infty(\omega+r))}} \ \exp\left\{ -\frac{2\ \frac{\kappa}{\nu}\ \omega\ P_\infty\ s^f}{s} \right\},\ s > 0,$$

where the constant C should be chosen as

$$C := \frac{(2\ \kappa\ \omega\ P_\infty^x\ s^f)^{1+2\frac{\kappa}{\nu}\ P_\infty(\omega+r)}}{\Gamma(1+2\frac{\kappa}{\nu}\ P_\infty(\omega+r))},$$

such that the mass of V_∞ is equal to one. Here, $\Gamma(\cdot)$ denotes the gamma function. We immediately observe that for large stock prices s, the distribution function asymptotically satisfies

$$V_\infty \sim \frac{1}{s^{2(1+\frac{\kappa}{\nu}\ P_\infty(\omega+r))}}.$$

Hence, the equilibrium distribution is described by a power-law.

Remark 4.

- *We have observed that the presence of high-frequency fundamentalists leads to power-law behavior in the stock price distribution. This coincides with earlier findings in the closely related Pareschi-Maldarella model [29]. Furthermore, the shape of the steady state (inverse-gamma distribution) is identical to wealth distributions observed in [6, 15] .*
- *The universal features which create power-law tails are multiplicative noise and additionally an external force on the microscopic level. In our case, this force is given by the fundamental value s^f of the fundamental trading strategy.*
- *The presence of chartists may also lead to fat-tails in the stock price distribution. Their estimated stock return was given by*

$$k^c(t,s) = U_\gamma\left(\frac{\kappa/\rho\ ED(t,f,s) + D(t)}{s}\right).$$

We assume that the value function is given by $U_\gamma(\cdot) = (\cdot) - \mathfrak{r}$, where $-\mathfrak{r} < 0$ is the reference point. Then, it is possible to observe a steady state distribution of inverse-gamma type. Thus, a fixed overestimation of risk also leads to power-law tails.

7 Numerics

In this section, we want to present some numerical examples of our mean field model. We always consider the final kinetic model, namely the diffusive mean field portfolio stock price model. Our simulations have been conducted with a standard Monte Carlo solver. Furthermore, we use the weighting function W and value function U_γ as defined previously.
First, we have a look at the price and portfolio dynamics in the case of long-term investors. Secondly, we consider the case of high-frequency trader. Detailed information of our parameter choice can be found in the appendix F.

Long-Term Investors In order to observe more realistic price behavior compared to our moment model, we introduce a time varying fundamental price $s^f(t)$. We choose a stationary log-normally distributed fundamental price, modeled by the following SDE

$$ds^f = 0.1\ s^f\ dW.$$

Again, W denotes the Wiener process and the integrals need to be interpreted in the Itô sense. In the case of a constant fundamental price, we observe oscillatory behavior (see figure

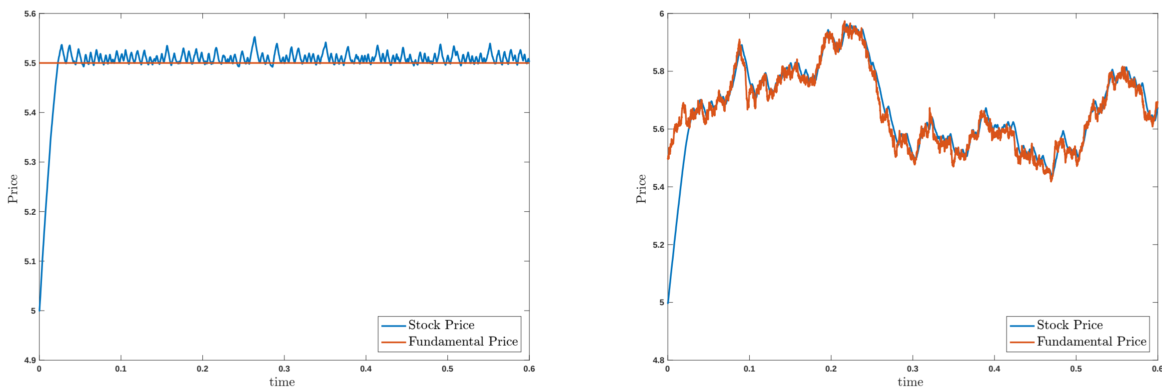


Figure 5: Stock price with a constant fundamental price s^f (left figure) and a time varying fundamental price (right figure).

