ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/280918570

Bounded Rationality and Incomplete Financial Markets

Book · January 2009

DOI: 10.13140/RG.2.1.2099.6966

CITATION
1

READS
101

1 author:

Soumitra K Mallick
Institute for Advanced Study

71 PUBLICATIONS **81** CITATIONS

SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Project

Nobel Physics Prize proposal (2)"String Matching Field Theory (HM(YM)) of Electronic Internetworked Genetic "sustainable" Coal Mining Stock Markets in spacextime". View project

Project

Mathematical Statistical Pareto distribution properties of Bombay Stock Exchange and Econophysics Implementation using Panel Data View project

All content following this page was uploaded by Soumitra K Mallick on 12 August 2015.

The user has requested enhancement of the downloaded file.

BOUNDED RATIONALITY & INCOMPLETE FINANCIAL MARKETS

Soumitra K. Mallick

December 7, 2009

Contents

Preface

This monograph grew out of my dissertation submitted to the faculty of the Graduate School of Arts and Sciences of New York University, in partial fulfilment of the requirements of the degree of Doctor of Philosophy.

I am particularly grateful to the members of my dissertation committee, Yaw Nyarko, Andrew Schotter, Aldo Rustichini and Christopher Flinn. Their advisement was invaluable.

Acknowledgement

The original dissertation of which this is an expanded and revised version would not exist without the introduction to the topics of General Equilibrium theory and Bounded Rationality made by Roy Radner, Mukul Majumdar, Heraklis Polemarchakis and Jaysri Dutta. Helpful comments and advice were provided by Jess Benhabib, Charles DeBartolome, Paolo Siconolfi, Shashi Murthy, Arijit Mukherji, Dilip Abreu, Charles Wilson and Kenneth French, the seminar participants at the Eastern Economic Association Meetings, New York, at Jadavpur University, at the Indian Statistical Institute, New Delhi and at a winter school of the Econometric Society at the Indian Institute of Management, Calcutta. Nicholas Yannelis, Glenn Ellison and Matthew Jackson provided valuable editorial advice in preparing this monograph at various stages, and the Indian Statistical Institute, Calcutta and the Indian Institute of Social Welfare & Business Management, Calcutta provided essential technical support, and the latter two provided some free time, for which this project has been completed.

Special thanks to my wife Prakriti and son Sandipan, to our parents and the rest of our families and all my friends,too many to enumerate, for all their constant support at various stages of this work. A number of new directions are being explored by Raffaela Giordano in the areas of budget deficits and alternative policy frameworks with Italian data, recently, which makes the exploration of the structural properties of markets begun here, already applicable to macroeconomic policy making. I thank her for allowing me to use chapter four out of our joint work. All remaining errors are, however, my sole responsibility.

Chapter 1

Introduction

The Neo-Classical Choice model under uncertainty assumes that agents are fully rational i.e. they have a well defined Von-Neumann-Morgenstern utility function over all the state-contingent consumption plans and they maximize this expected utility subject to their budget constraints to decide on their consumption-investment plan. Existence of a well defined utility function over all states in the future means that agents have to be able to compare and contrast the utility derived from consumption in each state over the future. Thus, if for e.g. there are 10 states at each date of the world and the agent makes decisions in the Arrow-Debreu sense over the whole future which comprises of 1000 dates , then finding out the expected utility requires comparing and contrasting all these plans. Certainly in the real world this number is much larger, hence the complexity of decisions much bigger. This fact has been noted by Radner(1974, 1982) who says - “If the Arrow-Debreu model is given a literal interpretation, then it clearly requires that the economic agents possess capabilities of imagination and calculation that exceed reality by many orders of magnitude...”.

It has been also observed under strategic experimental situations by Schotter et. al. (1990) that agents’ behavior do not conform with the rationality postulates. Since non-strategic behavior can be interpreted as the limit of strategic behavior as the size of the economy gets very large hence this observation can be easily extended to the case of a non-strategic Arrow-Debreu economy. Hence, there seems to be a definite requirement in a general equilibrium Arrow-Debreu model to relax the assumption of full rationality and to find a way of introducing "Bounded Rationality" i.e. "constraints on the information processing capacities of the actors" (Simon (1972)). In this monograph I introduce Bounded Rationality into a general equilibrium model under uncertainty (i.e. the Arrow-Debreu model) by relaxing the assumption of complete state-contingent preference orderings. I introduce the fact that agents are not born with a completely defined set of preferences within the context of an Arrow-Debreu model. In a recent study on the Indian Stock Markets by Sarkar, Roy, Mallick, Duttachaudhuri & Chakraborty (2001), extended in Sarkar, Mallick & Roy (2006,2003), it has been observed that during the 80s and 90s Indian Stock Markets exhibited substantial deviations from rationality with the presence of significant negative intercepts, significant positive weightage on dividends as opposed to profit, better fit of various nonlinear expectations models as opposed to linear expectations models of stock pricing and significant excess volatility in returns in the market. The well known paper of Prescott & Mehra (1985)(see also Kocherlakota (1996)) had shown that stock returns in the USA showed systematic premiums over and above that justified by share of profit attributable to them, after adjusting for risk premium. This has of date served as an anomaly, probably explainable by the bounded rationality model of this monograph. All of these features are suggestive of divergence from rationality either in the form of bounded rationality or in the form of speculative behavior in stock markets, relevant more so in the case of emerging economies such as India.

The introduction of Bounded Rationality into the Arrow-Debreu model can explain two things - (a) a microfoundation for Incomplete Markets in line with what Polemarchakis (1990) says - "It is an open question to derive the incompleteness of the asset market from primitive differences of information among individuals", and (b) a microfoundation for "institutions" or rule following behavior (discussed in for e.g. Radner & Rothschild (1975), Majumdar & Radner (1991)), within the context of an Arrow-Debreu economy. In this monograph I pursue the first investment response to bounded rationality i.e. active expenditure on learning to make decisions within the learnt decision horizon .

The introduction being the first essay, the second essay viz. “Bounded Rationality & Incomplete Markets” proceeds as follows. In this essay I take the following view of Incomplete Markets. I say that a market is incomplete if any agent in the economy cannot find assets in the market to transfer consumption between any two states (and/or dates) over which he desires to spread consumption . Here, the state space over which agents choose to participate in trades is endogeneously chosen. Due to heterogeneity amongst individuals this results in the fact that these “participation rates” become diverse (see for e.g. Siconolfi (1986) for discussion relating to an equilibrium model where these “participation rates” are exogenously given and are heterogeneous). Hence, this results in there being at least one agent who cannot find a partner to trade contingent consumption with, with respect to certain states and/or dates , even though he plans to consume quantities different from his endowments in those states (and/or dates) . I provide a two-date partial equilibrium model which shows that when agents have “bounded rationality” , and under certain additional conditions which are sufficient, heterogeneity in the initial period wealth distribution leads to incompleteness of asset markets. The same result would also hold, it is conjectured, if the heterogeneity was in the utility parameters or what can be interpreted as the degree of “risk aversion” of agents.

According to the view taken in this paper, Bounded Rationality means that agents cannot order preferences over contingent consumption based on their initial beliefs about the future states of the world. Thus, they do not know their optimizing objectives when they are born. This results in each agent having to “actively invest” in a learning technology prior to making contingent consumption decisions, so as to lay down a well-defined optimizing objective. This technology narrows down the “relevant” state-space towards the true state and helps agents in establishing their expected utility over this reduced state-space. The decision proceeds as follows: given the learning technology each agent computes the “value of learning” which is the “expected utility” rationally expected to be generated after taking into account information processing costs, and chooses the optimum number of states to learn about and makes optimizing decisions only with respect to this set of states .

Thus, the state space with respect to which each agent trades consumption at the initial date is endogenously determined. Since, the value function of learning varies with the initial period wealth, hence unequal wealth levels would lead to diverse learning and hence to diverse market participation and hence incomplete markets. If the utility parameters were to be heterogeneous, this value function would be different also, leading to heterogeneous choice of “participation rates” and hence, incomplete markets.

In the third essay titled “Incomplete Markets & Policy Control” I show how in an economy where agents have “Bounded Rationality” as in the previous essay, equilibrium might not exist, generically, if agents are heterogeneous with respect to their initial period wealth (and / or utility parameters). This provides a general equilibrium extension of the previous essay. Besides, it also provides for a case of wealth redistribution based on the generic nonexistence of equilibrium rather than on optimality properties (see for e.g. Atkinson & Stiglitz (1990)).

In the fourth essay titled “Incomplete Markets & Competition” I study a model whereby the degree of competition in the contingent commodity market is endogenously determined. Certain assumptions are made about the contingent commodity trading process in terms of the possible set of contracts which can be traded which makes it a restriction out of all possible markets and the disequilibrium resulting out of incompleteness in such markets are analyzed dynamically.A set of policy instruments are derived which are variables necessary to be controlled to ensure completeness and competitive equilibrium .

In my economy, agents are not born with a complete set of state contingent preferences over all possible states in the future. Each agent who is therefore infinitely lived is faced with the decision of allocating his resources over the infinite future, where there are a finite number of states at each date of the world. His rationality is bounded in the sense that he does not have a complete preference ordering over all possible states of the world. Hence, he knows that he cannot trade in the contingent commodities market because he cannot form optimal strategies in order to trade. If this is true of all agents in the world no trade will occur in the future. Hence, although it is not possible for him to know ex-ante which state of the world will occur at each date he can learn about each state of the world by spending resources on learning about the returns on assets which pays off in each state, and discard certain states from consideration as being improbable, then he could focus consideration on a subset of the states , and if this subset happens to be at most as large as the set of states over which his preference ordering is complete, then he can form optimal strategies and trade contingent consumption claims. This is the motivation for learning . However, since while learning agents do not have a well specified optimizing objective they have to use rules in order to learn. Since the construction of an optimizing objective depends on the learning rules to be followed the ability to participate in the market (which requires an optimizing objective) is determined by the rules being followed, hence, the existence of an Arrow-Debreu equilibrium which requires the participation of all agents in the economy can only be achieved if the rule followed in the economy results in establishing an optimizing objective for the entire economy. I show that under one type of rule Arrow-Debreu markets might exist while under an alternative set it might not.

This monograph is meant to model decision making under uncertainty in the process of bounded rationality , contingent commodity contracts which payoff in the future in "commodity money" and information which aggregates the intertemporal consumption-investment decisions in terms of the contingent consumption plans. The aggregate macro outcomes are discussed not in terms of values of variables perse but in terms of the market structure which arises as a result. Thus incompleteness is discussed as the outcome of contingent commodity trading plans mediated by perhaps a passive market maker, who merely observes whether complete markets, necessary for competitive prices and allocations to result, result with various probabilities. Thus incompleteness may be the macroeconomic outcome, as this monograph discusses. Economic policies thus get a new framework at least in terms of policy sequencing , which is beyond the scope of this monograph.

A running theme in this monograph is the notion of "commodity money" . All assets are pegged in terms of the "commodity money" . Hence, payoffs of assets are denominated in terms of this one good. This procedure of converting spot returns in bundles of goods into equivalent units of “commodity money” had been intrepreted by Geanokoplos & Polemarchakis (1985) as the “Gold Standard” of measuring financial returns. Mallick, Sarkar, Roy, Duttachaudhuri & Chakraborty (2006) have measured stock market returns in terms of fiat money instrumentalized by dividend and interest payments. As is evident such a conversion system, under flexible prices, assumes the existence of equilibrium spot prices with which such conversion can be achieved. The full impact of the failure of such fully flexible price regime, in the presence of incomplete markets, is beyond the scope of this monograph.

This monograph raises two different issues - one is about the process of learning in market economies where agents have bounded rationality , the other is about market participation in such economies. These issues become particularly relevant for emerging economies in countries moving towards a market based allocation system on the whole or trying to introduce some new markets, for these are problems associated with the successful implementation of the competitive market system.

REFERENCES

ARROW, K.J.& G. DEBREU , Existence of Equilibrium for a Competitive Economy, *Econometrica, 22: 265-290, 1954.*

GEANOKOPLOS, J. & H. POLEMARCHAKIS , Existence, regularity & constrained suboptimlity of competitive allocations when the asset market is incomplete in Heller, W.P., R.M. Starr & D.A. Starrett (eds.), Uncertainty , Information & Communication, Essays in honor of K.J. Arrow, Vol. III, Cambridge University Press, 1986.

KOCHERLAKOTA, N. , The equity premium : still a puzzle, *Journal of Economic Perspectives, 34, 42-71, 1996.*

MEHRA, R. & E. PRESCOTT , The Equity Premium: A Puzzle, *Journal of Monetary Economics, March, 15, 145-161, 1985.*

MALLICK, S.K. , Generic nonexistence of equilibrium with incomplete asset markets, *Quantitative Finance, 2007, submitted.*

−−−− ,A.SARKAR, K.K.ROY, T.DUTTACHAUDHURI & A.CHAKRABORTY , Dynamic financial pricing in emerging economies : an application to stock markets in India, *Journal of Asset Management, 6(7), 335-346, 2007.*

MAJUMDAR, M.K. & R. RADNER , Linear models of economic survival under production uncertainty ,*Economic Theory, 1: 13-30 , 1991.*

POLEMARCHAKIS, H., The Economic Implications of an Incomplete Asset Market, *American Economic Review, Papers & Proceedings, 1990 : 280-283.*

RADNER, R., Equilibrium under uncertainty , K.J. Arrow & Intriligator (eds.) Handbook of Mathematical Economics, Vol II, North Holland, Amsterdam, 1982.

− − −− , Market Equilibrium & uncertainty : Concepts and Problems, in, M.D. Intriligator & D.A. Kendrick (eds.) Frontiers of Quantitative Economics, Vol. II, North Holland, Amsterdam, 1974.

− − −− & M. ROTHSCHILD, On the allocation of effort,*Journal of Economic Theory, 10 : 358-376, 1975.*

SARKAR, A., K.K. ROY, S.K. MALLICK, T. DUTTACHAUDHURI, & A. CHAKRABORTY , Financial fragility, asset bubbles, and the real rate of growth: A study of the Indian economy 1970-2000, mimeo, Planning Commission, Govt. of India, New Delhi, `http://www.planningcommission.nic.in/reports/sereport/ser/scoiwel.pdf`, 2001.

S.K. MALLICK, A.SARKAR & K.K. ROY , Financial fragility in Indian stock markets : a cross section analysis, *Indian Economic Review, 2006, submitted*.

A.SARKAR, K.K. ROY & S.K. MALLICK , Financial fragility in Indian stock markets in the decade of the nineties - a time series study, in R.Jha ed. Indian Economic Reforms, Palgrave Macmillan, Basingstoke U.K., 2003.

SCHOTTER, A., K. WEIGELT & C. WILSON , A laboratory investigation of multiperson rationality and presentation effects, w.p. no. 90-24, C.V. Starr Center for Applied Economics, New York University, 1990.

SIMON, H.A., Theories of Bounded Rationality , in C.B. Mcguire & R. Radner (eds.) Decision and Organization, North Holland, Amsterdam, 1972.

ATKINSON, A.B. & J.E. STIGLITZ, Lectures on Public Economics, Mcgraw Hill Book Co., New York, 1980.

Chapter 2

Bounded Rationality & Incomplete Markets

2.1 Introduction

Radner (1974, 1982) has suggested two possible lines of extension of the Arrow-Debreu (1954) framework. It is noted that - "If the Arrow-Debreu model is given a literal interpretation, then it clearly requires that the economic agents possess capabilities of imagination and calculation that exceed reality by many orders of magnitude. Related to this is the observation that the theory requires in principle a complete system of contingent commodity and futures markets, which appears to be too complex, detailed and refined to have any practical significance". In line with the above suggestions I try to achieve the following goals in this paper:

1. Provide a theory of "bounded rationality" which according to Simon (1972) and as has been quoted by Radner (1975) are "Theories that incorporate constraints on the information processing capacities of the actor". Hence I shall provide a way of building into a neoclassical choice model under uncertainty the fact that agents' abilities to imagine and calculate are limited.[1]

[1]That agents' behaviors do not conform with the rationality postulates has been observed under experimental conditions by Schotter,Weigelt & Wilson (1990).

2. Provide an endogenous explanation as to why asset markets in the real world are incomplete rather than take as given an exogenously specified incomplete market structure (as is done in the existing literature like Radner (1972), Hart (1975), Geanokoplos & Polemarchakis (1986)).

3. Develop a link between the phenomena of bounded rationality and incomplete markets.

In a market economy, under uncertainty , agents have to allocate their consumption over the present and future dates . I assume that agents make decisions under "total ignorance". Since the future is uncertain, "total ignorance" implies any one of infinitely many possible states of the world might be realized at each date in the future. According to the Arrow-Debreu view, consumption of each good in each state at each date involves consumption of a particular contingent good. Thus choice of a future consumption plan involves comparing all these infinitely many contingent consumption plans which requires a huge ability to imagine and calculate. In a world where agents do not have such infinite comparison (or what I interpret as Simon's "information processing") capacity, it is natural that they will have to invest resources in learning about the various possibilities of states in the future, so as to narrow down the set of contingencies . This is accomplished with the help of some "learning technology "[2] Thus, learning results in establishing probability weights and utility measures with respect to each state within a certain endogenously chosen set of states .

According to the view taken in this essay, bounded rationality means that agents cannot order preferences over contingent consumption based on their initial beliefs about the states of the world. Thus, they do not know their optimizing objective (expected utilities in our case) when they are born. This results in each agent having to "actively invest" in an exogenous "learning technology " prior to making contingent consumption decisions, so as to lay down a well

[2]An example is getting forecasts from the weather service and understanding the implications on preferences under the various alternatives *a la* Arrow's famous paper on optimizing airline flights based on wind direction and speed.

specified optimizing objective . This technology narrows down the "relevant" state space towards the true state and helps agents in recovering their utility functions with respect to each state within this reduced state space . The agents then assign uniform probability weights to each state within this "relevant" state space and treat the rest of the states as having probability zero. This defines the optimizing objective for the expected utility criterion.

The information technology augments the agent's information processing capacity and causes expenditure of his resources. Thus, given that there is a fixed ability of agents to process information on their own, each agent first invests in information processing costs, and chooses the optimum number of states about which to learn and makes optimizing decisions only with respect to this set of states .

An important question to ask under this situation is what happens with respect to the rest of the states? Since, agents cannot formulate optimizing objectives with respect to those states they either follow some behavioral rule with regard to consumption in those states (see for e.g. Radner & Rothschild (1975)), or they only make decisions with respect to the states over which they have processed information. I take the second view in this paper which is based on the logical notion of "reductio ad absurdum".

A market is said to be incomplete, in the existing literature, when the set of assets available in the market does not span the entire state space. Such a view of incomplete markets considers both the state space and the available set of assets as exogenously given. Problems of nonexistence and suboptimality of equilibria in the goods market are discussed within such an asset market framework. However, assets are nothing but contracts between agents providing for risk sharing, hence assets arise out of individual decisions. Therefore, in a neoclassical framework the obvious question regarding incomplete markets which comes to one's mind is — why don't agents write contracts to cover all the contingencies? That an explanation in this regard is necessary has been noted by Polemarchakis (1990) - "It is an open question to derive the incompleteness of the asset market from primitive difference of information among individuals". However, instead of taking differences of information as the primitive I take a step backward to the fundamentals and provide an explanation in terms of agents' characteristics.