5). We have already seen this characteristic price behavior in our simulations of the moment model. The asymmetries in the peaks are caused by the noise of the Monte Carlo solver. The price behavior rapidly changes for a time varying fundamental price. The stock price follows the fundamental price but has some overshoots (see figure 5). These overshoots are essential to observe a fat-tail in the stock return distribution. To quantify this, we look at a quantile-quantile plot of logarithmic stock returns. We easily recognize that the stock return exhibits

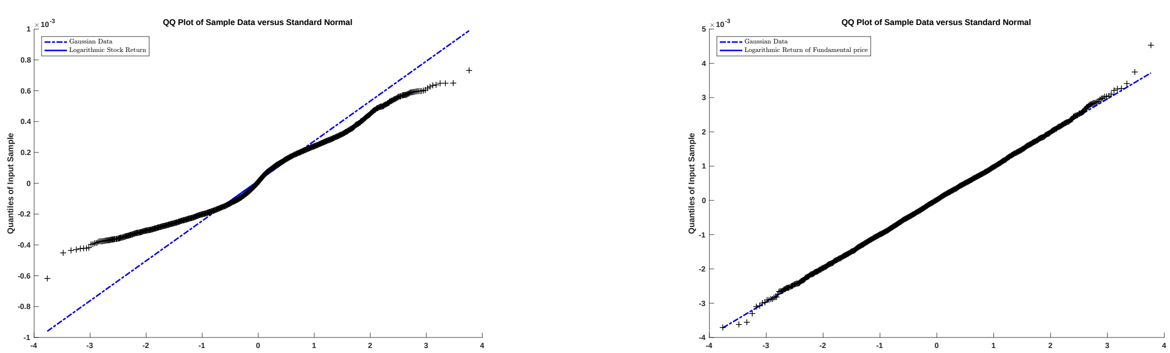


Figure 6: Quantile-quantile plot of logarithmic stock return distribution (left-hand side) and logarithmic return of fundamental prices (right-hand side).

heavy tails (see figure 6). In comparison to the stock return, the return of fundamental prices

is well fitted by a Gaussian distribution.
Due to the mesoscopic kinetic model, we can analyze the price and wealth distributions. In the previous paragraph, we could show that the stock price distribution is given by lognormal law. Our simulations (see figure 7) verify this result. In addition, we want to have a look at the

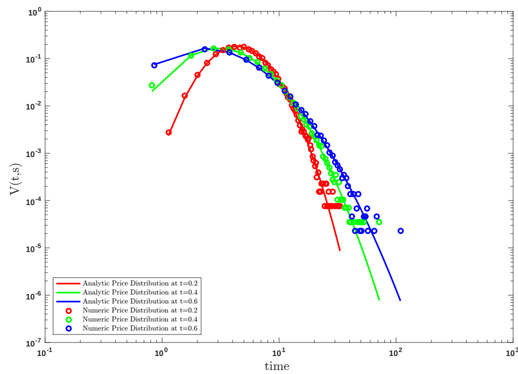


Figure 7: Stock price distribution.

marginal distribution $g(t, x)$ which describes the wealth of the stock portfolio. Unfortunately, we cannot give an analytic solution in the general case. Interestingly, the distribution of stock investments is well fitted by a normal distribution (see figure 8). In the special situation that

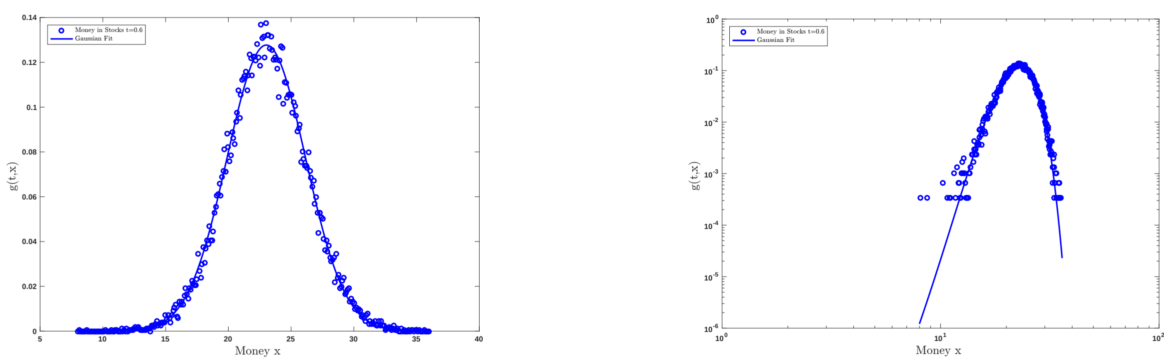


Figure 8: Distribution of the wealth invested in stocks with a gaussian fit. Left figure has a linear scale, whereas the right figure shows the distribution in log-log scale.

the aggregated estimate of stock return over bond return, denoted by K, is strictly positive or strictly negative, we can compute marginal distributions analytically. Then the marginal distribution admits log-normal behavior. For our example, we consider the case $K > 0$, thus, we observe the marginal distribution of wealth in bonds h. In order to ensure $K > 0$, we have set the fundamental stock price to $s^f \equiv 10$ and fixed the weight $\chi \equiv 1$. As figure 9 illustrates, our numerical simulations certify our analytic result.

High-Frequency Investors In the high-frequency investor case, we numerically observe a fat-tail (see figure 10). The fit by the inverse-gamma distribution reveals that the fit under-

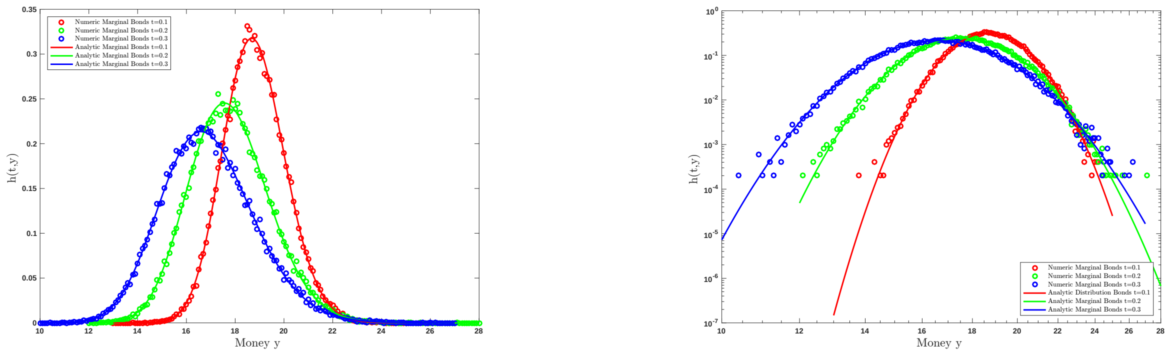


Figure 9: Distribution of the wealth invested in bonds plotted with the log-normal analytic self-similar solution.

estimates the tail probabilities. Furthermore, the wealth distributions are in both portfolios well-fitted by a Gaussian distribution as you can see in figure (11). The shape of the wealth coincides with the marginal portfolio distributions we computed in the long-term investor case.

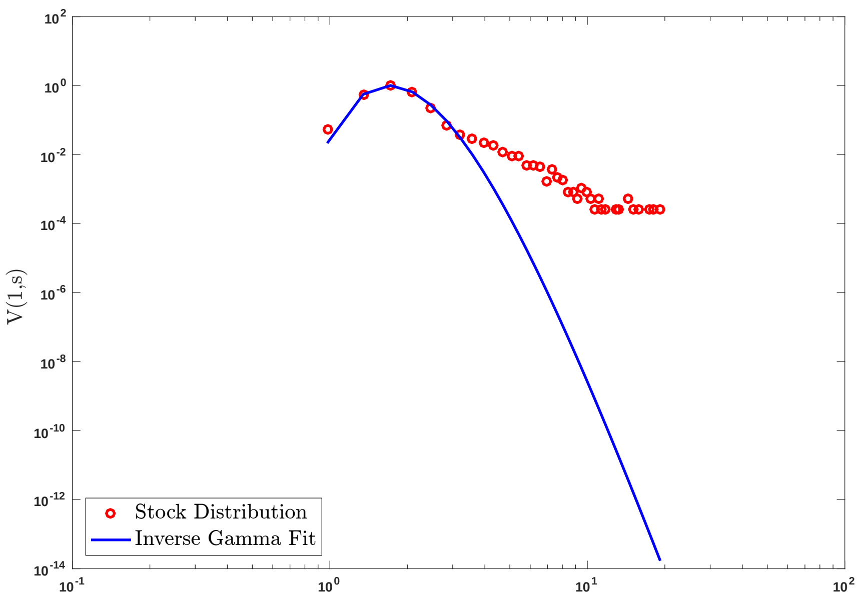


Figure 10: Stock price distribution.