In this essay I take the following view of incomplete markets. I say that a market is incomplete if any agent in the economy cannot find assets in the market to transfer consumption between any two states (and/or dates) over which he desires to spread consumption. Thus, contrary to the existing models every agent in our economy has an endogenously chosen, for reasons of bounded rationality, as has been discussed before, set of states (and/or dates) over which he wants to spread consumption. Due to heterogeneity amongst individuals this results in the fact that there is at least one agent who cannot find a partner to trade contingent consumption with, with respect to certain states (and/or dates) even though he wants to consume quantities different from his endowments in those states (and/or dates).

Incompleteness does not affect everybody. If an agent does not plan ex-ante for consumption different from his endowment in a certain state (and/or date) then nonexistence of assets to transfer consumption to that state (and/or date) does not affect him and does not affect the economy either if all the other agents' "relevant" state space does not include that state. Thus, nonexistence of assets paying off 100 years from now or paying off at summer temperatures of 100 degrees Celsius does not make the asset market incomplete because no one wants to trade claims today depending on those contingencies, as all agents in the world have "learnt" from history or from physics that living beyond 100 years or the earth's surface temperature reaching 100 degrees Celsius are zero probability events. While these are the very same reasons why no assets paying off in those states exist — because no one writes contracts contingent on those states .

The shape of the value function of learning will depend on the state-contingent wealth and utility profiles of each agent. Using a logarithmic utility example, I show that if the value function admits an interior maximum then heterogeneity in the initial period wealth is sufficient to cause incomplete markets generically. If the value function does not admit an interior maximum, then, agents who are optimizing by nature, resort to behavioral rules with respect to their choice of learning. I do not address incompleteness arising from such rule following behavior, in this essay.

Section 2 sets up a two period partial equilibrium learning , contingent consumption and incomplete market model. Section 3 characterizes the results using the logarithmic utility specification. Section 4 conjectures on a multi-period extension of this model. Section 5 is the conclusion.

2.2 The model

The model is as follows. There is no uncertainty at date 0, all the uncertainty is with respect to date 1. The uncertainty is with respect to the state-contingent utility measures and the state-contingent wealth levels. Since utilities are state-contingent the state which realizes determines the utility function of each agent. The economy started operating at $t = -\infty$. Agents are born at the beginning of $t = 0$ and will die at the end of $t = 1$. Agents have perfect memory with respect to events occurring prior to $t = 0$, but cannot analyze events on their own[3] I denote the σ-field generated by the set of events occurring upto $t = 0$ by $\mathcal{I}_0$. The result of bounded rationality is that agents are not able to form finer partitions associated with infinite dimensional state-space and to recover cardinal utility measures associated with consumption in each state, from information contained in $\mathcal{I}_0$. The state contingent wealth realizations are, however, recoverable.

2.2.1 The learning model

Learning involves reducing the possible state space to a smaller size so that it contains the true state and contingent contingent consumption plans with respect to this set of states can be ordered by the agent who has limited intelligence — which is necessary in order to formalize preferences over contingent consumption (bounded rationality) . This is in contrast to existing learn-

[3]This brings out the distinction between information storage (memory) and information processing (intelligence) capacities. Thus our agents have infinite memory but bounded intelligence. This distinction is not necessary for the purpose of our explanation of incomplete markets.

ing models like (Grossman, Kihlstrom & Mirman (1977), Blume & Easley (1984), Kiefer & Nyarko (1989), Nyarko (1990)) where agents use "passive" Bayes' Rule to learn about endowment distributions or about specific parameters when functional forms are known. The agents specified here are actively involved in information processing with respect to their optimizing objectives. Besides, these agents cannot use the dynamic programming approach like those models as the utility functions are not known ex-ante, hence, the value function cannot be defined beyond the present decision horizon .

Each agent i's initial belief at date 0 regarding the states of the world that might realize at date $t = 1$ is given by $\omega_i = \{1, 2, \ldots, \infty\}$[4]. $s = 1$ is the true state at each date, but this fact is not known to any agent. There is no uncertainty at $t = 0$, or in other words the agent's decision (which we will discuss shortly) is being made after the state of the world has realized at $t = 0$. Learning enables each agent to partition the initial state-space into "finer" partitions. If the agent i invests ℓ_1 (in physical units of the only consumption good) in learning (which I shall use interchangeably with "information processing") then his updated state partition ω_i^p becomes

$$\omega_1^p = \omega_i^{p'} \cup (\omega_i^{p'})^c$$

given,

$$\omega_i^{p'} = \{1, 2, \ldots, E[S_i|\{\mathcal{I}_0]\}$$

$$(\omega_i^{p'})^c = \{E[S_i + 1|\mathcal{I}_0], E[S_i + 2|\mathcal{I}_0], \ldots, \infty\}$$

such that:

$$P\{\text{some } s \in \omega_i^{p'} \text{occurring at } t = 1\} = 1 \tag{1}$$

and, the agent assigns uniform probability weight to each state within $\omega_i^{p'}$. where S_i the "decision horizon" is given by:

$$S_i = L_i(\ell_i) + \epsilon_i \tag{2}$$

[4]Initial beliefs ranging over an infinite state-space is to be interpreted as "total ignorance".

$L_i(.)$ is the "learning technology " available to agent i.

The "learning technology" works as follows. It uses the consumption good as input and produces a forecast of the possible state-space at date 1. In fact it partitions the initial infinite state-space to a finite set which contains the true state and the residual infinite set. Each agent i knows that the first set ($\omega_i^{p'}$ in (1)) contains the true state but does not know which one it is. They therefore attach probability 1 to the first set and probability zero to the other. These sets are as given in (1). However, the identity of the sets are not known to the agent before the investment. The agent knows the learning technology subject to some random error, i.e. the agent knows, subject to a random error with mean 0, how close to the true value the forecast will be consequent to a given level of input for e.g. suppose for some value of ℓ_i viz. $\ell_i', L_i(\ell_i') = 4$, the agent expects that the technology will churn out 4 integers (for notational simplification we assume this to be 4 consecutive integers starting at 1) and the true state will be one of these but does not know which 4 consecutive set of integers the machine will churn out, nor does he know which one out of these four is the true state. However, given the rational expectations framework of this paper the agents will be optimizing with respect to a set of states which consists of a consecutive set of integers starting at 1 the true state,in making the learning-investment decision which we will narrate in the next section.

The learning technology has the following properties:

$$L_i(.) \in I_+ - \{1\} \text{for} \ell_i > 0 \tag{2a}$$

$$\forall \ell_i, \ell_i' > 0 \text{s.t.} \ell_i' > \ell_i, L_i(\ell_i') \leq L_i(\ell_i) \tag{2b}$$

and $\forall \ell_i > 0 \text{s.t.} L_i(\ell_i) \geq 3 \exists \ell_i' \text{s.t.} \infty > \ell_i' > \ell_i$ and

$$L_i(\ell_i') = L_i(\ell_i) - 1 \tag{2c}$$

I_+ refers to the set of positive integers. Assumption (2a) is made so as to keep the choice model in discrete time and satisfy the convention of numbering states. It also implies that $s = 1$ being the true state (which we know but the agent does not know) it is never possible to have perfect knowledge priori. Assumption (2b) implies that an increase in learning reduces the size of the state space $\omega_i^{p'}$ (see (1)) and (2c) imply that the exogenous learning technology which agent i uses to learn about the possibility of various states occurring is a step function.

Also $\epsilon_i \; \forall \; i$ is a random error term with the property

$$E\{\epsilon_i | \mathcal{I}_0] = 0 \quad \forall i \tag{2d}$$[5]

Thus, (2) & (2d) imply,

$$E\{S_i | \mathcal{I}_0] = L_i(\ell_i) \tag{3}$$

agent i's beliefs about the relevant state space for decisions regarding $t = 1$ is restricted to ω_i, then since ω_i is infinite, the initial probability weights $\pi_i(s)$ - on each state s with respect to $t = 1$ is given by,

$$\pi_i(s) = P\{\ s \ \text{ occuring at } \ t = 1 \ \text{ for } \ s \in \omega_i\} = 0 \tag{4}$$

However, with the $\ell_i > 0$ the updated state-space $\omega_i^{p'}$ is countably finite and hence, the updated probability weights $\pi_i(s)$ - with respect to each state s within $\omega_i^{p'}$ becomes[6],

$$\pi_i(s) = P\{\ \text{state } s \ \text{ occurring at } \ t = 1 \ \text{ for } \ s \ \in \omega_i^{p'}\} = 1/E[S_i] > 0 \tag{5}$$

Notice that from (1) the effect of investment in learning by agent i i.e. $\ell_i > 0$ is that $P\{s \in \{ES_i + 1, \ldots, \infty\}$ occurs at $t = 1\} = 0.$

2.2.2 The choice model

There is one perishable consumption good in the economy which can be used for consumption or for information processing [7] Information processing

is private. The contingent consumption of the good by agent in state s at date t is denoted by $c_i^s(t)$. The price of the good at that 0

[6] I shall from now on drop the conditioning on $\mathcal{I}_0$ for national simplification. All expectations will have to be interpreted, however, with respect to this information set.

[7] See Allen (1990) for discussion with respect to information as a commodity

is normalized to 1 while its state contingent price in state s at $t = 1$ is given by $p(s) \geq 0$.

Each agent's decision at date $t = 0$ involves allocating his resources to consumption at $t = 0$, to learning with respect to $t = 1$ and purchasing contingent consumption for $t = 1$ with respect to the update state space ω_i^p. Agent i's state dependent utility function with respect to state s any date is given by the function $u_i^s(.)$ for $s \in \{1, 2, .s, ES_i\}$ as has already been stated $u_i^s(.)$ are not specified for $s \notin \{1, 2, \ldots, ES_i)$.

$u_i^s(.)$ has the following properties:

$$\partial u_i(.)/\partial \arg > 0 \tag{6a}$$

$$\partial^2 u_i(.)/\partial^2 \arg < 0 \tag{6b}$$

His rate of time discount is given by δ_i where $0 < \delta_i < 1$.

Since the state at date 0 when there is no uncertainty is 1 the utility function with respect to date 0 is $u_i^1(.)$. Also, since the utility functions for agent i are not defined for states beyond ES_i, and agents assign 0 probabilities to each of these states , the expected discounted utility with respect to $t = 1$ is given by :

$$\sum_{s=1}^{ES_i} \delta_i \pi_i(s) u_i^s(c_i^s(1))$$

where $\pi_i(s)$ is the updated probability beliefs as given by (3) & (5).

The price of the consumption good at date 0 is normalized to 1. Therefore, the cost of consumption at date 0 is given by $c_i^1(0)$, where, $c_i^1(0)$ is i's consumption at date 0, and the cost of learning is given by ℓ_i. Since, the price of the consumption with respect to date 1 is given by $\sum_{s=1}^{ES_i} p(s) c_i^s(1)$.

The value of wealth at date 0 for agent i is given by w_i^s, since as has been noted before the state that has realized at date 0 is $s = 1$, and the wealth process is assumed to be time stationary. Also, given that the wealth process is w_i^s the value of total wealth within the decision horizon at $t = 1$ is given by :

$$\sum_{s=1}^{ES_i} p(s) w_i^s$$

Since, as has already been described, each agent i treats all states $s > ES_i$ as having probability 0, hence he does not take these wealth realizations into consideration in his optimizing problem.

Therefore, the budget constraint of agent i is given by,

$$c_i^1(0) + \ell_i + \sum_{s=1}^{ES_i} p(s)(c_i^s(1) - w_i^s) - w_i^1 = 0.$$

I assume a rational expectations framework with respect to $t = 1$ and $s \leq ES_i$. Agent i's learning-investment decision at date 0, therefore, is :

$$\max_{(c_i),\ell_i \geq 0} u_i^1(c_i^1(0)) + \sum_{s=1}^{ES_i} \delta_i \pi_i(s) u_i^s(c_i^s(1))$$

$$s.t. c_i^1(0) + \ell_i + \sum_{s=1}^{ES_i} p(s)(c_i^s(1) - w_i^s) - w_i^1 = 0$$
$$ES_i = L_i(\ell_i) \in I_+ - \{1\}$$
$$\pi_i(s) = 1/ES_i \tag{7}$$

where, $\{c_i\}$ represents the set $\{c_i^1(0) \geq 0, c_i^s(1) \geq 0\}, c_i^s(0), c_i^s(1)$ represents his consumption at dates 0 & 1 state s, respectively, w_i^s is his state contingent endowment in state s, ES_i is given by (3), and $\pi_i(s)$ is given by (5).[8] [9]

Given the rational expectations framework, this problem is solved in two steps. First, the agent solves for $c_i(s)$ s given a particular value of ℓ_i, hence he gets $c_i^0(ES_i, (p(s)),$ $c_i^s(1; ES_i, p(s))$. Next, using this demand function he optimizes (7) to choose ℓ_i.

The choice of ℓ_i affects the decision problem in three ways - through the budget constraint, through the string of utility functions which define the agent's expected utility function and through the probability weights which are derived from the decision horizon. The final distribution of probability weights are uniform within

[8]According to (7) each agent behaves "as if" markets are complete within their decision horizon . This may not be fulfilled at equilibrium.

[9]Notice that this formulation information is being valued in terms of the expected utility that will be generated as a result of learning with the help of such information.

the states $s \leq ES_i(p(s))$. Thus, an increase in the decision horizon increases the number of consumption and wealth realizations under consideration while at the same time reducing the probability weights on the utility derived from consumption in each state.

There are two effects of learning in this model - updating probability beliefs and establishing preference ordering, which together determine each agent's optimizing objective . Notice that if instead of optimizing with respect to the updated state-space ω_i^p agent i were to optimize with respect to ω_i then the optimization problem (7) with ES_i replaced by ∞ is degenerate because each of the probability weights $\pi(s) = 0$. Thus, each agent's optimizing problem is degenerate even if the utility profile is known. On the other hand if i started with "partial ignorance" so that ω_i is finite then $\{u_i^s(c_i^s(1))\}$ not being $\mathcal{I}_0$ measurable would mean that (7) is unspecified.

Also, in order to make the choice set with respect to ℓ_i compact I need to close the choice set from below. By introducing the condition that the expected utility with respect to $t = 0$ is 0 when $\ell_i = 0$ (i.e. the state-space under consideration is infinite) no matter what the utility functions are, I have closed the choice set with respect to ℓ_i by 0 from below.

I write $[ES_i(p(s)), \{c_i(p(s))\}]$ as the solution to problem (7) for agent i at prices $\{p(s))\}$

2.3 Incomplete Market

Asset a is a contract which pays off 1 unit of the consumption good in state a and 0 units in all other states . Thus, if the agent wishes to transfer q units of consumption from state x to state y then his supply of asset x is q and demand for asset y is q.

The supply and demand at $t = 0$ for assets paying off in any state s at $t = 1$ at the price $p(s)$ arises in the following way :

if, $c_i^s(1; p(s)) - w_i(s) < 0$ for some s there is positive supply of asset s by agent i. if, $c_i^s(1; p(s)) - w_i(s) > 0$ for some s there is positive demand for assets s by agent i

$$(8a)$$

I shall say that *agent i faces an incomplete asset market at $t = 0$ at prices $\{p(s)\}$* if for some s there is positive supply or demand of asset s by agent j by supply or demand of asset s by any agent $i \neq j$, does not exist.

$$(8b)$$

I shall say that at $t = 0$ supply or demand for asset s by any agent $i \neq j$ does not exist (to be contrasted with supply or demand for asset s is zero) iff

$$c_i^s(1; p(s)) \text{ is undefined for all agents } i \neq j \tag{8c}$$

I shall say that $c_i^s(1; p(s))$ is undefined for agent i iff,

$$s > ES_i(p(s)) \tag{8d}$$

I shall say that *the contingent commodity market at $t = 0$ is incomplete at prices $\{p(s)\}$ if there exists at least one agent j such that j faces an incomplete asset market at $t = 0$ at prices $\{p(s)\}$.*

$$(9)$$

2.4 The logarithmic utility case

I consider an example where agents are heterogeneous only with respect to wealth distributions and utility functions. State 1 is realized at date 0 before making decisions. The utility profiles are given by,

$$u_i^s(.) = \alpha_i^s \ln(.), 0 < \alpha_i^s < 1, s \geq 1 \tag{10}$$

Thus, (7) becomes,

$$\max_{\{\ell_i \geq 0, c_i\}} \alpha_i^1 \ln(c_i^1(0)) + \sum_{s=1}^{ES_i} \delta_i \pi_i(s) \alpha_i^s \ln(c_i^s(1))$$

s.t. the constraints defined in (7)

$$(7')$$

where, $c_i^1(0)$ is consumption at date 0 where $s = 1$ is realized and the wealth that is realized is therefore w_i^1 and $c_i^s(1)$ is consumption at date 1 in state s.[10]

I first solve (7') for $c_i^1(0)$ and $c_i^s(1)$ holding $ES_i(p(s))$ fixed as given; I then substitute these values of $c_i^s(.)s$ in the expected utility function (the budget constraint has already been satisfied) and use the learning technology function to solve for ℓ_i. The following results are obtained.

THEOREM: *The value of information* - $V_i(\ell_i)$ - *at date 0 to agent* i *is given by,*

$$V_i(\ell_i) = \alpha_i^1 \ln(\alpha_i^1 Z_i(\ell_i)) + \sum_{s=1}^{L_i} \delta_i \alpha_i^s (L_i)^{-1} \ln(\alpha_i^s (p(s) L_i^{-1} Z_i(\ell_i)) \qquad (11a)$$

where,

$$Z_i(\ell_i) = \{w_i^1 - \ell_i + \sum_{s=1}^{L-1} p(s) w_i^s\}\{\alpha_i^1 + \sum_{s=1}^{L_i} \alpha_i^s L_i^{-1}\}^{-1}$$

$$L_i = L_i(\ell_i)$$

Proof:
The first order conditions for problem (7') treating ℓ_i as given, are as follows,

$$c_i^1(0) = \alpha_i^1/\lambda_i,$$

$$c_i^s(1) = \alpha_i^s/(\lambda_i(s) L_i),$$

where, λ_i is the Lagrange multiplier for agent i with respect to the budget constraint. Using the budget constraint to solve for λ_i and using (3) and (5), I substitute for $c_i^1(0), \{c_i^s(1)\}$, ES_i and $\pi_i(s)$ in the objective function to get the above result.Q.E.D.

DEFINITION 1: $\ell_i^*(p(s))$ *is an optimal choice of learning by agent* i *at*

prices $(p(s))$ *iff,*

$$V_i(\ell_i^*(p(s))) \geq V_i(\ell_i(p(s)) \forall \ell_i(p(s)) \geq 0 \qquad (11b)$$

---

[10]The discrete state space makes the investment problem a clay-putty type flow input-point output model.

OBSERVATION 1: From (11a) it is clear that the shape of $V_i(\ell_i)$

depends on $\{\alpha_i^s\}$ and $\{w_i^s\}$. There is a range of values of ℓ_i given a particular series of the utility and wealth parameters for which the value function is concave. If the maximized value of $V_i(\ell_i)$ for this set of values for ℓ_i is a global maximum then (11b) has an interior maximum. In this paper we deal with this case only. If, however it so happens that $V_i(\ell_i)$ does not have an interior maximum then it is plausible that agents will be forced to follow "behavioral rules" as in for e.g. Radner et.al. (1975), with respect to their learning decisions. In this paper I assume an interior solution.

Also, notice from (11a) & (11b) implicitly that $ES_i^*(p(s))$ (the decision horizon corresponding to the choice of learning $\ell_i^*(p(s)))$ depends on the wealth and utility profiles $\{w_i^s\}$ and $\{\alpha_i^s\}$ respectively. Thus, heterogeneity in either one of these two will generate distinct "decision horizons" $ES_i^*(p(s))$. I do not need heterogen learning technology to drive the results. Of course learning technologies may very well be another source of heterogeneity in this economy.

OBSERVATION 2:

Another point to note from Proposition 1 is that prices affect consumption in any state at each date through two channels. One is direct, operating through the $\{p(s)\}$ terms in (11a). The other operates indirectly though the $ES_i^*(p(s))$ term from (11a) and (11b). This measure $\partial ES_i^*(p(s))/\partial(p(s))$ can be termed as the "elasticity of expectations" with respect to prices as is discussed in Temporary Equilibrium models (see for e.g. Grandmont (1977)). This elasticity depends on the parameters α_i and the (w_i^s) and of course on the parameters of the learning technology or what may be termed the "bounded rationality parameters" and provides a microfoundation for the elasticity of expectations.

THEOREM 2: *If, $V_i(\ell_i)$ admits an interior optimal choice of learning $\forall_i$ and if, $w_i^1 \neq w_j^1$ for $i \neq j$, then the contingent commodity market at $t = 0$ is generically incomplete at finite, non-zero prices $(p(s))$.*

Proof:

From (11a) $\partial V_i/\partial w_i^1 > 0$ if L_i is finite i.e. if $V_i(\ell_i)$ admits an interior maximum (optimal choice of learning according to definition 1) (see also observation (1) above). Hence, $V_i(\ell_i) \neq V_j(\ell_j)$ if $w_i^1 \neq w_j^1$.

This implies from (11b) that $\ell_i^*(p(s)) \neq \ell_j^*(p(s))$. This, implies that $ES_i^*(p(s)) \neq ES_j^*(p(s))$. Further definitions (8a) -(9) imply that the contingent commodity market is generically incomplete (i.e. except in those probability zero cases where demand and supply of assets (see definition of incompleteness) by the agent i for whom $ES_i^*(p(s))$ is greater, for states k such that $ES_i^*(p(s)) > k \geq ES_j^*(p(s))$, are equal to zero). [11] Q.E.D.

2.5 Multiperiod extension

I now extent the model of the previous part to provide for the fact that agents 1 and 2 are infinitely lived, so that now information is necessary not only with respect to infinitely many states but also with respect to infinitely many time periods. The information set remains the same at date 0, as in the two period case, when decisions are being made. Thus this is also a static decision model i.e. all decisions are being made at $t = 0$ after state 1 is realized at that date.

2.5.1 The learning model

Learning involves reducing the possible state space with respect to each period in a certain time horizon to a smaller size so that the true state at each of these dates are contained within this set and consumption plans with respect to this set at each of these dates can be ordered by the agent who has limited intelligence - in order to formalize a well specified maximizing objective.

Each agent i's initial belief at date 0 regarding states

of the world that might realize at any date $t > 0$ is given by $\omega_i(t) = \{1, 2, \ldots, \infty\}$. There is no uncertainty at $t = 0$, when decisions are being made. If the agent invests $\ell_i(t)$ (in units of the consumption good) with respect to date t then his updated state partition becomes:

[11] It is not necessary for this proposition to hold that all agents' L_is be finite. If at least one agent's L_i is finite this proposition will hold. However, due to Bounded Rationality (7) is not defined if L_i is infinite.

$$\omega_i(t) = \omega_i^p(t) \cup (\omega_i^p(t))^c$$
$$\text{given,}\omega_i^p(t) = \{1, 2, \ldots, E[S_i(t)|\mathcal{I}_0]\}$$
$$(\omega_i^p(t))^c = \{E[S_i(t) + 1|\mathcal{I}_0], E[S_i(t) + 2|\mathcal{I}_0], \ldots, \infty\}$$

such that $P\{s \in \{ES_i + 1, \ldots, \infty\}$ occurs at $t\} = 0$.

$$\tag{12}$$

where, the probability weights are uniform within each set.

and,

$$S_i(t) = L_i^t(\ell_i(t)) + \epsilon_i(t) \tag{13}$$

where, L_i^t is the learning technology used to learn about the uncertainty with respect to date t, which takes account of the fact that the random variables(s) generating the states at different dates may not identically distributed. The learning technology with respect to each date has the properties (2a) - (2c).

Also, the date specific random term $\epsilon_i(t)\forall_i$ has the property,

$$E[\epsilon_i(t)|\mathcal{I}_0] = 0 \forall t \tag{14}$$

The prior probabilities with respect to each state at each date will be according to (5) equal to zero.

Equations (13) & (14) imply,

$$ES_i(t) = L_i^t(\ell_i(t)) \tag{13'}$$

After investment in learning $\ell_i(t) > 0$ with respect to date t, the updated probability weights $\pi_i(t)$ respect to each state in $\omega_i^p(t)$ will be given by,

$$\pi_i(t) = 1/E[S_i(t)] > 0 \tag{15}$$

where as before I have dropped the conditioning on the information set for notational simplification.

Notice that for any date t if $\ell_i(t) > 0$ then

$$P\{s \in \{E(S_i(t) + 1), \ldots, \infty\} \text{ occurs at } t\} = 0.$$

2.5.2 The choice model

There is one perishable consumption good in the economy which can be used for consumption or for information processing . Information processing is private. The exogenous endowment of the good which agent i receives at date t in state s is given by $w_i^s(t) \in R_+$ (the set of Positive Real Numbers). Since the true state is $s = 1$ at each date, which the agents do not know, as before the wealth at date $t = 0$ is given by $w_i^1(0)$. The contingent consumption of the good by agent i in state s at date t is denoted by $c_i^s(t)$. The price of the good at date 0 is normalized to 1 while its state contingent price in state s at date t is given by $p^s(t) \geq 0$.

Each agent's decision at date $t = 0$ involves allocating his resources over consumption at $t = 0$, learning at date $t = 0$ and contingent consumption over a time horizon with respect to states contained in $\omega_i^p(t)$ for each date t in such set.

Agent i's utility profile (which again is unknown to the agent exante) is given as in the choice model and follows properties (6a) - (6b). His rate of time preference at date t is given by δ_i^t.

I again assume a rational expectations framework with respect to $t \leq T_i$ and $s \leq ES_i$, where T_i the time horizon will be endogeneously chosen. Agent i's learning-investment decision at date 0, therefore, is:

$$Max_{\{c_i \geq 0, \ell_i \geq 0), (T_i \geq 1)} u_i^1(c_i^1(0)) + \sum_{t=1}^{T_i} \sum_{s=1}^{ES_i(t)} \delta_i^t \pi_i(s) u_i^s(c_i^s(t))$$

$$s.t. c_i^0 + \sum_{t=1}^{T_i} \ell_i(t) + \sum_{t=1}^{T_i} \sum_{s=1}^{ES_i(t)} p^s(t)(c_i^s(t) - w_i^s(t)) - w_i^1(0) = 0$$

$$ES_i(t) = L_i^1(\ell_i(t)) \in I_+ - \{1\} \forall i$$

$$T_i \in I_+ \forall i$$

$$\pi_i(t) = 1/ES_i(t) \tag{16}$$

where $\{c_i \geq 0\}$ represents $\{c_i^1(0) \geq 0, \{c_i^s(t) \geq 0\}\}$ and $\{\ell_i \geq 0\}$ represents $\{\ell_i(t) \geq 0\}$ for $s \leq ES_i(t)$ within each date $t \leq T_i$.[12]

[12]See footnote (9). Also here I could have chosen to build the model in continuous time without the integer constraint on time. Such a model would provide

Given the rational expectations framework, this problem is solved in three steps. First, the agent solves for $c_i^1(0), \{c_i^s(t)\}$ given a particular sequence $\{\ell_i(t)\}$ for a given T_i. This gives the consumption sequence. Next, using these conditional demands in the expected utility function and using (13), given a certain T_i the agent solves for the choice of $ES_i(t)s$. This gives $ES_i(T_i, (p(t))$ for each date t. Finally, using these conditional initial consumption , contingent consumption and participation functions in the expected utility , the agent solves for T_i.

The choice of the information sequence and the "time horizon" [13] affects the decision problem as before through three channels - through the budget constraint, through the utility profile (over time and state horizons) and through the probability weights on the utility profile.

I write $T_i^*(p^s(t)), ES_i^*(p^s(t)), c_i(0, (p^s(t))), c_i^s(t, (p^s(t)))$ as the final choice of time horizon, state horizon for date t, consumption at date 0, and contingent consumption at date t respectively, where the asterisk on c's have been sup

2.6 Incomplete Market

Asset a is defined as before.

The supply and demand at $t = 0$ for assets paying off in any state s at date $u > 0$ at prices $(p^s(t))$ arises in the following way:

If, $c_i^s(u, (p^s(t)) - w_i^s < 0$ for some i there is positive supply of asset s with respect to date u, by agent i if, $c_i^s(u, (p^s(t)) - w_i^s > 0$ for some i there is positive demand for asset s with respect to date u, by agent i (17a)

I shall say the *agent j faces an incomplete asset market at $t = 0$ at prices* $\{(p^s(t))\}$ if for some s with respect to some date u (referred to as (s, u)), there is positive supply or demand for asset (s, u) by agent j but supply or demand for asset (s, u) by any agent $i \neq j$, does not exist. (17b)

---

an explanation, in fact, for the fact that agents make periodic decisions in a continuously evolving world.

[13] See Merton (1982) for an exposition with respect to the different kinds of time horizons under uncertainty .

I shall say that at $t = 0$ supply or demand for asset (s, u) any agent $i \neq j$ does not exist iff $c_i^s(u, (p^s(t)))$ is undefined for all agents $i \neq j$ (17c)

I shall say that $c_i^s(u, (p^s(t)))$ is undefined for agent i iff,

$$s > ES_i((p^s(t))) \text{ for } u \leq T_i^*((p^s(t))$$

$$\text{or}, u > T_i^*((p^s(t))) \qquad (17d)$$

I shall say that the *contingent commodity market at* $t = 0$ *is incomplete at prices* $\{(p^s(t))\}$ if there exists at least one agent j such that j faces an incomplete asset market at $t = 0$ at prices $\{(p^s(t))\}$.

(18)

2.7 The dynamic logarithmic utility case

Some results are now characterized for the utility profile given under:

$$u_i^s(.) = \alpha_i^s \ln(.), 0 < \alpha_i^s < 1 \forall s$$

THEOREM 3: *The multi-period value of information* - $V_{i,t}(\{\ell_i(t)\}, T_i)$ -

at $t = 0$ *to agent* i *is given by,*

$$V_{i,0}(\{\ell_i(t)\}, T_i) = \alpha_i^1 \ln(\alpha_i^1 Z_i(\sum^{T_i} \ell_i(t)) + \sum_{t=1}^{T_i} \sum_{s=1}^{ES_i(t)} \delta_i^t \alpha_i^s (L_i)^{-1} \ln(\delta_i^t \alpha_i^s (L_i p^s(t))^{-1} Z_i(\sum^{T_i} \ell_i(t)))$$

where,

$$L_i = L_i(\ell_i(t))$$

$$Z_i(\sum^{T_i} \ell_i(t)) = \{w_i^1(0) - \sum_{t=1}^{T_i} \ell(t) + \sum_{t=1}^{T_i} \sum_{s=1}^{ES_i(t)} p^s(t) w_i^s(t)\}\{\alpha_i^1 + \sum_{t=1}^{T_i} \sum_{s=1}^{ES_i(t)} \delta_i^t \alpha_i^s (L_i)^{-1}\}^{-1}$$

Proof: Similar to Theorem 1.Q.E.D.

DEFINITION 2: $\ell_i^*(t;\{p^s(t)\})$ *is an optimal choice of learning and* $T_i^*(\{p^s(t)\}$ *is the corresponding optimal choice of time horizon at prices* $\{p^s(t)\}$ *iff,*

$$V_{i,t}(\{\ell_i^*(t;\{p^s(t)\})\}, T_i^*(\{p^s(t)\})) \geq V_{i,t}(\{\ell_i(t;\{p^s(t)\})\}, T_i(\{p^s(t)\}))\forall \ell_i(.,.) \geq 0$$
$$\& \;\; T_i(.) \geq 1. \tag{19}$$

OBSERVATION 3:

From the form of value function in Theorem 3 it is clear that if $\ell_i(t)$*s and* T_i*s are chosen simultaneously there exists a problem of indeterminacy with respect to the optimal choice of the* $\ell_i(t)$ *in each period* t *within the decision horizon* T_i^* *and the choice of* T_i^**.*[14]

2.8 Conclusion

I have provided an interpretation of bounded rationality in a neo-classical choice model under uncertainty , a microfoundation for incomplete markets and shown a link between the two through a partial equilibrium analysis. For the logarithmic utility case I have shown that if the value function with respect to learning admits an interior maximum then heterogeneity in the initial period wealth distribution is sufficient to cause the asset market structure to be incomplete, generically. If the value function does not admit such an interior maximum I have hypothesized that agents will be forced to follow "behavioral rules" with respect to learning even though they are optimizing by nature. I leave this line of analysis, viz. relating incompleteness of assets markets to agents following fundamental nontradeable behavioral rules, for the next two essays.

[14]An earlier referee has pointed out as follows:"...One interesting aspect of the many period case is that a player must choose the time horizon over which he formulates his choice problem.This may lead to a further connection between bounded rationality and incomplete markets as differing players choose different time horizons .The author offers no discussion of this issue."This I leave for the future without jeopardizing the basic thrust of this monograph which is to introduce the essence of incomplete market dynamics through a two period model and assuring future researchers that the problems are far from over.

I have maintained the rational expectations approach to decisions under uncertainty but with respect to endogenously determined state and time horizons. This maintains the Arrow-Debreu framework intact within the bounds of these endogenously determined horizons. This has made it possible to value investment in information in terms of the expected utility

that it generates after utility functions have been learnt about using such information and optimizing choices have been made.

This is a static model, in keeping with the Arrow-Debreu model. How incomplete market structure evolves when learning choices are being made periodically, and the information "environment" is evolving, is a question of further study.

In order to study the existence and optimality properties of equilibria in an incomplete market economy under our explanation of incompleteness a general equilibrium model is necessary. Structural dynamics which take such structure formation endogenously, as the results already show, will require quite intensive analysis. This essay concentrates on a partial approach for the objective here is to open up the definition of incompleteness both in cross section as well as in time, which contemporally has wide ranging applicability in globalization strategies.

The information technologies discussed here were private and there were no externalities involved. A serious issue with regard to information technologies is the presence of non-convexities in choice sets arising out of decisions based on such information sets.This arises as a result of information externalities (see Radner & Stiglitz (1984)). Policy analysis based on my model requires consideration of these externalities and the added non-convexities, which this monograph does not attempt to do.

REFERENCES

ARROW, K.J., Essays in the Theory of Risk Bearing, Chicago: Markham, Amsterdam: North Holland, 1971.

ARROW, K.J. & G. DEBREU , Existence of Equilibrium for a Competitive Economy, *Econometrica 22: 265-290, 1954.*

ALLEN, B. , Information as an Economic Commodity, *American Economic Review, Papers & Proceedings: 268-273, 1990.*

– – –– , The Demand for (Differentiated) Information, *Review of Economic Studies, 53: 311-323, 1986.*

BILLINGSLEY, P., Probability and Measure, John Wiley & Sons. 2nd. edition, 1986.

BLUME, L. & D. EASLEY, Rational Expectations Equilibrium : An Alternative Approach, *Journal of Economic Theory, 34: 116-129, 1984.*

CASS, D. , Competitive Equilibrium with Incomplete Financial Markets, *W.P. no. 84-09. C.A.R.E.S.S., University of Pennsylvania, 1984.*

DEBREU, G., Theory of Value, New Haven: Yale University Press, 1959.

GRANDMONT, J.M., Money & Value, A Reconsideration of Classical & Neoclassical Monetary Theories, *Econometric Society Monograph no.5, Cambridge University Press & Editions De La Maison Des Sciences De L'Homme, Paris, 1983.*

– – –– , Temporary General Equilibrium Theory , *Econometrica, 45: 535-572, 1977.*

GROSSMAN, S.J., R.E. KIHLSTROM & L.J. MIRMAN, A Bayesian Approach to the Production of Information & Learning By Doing, *Review of Economic Studies, 44: 533-547, 1977.*

GROSSMAN, S.J., A Characterization of the Optimality of Equilibrium with Incomplete Markets, *Journal of Economic Theory, 15: 1-15, 1977.*

GEANOKOPLOS, J. & H. POLEMARCHAKIS, Existence, regularity and constrained sub-optimality of competitive equilibrium when the asset market is incomplete, in *Heller, W.P., R.M. Starr & D.A. Starr (eds.) Uncertainty , Information & Communication, Essays in honor of K.J. Arrow, Vol. III, Cambridge University Press, 1986.*

HART, O., On the Optimality of Equilibrium when the Market Structure is incomplete, *Journal of Economic Theory, 11: 418-443, 1975.*

KIEFER, N.M. & Y. NYARKO ,Optimal Control of an Unknown Linear Process with Learning , *International Economic Review, 30: 571-586, 1989.*

KNIGHT, F.H., Risk, Uncertainty & Profit, University of London, London, 1921.

KOHN, M.G. & S. SHAVELL, The Theory of Search, *Journal of Economic Theory, 9:93-123, 1974.*

LUCAS, R.E., Asset prices in an Exchange Economy, *Econometrica 46: 1429-1445, 1978.*

MAJUMDAR, M.K., A Note on Learning and Optimal Decisions with a Partially Observable State Space , *i*n L.J. Mirman & D.F. Spulber (eds.), Essays in the Economics of Renewable Resources, North Holland, Amsterdam, 1982.

MERTON, R., Macroeconomic Theory of Investment , in *K*.J. Arrow & M. Intriligator (eds.) Handbook of Mathematical Economics, Vol. II., 1982

NYARKO, Y., Learning in Misspecified Models and the Possibility of Cycles, *W*.P. no 90-03, C.V. Starr Center of Applied Economics, New York University, 1990.

POLEMARCHAKIS, H., The Economic Implications of a Incomplete Asset Market, *A*merican Economic Review, Papers & Proceedings, 1990: 280-283, 1990.

RADNER, R., Equilibrium under Uncertainty , *K*.J. Arrow & M. Intriligator (eds.) Handbook of Mathematical Economics, Vol II., 1982

– – –– , Satisficing, *J*ournal of Mathematical Economics, 2: 253-262, 1975.

– – –– , Market Equilibrium & Uncertainty : Concepts and Problems, in *M*.D. Intriligator & D.A. Kendrick (eds.) Frontiers of Quantitative Economics, Vol. II, North Holland, Amsterdam, 1974.

–––– , Existence of Equilibrium of Plans, Prices and Price Expectations in a Sequence of Markets, *E*conometrica, 40: 289-303, 1972.

– – –– & M. ROTHSCHILD , On the Allocation of Effort, Journal of Economic Theory, 10: 358-376, 1975.

– – –– & J.E. STIGLITZ, A Non-Concavity in the Value of Information, in M. Boyer & R.E. Kihlstrom (eds.) Bayesian Models in Economic Theory, Elsevier Publishers, B.V., 1984.

SCHOTTER, A., K. WEIGELT & C. WILSON, A laboratory investigation of multiperson rationality and presentation effects, *W.P. no. 90-24, C.V. Starr Center for Applied Economics, New York University, 1990.*

STIGLITZ, J.E., The Inefficiency of Stock Market Equilibrium , Review of Economic Studies, 49: 241-261, 1982.