In the previous section, we could compute an admissable steady state distribution analytically. We have observed that the inverse-gamma distribution is a steady state, which is asymptotically well characterized by a power-law for large stock prices s. In order to compute the steady state numerically, the constants r and D must be chosen in an unrealistic manner to guarantee a steady state in the portfolio dynamics. Furthermore, we do not con-

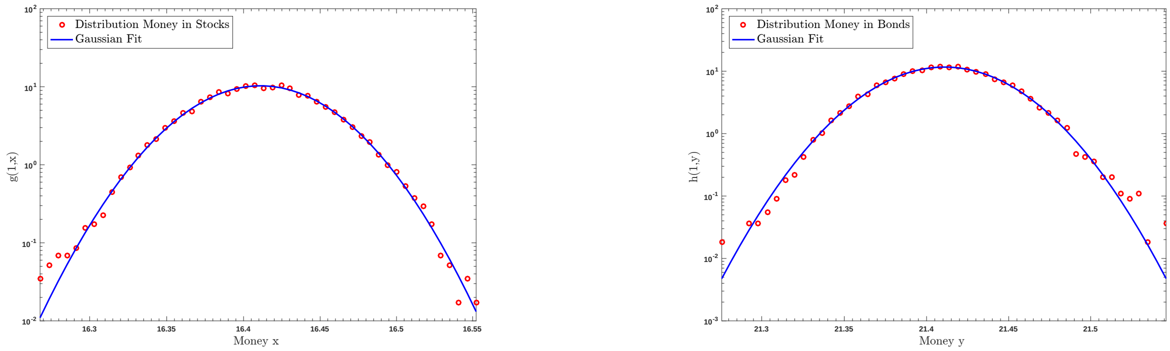


Figure 11: Distributions of the wealth invested in stocks (left-hand side) and wealth invested in bonds (right-hand side) at $t = 1$.

sider the diffusive portfolio equation, but instead the mean field portfolio. In addition, the value function has been chosen as the identity and the weight is fixed as $\chi \equiv 1$. Figure (12)

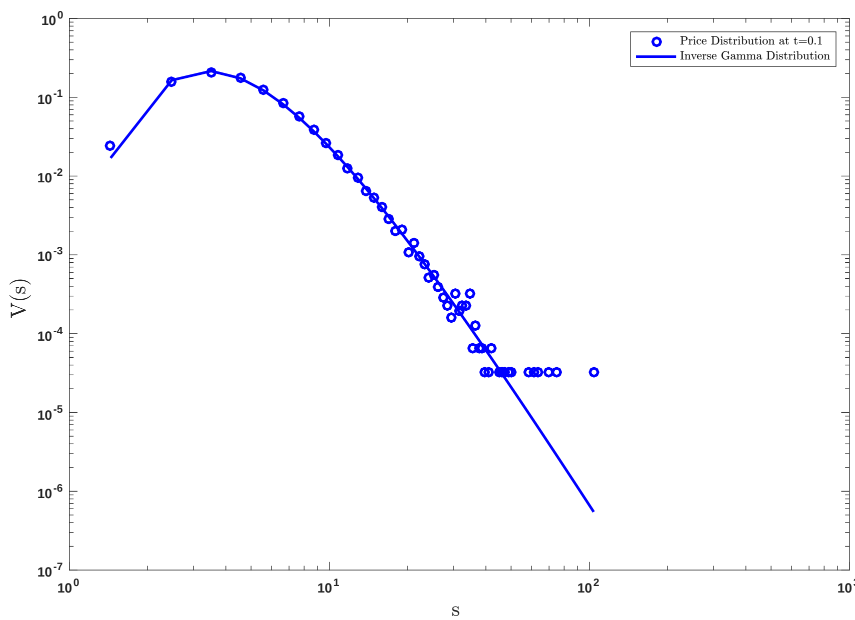


Figure 12: Steady state distribution.

shows that the stock price distribution converges to the analytically computed steady state of inverse-gamma type.

8 Conclusion

The starting point of our investigation was a microscopic portfolio model coupled with the macroscopic stock price equation. Each financial agent was equipped with an optimization problem in order to derive his investment decision. We used the MPC approach in a game theoretical concept to simplify and solve the optimization. The MPC approach has given us a mathematical connection between the two economic concepts of rational and boundedly rational financial agents. Then, starting from the feedback controlled model, we derived a kinetic model. In three modeling stages we derived the mean field portfolio model, the diffusive mean field portfolio model and the diffusive mean field portfolio stock price model.