SIMON, H.A., Theories of Bounded Rationality , in C.B. Mcguire & R. Radner (eds.) Decision and Organisation, North Holland, Amsterdam, 1972.

WERNER, J., Equilibrium in Economics with Incomplete Financial Markets, Journal of Economic Theory, 36: 110-119, 1985.

Chapter 3

Incomplete Markets & Policy Control

3.1 Introduction

Traditionally policies for intervening in market failure call for wealth redistribution in order to achieve full or approximate Pareto Optimality (see for e.g. Atkinson & Stiglitz (1980)). In this essay I show how a case for wealth redistribution arises due to the nonexistence of equilibrium. Although I do not develop the full thrust of the argument, which is why market participation remains exogenous in this analysis, I provide an outline of the logic with a plausible model.

I study an Arrow-Debreu type of economy where agents have to first acquire costly information before participating in trade and are limited in their participation in this trading process to the extent of such information. I motivate this by making the assumption that agents have "Bounded Rationality". Bounded Rationality means that agents have limited information processing capacities (Simon (1972), Radner (1975)). I interpret this in an Arrow-Debreu economy, as in the previous essay, as meaning that agents cannot establish state-contingent preferences over contingent consumption based on their initial beliefs about the states of the world. Thus, they do not know their optimizing objective when they are born. This results in each agent having to actively "invest" in a costly exogenous “learning technology ” (for e.g. information technology) prior to making contingent consumption decisions in an optimizing fashion, so as to lay down a well specified optimizing objective. This provides a rationale for pre-trading investment in costly information processing by each agent in a Rational Expectations framework (see Grossman & Stiglitz (1981) for the necessity to provide such motivation. Also see Verrechia (1982) and VanZaandt (1989) for alternative motivations).

This learning technology narrows down the “relevant” state space towards the true state and helps agents in recovering their utility functions with respect to each state within this reduced state space. The agents assign uniform probability weights to each state within this “relevant” state space and treat the rest of the states as having probability zero. This defines the optimizing objective according to the expected utility criterion. With respect to contingent consumption in the rest of the states agents follow some “rule” because their preferences are not defined with respect to those states (this is in contrast to the “optimal learning” literature — see for e.g. Rothschild (1974), Nyarko (1992)). This feature of Bounded Rationality will be crucial for our non-existence result as we shall see later on.

This paper is in a Rational Expectations framework. Given the learning technology, each agent computes the value of learning, which is the expected utility “rationally expected” to be generated after taking into account learning (information processing) costs, and chooses the optimum number of states about which to learn and makes optimizing decisions only with respect to this set of states. With respect to the rest of the states agents follow the rule of “consuming their endowments”.

In a two date world, where uncertainty is only with respect to the second date, I shall say that given a particular set of prices, markets are incomplete at the initial date if different agents participate in trade with respect to assets pertaining to different state - spaces. It is this heterogeneous “market-participation” at a given price vector which gives rise to incomplete markets at that set of prices. I will argue that if markets are incomplete under certain conditions for all finite nonnegative price vectors, then an Arrow-Debreu equilibrium does not exist for this economy.

This essay shows in the logarithmic utility case that, if agents are heterogeneous with respect to their initial wealth levels markets will be incomplete generically. From the definition of incomplete markets given here[1] will follow that if agents are "sufficiently" heterogeneous with respect to their initial wealth levels - Arrow-Debreu equilibrium will not exist in this economy, generically.

3.2 The model

There are $N > 1$ heterogeneous agents in this economy. Agents differ in terms of their preferences and endowments. There are only two periods.

There is no uncertainty at date 0 (the first period of our analysis), all the uncertainty is with respect to date 1. The uncertainty is with respect to the state-contingent utility measures and the state-contingent wealth levels. Since utilities are state-contingent the state which realizes determines the utility function of each agent. The agents are born at the beginning of $t = 0$ and will die at the end of $t = 1$. The result of bounded rationality is the agents do not have cardinal utility measures associated with consumption in each state within $\mathcal{I}_0$ (the initial information set). The state contingent wealth realization are, however, recoverable.

3.2.1 The learning model

Learning involves reducing the possible state space to a smaller size so that it contains the true state and contingent consumption plans with respect to this set of states can be ordered by the agent in order to formalize preferences over contingent consumption (bounded rationality) .

Each agent i's information set is given by $\omega_i = \{1, 2, \ldots, k\}$ (let's say that k is a very large number, loosely speaking). $s = 1$ is the true state at $t = 1$, but this fact is not known to any agent. There is no uncertainty regarding $t = 0$ when $s = 1$ is already realized.

[1] I thank Yaw Nyarko for this point that the definition of incompleteness of this monograph is different from the spanning definition usually used but it has been motivated empirically

Learning enables each agent to partition the initial state-space into "finer" partitions (Aumann (1976), Geanokoplos (1988), and Allen (1990)) and recover state-contingent utility measures with respect to states having positive probability. If the agent i invest ℓ_i (in physical units of the only consumption good) in learning then his updated state partition ω_i^p becomes,

$$\omega_i^p = \omega_i^{p'} \cup (\omega_i^{p'})^c$$
$$\text{given, } \omega_i^{p'} = \{1, 2, \ldots, ES_i\}$$
$$\text{such that P\{some s } \in \omega_\mathrm{i}^{\mathrm{p}'} \text{ occurs at t\}} = 1 \tag{1}$$

and, the agent assigns uniform probability weight to each state within $\omega_i^{p'}$,

where ES_i the decision horizon of agent which is also is "market participation" rate given by:

$$ES_i = L(\ell_i) \tag{2}$$

$L(.)$ is the "learning technology " available to all the agents, the "learning technology " works as follows. It uses the consumption good as input and produces a forecast of the possible state-space at date 1. In fact it partitions the initial state-space into two subsets. Each agent i knows that the first set ($\omega_i^{p'}$ in (1))) constrains the true state but does not know which one it is. He therefore attaches probability 1 to the first set and probability zero to the other. These sets are as given in (1). However, neither the membership of these sets nor the state-contingent utility functions are known to any agent before the investment .

The agents know the learning technology i.e. how close to the true value the forecast will be consequent to a given level of input for e.g. for $\ell_i = 10$ $L(\ell_i) = 4$ means that for 10 units of input the forecast will be of four states one of which will be the true state, but do not know which four states these will be. However, given the Rational Expectations framework of this paper the agents will be optimizing with respect to a set of states which consists of a consecutive set of integers starting at 1 - the true state and the true utility profile over this set of states , in making the investment in learning decision which we will narrate in the next section.

The learning technology has the following properties:

$$L(0) = k \tag{3a}$$

$L(.) \in I_+ - \{1\}$ for $\ell_i > 0, I_+$ being the set of positive integers (3b)

$$\forall \ell_i, \ell_i' > 0 \text{s.t.} \ell_i' > \ell_i, L(\ell_i') \leq L(\ell_i) \tag{3c}$$

$$\forall\ \ell_i > 0 \text{ s.t.} L(\ell_i)\ \geq 3\ \exists \ell_i' \text{ s.t.} \infty >\ \ell_i'\ >\ \ell_i \text{ and } L(\ell_i') = L(\ell_i) - 1 \tag{3d}$$

The prior probability beliefs over the whole state-space ω is uniform, for all agents. With $\ell_i > 0$ the updated state-space for agent i is $\omega_i^{p'}$, and the updated probability weights — $\pi_i(s)$ -with respect to each state s within $\omega_i^{p'}$ becomes,

$$\pi_i(s) = P\{\text{ state } s \text{ occurring at } t = 1 \text{ for } s\ \in \omega_i^{p'}\} =\ 1/ES_i \tag{4}$$

3.2.2 The choice model

There is one perishable consumption good in the economy which can be used for consumption or for information processing . Information processing is private. Consumption by agent i at $t = 0$ is denoted by $c_i^1(0)$ and contingent consumption in state s at $t = 1$ is denoted by $c_i^s(1)$. Agent i's endowment at $t = 0$ is w_i^1 and in state s at $t = 1$ is w_i^s. The price of the good at date 0 is normalized to 1 while its state contingent price in state s at $t = 1$ is given by $p(s) \geq 0$.

Each agent's decision at date $t = 0$ involves allocating his resources to consumption at $t = 0$, to learning with respect to $t = 1$ and purchasing contingent consumption for $t = 1$ with respect to the updated state space ω_i^p. Agent i's state dependent utility function with respect to state s is given by the function $\alpha_i^s \ln(.)$ for $s \in \{1, 2, \ldots, ES_i\}$.

His rate of time discount is given by δ_i where $0 < \delta_i < 1$. Agent i's learning-consumption decision at date 0 is:

$$\max_{\{\ell_i \geq 0, c_i \geq 0\}} \alpha_i^1 \ln(c_i^1(0)) + \sum_{s=1}^{S_i} \delta_i \pi_i(s) \alpha_i^s \ln(c_i^s(1))$$

$$\text{s.t.}c_i^1(0) + \ell_i + \sum_{s=1}^{s_i} p(s)(c_i^s(1) - w_i^s) - w_i^1 = 0$$
$$\pi_i(s) = 1/ES_i = 1/L(\ell)$$

$$c_i^s(1) = w_i^s \forall s > ES_i \tag{5}$$

where, $\{c_i \geq 0\}$ represents the set $\{c_i^1(0) \geq 0, c_i^s(1) \geq 0 \forall s\}$

3.3 Incomplete Market

Asset s is a contract which pays off 1 unit of the consumption good in state s and 0 units in all other states .

Let, $c_i^s(1;(p(s)))$ denote the choice of contingent consumption by agent i for state s at date 1 given the price vector $(p(s))$.

I shall say that at $t = 0$, $c_i^s(1;(p(s))) - w_i(s) < 0$ for some $i <=>$ positive supply of asset s at $(p(s))$ by agent i, $c_i^s(1;(p(s))) - w_i(s) > 0$ for some $i <=>$ positive supply of asset s at $(p(s))$ by agent i, I shall say the $c_i^s(1;(p(s)))$ is undefined for agent i at $(p(s))$ iff,

$$s > ES_i((p(s))) \tag{6b}$$

where $ES_i((p(s))) = L(\ell_i((p(s)))$ and $\ell_i((p(s)))$ is the choice of ℓ_i at prices $(p(s))$.

I shall say that *agent i faces an incomplete asset market at $t = 0$ at prices* $(p(s))$ iff for some s there is positive supply or positive demand of asset s by agent i but supply or demand of asset s by any agent $j \neq i$ does not exist, at those prices. (6c)

I shall say *the asset market at $t = 0$ is incomplete at prices* $(p(s))$ iff there exists at least one agent i such that i faces an incomplete asset market at $t = 0$ at prices $(p(s))$. (7)

3.4 Equilibrium

DEFINITION 1:

Arrow-Debreu Equilibrium at $t = 0$ in this economy is defined as follows:

An Arrow-Debreu equilibrium is a set of finite nonnegative prices (1 at date 0 and $(p^(s))_s \in (1, 2, \ldots, k)$ at date 1), investment in learning (ℓ_i), initial consumption choices $(c_i^s(0))$ and contingent consumption choices $(c_i^s(1))$ at $t = 1$ for $s \in \{1, 2, \ldots, k\} \forall i$ such that:

(a) given (p(s)), $\ell_i, (c_i^s(1))_{s \in \{1,2,\ldots,k\}}$ solves problem (5) for each agent i,

(b) $\sum_i (c_i^s(0) + \ell_i) = \sum_i w_i^1$, and

$$\sum_i c_i^s(1) = \sum_i w_i^s \forall s \in \{1, 2, \ldots, k\} \tag{8}$$

PROPOSITION

If the asset market at $t = 0$ is incomplete with respect to all finite nonnegative price vectors, then equilibrium does not exist for this economy .

Proof: see Appendix.Q.E.D.

The value of learning - $V_i(\ell_i)$ - at date 0 to agent i is given by,

$$V_i(\ell_i) = \alpha^1 \ln(\alpha^1 Z_i(\ell_i) + \sum_{s=1}^{S_i} \delta_i \alpha^s (S_i)^{-1} \ln(\alpha^s (p(s) S_i)^{-1} Z_i(\ell_i)) \tag{9}$$

$$\text{where, } Z_i(\ell_i) = \{w_i^1 - \ell_i + \sum_{s=1}^{S_i} p(s) w_i^s\}\{\alpha^1 + \sum_{s=1}^{S_i} \alpha^s S_i^{-1}\}^{-1}$$

Note : The right hand side is the indirect utility of learning and is derived from (5).

DEFINITION 2

$\ell_i^*((p(s)))$ is an *optimal choice of learning by agent* i *at prices* $(p(s))$ *iff,*

$$V_i(\ell_i^*(p(s))) \geq V_i(\ell_i(p(s))) \forall \ell_i(p(s))) \geq 0 \tag{10}$$

THEOREM

If, $S_i((p(s))) < k \forall i$ *and if* $w_i^1 >> w_j^1$ *or* $w_j^1 >> w_i^1$ *for* $i \neq j$*, then the asset market at* $t = 0$ *is generically incomplete at finite, non-negative prices* $(p(s))$.

Proof: see Appendix.Q.E.D.

3.5 Policy control

COROLLARY

If $S_i(p(s)) < k \forall i$ *and if* $w_i^1 >> w_j^1$ *or* $w_j^1 >> w_i^1$ *for* $i \neq j$ *then an Arrow-Debreu equilibrium does not exist for this economy, generically.*

Proof: see Appendix. Q.E.D.

NOTE: The requirements of the theorem are too strong. If only one agent i has an interior choice of "participation rate" i.e. $S_i < k$ and $S_j > k \forall j \neq i$ then the theorem will hold without any condition placed on the wealth distribution. If at least two agents have interior choices of "participation rates" then the theorem will hold with the condition placed on wealth distribution. Hence, what I have provided are sufficient conditions for the policy control.

APPENDIX

Proof of Proposition:

Suppose agent j is the agent for whom $S_j(p(s)) > S_i(p(s)) \forall i \neq j$ (if there are more than one agent j who have the largest S_js then the extension is obvious).

If for any finite nonnegative price vector the market is incomplete by definitions (6a) - (7) then for the corresponding price vector for the states s such that:

$$\max_{i \in \{1,2,\ldots,N\}} \{S_i(p(s))\} \geq s > \max_{i \in \{1,2,\ldots,j-1,j+1,\ldots,n\}} \{S_i(p(s))\}$$

$$c_j^s(1;(p(s))) - w_j^s \neq 0.$$

But, by (5) $c_i^s(1;(p(s))) - w_i^s = 0 \forall i \neq j$

Now, if this is true for all finite nonnegative price vectors then for any finite nonnegative price vector $(p(s))$

$$\sum_i c_i(1;(p(s))) \neq \sum_i w_i^s$$

which violates condition (b) of the definition of Arrow-Debreu equilibrium (see (8) above).

Proof of Theorem:

I provide a sketch of the proof. For any finite non-negative price vector from (11a) $\partial V_i / \partial w_i^1 > 0$ if $S_i((p(s))) < k$ (otherwise $c_i^s = w_i^s \forall s$ by (5)). Hence, $V_i(\ell_i) \neq V_j(\ell_i)$ if $w_i^1 \neq w_i^1$. This implies from (11b) that the optimal choice of learning (see definition 2) $\ell_i^*(p(s)) \neq \ell_j^*(p(s))$ generically in the class of differentiable functions which follows from the fact that a function is the integral of its derivative (Fundamental Theorem of Calculus) and if the integrals i.e. the original value functions are different then the derivatives and hence the solution to the maximizing equation have to be different in value,(provided given the integer values of states in assumptions (3b)-(3d), $V_i(\ell_i)$ and $V_j(\ell_j)$ are sufficiently far apart (i.e. w_i^1 is sufficiently different from w_j^1) such that $\ell_i^*(p(s))$ and $\ell_j^*(p(s))$ are distinct integers). These conditions , imply that $S_i^*(p(s)) \neq S_j^*(p(s))$. Further definitions (6a)-(7) imply that the contingent commodity market is generically incomplete (i.e. except in those probability zero cases where demand

or supply of assets (see the definition) by the agent i for whom $ES_i^*(p(s))$ is greater, for states m such that $ES_i(p(s)) \geq m > ES_j(p(s))$, are equal to zero).$ES_i$ and S_i refer to the same integers as in the previous essay.Q.E.D.
Proof of Corollary:

Follows immediately from the Theorem and the Proposition.Q.E.D.

REFERENCES

ARROW, K.J. & G.DEBREU, (1954), Existence of Equilibrium for a Competitive Economy, *Econometrica, 22: ;265-290.*

ATKINSON, A.B. & STIGLITZ, J.E., (1980), Lectures on Public Economics, McGraw Hill Book Company, New York.

ALLEN, B., (1990), Information as an economic commodity, *American Economic Review, Papers & Proceedings : 268-273.*

AUMANN, R.J., (1976), Agreeing to Disagree, *The Annals of Statistics, 4: 1236-1239.*

GEANOKOPLOS, J. (1988), Common Knowledge, Bayesian Learning & Market Speculation with Bounded Rationality , mimeo, Yale University.

GROSSMAN, S.J., R.E. KIHLSTROM & L.J. MIRMAN (1977), A Bayesian Approach to the production of Information & Learning By Doing, *Review of Economic Studies, 44:533-547.*

GROSSMAN, S.J. & J.E. STIGLITZ (1976), Information & Competitive Price Stystems, *American Economic Review, 66, 246-253.*

GEANOKOPLOS, J. & H. POLEMARCHAKIS, (1986), Existence, Regularity ad Constrained Suboptimality of Competitive Allocations when the Asset Market is Incomplete, in Starr, R., Heller, and Starett, D. (eds.) Essays in Honor of Kenneth J. Arrow, Vol. III, Cambridge University Press, New York.

NYARKO, Y., (1992), Learning in Misspecified Models and the Possibility of Cycles, *Journal of Economic Theory, 55: 416-427, 1991.*

RADNER, R., (1975), Satisficing, *Journal of Mathematical Economics, 2: 253-262.*

ROTHSCHILD, M., (1975), A two-armed bandit theory of market pricing, *Journal of Economic Theory, 9: 185-202.*

SIMON, H.A., (1972), Theories of Bounded Rationality , in *C.B. Mcguire & R. Radner (eds.) Decision & Organization, North Holland, Amsterdam.*

VANZAANDT, T., (1989), The Private Acquisition of Information with Individual Risks, mimeo, AT&T Bell Laboratories.

VERRECHIA, R., (1982), Information Acquisition in a Noisy Rational Expectations Economy, *Econometrica, 50, 1415-1430.*

Chapter 4

Incomplete Markets & Competition

4.1 Introduction

In the previous three chapters we had motivated the model for endogenous acquisition of private information by market participants in contingent commodity markets by considering economies with bounded rationality defined in the first two essays.This fundamental property of such economies results in participation in markets being endogenous and hence competition in contingent commodity markets in such economies evolve endogenously. While heterogeneity is necessary for trades to occur it is shown that the same heterogeneity may result in contingent commodity markets being incomplete resulting in market failure . Such market failure can however be corrected by the planner redistributing initial period wealth where wealth distribution is the source of such heterogeneity. These complications and necessity of active policy controls arise in otherwise monetized economies where the presence of Arrow securities and spot prices allows the reduction of commodity bundles into equivalent numeraire units in spot markets . Thus controls are targeted at futures markets or commodity markets contingent on future dates . In deriving these results however, we had made assumptions regarding the individual consumption-saving-investment problems by assuming solutions to such problems exist automatically. This however, may not be so as the present essay shows. In this essay we take up explicitly the individual decision problems in economies within the bounded rational class that are being considered, and also consider the market participation and aggregate efficiency properties in terms of existence of equilibrium . This we do by considering the evolution of competition in such contingent commodity markets.

Competition in a market depends on the number of participants (buyers and sellers) in that particular market. Since participation in a market is costly it seems natural that agents will consider the costs and benefits of participating as either buyers or sellers in the various commodities of that market, and not everybody can participate. Hence, there will arise various degrees of competition depending upon to what extent there is participation in the market. In the case of goods market under certainty papers like Hart(1980), Makowski(1980) discuss the endogenous choice of a bundle of goods and the nature of competition that results. In the context of industries with contingent commodity trades like contingent commodity, there will arise various degrees of competition with respect to the various types of contingent commodities. The purpose of this paper is to derive this endogenous participation rate from the consideration of the costs of such participation and expected utility of such participation with respect to any particular contingent commodity market. Since participation indexes competition, this paper provides a model for endogenously determined competition in contingent commodity markets (also see Mallick (1993, 2005)).

Uncertainty arises from ignorance - 'Knightian' uncertainty. Thus, under uncertainty agents invest in both information and contingent commodity to maximize expected utility of state contingent consumption (for a review of investment under uncertainty in non bounded rationality and non contingent commodity economies see Dixit & Pindyck (1994)). The state space when no uncertainty has been resolved is the set of strictly positive integers. Investment in information reduces the dimension of uncertainty or the number of possible states of nature facing an individual and each agent plans for contingent consumption with respect to this reduced number of possible states of nature. While considering investment in information the maximum expected utility derivable from insuring against this possible set of states and the cost of learning which reduces the size of uncertainty to this set of states is taken into account. Since with contingent commodities, level of learning determines the number of state-contingent commodities or the various types of contingent commodities agents choose to buy. In a two period model heterogeneity in initial wealth levels and complete ordering of agents in terms of intertemporal wealth levels may cause difference in the demand or supply of various agents with respect to the level of learning and hence access to the different types of contingent commodities, which determines participation in the various contingent commodity markets, hence causing various degrees of competition with respect to various types of contingent commodities, and to failure of contingent commodity asset markets due to deficient demand or supply.

The incidental interest of this paper lies in adapting a optimization program used by Anderson & Takayama (1979) in solving nonconvex optimization problems to the integer valued choice set and stepwise participation (learning technology) . The dependence of the parametric results in reduced form depend on the logarithmic utility and stepwise technology specification in the economy being studied. To what extent the results carry over to other types of economies is not investigated here. Also,the results are all sufficiency conditions generating the nature of aggregate competition being studied. Hence, the sensitivity of the results to the preference and technology specifications can be the subject matter of other studies. Besides, certain policy implications are derived which implements efficiency of the contingent commodity markets in the aggregate. Section 2 sets up the two period model, Section 3 provides the characterization in the logarithmic utility case, Section 4 concludes.

4.2 The model

There are finite number of agents $N > 1$ in this economy (1)

Agents plan over two periods date 0 and 1. There is no uncertainty at date 0 (the first period), all the uncertainty is at date 1.

The uncertainty is with respect to the state of the world.

4.2.1 Investment in information

Each agent i's initial belief at date 0 regarding the states of the world (which are ordered) that might realize at date t=1 is given by $s = 1, 2, \ldots, \infty$, $s = 1$ is the true state at each date, but this fact is not known to any agent. Learning partitions the initial state-space into finer partitions. If the agent i invest l_i (in physical units of the only consumption good) in learning (which I shall use interchangeably with information processing) then his updated state partition Ω_i^P becomes

$$\Omega_i^P = \Omega_i^{P'} U (\Omega_i^{P'})^c$$

given $\Omega_i^{P'} = \{1, 2, , s_i\}$

$$(\Omega_i^{P'})^c = \{s_i + 1, s_i + 2, , \infty\}$$

such that $P\{$ Some $s \epsilon \Omega_i^{P'}$ occurring at $t = 1\}$

$$= 1$$

and the agent assigns uniform probability weight to each state within $\Omega_i^{P'}$ (2)

where s_i the index of residual uncertainty is given by :

$$s_i = L(l_i) \tag{3}$$

L(.) is the private learning technology available to agent i. The learning technology works as follows. It uses the consumption good as input and produces a forecast of the possible state-space at date 1. In fact it partitions the initial infinite state-space to a finite set, which contains the true state, and the residual infinite set. Each agent i knows that the first set ($\Omega_i^{p'}$ in (2)) contains the true state but does not know which one it is. They therefore attach probability 1 to the first set and probability zero to the other. These sets are as given in (2).

The learning technology has the following properties :

$$L(0) = \infty, \min\{x : L(x) = 1\} = \infty \tag{3a}$$

$$L(.)\epsilon I_+ \text{ for } l_i > 0 \tag{3b}$$

$$\forall l_i', l_i > 0 \;\; s.t. \;\; l_i > l_i', L(l_i') \geq L(l_i) \tag{3c}$$

and,

$$\forall l_i' > 0 \& L(l_i) \geq 3 \exists l_i' : \infty > l_i' > l_i$$

and,

$$L(l_i') = L(l_i) - 1 \tag{3d}$$

I_+ refers to the set of positive integers. Assumption (3b) is made so as to keep the choice model in discrete state space and satisfy the convention of numbering states . Assumption (3a) implies the s=1 being the true state it is never possible to have perfect knowledge a priori with finite wealth. Assumption (3c) implies that an increase in learning reduces the size of the state space in steps and (3d) implies that the exogenous learning technology which agent i uses to learn about the possibility of various states occurring is a stepwise linear function of the Leontief type. With the investment in information $l_i > 0$ the updated state space $\Omega_i^{p'}$ is finite and hence, the posterior probability weights - $\pi_i(s)$ - with respect to each states within $\Omega_i^{p'}$ becomes, $\pi_i(s) = P\{$ State s occurs at t = 1 for $s\epsilon\Omega_i^{P'}\} =$

$$\frac{1}{s_i} > 0 \tag{4}$$

Notice that from (2) the effect of investment in learning by agent i i.e. $l_i > 0$ is that the probability of occurrence of the residual set of states in zero i.e. P $\{\ s\epsilon\{s_i, s_i + 1, , \infty\}\text{occurs at } t = 1\} = 0$

The optimal investment in information is defined as follows.

DEFINITION 1: Let $V_i(L_i((p(s)))$ be the maximum expected utility derivable from consumption to agent i, where $L_i((p(s)))$ is a feasible choice of horizon at any price vector $(p(s))$. $L_i^*((p(s)))$ is an optimal choice of information by agent i at prices $(p(s))$ iff,

$$V_i(L_i^*(p(s))) \geq V_i(L_i(p(s))), \forall \text{ such } L_i((p(s)))$$

the choice model which gives rise to this value is described next.

4.2.2 Contingent consumption

There is one perishable good in the economy which can be used for consumption or for information processing . Information processing is private. The contingent consumption of the good by agent i in state s at date t is denoted by $c_i^s(t)$. The price of the good at date 0 is normalized to 1 while its state contingent price in state s at t = 1 is given by $p(s) \geq 0$.

Each agent's decision at date t = 0 involves allocating his resources to consumption at t = 0, to learning with respect to t = 1 and purchasing contingent consumption for t = 1 with respect to the updated state space $\Omega_i^{p'}$. Agent i's state dependent utility function with respect to state s at any date is given by the function $u_i^s(.)$ for $s\epsilon\{1, 2, , s_i\}$.

Each $u_i^s(.)$ has the following properties :

$u_i^s(c_i^s(.))$ is continuous over $R+$ and

$$u_i^s(0) \leq 0 \quad \forall \quad i, s \tag{5a}$$

$$du_i^s(.)/d \quad arg > 0 \tag{5b}$$

$$d^2u_i^s(.)/d \quad arg^2 < 0 \tag{5c}$$

(arg=argument)

Agent i's rate of time discount is given by δ where $0 < \delta < 1$. Since the state at date 0 when there is no uncertainty is 1 the utility function with respect to date 0 is $u_i^1(.)$.

Since agents assign 0 probabilities to each of the states $\{s : s > s_i\}$, the expected discounted utility with respect to t = 1 is given by

$$\sum_{s=1}^{s_i} \delta\pi_i(s)u_i^s(c_i^s(1))$$

where $\pi_i(s)$ is the updated probability beliefs as given by (4).

The price of the good at date 0 is normalized to 1. Therefore, the cost of consumption at date 0 is given by $c_i^1(0)$, where $c_i^1(0)$ is i's consumption at date 0, and the cost of learning is given by l_i. The price of the consumption good in state s at date 1 is given by p(s), and hence the total cost of consumption with respect to date 1 is given by

$$\sum_{s=1}^{s_i} p(s)c_i^1(1)$$

The level of wealth at date o for agent i is given by $w_i^{0,1}$.

The wealth process at $t = 1$ is $\{w_i^{1,s}\}$ where s is a state and hence the value of total wealth within the decision horizon at $t = 1$ is given by

$$\sum_{s=1}^{s_i} p(s) w_i^{1,s}$$

All wealth levels for any agent are bounded

$$(5d)$$

Since, as has already been described, each agent i treats all states $s > s_i$ as having probability 0, hence he does not take these wealth realizations into consideration in his optimizing problem.

Therefore, the budget constraint of agent i is given by,

$$c_i^1(0) + l_i + \sum_{s=1}^{s_i} p(s)(c_i^s(1) - w_i^{1,s}) - w_i^{0,1} = o$$

Agent i's learning-investment decision at date 0, therefore, is :

$$\max_{\{(c_i \geq 0), l_i \geq 0\}} u_i^1(c_i^1(0) + \sum_{s=1}^{s_i} \delta \pi_i(s) u_i^s(c_i^s(1))$$

$$s.t. c_i^1(0) + l_i + \sum_{s=1}^{s_i} p(s)(c_i^s(1) - w_i^{1,s}) - w_i^{0,1} = 0$$

$$s_i = L(l_i)$$

$$\pi_i(s) = \frac{1}{s_i} \tag{6}$$

where, $(c_i \geq 0)$ represents the set $\{c_i^1(0) \geq 0, \{c_i^s(1) \geq 0\}\}, c_i^1(0), c_i^s(1)$ represents his consumption at date 0 and date 1 state s respectively, $w_i^{1,s}$ is his state contingent endowment is state s, s_i is given by (3), and $\pi_i(s)$ is given by (4).

4.2.3 Endogenous competition

The contingent consumption by agent i of the only good at date 1 state s is denoted by $c_i^s(1, (p(s))$, where $(p(s))$ is the price vector.

The imperfectly competitive contingent commodity market is characterized as follows :

Contingent commodity s is a contract which pays off 1 unit of the consumption good is state s and 0 units in all other states

. Thus, if the agent wishes to transfer q units of consumption from state a to state b then his supply of contingent commodity a is q and demand for contingent commodity b is q.

The supply and demand at t = 0 for contingent commodity paying off in any state η at t = 1 at any set of prices $(p(s))$ arises in the following way:

if, $c_i^\eta(1; (p(s)) - w_i^{1,\eta} < 0$ for some i there is positive supply of contingent commodity η by agent i

If, $c_i^\eta(1; (p(s)) - w_i^{1,\eta} > 0$ for some i there is positive demand for contingent commodity η by agent i

$$(7a)$$

I shall say that at t = 0 supply or demand for contingent commodity η by any agent $i \neq j$ does not exist (to be contrasted with supply or demand for contingent commodity η is zero) iff

$c_i^\eta(1; (p(s))$ is undefined for all agents

$$i \neq j \tag{7b}$$

I shall say that $c_i^\eta(1; p(s))$ is undefined for agent i iff

$$\eta > s_i^*(p(s)) \tag{7c}$$

where $s_i^*(p(s))$ is the optimal choice of learning by agent i at prices $(p(s))$ (see Definition (1)).

I shall say that agent j faces an imperfect contingent commodity market at t = 0 at prices $(p(s))$ if for some s there is positive supply or demand of contingent commodity s by agent j but supply or demand of contingent commodity s by any agent $i \neq j$, does not exist. (7d)

I shall say that the contingent commodity market at t = 0 is imperfect at prices $(p(s))$ if there exists at least one agent j such that j faces an imperfect contingent commodity market at t = 0 at prices $(p(s))$ (8)

This model of markets describes markets where agents do not trade on contingent commodities which they know, privately, will not occur, but do not reveal such information. This is in contrast with rational expectations models as such private information should be revealed through the pricing mechanism. The results of this paper depend on this structural assumption. The absence of punishment mechanisms to enforce truth revelation which this model therefore helps analyze (on this see Dubey, Geanokoplos & Shubik (2001)), is a particular intrepretation of this assumption.This would also depend on the particular type of contingent contracts being traded for e.g.term plans in contingent commodity provide for trading in contingent commodity contracts which pay off after a particular term of paying premium and there is no verification machanism for publicly determining the time over which the annuity will be actually payable, as this depends on private information about the life span.On such problems of valuing investment in a firm under equilibrium see Dreze(1985).The model of this paper could therefore be considered to be a disequilibrium model,[1]

4.3 Endogenous evolution of competition

Agents are heterogeneous only with respect to wealth levels. State 1 is realized at date 0 before agents make decisions. The utility profiles are given by,

$$u_i^s(.) = \ln(.), \qquad s \geq 1$$

The individual optimizing problem takes the form :

[1]A related but alternative notion, of using a fixed rate of deficit, has been discussed by Raffaela Giordano at the Bank of Italy, where the optimal GDP depends on the extent of the parameterized budget deficit (Gennari,Giordano & Momigliano (2002)) where a particular type of disequilibrium is considered in the context of the EMU.

$$\max_{\{l_i \geq 0, (c_i \geq 0)\}} \ln(c_i^1(0) + \sum_{s=1}^{s_i} \delta\pi_i(s) \ln(c_i^s(1))$$

$$s.t. c_i^1(0) + l_i + \sum_{s=1}^{s_i} p(s)(c_i^s(1) - w_i^{1,s}) - w_i^{0,1} = 0$$

$$s_i = L(l_i)$$

$$\pi_i(s) = \frac{1}{s_i} \tag{6'}$$

Where, $c_i^1(0)$ is consumption at date 0 when s = 1 is realized and the wealth that is realized is therefore $w_1^{0,1}$ and $c_i^s(1)$ is consumption at date 1 in state s.

DEFINITION 2:

The function $f(x, \alpha) : x \epsilon X \in R^n$ and α is a parameter vector is said to be subdifferentiable in α, if it is convex in α, and if for each $\alpha^1 \exists$ a vector, say $f_\alpha(x, \alpha^1)$, s.t. the inequality $f(x, \alpha^0) - f(x, \alpha^1) \geq f_\alpha(x, \alpha^1)(\alpha_0 - \alpha_1) \forall x \in X$ holds $\forall \alpha^0$. If f is concave, the inequality is reversed. (Anderson & Takayama (1979)).

In deriving our results I shall make use of this proposition which is adapted from Anderson & Takayama (1979), (Proposition 1 case (v))

PROPOSITION :

Consider the problem

$$(M) \mathrm{Max}_x f(x, \alpha)$$

$$s.t. g_j(x, \beta, \gamma) \equiv \beta_j - h_j(x, \gamma) \geq 0, \quad j = 1, 2, , m$$

and $x \epsilon X \subset R^n$, where f and $g_j s$ are real valued functions and where α, β and γ signify vectors of shift parameters.

Let, A be the set of (α, β, γ) in which the solution of (M) exists. I assume $A(\& X)$ is non-empty. Given (α, β, γ) let $x(\alpha, \beta, \gamma)$ denote the solution of (M). Corresponding to a shift of parameters I define the function x^j by

$$x^j = x(\alpha^j, \beta^j, \gamma^j)$$

Suppose, f is convex and subdifferentiable in α with f_α denoting the subgradient vector with respect to α.

Then, if

$$g(x^1, \beta, \gamma^0) \geq (\gamma^1 - r^0).x^1 \quad (i)$$

$$g(x_0, \beta, \gamma^1) \geq (\gamma^1 - \gamma^0).x_0 \quad (ii)$$

$$f_\alpha(x^0, \alpha^0)\triangle\alpha_1 + \lambda_1^1\triangle\gamma^1 x_0 \leq F(\alpha^1, \beta^1, \gamma^1) - F(\alpha^0, \beta^0, \gamma^0) \leq$$
$$f_\alpha(x^1, \alpha^1)\triangle\alpha_1 - \lambda_1^0\triangle\alpha_1 - \lambda_1^0\triangle\gamma^1 x^1 \quad (I)$$

$$0 \geq \triangle\alpha_1\{-f_\alpha(x^1, \alpha^1) + f_\alpha(x^0, \alpha^0)\} + \triangle\gamma_1(\lambda_1^0 x^1 + \lambda_1^1 x^0) \quad (II)$$

$$\text{if} \quad g(x^2, \beta, \gamma^0) \geq (\gamma^0 - \gamma^2)x^2 \quad (iii)$$

$$g(x^0, \beta, \gamma^2) \geq (\gamma^0 - \gamma^2)x^0 \quad (iv)$$

$$f_\alpha(x^0, \alpha^0)\triangle\alpha_2 + \lambda_2^0\triangle\gamma_2 x^2 \leq F(\alpha^2, \beta^2, \gamma^2) - F(\alpha^0, \beta^0, \gamma^0) \leq$$
$$f_\alpha(x^2, \alpha^2)\triangle\alpha_2 - \lambda_2^1\triangle\gamma_2 x^0 \quad (III)$$

$$0 \geq \{-f_\alpha(x^2, \alpha^2) + f_\alpha(x^0, \alpha^0)\}\triangle\alpha_2 + \triangle\gamma_2(\lambda_2^0 x^2 + \lambda_2^1 x^0) \quad (IV)$$

$$\text{if,} \quad g(x^1, \beta^0, \gamma^0) \geq (\beta^0 - \beta^1) + (\gamma^1 - \gamma^0)x^1 \quad (v)$$

$$g(x^0, \beta^1, \gamma^1) \geq (\beta^1 - \beta^0) + (\gamma^1 - \gamma^0)x^0 \quad (vi)$$

$$f_\alpha(x^0, \alpha^0)\triangle\alpha_1 + \lambda_3^1\triangle\beta_1 + \lambda_3^1\triangle\gamma_1 x^0 \leq F(\alpha^1, \beta^1, \gamma^1) - F(\alpha^0, \beta^0, \gamma^0) \leq$$
$$f_\alpha(x^1, \alpha^1)\triangle\alpha_1 + \lambda_3^0\triangle\beta_1 - \lambda_3^0\triangle\gamma_1 x^1 \quad (V)$$

$$0 \geq \triangle\alpha_1\{-f_\alpha(x^{1'}, \alpha^1) + f_\alpha(x^0, \alpha^0)\} + \triangle\beta_1(\lambda_3^1 - \lambda_3^0)$$
$$+\triangle\gamma_1(\lambda_3^0 x^1 + \lambda_3^1 x_0) \quad (VI)$$

where, $F(\alpha^i, \beta^i, \gamma^i) = \max_{x_i} f(x^i)$ s.t. the budget constraint where α^i, β^i, γ^i are the parameters, $\forall i, j, \lambda_j^i \epsilon R_+^m$ and f_α is given in definition (2). (where whether x^j represents the function $x(\alpha^j, \beta^j, \gamma^j)$ or the variable x^j is clear from the context.

Proof : See Appendix

The following Lemmas are necessary in order to apply the above Proposition in the proof of the theorem.