We first considered a macroscopic perspective and derived and analyzed a three-dimensional moment model. We have seen that this simple ODE model can already replicate financial data exhibiting prominent features like oscillatory solutions, booms and crashes. Then, the three kinetic portfolio models have been analyzed to discover insights in the portfolio distribution. The marginal distributions of wealth in bonds or wealth in stocks can be characterized by a lognormal distribution in special cases. These findings have been supported by our simulations. We have employed the diffusive mean field portfolio stock price model to investigate the price behavior. In the case of long-term investors, the price distribution is given by a lognormal law. In addition, we have computed a steady state of inverse gamma type in the stock price distribution for high-frequency trader. We have seen that for large stock prices the distribution asymptotically satisfies a power-law.
The possibility to analyze the underlying distribution functions means a huge advantage of the chosen kinetic approach. We want to conclude that kinetic modeling is a good tool to reveal insights in the stock price and wealth distributions.

Our model provides an explicit explanation for the creation of power-laws in the stock price distribution. Furthermore, we have observed fat-tails in the stock return distribution, which is a universal feature in real return data. Interestingly, we do not observe fat-tails in the portfolio distribution. This is a surprising result as one could expect to see the same distribution in the stock price and portfolio dynamics. A likely reason is the reallocation of wealth between the two assets which leads to a balance of wealth in both portfolios. We need to remark that regarding the wealth distribution, our model fails to replicate power-laws present in real financial data.

In order to observe a power-law in the wealth distribution, there are several possible model extensions. One idea is to add earnings to the microscopic model. Thus, one would add an external force on the microscopic level. This might give a fat-tail in the portfolio distribution. Alternatively, one could introduce wealth interactions among agents. There are several kinetic models which consider wealth distributions where a power-law has been observed [6, 15, 9, 37]. These models consider interacting financial agents who are performing binary trades. We leave this question open for further research.

Acknowledgement

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Appendix

A

Moment model We summarize the solutions of the moment model in the following propositions.

Proposition 1. *In special cases, we can compute solutions of our stock price equation. We assume constant weights χ, a fixed frequency $\rho \equiv 1$ and assume that the utility function is described by the identity.*

- *Fundamentalists alone ($\chi = 1$): The stock price equation reads*

$$\dot{S} = \begin{cases} \kappa\ (\omega\ s^f - (\omega + r)\ S)\ Y, & \frac{\omega s^f}{\omega + r} > S, \\ \kappa\ (\omega\ s^f - (\omega + r)\ S)\ X, & \frac{\omega s^f}{\omega + r} < S. \end{cases}$$

  This equation seems reasonable, so the investor shifts his capital into stocks if he expects a positive stock return, and vice versa. The solution is given by

$$S(t) = \begin{cases} (1 - \exp\{-\kappa\ (\omega + r)\ \int_0^t Y(\tau)\ d\tau)\})\ \frac{\omega\ s^f}{\omega + r} + S(0)\ \exp\{-\kappa\ (\omega + r)\ \int_0^t Y(\tau)\ d\tau\}, \\ \quad \text{for } \frac{s^f}{\omega + r} > S, \\ (1 - \exp\{-\kappa\ (\omega + r)\ \int_0^t X(\tau)\ d\tau)\})\ \frac{\omega\ s^f}{\omega + r} + S(0)\ \exp\{-\kappa\ (\omega + r)\ \int_0^t X(\tau)\ d\tau\}, \\ \quad \text{for } \frac{s^f}{\omega + r} < S. \end{cases}$$

  Hence, the price is driven exponentially fast to the steady state $S_\infty = \frac{\omega\ s^f}{\omega + r}$.

- *Chartists alone ($\chi = 0$): We get*

$$\dot{S} = \begin{cases} \frac{\kappa\ D\ Y}{1 - \kappa\ Y} - \frac{r\ \kappa Y}{1 - \kappa Y}\ S, & \text{for } \kappa\ D\ Y + D\ (1 - \kappa\ Y) > S, \\ \frac{\kappa\ D\ X}{1 - \kappa\ X} - \frac{r\ \kappa X}{1 - \kappa X}\ S, & \text{for } \kappa\ D\ X + D\ (1 - \kappa\ Y) < S. \end{cases}$$

  The solution is given by

$$S(t) = \begin{cases} \left(1 - \exp\left\{-r\ \kappa\ \int_0^t \frac{Y(\tau)}{1 - \kappa Y(\tau)}\ d\tau\right\}\right)\ \frac{D}{r} + S(0)\ \exp\left\{-r\ \kappa\ \int_0^t \frac{Y(\tau)}{1 - \kappa Y(\tau)}\ d\tau\right\}, \\ \quad \text{for } \kappa\ D\ Y + D\ (1 - \kappa\ Y) > S, \\ \left(1 + \exp\left\{-r\ \kappa\ \int_0^t \frac{X(\tau)}{1 - \kappa\ X(\tau)}\ d\tau\right\}\right)\ \frac{D}{r} + S(0)\ \exp\left\{-r\ \kappa\ \int_0^t \frac{X(\tau)}{1 - \kappa\ X(\tau)}\ d\tau\right\}, \\ \quad \text{for } \kappa\ D\ X + D\ (1 - \kappa\ Y) < S. \end{cases}$$