Lemma 1 :

Suppose conditions 3(a) - 3(d) hold. Then, $L^{-1}(L_i) = \min\{x \epsilon R_+ : L(x) = L_i\}$ is bounded, monotonically decreasing & continuous for $L_i \epsilon \{I_+/1\} \;\; U\{\infty\}$

Proof : See Appendix.

4.4 Value of information

Investment in information improves the accuracy of the forecast by reducing the dimension of uncertainty i.e. the number of states to which the agent attaches positive probability. This reduces the dimension of the second period wealth vector considered in the budget constraint. At the same time it increases the probability weight on the utility derived from each state contingent consumption . Thus, the reduction in the dimensionality of the demand vector for state contingent consumption has to be compared with the shift in the demand levels with respect to each of the state contingent commodities within the possible set of states for valuing various levels of investment in information.

Lemma 2 : The value of information $V(L^{-1}(L_i); \{p(s)\}, w_i^{0,1})$ at date 0 to agent i is given by,

$$V(L^{-1}(L_i); \{p(s)\}, w_i^{0,1}) = \ln(\chi(L^{-1}(l_i)))$$

$$+\sum_{s=1}^{L_i} \delta(L_i)^{-1} \ln((p(s)L_i)^{-1}\chi(L^{-1}(L_i)))$$

where,

$$\chi(L^{-1}(L_i)) = \frac{\{w_i^{0,1} - L^{-1}(L_i) + \sum^{L_i} p(s)w^{1,s}\}}{2}$$

$$L_i = L(l_i)$$

$$L^{-1}(L_i) = \min\{x \epsilon R_+ : L(x) = L_i\}$$

Proof : See Appendix.

4.5 Continuous Extension of the problem

The function

$$u(\{c_i^s\}, L^{-1}(\theta_i)), \theta_i \epsilon I_+$$

is discontinuous with respect to l_i whose range is R_+. Hence, I have to transform the utility function into a continuous extension with respect to l_i. Consider the following extension of the function $u(\{c_i^s\}, L^{-1}(\theta_i))$ denoted by $\tilde{u}(\{c_i^s\}, l_i)$ as follows :

$$\tilde{u}(\{c_i^s\}, l_i) = u(\{c_i^s\}, l_i) \text{if, } l_i = L^{-1}(\theta_i) \text{for some } \theta_i \epsilon I_+$$
$$= \tau u(\{c_i^s\}, l_i^{-1}) + (1-\tau) u(\{c_i^s\}, l_i^{+1}),$$
$$\text{if } l_i \neq L^{-1}(\theta_i) \text{for any } \theta_i \epsilon I_+.$$
$$\text{where, } l_i = \tau l_i^{-1} + (1-\tau) l_i^{+1} \text{for some } \tau : 0 < \tau < 1$$
$$\text{and, } l_i^{-1} = L^{-1}(\theta_i - 1) \text{for some } \theta_i \epsilon I_+$$
$$\text{and } l_i^{+1} = L^{-1}(\theta_i)$$

The maximization problem $(6')$ can be restated using $\tilde{u}()$ as the utility function as follows :

$$\max_{\{(c_i \geq 0), l_i \geq 0\}} \tilde{u}(\{c_i^s\}, l_i, \pi_i)$$
$$s.t. \;\; w_i^{0,1} + \sum^{L_i} p(s)(w_i^{1,s} - c_i^s(1)) - c_i^1(0) - l_i \geq 0$$
$$\pi_i = 1/L(l_i) \tag{6''}$$
$$\text{where, } \tilde{u}(\{c_i^s\}, l_i, \pi_i) = u(\{c_i^s\}, l_i, \pi_i) \text{if } \tilde{u}(\{c_i^s\}, l_i) = u(\{c_i^s\}, l_i)$$
$$= \tau u(\{c_i^s\}, l_i^{-1}, \pi_i) + (1-\tau) u(\{c_i^s\}, l_i^{+1}, \pi_i)$$
$$\text{if } \;\; \tilde{u}(\{c_i^s\}, l_i) = \tau u(\{c_i^s\}, l_i^{-1}) + (1-\tau) u(\{c_i^s\}, l_i^{+1})$$
$$\text{and, } \;\; u(\{c_i^s\}, L^{-1}(\theta_i), \pi_i) = \ln c_i^1(0) + \frac{1}{L(l_i)} \sum_{s=1}^{\theta_i} \ln c_i^s(1)$$

I shall establish the equivalence between $(6')$ and $(6'')$ in the following lemma.

Lemma 3: $(\{c_i*\}, l_i^*)$ is a local optimum of $(6'') \Leftrightarrow (\{c_i^*\}, l_i^*)$ is a local optimum of $(6')$.

Note (2) : In proving this lemma I have used concavity of $u(\{c_i^s\}, l_i)$ with respect to $\{c_i^s\}$ only.

Note (3): If $u(\{c_i^s\}, L^{-1}(\theta_i))$ is concave (convex) with respect to a certain range of $\theta_i \epsilon \; I_+, u(\{c_i^s\}, l_i)$ is concave (convex) with respect to to l_i lying within the range of $L^{-1}(\theta_i)$ for such θ_i. Hence I shall only concern ourselves with problem $(6')$ where, the choice set of l_i is restricted to $L^{-1}(\theta_i), \theta_i \epsilon I_+$.

4.6 Individual's problem

As discussed in sec (3.2) a change in the investment in information by any agent alters both the present value of wealth as also the probability weights on the set of contingent consumption , the dual impact of which given a particular learning technology and the probability updating rule (4) depends on the shape of the state contingent utility functions, the distribution of state contingent wealth over the state-space as also the particular price vector under consideration. A sufficient set of conditions which ensures an interior solution to the above investment problem is derived in the next two lemmas.

Lemma 4 : If conditions 3(a) - 3(d) hold with respect to L(.)

$$\text{and if } \ (i) w_i^{1,s} = w^{1,s} > 0 \ \ for s \leq \overline{s}_i < \infty$$

$$= 0 \quad \text{for } s > \overline{s}_i, \quad \forall i$$

where, $\{w^{1,s}\}_{s=1}^{\infty}$ *is a sequence of strictly positive real numbers and (iii)* $\infty > p(s) > 0 \forall s$

Then, the subgradient of $u(\{c_i^s\}, l_i)$ denoted by $u_i(\{c_i^s\}, l_i)$ and given by

$$u_l(\{c_i^s\}, l_i) = \ [\frac{\delta}{L(l_i) - 1} \sum^{L(l_i)-1} \ln c_i^1(s) - \frac{\delta}{L(l_i)} \sum^{l(l_i)} \ln c_i^1(s)] \frac{\triangle L}{\triangle l_i}]$$

exists.

Where, $l_i = L^{-1}(\theta_i)$ for some $\theta_i \epsilon I_+$ and

$$\triangle L = \theta_i - (\theta_i - 1),$$

$$\triangle l_i = L^{-1}(\theta_i) - L^{-1}(\theta_i - 1)$$

Proof : see Appendix.

Note (4) : The subgradient $u_l(\{c_i^s\}, l_i)$ defined in Lemma 4 is the maximum of all the subgradients of $u(\{c_i^s\}, \quad)$ at l_i, since $u(\{c_i^s\}, \quad)$ is convex in l_i (hence concave in L_i) by the conditions of the Lemma (see Rockfeller (1970) pg. 229)

Lemma 5:

If $\exists$ an integer $L_i^0 \geq 2 \quad s.t.$

$$\prod^{L_i^0-1} (p(s))^\delta > p(l_i^0)^{\delta(L_i^0-1)}$$

and

$$\prod^{L_i^0} (p(s))^\delta \leq p(L_i^0+1)^{\delta L_i^0}$$

then, L_i^0 is an interior optimum for agent i if

1. conditions 3(a) - 3(b) hold w.r.t. L(.)
2. $w_i^{1,s} = w^{1,s} \quad \text{for} \quad s \leq \overline{s}_i < \infty$

   $$= 0 \qquad \text{for} \quad s > \overline{s}_i, \forall i$$

   where $\{w^{1,s}\}_{s=1}^{\infty}$ is a sequence of finite strictly positive real numbers.
3. $\infty > p(s) > 0 \forall s$
4. $w_i^{0,1} + \sum^{L_i^0-1} p(s) w_i^{1,s} - L^{-1}(L_i^0 - 1) > 0$ and
5. $\frac{w_i^{0,1} + \sum^{L_i^0} p(s) w_i^{1,s}}{2(L_i^0+1)} \geq L^{-1}(L_i^0) - \frac{L_i^0}{2(L_i^0+1)} L^{-1}(L_i^0+1)$

Proof: see Appendix

Note (5) : The inequality (C*) is the proof is redundant for the purpose of this Lemma as I need to compare the value w.r.t. L_i^0 with $L_i^0 - 1$ and $L_i^0 + 1$ for local optimality.

Note (6) : Conditions (1), (4) and (5) are restrictions on the shape of the learning technology . Conditions (2) requires that given any common probability distribution over the states the expected wealth level of any agent at t=1 is bounded.

Under the conditions which guarantee the existence of an interior optimal choice of investment in information for all agents I discuss market failure .

A similar approach to optimization has been studied recently by Milgrom & Segal (2002) where the choice sets for optimization may not be convex.Their characterization of local optimality conditions is similar to the approach of this paper.Besides, their approach of using left and right hand derivatives is similar to the approach of using left and right hand subgradients of this paper.However, they have neither used the integer choice set problem specifically nor used a stepwise technology in their objective function as in this paper.Their notion of ”equidifferentiability” requires using an asymptotic approximation for slopes whereas this paper’s procedure consists of convexification at steps and then deleting the nonrelevant optima once tangency has been established.Also, see the references therein particularly Sah & Zhao (1998) and Kim (1993)for similar differences.I would say that this paper also makes certain technical contributions in valuing and aggregating investment independently of the above papers.

4.7 Competition and Market Failure

In order to keep the ordering of agents in terms of wealth levels complete I impose the following condition on the two period wealth levels :

$$\text{if}, w_i^{0,1} > w_j^{0,1} \text{for any} \quad i \neq j \quad \text{then} \quad \overline{s}_i > \overline{s}_j, \forall j \leq N \tag{11}$$

THEOREM :

Even if the conditions of Lemma 5 are satisfied for all agents $i \leq N$, the contingent commodity market is imperfect at $t = 0$, if

$$\mid w_j^{0,1} - w_i^{0,1} \mid > 1 \qquad \forall j \neq i,$$
$$\min_i \overline{s}_i \leq N$$

and (11) holds.

Proof: see Appendix

Note (7) : The particular notion of market clearing I have in mind when I defined imperfect contingent commodity markets is one with restricted participation. i.e.

$$\sum_{i \epsilon N_1(0,1)} (w_i^0 - c_i^{1^*}(0, p(s))) - l_i^*$$
$$= \sum_{i \epsilon N_2(0,1)} (c_i^{1*}(0, (p(s))) - w_i^{0,1}) + l_i^*$$

and for any $s \leq \max_i L_i^0(p(s)))$ for date 1 :

$$\sum_{i\epsilon N_1(1,s)} (w_i^{1,s} - c_i^{s*}(1, p(s))) = \sum_{i\epsilon N_2(1,s)} (c_i^{s*}(1, (p(s))) - w_i^{1,s})$$

where, $N_1(t, s)$ and $N_2(t, s)$ are the set of agents participating with respect to market s at date t, with positive supply of contingent commodity s and positive demand for contingent commodity s, respectively, at prices $\{p(s)\} > 0$ (see 7(a)). Here, I have ignored $\{(p(s)\}$ in the argument for notational simplification.

Note (8) : It is not necessary for our results to hold for there to be only 1 agent with wealth level of class i for each i. Our results hold with more than 1 agent in each class as long as there are more than 1 class , but the number of classes is finite.

4.8 Conclusion

I have provided a model whereby the degree of competition in contingent commodity markets, is endogenously determined. Competition ensures full employment of all resources and therefore efficiency. This endogenous derivation of the degree of competition is necessary for at least two reasons. Firstly, while in existing oligopoly theory competition is treated as exogenous, either of the perfectly competitive variety or of the monopolistic variety, a variable which would depend both on the cost of entry and the benefits of entry which each agent expects to face in entering such markets and hence the consequential number of agents who decide to participate in the various markets. Since the decision to participate is made by rational agents from the expected costs and benefits of participation it seems natural to think that the participation in markets and hence the corresponding nature of competition will be derived endogenously from the model rather than being exogenously specified. I have provided one such model where agents behave competitively. Secondly, market failure at any finite positive price can arise from non-participation of agents in a market such that there is only demand for contingent commodity but not supply or vice versa which requires the analysis of the degree of competition . In this paper I have formalized a model of contingent commodity markets where agents use information and contingent commodity as two tools for smoothing risk, and there by the number of participants varies across markets for endogenous reasons.

This analysis has brought out a vital feature of contingent commodity markets which is that if initial wealth distribution is asymmetric across agents and the ordering in terms of wealth levels (actual or expected as the case may be) is preserved over time then contingent commodity markets may fail in the sense that not all agents can participate in all markets leading to deficient demand or supply.This provides the framework for economic policies with respect to emerging contingent commodity markets and their corresponding economies which are being studied here.

4.9 Appendix

Proof of Proposition:

The saddle-point condition for the constrained maximization problem (M) for $(\alpha^0, \beta^0, \gamma^0)$ is written as :

(SP) $\exists x^0 \epsilon X$ and $\lambda^0 \epsilon R^m, \quad \lambda^0 \geq 0$ s.t.

$$\phi(x, \lambda^0; \alpha^0, \beta,^0, \gamma^0) \leq \phi(x^0, \lambda^0; \alpha^0, \beta^0, \gamma^0) \leq \phi(x^0, \lambda, \alpha^0, \beta^0, \gamma^0)$$

$\forall x \epsilon X$ and $\lambda \epsilon R^m$ with $\lambda \geq 0$, where

$$\phi(x, \lambda; \alpha, \beta, \gamma) \equiv f(x, \alpha) + \lambda.g(x, \beta, \gamma)$$

As is well known if (SP) holds then x^0 is a solution of (M) automatically.

From the above saddle-point condition, I obtain

$$f(x^0, \alpha^0) - f(x, \alpha^0) \geq \lambda^0.g(x, \beta^0, \gamma^0), \forall x \epsilon X \qquad (a)$$

Step I : Now, if

$$f(x, \alpha^0) - f(x, \alpha^1) = f_\alpha(x, \alpha^1)(\alpha^0 - \alpha^1) \qquad (b)$$

Then, adding (a) and (b) I have

$$f(x^0, \alpha^0) - f(x^1, \alpha^1) \geq f_\alpha(x^1, \alpha^1)(\alpha^0 - \alpha^1) + \lambda_1^0 g(x^1, \beta^0, \gamma^0) \qquad (c)$$

Also,

$$f(x^1,\alpha^1) - f(x^0,\alpha^0) \geq f_\alpha(x^0,\alpha^0)(\alpha^1-\alpha^0) + \lambda_1^1 g(x^0,\beta^1,\gamma^1) \qquad (d)$$

Adding (c) and (d), I have

$$\begin{aligned} 0 \geq (\alpha^1-\alpha^0)\{-f_\alpha(x^1,\alpha^1) + f_\alpha(x^0,\alpha^0)\} + \lambda_1^0 g(x^1,\beta^0,\gamma^0) \\ +\lambda_1^1 g(x^0,\beta^1,\gamma^1) \end{aligned} \qquad (e)$$

Conditions (i), (ii) and (c) and (d) imply,

$$f(x^0,\alpha^0) - f(x^1,\alpha^1) \geq -f_\alpha(x^1,\alpha^1)(\alpha^1-\alpha^0) + \lambda_1^0(\gamma^1-\gamma^0)x^1$$

$$f(x^1,\alpha^1) - f(x^0,\alpha^0) \geq f_\alpha(x^0,\alpha^0)(\alpha^1-\alpha^0) + \lambda_1^1(\gamma^1-\gamma^0)x^0$$

which implies,

$$\begin{aligned} f_\alpha(x^0,\alpha^0)\triangle\alpha_1 + \lambda_1^1\triangle\gamma_1 x^0 \leq F(\alpha^1,\beta^1,\gamma^1) - F(\alpha^0,\beta^0,\gamma^0) \\ \leq f_\alpha(x^1,\alpha^1)\triangle\alpha_1 - \lambda_1^0\triangle\gamma_1 x^1 \\ (\text{where}\triangle\alpha_1 = \alpha^1-\alpha^0, \triangle\gamma_1 = \gamma^1-\gamma^0) \end{aligned} \qquad (1)$$

Conditions (i), (ii) and (e) imply,

$$\begin{aligned} 0 &\geq \triangle\alpha_1\{-f_\alpha(x^1,\alpha^1) + f_\alpha(x^0,\alpha^0)\} + \lambda_1^0\triangle\gamma_1 x^1 + \lambda^1\triangle\gamma_1 x^0 \\ &\geq \triangle\alpha_1\{-f_\alpha(x^1,\alpha^1) + f_\alpha(x^0,\alpha^0)\} + \triangle\gamma_1(\lambda_1^0 x^1 + \lambda_1^1 x^0) \end{aligned} \qquad (2)$$

Step II : If

$$f(x,\alpha^0) - f(x,\alpha^2) = f_\alpha(x,\alpha^2)(\alpha^0-\alpha^2) \qquad (b')$$

Then, adding (a) and (b') I have

$$f(x^0,\alpha^0) - f(x^2,\alpha^2) \geq f_\alpha(x^2,\alpha^2)(\alpha^0-\alpha^2) + \lambda_2^0 g(x^2,\beta,\gamma^0) \qquad (c')$$

also,

$$f(x^2,\alpha^2) - f(x^0,\alpha^0) \geq f_\alpha(x^0,\alpha^0)(\alpha^2-\alpha^0) + \lambda_2^1 g(x^0,\beta,\gamma^2) \qquad (d')$$

Adding (c') and (d'), I have

$$0 \geq (\alpha^2 - \alpha^0)\{-f_\alpha(x^2, \alpha^2) + f_\alpha(x^0, \alpha^0)\} + \lambda_2^0 g(x^2, \beta, \gamma^0)$$
$$+\lambda_1^2 g(x^0, \beta, \gamma^2) \qquad (e')$$

Conditions (iii), (iv) and (c') and (d') imply,

$$f(x^0, \alpha^0) - f(x^2, \alpha^2) \geq f_\alpha(x^2, \alpha^2)(x^0 - \alpha^2) + \lambda_2^0(\gamma^0 - \gamma^2)x^2$$

$$f(x^2, \alpha^2) - f(x^0, \alpha^0) \geq f_\alpha(x^0, \alpha^0)(\alpha^2 - \alpha^0) + \lambda_2^1(\gamma^0 - \gamma^2)x^0$$

which implies,

$$f_\alpha(x^0, \alpha^0)\triangle\alpha_2 + \lambda_2^0 \triangle\gamma_2 x^2 \leq F(\alpha^2, \beta^2, \gamma^2) - F(\alpha^0, \beta^0, \gamma^0)$$
$$\leq f_\alpha(x^2, \alpha^2)\triangle\alpha_2 - \lambda_1^2 \triangle\gamma_2 x^0 \qquad (3)$$

$$(\text{where,}\triangle\alpha_2 = \alpha^2 - \alpha^0, \triangle\gamma_2 = \gamma^2 - \gamma^0)$$

Conditions (iii), (iv) and (e') imply,

$$0 \geq \triangle\alpha^2\{-f_\alpha(x^2, \alpha^2) + f_\alpha(x^0, \alpha^0)\} + \lambda_2^0 \triangle\gamma_2 x^2 + \gamma_1^2 \triangle\gamma_2 x^0$$
$$\geq \triangle\alpha_2\{-f_\alpha(x^2, \alpha^2) + f_\alpha(x^0, \alpha^0)\} + \triangle\gamma_2(\lambda_2^0 x^2 + \lambda_1^2 x^0) \qquad (4)$$

<u>Step III :</u>
If,

$$f(x, \alpha^0) - f(x, \alpha^1) = f_\alpha(x, \alpha^1)(\alpha^0 - \alpha^1) \qquad (b)$$

Then adding (a) and (b)

$$f(x^0, \alpha^0) - f(x^{1'}, \alpha^1) \geq f_\alpha(x^{1'}, \alpha^1)(\alpha^0 - \alpha^1) + \lambda_3^0 g(x^{1'}, \beta^0, \gamma^0) \qquad (c'')$$

Also,

$$f(x^{1'}, \alpha^1) - f(x^0, \alpha^0) \geq f_\alpha(x^{1'}, \alpha^1)(\alpha^1, \alpha^0) + \lambda_3^1 g(x^0, \beta^1, \gamma^1) \qquad (d'')$$

Adding (c'') and (d'') I have,

$$0 \geq (\alpha^1 - \alpha^0)\{-f_\alpha(x^{1'}, \alpha^1) + f_\alpha(x^0, \alpha^0) + \lambda_3^0 g(x^1, \beta^0, \gamma^0)$$
$$+\lambda_3^1 g(x^0, \beta^1, \gamma^1) \qquad (c'')$$

Conditions (v), (vi), (c'') and (d'') imply,

$$f(x^0, \alpha^0) - f(x^{1'}, \alpha^1) \geq f_\alpha(x^{1'}, \alpha^1)(\alpha^0 - \alpha^1)$$
$$+\lambda_3^0(\beta^0 - \beta^1) + \lambda_3^0(\gamma^1 - \gamma^0)x^{1'}$$
$$f(x^{1'}, \alpha^1) - f(x^0, \alpha^0) \geq f_\alpha(x^0, \alpha^0)(\alpha^1 - \alpha^0)$$
$$+\lambda_3^1(\beta^1 - \beta^0) + \lambda_3^1(\gamma^1 - \gamma^0)x^0$$

which implies,[2]

$$f_\alpha(x^0, \alpha^0)\triangle)\alpha_1 + \lambda_3^1\triangle\beta_1 + \lambda_3^1\triangle\gamma_1 x^0 \leq F(\alpha^1, \beta^1, \gamma^1) - F(\alpha^0, \beta^0, \gamma^0)$$
$$\leq f_\alpha(x^{1'}, \alpha^1)\triangle\alpha_1 + \lambda_3^0\triangle\beta_1 - \lambda_3^0\triangle\gamma_1 x^{1'} \tag{5}$$

Conditions (v), (vi) and (e'') imply,

$$0 \geq \triangle\alpha_1\{-f_\alpha(x^{1'}, \alpha^1) + f(x^0, \alpha^0)\}$$
$$-\lambda_3^0\triangle\beta_1 + \lambda_3^0\triangle\gamma_1 x^{1'} + \lambda_3^1\triangle\beta_1 + \lambda_3^1\triangle\gamma_1 x^0$$
$$\geq \triangle\alpha_1\{-f_\alpha(x^{1'}, \alpha^1) + f_\alpha(x^0, \alpha^0)\}$$
$$+\triangle\beta_1(\lambda_3^1 - \lambda_3^0) + \triangle\gamma^1(\lambda_3^0 x^{1'} + \lambda_3^1 x^0) \tag{6}$$

(where $\triangle\beta_1 = \beta_1 - \beta_0$) Q.E.D.

Proof of Lemma 1:

$$L^{-1}(L_i) : \{I_+/I\}U\{\infty\} \rightarrow R_+^1 \quad by\ (3b)$$

Also, $L^{-1}(.)\text{exists} \ \ \forall L_i \epsilon I_+$ by (3a) and (3d)

$L^{-1}(.)$ is monotonic :

by (3c) if $l'_i > l_i$ then $L(l'_1) \leq L(l_i)$

now, suppose $L_i''' > L_i^{''} > L'_i$

and, $\min\{x : L(x) = L_i''\} < \min\{x : L(x) = L_i'\}$

but, $\min\{x : L(x) = L_i'''\} < \min\{x : L(x) = L_i''\}$

Also,

Suppose, $min\{x : L(x) = L_i^{'}\} = x^{'}$

$$min\{x : L(x) = L_i^{''}\} = x^{''}$$

[2]The general iterative procedure discussed in this proof can be extended to an algorithm for computing equilibria with integer constraints of course in such a case the model has to be closed by suitable redistribution.

$$min\{x : L(x) = L_i^{'''}\} = x^{'''}$$

$$x'' < x'$$

Then,

$$x''' > x''$$

$\because$ Either $x''' < x'$ but $x''' > x''$ or $x''' = x'$ or $x''' > x'$

now, if $x''' < x'$, then by 3(c) $L_i^{'''} \geq L_i^{'}$ but $L_i^{'''} \leq L_i^{''}$ - a contradiction

$$\therefore L_i^{''} > L_i^{'}$$

If, $x''' = x'$, then $L_i^{'''} = L_i^{'}$ - again a contradiction

$$\therefore L_i^{'''} > L_i^{''} > L_i^{'} \Rightarrow L_i^{'''} > L_i^{'}$$

if, $x''' > x'$, then by (3c) $L_i^{'''} \leq L_i^{'}$, again a contradiction.

Hence, L^{-1} is monotonically decreasing.

<u>$L^{-1}(.)$ is bounded :</u>

$\because L^{-1}(.)$ is monotonically decreasing, and I_+ is ordered therefore the maximum and minimum are attained at the minimum and maximum of I_+ Now, By (3a), $L^{-1}(\infty) = 0$ and by (3d), $L^{-1}(2)$ is bounded.

Hence, $L^{-1}(.)$ is bounded over $I_+ - \{1\}$

<u>L^{-1} is continuous :</u>

$L(.) : R \rightarrow \{I_+/I\}U(\infty\}$ by 3(a) and 3(b)

Now, both R and $I_+U\{\infty\}$ are metric spaces.

$\because$ If $L(.)$ is a continuous 1-1 mapping, then it can be shown that $L^{-1}(.)$ is continuous. (Theorem 4.17 pg. 90 Rudin (1976)).

Now, pick any point x in R

Pick any $\epsilon > 0$

Suppose, $0 < \epsilon < 1$

Then, by (3c) $\exists \infty > \delta > 0$ s.t. $\forall y \epsilon R_+$:

$$| y - x | < \delta,$$

$$| L(y) - L(x) | < \epsilon$$

Suppose, I pick any $\infty > \epsilon \geq 1$

then by 3(c) and 3(d) $\exists \infty > \delta > 0$ s.t. $\forall y \epsilon R_+$:

$$| y - x | < \delta,$$

$$| L(y) - L(x) | < \epsilon$$

This holds $\forall x \epsilon R_+$ (by 3(b)).

Hence, by definition $L(l_i) : R_+ \rightarrow \{I_+/1\}U\{\infty\}$ is continuous, $L(L^{-1}(.))$ is a 1-1 mapping. Hence $L^{-1}(.)$ is continuous.

Q.E.D.

Proof of Lemma 2:

$V(L^{-1}(L_i), \{p(s)\}, w_i^{0,1})$ is the indirect utility for agent i given the choice of investment in information $L^{-1}(L_i)$. Now, given a particular choice of L_i and therefore $L^{-1}(L_i)$ (see 3(a)) the value of optimal consumption at the given price vector $\{p(s))\}$ is given by,

$$c_i^{s*}(1, \{p(s)\}) = \frac{\chi(L^{-1}(L_i))}{p(s)L_i}$$

$$c_i^{1*}(0, \{p(s)\}) = \chi(L^{-1}(L_i))$$

Replacing in the utility function gives the result Q.E.D.

Proof of Lemma 3:

First I shall prove that if $(\{c_i^*\}, l_i^*)$ is a local maximum of $\tilde{u}(\{c_i^*\}, .)$ then it is necessary that $l_i^* = L^{-1}(\theta_i^*)$ for some $\theta_i^* \epsilon I_+$.

Suppose, $l_i^* \neq L^{-1}(\theta_i)$ for any $\theta_i \epsilon I_+$ Then, $l_i^* = \tau^*(l_{i*}^{-1}) + (1 - \tau*)(l_i^{*+1})$ for some $\theta_i^* \epsilon I_+, \ \ 0 < \tau* < 1$. Hence, from (9),

$$\tilde{u}(\{c_{i*}\}, l_i^*) = \tau^* u(\{c_i^*\}, L^{-1}(\theta_i^* - 1)) + (1 - \tau^*) u(\{c_i^*\}, L^{-1}(\theta_i^*))$$

Now, either $u(\{c_i^*\}, L^{-1}(\theta_i^*)) \geq u(\{c_i^*\}, L^{-1}(\theta_i^* - 1))$

Hence, either

$$u(\{c_i^*\}, L^{-1}(\theta_i^*)) \geq \tilde{u}(c_i^*\}, l_i^*)$$

or $u(\{c_i^*\}, L^{-1}(\theta_i^* - 1)) \geq \tilde{u}(\{c_i^*\}, l_i^*)$

In every case the maximum is attained at $l_i^* = L^{-1}(\theta_i^*)$ for some $\theta_i^* \epsilon I_+$. Hence, it is not possible that $l_i^* \neq L^{-1}(\theta_i)$ for any $\theta_i \epsilon I_+$.

Now, from the budget constraint in $(6'')$ if $l_i' = L^{-1}(\theta_i)$ and , $l_i^{1'} < l_i : L(l_i^*) = \theta_i$, then l_i^* cannot be an optimal choice of l_i due to the fact that $u(., l_i)$ and hence $\tilde{u}(., l_i)$ is concave in $\{c_i^s\}$. Hence, the optimal choice of l_i can only be such that $l_i^* = L^{-1}(\theta_i)$ for some $\theta_i \in I_+$

Now, since

$$\tilde{u}(\{c_i^s\}, l_i, \pi_i) = u(\{c_i^s\}, l_i, \pi_i) \quad \forall l_i$$

$$l_i = L^{-1}(\theta_i) \quad \text{for} \quad \theta_i \in I_+.$$

Hence, if l_i^* = Arg $\max_{l_i} u(\{c_i^*\}, l_i)$ s.t. the b.c. in ($6''$),
Then, l_i^* = Arg $\max_{l_i} u(\{c_i^*\}, L^{-1}(\theta_i *))$ s.t. the b.c. in ($6'$), and

$$l_i^* = L^{-1}(\theta_i *), \theta_i^* \epsilon I_+$$

To prove the converse Suppose, $(\{c_i^*\}, L^{-1}(\theta_i^*))$ solves $(6')$
But, $(\{c_i^*\}, L^{-1}(\theta_i^*))$ does not solve $(6'')$

$$\because \exists(\{\tilde{c}_i\}, \tilde{l}_i), \tilde{l}_i \neq L^{-1}(\theta_i^*)$$

s.t.

$$u(\{\tilde{c}_i\}, \tilde{l}_i, \pi_i(\tilde{l}_i)) >$$

$$u(\{c_i^*\}, l_i^*, \pi_i(l_i^*))$$

and $\tilde{l}_i$ is feasible where,

$$\pi_i(\tilde{l}_i) = \frac{1}{L(\tilde{l}_i)}$$

$$\pi_i(l_i^*) = \frac{1}{L(l_i^*)}$$

Now by 3(b) $L(\tilde{l}_i) = \tilde{\theta}_i \epsilon I_+$
Now $L(\tilde{l}_i) \neq \theta_i^*$, for by the concavity of $u(.,.)$ with respect to $\{c_i\}$, if $L(\tilde{l}_i) = \theta_i^*$, then $\tilde{l}_i > L^{-1}(\theta_i^*)$, from the budget constraint this implies that $u(\{c_i^*\}, l_i^*, \pi_i(l_i^*))$ cannot be an optimum of $(6')$.

$$L(\tilde{l}_i) = \tilde{\theta}_i < \theta_i^* \text{by3(c)}$$

But,

$$\tilde{u}(\{\tilde{c}_i\}, L^{-1}(\tilde{\theta}_i), \pi_i(\tilde{l}_i)) > \tilde{u}(\{c_i^*\}, L^{-1}(\theta_i^*), \pi_i(l_i^*))$$

$$\Rightarrow u(\{\tilde{c}_i\}, L^{-1}(\tilde{\theta}_i), \pi_i(\tilde{l}_i)) > u(c_i^*\}, L^{-1}(\theta_i^*), \pi_i(l_i^*))$$

which contradicts the fact that $(c\{c_i^*\}, L^{-1}(\theta_i^*))$ solve ($6'$)

Hence the Lemma.

Q.E.D.

Proof of Lemma 4:

I pick any set of prices $\{p(s)\} > 0$. For notational simplification I drop prices from the argument. From definition 2 of subgradient, I have to first check whether $u(\{c_i^s\}, l_i)$ is convex or concave.

Now, $u(\{c_i^s\}, l_i)$ is convex in l_i if for any $\infty > l'_i > l_i$

$$u(\{c_i^s\}, \alpha l_i + (1-\alpha)l_i')) \leq \alpha u(\{c_i^s\}, l_i) + (1-\alpha)u(\{c_i^s\}, l_i')$$

where, $L(l_i) = L_i$, (the inequality is reversed if $u(.,.)$ is concave) and $0 < \alpha < 1$. Let, l_i' be close enough s.t. $L(l_i') = L(l_i)\ or\ L(l_i') = L(l_i) - 1$. If l_i' is greater the convexity or concavity can be derived by induction from the fact that $L^{-1}(.)$ is monotonic and continuous (Lemma 1 - conds. 3(a) - 3(d) hold).

Case a : If, $L(l_i^{'}) = L(l_i)$

then, $u(\{c_i^s\}, l_i^{'}) = u(\{c_i^s\}, l_i)$

$u(\{c_i^s\}, \alpha l_i + (1-\alpha)l_i') = u(\{c_i^s\}, l_i)$ $(L^{-1}(.)$is continuous from Lemma 1)

$$u(\{c_i^s\}, l_i)$$

is both concave and convex.

Case b: If, $L(l_i') = L(l_i) - 1$

Then,

$$u(\{c_i^s\}, \alpha l_i) + (1-\alpha)l_i') - \ln c_i^1(0) + \sum_{s=1}^{L_i-1} \frac{\delta}{L_i - 1} \ln c_i^s(1)$$

now,

$$\alpha u(\{c_i^s\}, l_i) + (1-\alpha)u(\{c_i^s\}, l_i^{'})$$

$$= \alpha\{\ln c_i^1(0) + \sum^{L_i} \frac{\delta}{L_i} \ln c_i^s(1)\} + (1-\alpha)\{\ln c_i^1(0) +$$

$$\sum_{s=1}^{L_i-1} \frac{\delta}{L_i - 1} \ln c_i^s(1)$$

$$= \ln c_i^1(0) + \alpha \sum^{L_i} \frac{\delta}{L_i} \ln c_i^s(1) + (1-\alpha) \sum^{L_i-1} \frac{\delta}{L_i - 1} \ln c_i^s(1)$$

$u(.,l_i)$ is convex in l_i iff :

$$\ln c_i^1(0) + \alpha \sum^{L_i} \frac{\delta}{L_i} \ln c_i^s(1) + (1-\alpha) \sum^{L_i-1} \frac{\delta}{L_i - 1} \ln c_i^s(1) \leq \ln c_i^1(0) +$$

$$\sum^{L_i-1} \frac{\delta}{L_i - 1} \ln c_i^s(1)$$

$u(.,.)$ is convex in l_i iff :

$$\frac{\delta}{L_i} \sum^{L_i} \ln c_i^s(1) \leq \frac{\delta}{L_i - 1} \sum^{L_i-1} \ln c_i^s(1)$$

$$or, \quad \sum^{L_i} \ln(c_i^s(1))^{\frac{1}{L_i}} \leq \sum^{L_i-1} \ln(c_i^s(1))^{\frac{1}{L_i-1}} \quad [\quad \delta > 0]$$

$$\infty > w_i^{1,s} > \quad 0 \quad fors \quad \leq \overline{s}_i < \infty$$

$$= 0 \quad for \quad s > \overline{s}_i$$

then $\{c_i^s(1)\}$ is bounded $\forall \{p(s)\} > 0$

$c_i^s(1)$ is bounded below by 0, $c_i^s(1)$ is bounded above by $(w_i^{0,1} + \sum_{s=1}^{\infty} p(s) w_i^{1,s})/p(s)$

$$= \frac{(w_i^{0,1} + \sum_{s=1}^{\overline{S}_i} p(s) w_i^{1,s})}{p(s)} \qquad (from (6))$$

Denote, $c_n = \prod^n c_i^s(1)$
then, $c_{n+1} = \prod^{n+1} c_i^s(1)$
if, $\{c_i^s(1)\} > 0$, then

$$\lim_{n \to \infty} \sup (c_n)^{1/n} \leq \lim_{n \to \infty} \sup \frac{c_{n+1}}{c_n}$$

$$\Rightarrow \lim_{n \to \infty} \prod^n (c_i^s(1))^{1/n} \leq \lim_{n \to \infty} \sup (c_i^{n+1}(1))$$

$$\lim_{n \to \infty} \inf (c_i^{n+1}(1)) \leq \lim_{n \to \infty} \inf \prod^n (c_i^s(1))^{1/n}$$

[Theorem 3.37 Rudin (1976), pg.68]

Now, $\because$ the sequence $\{c_i^s(1)\}_{s=1}^{\infty}$ is real and bounded (see above) by Weirstrass Theorem (Rudin (1976), pg.68))

$$\lim_{n \to \infty} c_i^n(1)$$

exists.

Hence, $\lim_{n\to\infty} sup\ c_i^{n+1}(1)$ =$\lim_{n\to\infty} inf\ c_i^{n+1}(1)$

$$= \lim_{n\to\infty} c_i^{n+1}(1)$$

$$= \lim_{n\to\infty} \sup \prod^{n} (c_i^s(1))^{1/n}$$

$$= \lim_{n\to\infty} \inf \prod^{n} (c_i^s(1))^{1/n}$$

$$= \lim_{n\to\infty} \prod^{n} (c_i^s(1))^{1/n}$$

Thus, $\lim_{n\to\infty} \prod^n (c_i^s(1))^{1/n}$ exists in R

Hence, $\exists$ a sequence of $(c_i^s(1))$s.t.

$$\prod^{n-1} (c_i^s(1))^{1/n-1} \geq \prod^{n} (c_i^s(1))^{1/n}$$

For such a sequence, $\because$ $\ln$ is a monotonically increasing function.

$$\ln \prod^{n-1} (c_i^s(1))^{1/n-1} \geq \ln \prod^{n} (c_i^s(1))^{1/n}$$

$$\Rightarrow \frac{1}{n-1} \sum^{n-1} \ln(c_i^s(1)) \geq \frac{1}{n} \sum^{n} \ln(c_i^s(1))$$

Hence, $u(.,.)$ is convex in l_i for such a sequence of $\{c_i^s(1)\}$.