- *Chartists and fundamentalists with a constant weight $\chi \in (0, 1)$: The corresponding stock price equation reads*

$$\dot{S} = \begin{cases} \kappa\ (\chi \omega s^f + (1 - \chi)\ D - (r + \chi \omega)\ S)\ \frac{Y}{1 + (1 - \chi)\kappa\ Y}, & K(S) > 0, \\ \kappa\ (\chi \omega s^f + (1 - \chi)\ D - (r + \chi \omega)\ S)\ \frac{X}{1 + (1 - \chi)\kappa\ X}, & K(S) < 0. \end{cases}$$

The solution is given by

$$S(t) = \begin{cases} (1-\exp\{-\kappa\ (\chi\omega + r)\ \int\limits_0^t \frac{Y(\tau)}{1+\kappa(1-\chi)Y(\tau)}\ d\tau)\})\ \frac{\chi\omega\ s^f+(1-\chi)\ D}{\chi\omega+r} \\ \quad +S(0)\ \exp\{-\kappa\ (\chi\omega + r)\ \int\limits_0^t \frac{Y(\tau)}{1+\kappa(1-\chi)Y(\tau)}\ d\tau\}, \quad \textit{for } K(S) > 0, \\ (1-\exp\{-\kappa\ (\chi\omega + r)\ \int\limits_0^t \frac{X(\tau)}{1+\kappa(1-\chi)X(\tau)}\ d\tau)\})\ \frac{\chi\omega\ s^f+(1-\chi)\ D}{\chi\omega+r} \\ \quad +S(0)\ \exp\{-\kappa\ (\chi\omega + r)\ \int\limits_0^t \frac{X(\tau)}{1+\kappa(1-\chi)X(\tau)}\ d\tau\}, \quad \textit{for } K(S) < 0. \end{cases}$$

Proposition 2. *For the wealth evolution, we consider the stock and bond portfolio separately.*

- *In the stock portfolio, the wealth evolution is given by*

$$\dot{X} = \begin{cases} (\kappa\ K(S)\ Y + \frac{D}{S})\ X + K(S)\ Y, & \textit{for } K(S) > 0, \\ (\kappa\ K(S)\ X + \frac{D}{S})\ X + K(S)\ X, & \textit{for } K(S) < 0. \end{cases}$$

The solution is given by

$$X(t) = \begin{cases} X(0) \exp\left\{\int\limits_0^t \kappa\ K(S)Y\ + \frac{D}{S}\ d\tau\right\} + \left(1 - \exp\left\{-\int\limits_0^t \kappa\ K(S)Y\ d\tau\right\}\right) \frac{1}{\kappa}, \\ \quad \textit{for } K(S) > 0, \\ \frac{X(0)\ \exp\left\{\int\limits_0^t K(s)+\frac{D}{S}\ d\tau\right\}}{1+\kappa \int\limits_0^t K(S)\ \exp\left\{\int\limits_0^\zeta K(S)+\frac{D}{S}\ d\zeta\right\}\ d\tau}, \quad \textit{for } K(S) < 0. \end{cases}$$

- *The bond portfolio is given by*

$$\dot{Y} = \begin{cases} r\ Y - K(S)\ Y, & \textit{for } K(S) > 0, \\ r\ Y - K(S)\ X, & \textit{for } K(S) < 0, \end{cases}$$

with the solution

$$Y(t) = \begin{cases} Y(0)\ \exp\left\{\int\limits_0^t (r - K(S))\ d\tau\right\}, & \textit{for } K(S) > 0, \\ \exp\{r\ t\} \left(Y(0) - \int\limits_0^t K(S)\ X \exp\{-r\ \tau\}\ d\tau\right), & \textit{for } K(S) < 0. \end{cases}$$

B

Parameters of Simulation If not indicated differently the parameters are set to:

Δt	0.0001	κ	0.1
D	0.01	ν	5
r	0.01	S_0	5
α	0.5	Y_0	20
ω	20	X_0	20
s^f	5.5	T_{end}	3
γ	0.35	ρ	1

C

Marginals of mean field portfolio model For $K > 0$, we get a closed equation for h:

$$\partial_t h(t,y) + \partial_y \left(\left(r - \frac{K(S(t))}{\nu} \right) y \; h(t,y) \right) = 0.$$

This advection equation can also be solved by the lognormal density function

$$h(t,y) = \frac{\hat{c}}{\sqrt{\pi} y} \; \exp \left\{ - \left(\log(y) - \int\limits_0^t r - \frac{K(S(\tau))}{\nu} \; d\tau \right)^2 \right\}, \; \hat{c} > 0,$$

which can be verified by simple computations.