Now, Let us denote

$$u_i(\{c_i^s\}, l_i) =$$

$$\frac{[\ln c_i^1(0) + \sum^{L_i} \frac{\delta}{L_i} \ln c_i^s(1) - \ln c_i^1(0) - \sum^{L_i-1} \frac{\delta}{L_i-1} \ln c_i^s(1)]}{L_i - (L_i - 1)} \frac{\triangle L_i}{\triangle l_i}$$

$$= [\sum^{L_i} \frac{\delta}{L_i} \ln c_i^s(1) - \sum^{L_i-1} \frac{\delta}{L_i - 1} \ln c_i^s(1)] \frac{\triangle L_i}{\triangle l_i}$$

[This is one of the sub-gradients, see Note (4)]

$$u_i(.,.)(l_i^1 - l_i^0) = (\frac{\delta}{L_i} \sum^{L_i} \ln c_i^s(1) - \frac{\delta}{L_i - 1} \sum^{L_i-1} \ln c_i^s(1))(-1)$$

$$\text{Let,} \quad l_i^0 = L^{-1}(L_i^0), \quad l_i^1 = L^{-1}(L_i^0 - 1)$$

$$\text{Now,}\quad u(\{c_i^s\}, l_i^0) = \ln c_1^1(0) + \frac{\delta}{L_i^0} \sum^{L_i} \ln c_i^s(1)$$

$$u(\{c_i^s\}, l_i^{'}) = \ln c^1(0) + \frac{\delta}{L_i^0 - 1} \sum^{L_i - 1} \ln c_i^s(1)$$

$$u(\{c_i^s\}, l_i^1) - u(\{c_i^s\}, l_i^0)$$

$$= \frac{\delta}{L_i^0 - 1} \sum^{L_i^0 - 1} \ln c_i^s(1) - \frac{\delta}{L_i^0} \sum^{L_i^0} \ln c_i^s(1) \geq 0$$

$$u(\{c_i^s\}, .) \text{is convex in } l_i$$

$$u(\{c_i^s\}, l_i^1) - u(\{c_i^s\}, l_i^0) = u_l(., l_i^0)(l_i^1 - l_i^0)$$

Similar arguments show,

$$u(\{c_i^s\}, l_i^2) - u(\{c_i^s\}, l_i^1) = u_l(., l_i^1)(l_i^2 - l_i^1)$$

Hence,

$$u(\{c_i^s\}, l_i^2) - u(\{c_i^s\}, l_i^0) \geq u_l(., l_i^0)(l_i^2 - l_i^0)$$

Hence, by definition 2 $u(\{c_i^s\}, l_i)$ is subdifferentiable in l_i and the subgradient is given by

$$u_l(\{c_i^s\}, l_i) = [\frac{\delta}{L_i - 1} \sum^{L_i - 1} \ln c_i^s(1) - \frac{\delta}{L_i} \sum^{L_i} \ln c_i^s(1)] \frac{\triangle L_i}{\triangle l_i}$$

Here, $\triangle L_i = L_i - (L_i - 1)$ and $\triangle l_i = L^{-1}(L_i) - L^{-1}(L_i - 1)$, where $L_i = L(l_i)$

This holds for any prices $\infty > \{p(s)\} > 0.$ Q.E.D.

Proof of Lemma 5:

Let us denote,

$$\Phi(c_i, \lambda_i, L_i, w_i^{0,1}, L_i) \equiv \ln c_i^1(0) + \sum^{L_i} \frac{1}{L_i} \ln c_i^s(1)$$

$$+\lambda_i[w_i^{0,1} + \sum^{L_i} p(s)(w_i^{1.s} - c_i^s(1)) - c_i^1(0) - L^{-1}(L_i)]$$

where, $\phi(c_i, \lambda_i, L_i, w_i^{0,1}, L_i)$ is as given in the proof of the proposition and the maximization problem (M) corresponds to our $(6')$ with L_i held as a parameter, where c_i is the vector $(c_i^1(0), (c_i^s(1)_1^{L_i})$

Now, if $\exists$ a c_i^*, λ_i^* at prices $\{p(s)\}$, such that

$$\Phi(c_i, \lambda_i^*, L_i, w_i^{0,1}, L_i) \leq \Phi(c_i^*, \lambda_i^*, L_i, w_i^{0,1}, L_i)$$

$$\leq [\Phi(c_i^*, \lambda_i, L_i, w_i^{0,1}, L_i)$$

then, $\{c_i^*\}$ is necessarily a solution to $(7')$ given $L_i, w_i^{0,1}$ and $\{p(s)\}$.
Now, for any $\lambda_i \geq 0, c_i^*$ as derived in Lemma 2 satisfies,

$$\Phi(c_i, \lambda_i^*, L_i, w_i^{0,1}, L_i) \leq \Phi(c_i^* \lambda_i^*, L_i, w_i^{0,1}, L_i)$$
$$\leq \Phi(c_i^*, \lambda_i, L_i, w_i^{0,1}, L_i)(SP*)$$

[$\because$ $f(x, \alpha)$ is concave in x and g(.) is linear in x-see foot note 3 of Anderson et.al.(1979)]

Now, denote in terms of the Proposition :

as, $x_i^0 \epsilon R^{L_i^0+1}$ the vector $(c_i^{1*}(0; L_i^0, w_i^{0,1}, L_i^0), (c_i^{s*}(1; L_i^0, w_i^{0,1}, L_i^0))_{s=1}^{L_i^0})$

as, $x_i^1 \epsilon R^{L_i^0}$ the vector $(c_i^{1*}(0; L_i^0-1, w_i^{0,1}, L_i^0-1), (c_i^{s*}(1; L_i^0-1, w_i^{0,1}, L_i^0-1))_{s=1}^{L_i^0-1})$

as, $x_i^2 \epsilon R^{L_i^0+2}$ the vector $(c_i^{1*}(0; L_i^0+1, w_i^{0,1}, L_i^0+1), (c_i^{s*}(1; L_i^0+1, w_i^{0,1}, L_i^0+1))_{s=1}^{L_i^0+1})$

as, α_i^0 the scalar L_i^0

as, α_i^1 the scalar $L_i^0 - 1$

as, α_i^2 the scalar $L_i^0 + 1$

as, β_i the scalar $w_i^{0,1}$

as, γ_i^0 the scalar L_i^0

as, γ_i^1 the scalar $L_i^0 - 1$

as, γ_i^2 the scalar $L_i^0 + 1$

as, $f(x, \alpha)$ the maximand and

as, $g(x, \beta, \gamma)$ the budget constraint in (6′).

From Lemma 4, $f(x, \alpha^0) - f(x, \alpha^1) = f_\alpha(x, \alpha^0)(\alpha^0 - \alpha^1)$, by virtue of assumptions (i), (ii) and (iii). Hence, condition (b) in the proof of the proposition is satisfied.

Also, by virtue of (SP*), condition (a) is satisfied.

Also, it can be easily verified by writing down the budget constraint that conditions (i) and (ii) of the proposition are satisfied, if (iv) holds.

Similarly, by virtue of assumptions (iv) and (v) it can be shown that conditions (iii) and(iv) of the proposition hold.

equations (1) - (4) of the proposition hold.

Now, x_i^0 is the local optimal choice for the set of parameters $(\alpha_i^0, \beta_i, \gamma_i^0)$

Iff,

$$F(\alpha_i^1, \beta_i, \gamma_i^1) - F(\alpha_i^0, \beta_i, \gamma_i^0) \leq 0(A)$$

and

$$F(\alpha_i^2, \beta_i, \gamma_i^2) - F(\alpha_i^0, \beta_i, \gamma_i^0) \leq 0(B) \quad \text{[see defn(1)]}$$

which using the upper bounds of the differences in equations (1) and (3) can be written down as :

$$f_\alpha(x_i^1, \alpha_i^1)\triangle\alpha_{i,1} - \lambda_i^0 \triangle\gamma_{i,1} x_i^1 \leq 0(A')$$

$$f_\alpha(x_i^2, \alpha_i^2)\triangle\alpha_{i,2} - \lambda_i^2 \triangle\gamma_{i,2} x_i^0 \leq 0(B')$$

and from equations (2) and (4) :

$$0 \geq \triangle\alpha_{i,1}\{-f_\alpha(x_i^1, \alpha_i^1) + f_\alpha(x_i^0, \alpha_i^0)\} + \triangle\gamma_{i,1}(\lambda_i^0 x_i^1 + \lambda_i^1 x_i^0)(C)$$

$$0 \geq \triangle\alpha_{i,2}\{-f_\alpha(x_i^2, \alpha_i^2) + f_\alpha(x_i^0, \alpha_i^0)\} + \triangle\gamma_{i,2}(\lambda_i^0 x_i^2 + \lambda_i^1 x_i^0)(D)$$

Using Lemma 2 to replace x_i^0 and Lemma 4 for f_α, I get the following four sufficient conditions.

$$\ln \prod^{L_i^0-1} \{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0-1}} - \ln \prod^{L_i^0} \{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0}}$$

$$+\lambda_i^0\{\frac{w_i^{0,1} + \sum^{L_i^0-1} p(s) w_i^{1,s} - L^{-1}(L_i^0 - 1)}{2p(s)}\} \leq 0(A*)$$

$$\ln \prod^{L_i^0+1} \{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0+1}} - \ln \prod^{L_i^0} \{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0}}$$

$$-\lambda_i^2\{\frac{w_i^{0,1} + \sum^{L_i^0} p(s) w_i^{1,s} - L^{-1}(L_i^0)}{2L_i^0}\} \leq 0(B*)$$

$$\ln \prod^{L_i^0-2} \{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0-2}} - \ln \prod^{L_i^0} \{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0}}$$

$$+\lambda_i^0\{\frac{w_i^{0,1} + \sum^{L_i^0-1} p(s) w_i^{1,s} - L^{-1}(L_i^0 - 1)}{2(L_i^0 - 1)}$$

$$+\lambda_i^1\{\frac{w_i^{0,1}+\sum^{L_i^0} p(s)w_i^{1,s}-L^{-1}(L_i^0)}{2(L_i^0)}\}\geq 0(C*)$$

$$\ln\prod^{L_i^0-1}\{\frac{1}{p(s)}\}^{\frac{\delta}{L_1^0-1}}-\ln\prod^{L_i^0+1}\{\frac{1}{p(s)}\}^{\frac{\delta}{L_i^0+1}}$$

$$+\lambda_i^0\{\frac{w_i^{0,1}+\sum^{L_i^0+1} p(s)w_i^{1,s}-L^{-1}(L_i^0+1)}{2p(s)}\}$$

$$+\lambda_i^2\{\frac{w_i^{0,1}+\sum^{L_i^0} p(s)w_i^{1,s}-L^{-1}(L_i^0)}{2p(s)}\}\leq 0(D*)$$

where, $(A*)$ holds $\forall s \leq L_i^0$ and $(D*)$ holds $\forall s \leq L_i^0$ Using the fact that $\lambda'_i\ s \geq 0$ in $(B*)$ gives the condition

$$L_i^0\quad s.t.\prod^{L_i^0}\{p(s)\}^{\delta}\leq (p(L_i^0+1))^{\delta L_i^0}$$

and using (B*) and (D*) to solve for λ_i^0 (treating them as holding with equality) and then using (A*) I have

$$L_i^0\geq 2\quad s.t.\prod^{L_i^0-1}\{p(s)\}^{\delta}\geq (p(L_i^0))^{\delta(L_i^0-1)}$$

Thus, if $\exists L_i^0 \geq 2, \quad s.t.$

$$\prod^{L_i^0}\{p(s)\}^{\delta}\leq (p(L_i^0+1))^{\delta L_i^0}$$

and,

$$\prod^{L_i^0-1}\{p(s)\}^{\delta}>(p(L_i^0))^{\delta(L_i^0-1)}$$

and conditions (i) -(v) hold, then L_i^0 is an interior optimum for agent i. Q.E.D.

Proof of Theorem :

From (7a) - (8) the contingent commodity market at $t = 0$ is imperfect at prices $\{p(s)\} > 0$ if $\exists$ at least one agent $i*$, s.t.

$$L_{i*}^0(\{p(s)\})>L_{i*}^0(\{p(s)\})\quad\forall i\neq i*$$

and,

$$c_{i*}^s(1,\{p(s)\})-w^{1,s}>0,\quad s=L_{i*}^0(\{p(s)\})$$

Since, N is a finite set of integers and (11) holds the agents can be ordered in terms of decreasing wealth by decreasing order of initial wealth levels.

Let w.l.o.g.,

$$w_N^{0,1} < w_{N-1}^{0,1} < w_{N-2}^{0,1} << w_2^{0,1} < w_1^{0,1}$$

I shall show that at prices $\{p(s)\} > 0$,

$$L_N^0(\{p(s)\}) > L_i^0(\{p(s)\})\forall i \neq N$$

and,

$$c_N^s(1, \{p(s)\}) - w^{1,s} > 0, s = L_N^0(\{p(s)\})(*)$$

Now, by the conditions of the theorem,

$$w_j^{0,1} - w_i^{0,1} > 0 \quad \forall j < i \forall 1 < i \leq N,$$

in terms of the above ordering of agents.

I shall show that this is sufficient for (*) to hold, if the conditions of Lemma 5 are satisfied, $\min_i s_i \leq N$ and (11) holds $\forall i, j \leq N$

Now,

$$L_N^0(\{p(s)\}) > L_i^0(\{p(s)\}), i \neq N$$

if,

$$\phi(c_N^*, \lambda_N, L_N^0, w_N^{0,1}, L_N^0) > (c_i^*, \lambda_N, L_i^0, w_i^{0,1}, L_i^0)$$

$$\forall \lambda_N \geq 0$$

[Notice that at c_i^* for any L_i the budget constraint in (7') is fulfilled with equality due to the strictly monotonically increasing nature of the utility function and the perfect divisibility of the consumption good, hence,

$$\phi(c_i^*, \lambda_i, L_i^0(\{p(s)\}), w_i^{0,1}, L_i^0) \equiv$$

$$V(L^{-1}(L_i^0), \{p(s)\}_i, w_i^{0,1})$$

of Lemma $2 \ \ \forall \lambda_i \geq 0]$

for any agent $i < N$ and $i + 1$,

$$\phi(c_{i+1}^*, \lambda_i, L_i^0 - 1, w_{i+1}^{0,1}, L_i^0 - 1) - \phi(c_i^*, \lambda_i, L_i^0, w_i^{0,1}, L_i^0) > 0$$

at prices $\{p(s)\}$.

(where as before $\{c_i^*\}$ is the optimal choice of consumption when choice of L_i is L_i^0 by agent i).

iff, from equation (5) of the proposition,

$$u_l(c_i*, L_i^0)(L_i^0 - 1 - L_i^0) + \lambda_i(w_{i+1}^{o,1} - w_i^{0,1}) + \lambda_i(L_i^0 - 1 - L_i^0) > 0$$

[where, $u_l(.,.)$ is given by Lemma 4]

or iff,

$$-u_l(c_i^*, L_i^0) + \lambda_i(w_{i+1}^{0,1} - w_i^{0,1}) + \lambda_i(-1) > 0$$

or iff,

$$w_{i+1}^{0,1} - w_i^{0,1} > \frac{u_l(c_i^*, L_i^0) + \lambda_i}{\lambda_i} = \frac{u_l(c_i^*, L_i^0)}{\lambda_i} + 1(A)$$

if, $u_l(.,.) \leq 0$ condition (A) is satisfied if

$$w_{i+1}^{0,1} - w_i^{0,1} > 1.$$

Now, $u_l(c_i^*, L_i^0) \leq 0 \quad u_i(\{c_i^*\}, .)$ is convex in l_i (see lemma 4)

for $\{c_i^*\}$ as derived in Lemma 2 for $L_i = L_i^0$.

$$(A) \ \text{ becomes } \prod^{L_i^0 - 1} (p(s))^{\delta/L_i^0 - 1} > (p(L_i^0))^{\delta}$$

which is fulfilled from Lemma 5, $\because$ L_i^0 is the optimal choice of L_i by i.

Now since this is true for any agent i, this is true for agents N and $N-1$.

Thus, at any prices $\{p(s)\}$

$$L_N^0(\{p(s)\}) > L_i^0(\{p(s)\}) \forall i \neq N$$

(by induction)

Now, from assumptions (3a), (6d) and condition (ii) of Lemma 5, $L_i^0 \geq 2 \forall i$. hence, $L_N^0(\{p(s)\}) > N$. Also, $N : \bar{s}_N = \min_i \bar{s}_i$ Hence, $w^{1, L_N^0(\{p(s)\})} = 0$ for agent $N, \bar{s}_N \leq N$.

Now, $c_N^{s*}(1, \{p(s)\}) = 0$ for $s = L_N^0(\{p(s)\})$ is not possible.

since, $V(L^{-1}(L_N^0), \{p(s)\}, w_N^{0,1}) = -\infty$ if $c_N^{s*}(1, \{p(s)\}) = 0$

and, $-\infty < V(L^{-1}(\overline{s}_N), \{p(s)\}, w_N^{0,1})$,
since by virtue of assumption (iv) Lemma 5,
$\chi(L^{-1}(\overline{s}_N)) > 0$ for agent N, hence from Lemma 2,
$c_N^{s*}(L^{-1}\ (\overline{s}_N), \{p(s)\}, w_N^{0,1}) > 0$.

Hence, $c_N^{s*}(1, \{p(s)\}) > 0$ for $s = L_N^0(\{p(s)\})$ This implies, $c_N^{s*}(1, \{p(s)\}) - w^{1.s} > 0$ for $s = L_N^0(\{p(s)\})$.

Thus, the two parts of condition (*) of this Theorem are proved for agent N.

Hence the theorem. Q.E.D.

REFERENCES

ANDERSON, R.K. & A. TAKAYAMA : Comparative Statics with Discrete Jumps in Shift Parameters, or, How to do Economics on the Saddle (-Point), *Journal of Economic Theory, 21, 491-509, 1979.*

DIXIT, A. & R.S. PINDYCK : Investment under Uncertainty , Princeton University Press, Princeton, 1994.

DREZE, J.H. : (Uncertainty and) the firm in General Equilibrium Theory , *Economic Journal (Supplement), 1-20, 1985.*

DUBEY, P., J. GEANOKOPLOS and M. SHUBIK : Default and Punishment in General Equilibrium , Cowles Foundation Discussion Paper 1304, Yale University, 2001.

GENNARI, E., R. GIORDANO & S. MOMIGLIANO : Optimal debt maturity under EMU, Bank of Italy, 2002.

HAHN, F. : Excess Capacity and Imperfect Competition, ch. 19 in F. Hahn : Money, Growth and Stability, Cambridge University Press, Cambridge, 1985.

HART, O.D. : Perfect Competition and Optimal Product Differentiation, *Journal of Economic Theory, 22, 279-312, 1980.*

KIM, T. : Differentiability of the Value Function : A new characterization, *Seoul Journal of Economics, 6, 257-265, 1993.*

MAKOWSKI, L. : Perfect Competition, the Profit Criterion, and the Organization of Economic Activity, *Journal of Economic Theory, 22, 222-242, 1980.*

MILGROM, P. & I. SEGAL : Envelope theorems for arbitrary choice sets, *Econometrica, 70, 583-601, 2002.*

ROCKAFELLER, R.T. : Convex Analysis, Princeton University Press, Princeton, N.J., 1970.

RUDIN,W. : Principles of Mathematical Analysis, Mc. Graw Hill Book Co., New York, 3rd edn., 1976.

SAH, R. & L. ZHAO : Some Envelope Theorems for Integer and Discrete Choice Variables, *International Economic Review, 39, 623-634, 1998.*

View publication stats