D

Marginals of diffusive mean field portfolio model In the case $K > 0$, we obtain for h the equation

$$\partial_t h(t,y) + \partial_x \left(\left[r - \frac{K(S(t))}{\nu} \right] y \; h(t,y) \right) = \frac{1}{2 \; \nu^2} \partial_y^2 (y^2 \; h(t,y)).$$

Again, we consider the scaling $\bar{h}(t,\bar{y}) = y \; h(t,y), \; \bar{y} = \log(y)$ and define $e(t) := r - \frac{K(S(t))}{\nu}$. Simple computations reveal that $\bar{h}$ satisfies

$$\partial_t \bar{h}(t,\bar{x}) + \left(e(t) - \frac{1}{2 \; \nu^2} \right) \; \partial_{\bar{y}} h(t,\bar{y}) = \frac{1}{2 \; \nu^2} \partial_{\bar{y}}^2 \bar{h}(t,\bar{y}).$$

We define $E(t) := \int\limits_0^t e(\tau) \; d\tau + c_2, \; c_2 > 0$ and

$$\bar{h}(t,\bar{y}) = \frac{1}{(2 \; (\frac{t}{\nu^2} + c_1) \; \pi)^{\frac{1}{2}}} \exp \left\{ - \frac{(\bar{y} + \frac{(\frac{t}{\nu^2} + c_1)}{2} - E(t))^2}{2 \; (\frac{t}{\nu^2} + c_1)} \right\}, \; c_1 > 0,$$

solves the previous convection-diffusion equation. Then, reverting to the original variables, we observe a lognormal law.

$$h(t,y) = \frac{1}{y \; (2(\frac{t}{\nu^2} + c_1) \; \pi)^{\frac{1}{2}}} \exp \left\{ - \frac{\left(\log(y) + \frac{\frac{t}{\nu^2} + c_1}{2} - E(t) \right)^2}{2 \; (\frac{t}{\nu^2} + c_1)} \right\}, \; c_1 > 0.$$

E

Asymptotic limit of Boltzmann model We expand the test function $\phi(x', y')$ in a Taylor series up to order two.

$$\begin{aligned} \phi(x',y') - \phi(x,y) = (x' - x) \; \frac{\partial \phi(x,y)}{\partial x} + (y' - y) \; \frac{\partial \phi(x,y)}{\partial y} + \\ (x' - x) \; (y' - y) \; \frac{\partial^2 \phi(x,y)}{\partial y \partial x} + \frac{1}{2} (y' - y)^2 \; \frac{\partial^2 \phi(x,y)}{\partial y^2} + \\ \frac{1}{2} (x' - x)^2 \; \frac{\partial^2 \phi(x,y)}{\partial x^2} + R(x,y). \end{aligned}$$

Here, R denotes the remainder of the Taylor series. The right-hand side of our kinetic equation is then given by:

$$
\begin{aligned}
(L(f),\phi) = &\left\langle \theta \int a \left[x \ \left(\kappa \ ED + \frac{D}{S}\right) + u_\eta^*(t,x,y,S)\right] \frac{\partial \phi(x,y)}{\partial x} f(t,x,y) \ dxdy \right\rangle + \\
&\left\langle \theta \int a \ [y \ r - u_\eta^*(t,x,y,S)] \ \frac{\partial \phi(x,y)}{\partial y} f(t,x,y) \ dxdy \right\rangle + \\
&\left\langle \theta \int \left[(x'-x) \ (y'-y) \ \frac{\partial^2 \phi(x,y)}{\partial y \partial x} + \frac{1}{2}(y'-y)^2 \ \frac{\partial^2 \phi(x,y)}{\partial y^2} \right] \ f(t,x,y) \ dxdy \right\rangle + \\
&\left\langle \theta \int \left[\frac{1}{2}(x'-x)^2 \ \frac{\partial^2 \phi(x,y)}{\partial x^2} + R(x,y) \right] \ f(t,x,y) \ dxdy \right\rangle .
\end{aligned}
$$

We make the following scaling assumptions:

$$
\theta = \frac{1}{\epsilon}, \quad a = \epsilon.
$$

The interaction operator is consequently given by:

$$
\begin{aligned}
(L(f),\phi) = &\int \left[x \ \left(\kappa \ ED + \frac{D}{S}\right) + u^*(x,y,S)\right] \ \frac{\partial \phi(x,y)}{\partial x} f(t,x,y) \ dxdy \\
&+ \int [y \ r - u^*(x,y,S)] \ \frac{\partial \phi(x,y)}{\partial y} f(t,x,y) \ dxdy \\
&+ \int \left[\epsilon \ x \ \left(\kappa \ ED + \frac{D}{S}\right) \ y \ r - \frac{1}{\nu^2}(H(-K)x + H(K)y)^2 \right] \\
&\quad \frac{\partial^2 \phi(x,y)}{\partial y \partial x} \ f(t,x,y) \ dxdy \\
&+ \int \frac{1}{2}[\epsilon y^2 r^2 - 2 \ \epsilon y r u^*(x,y,S) + \epsilon u^*(x,y,S) + \frac{1}{\nu^2}(H(-K)x + H(K)y)^2] \ \frac{\partial^2 \phi(x,y)}{\partial y^2} \ f(t,x,y) \ dxdy \\
&+ \int \frac{1}{2}[\epsilon \left(x \left(\kappa \ ED + \frac{\bar{D}}{S}\right) + u^*(x,y,S)\right)^2 + \frac{1}{\nu^2}(H(-K)x + H(K)y)^2] \frac{\partial^2 \phi(x,y)}{\partial x^2} f(t,x,y) \ dxdy \\
&+ R_\epsilon(t).
\end{aligned}
$$

Here, we have used the fact that our random variable has zero mean. We assume that the remainder

$$
R_\epsilon(t) := \left\langle \theta \int R_\epsilon(x,y,\gamma) \ f(t,x,y) \ dxdy \right\rangle ,
$$

vanishes in the limit $\epsilon \to 0$. Consequently, our integral operator simplifies to

$$
\begin{aligned}
(L(f),\phi) = &\int \left[x \ \left(\kappa \ ED(t) + \frac{D}{S}\right) + u^*(x,y,S)\right] \ \frac{\partial \phi(x,y)}{\partial x} f(t,x,y) \ dxdy + \\
&\int [y \ r - u^*(x,y,S)] \ \frac{\partial \phi(x,y)}{\partial y} f(t,x,y) \ dxdy + \\
&\int \frac{1}{2}\frac{1}{\nu^2}(H(-K)x + H(K)y)^2 \left[\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} - \frac{\partial^2 \phi(x,y)}{\partial y \partial x} \right] f(t,x,y) \ dxdy,
\end{aligned}
$$

as $\epsilon \to 0$. Then we integrate by parts and observe the weak form of the following Fokker-Planck equation

$$
\begin{aligned}
&\frac{\partial}{\partial t} f(t,x,y) + \frac{\partial}{\partial x}\left(\left[x\ \left(\kappa\ ED(t) + \frac{D}{S}\right) + u^*(x,y,S)\right]\ f(t,x,y)\right) + \\
&\frac{\partial}{\partial y}\left([y\ r - u^*(x,y,S)]\ f(t,x,y)\right) + \frac{1}{\nu^2\ 2}\frac{\partial^2}{\partial x \partial y}((H(-K)x + H(K)y)^2\ f(t,x,y)) \\
=&\frac{1}{\nu^2\ 2}\frac{\partial^2}{\partial x^2}((H(-K)x + H(K)y)^2\ f(t,x,y)) + \frac{1}{\nu^2\ 2}\frac{\partial^2}{\partial y^2}((H(-K)x + H(K)y)^2\ f(t,x,y)).
\end{aligned}
$$

F

The parameters of our simulations are set to:

Random Fundamental Price

Δt	0.0001	κ	0.4
D	0.01	ν	5
r	0.01	S_0	5
α	0.5	Y_0	20
β	0.65	X_0	20
ω	80	T_{end}	0.6
γ	0.55	Samples	$3 \cdot 10^4$
ρ	$\frac{2}{3}$		

Computation of Marginal

Δt	0.0001	κ	0.1
D	0.01	ν	5
r	0.01	S_0	5
χ	1	Y_0	20
s^f	10	X_0	20
ω	20	T_{end}	0.3
γ	0.35	Samples	$5 \cdot 10^4$
ρ	$\frac{2}{3}$		

High-frequency trader

Δt	0.001	κ	0.4
D	0.01	ν	5
r	0.01	S_0	5
α	1	Y_0	20
β	0.2	X_0	20
ω	80	T_{end}	1
γ	0.55	Samples	$5 * 10^3$
ρ	$\frac{2}{3}$		

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