Robust Mathematical Formulation and Implementation of Agent-Based Computational Economic Market Models

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Abstract

Monte Carlo Simulations of agent-based models in science and especially in the economic literature have become a widely used modeling approach. In many applications the number of agents is huge and the models are formulated as a large system of difference equations. In this study we discuss four numerical aspects which we present exemplified by two agent-based computational economic market models; the Levy-Levy-Solomon model and the Franke-Westerhoff model. First, we discuss **finite-size effects** present in the Levy-Levy-Solomon model and show that this behavior originates from the scaling within the model. Secondly, we discuss the impact of a low-quality **random number generator** on the simulation output. Furthermore, we discuss the **continuous formulation** of difference equations and the impact on the model behavior. Finally, we show that a continuous formulation makes it possible to employ correct **numerical solvers** in order to obtain correct simulation results.

We conclude that it is of immanent importance to simulate the model with a large number of agents in order to exclude finite-size effects and to use a well tested pseudo random number generator. Furthermore, we argue that a continuous formulation of agent-based models is advantageous since it allows the application of proper numerical methods and it admits a unique continuum limit.

Keywords: agent-based models, Monte Carlo simulations, finite size effects, time scaling, continuous formulation, continuous limit, random number generator, numerical solver

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1 Introduction

In the past decade Monte Carlo simulations of agent-based models have become a highly utilized simulation and modeling tool in various disciplines [23]. Starting from rather classical applications in physics, agent-based simulations have been effectively applied in the context of engineering, life sciences and socio-economic applications. The popularity of this approach is due to the simplicity of studying statistical quantities of rather complex microscopic dynamics. Furthermore, the increasing computational power and the perfect parallel scaling of Monte Carlo simulations on large computer systems has enabled scientist to study larger and more complex agent models.

Monte Carlos simulations originate from statistical physics and have been first applied to particle transport by Enrico Fermi in the 1930s and in the 1940s by Stanislaw Ulam and John von Neumann. The Monte Carlo method has been further developed by Metropolis and Rosenbluth in the 1950s. Early examples of Monte Carlo simulations of agent-based models beyond the field of statistical physics are [53, 35] in life sciences and [17, 55, 58] in socio-economics.

In this work we focus on agent-based computational economic market (ABCEM) models. The model by Stigler [58] may be seen as the first ABCEM model but generally the model by Kim and Markowitz [38, 54] is referred to as the first modern ABCEM model. ABCEM models are a notable class in the research field econophysics. The general goal of ABCEM models is to reproduce persistent statistical features present in financial data all over the world which are known as stylized facts. Possible research questions are: evaluate the kind of stylized facts microscopic agent dynamics create; and estimate the impact of regulations on a financial market. Thanks to Monte Carlo simulations, it is possible to study the time evolution of statistical quantities such as the wealth distribution or the stock return distribution. The importance of agent-based modeling in economics has been emphasized by several authors [24, 33, 59].

Many ABCEM models are afflicted by numerical problems [22, 68, 40, 32]. These works document finite-size effects in several ABCEM models, i.e. these models show the desired effects for small numbers of agents (e.g. in the order of 100 agents) but effects of a different nature for larger numbers of agents (e.g. in the order of 10,000 agents). Obviously, this is an undesirable characteristic as one expects the quality of the results to improve for an increasing number of agents. In addition, the sensitivity of many ABCEM models w.r.t. their parameters, observed in a tendency towards blow ups for particular choices of model parameters, motivated this work. The deeper study of these phenomena has revealed that this behavior often originates from the time stepping scheme of difference equations which can be viewed as explicit Euler discretizations with a normalized time step of one. It is a well known property of the explicit Euler method to perform poorly in the case of stiff differential equations. For this reason we believe that a time continuous formulation of difference equations helps understanding several numerical issues. Albeit we have to recognize that starting from a difference equation, the continuous formulation, respectively the continuum limit, is not uniquely defined. To our knowledge, this approach is the first work within the ABCEM community addressing this important issue.

For these reasons, the goal of this work is to give a rather general overview of numerical aspects in ABCEM models and to present strategies for a robust mathematical formulation of ABCEM models. Clearly these numerical aspects apply to a wider class of agent-based models. In this work we analyze the following numerical issues:

1. **Finite-size effects**: The existence of finite-size effects in ABCEM models has been documented in several references [22, 68, 40, 32]. In order to test each ABCEM model with respect to finite-size effects, we recommend to perform simulations with a large number of agents.

2. **Pseudo random number generator**: Many ABCEM models require large amounts of pseudo random numbers. We show in this work that a low quality pseudo random number generator may drastically influence the model output.

3. **Continuous formulation and limit**: We discuss the connections of difference equations and differential equations. In particular, we emphasize that the continuum formulation of a difference equation is not unique. In addition, we claim that a continuous models e.g. ordinary differential equations have several advantages in comparison to difference equations.

4. **Numerical solver**: Examples of a numerical solvers are particular choices of discretization schemes of differential equations or root finding algorithms. We highlight that the choice of numerical solvers is paramount, e.g. for the solution of stiff ordinary differential equations either explicit methods with sufficiently small timesteps or implicit methods are required.

One may summarize the numerical aspects 1. and 2. as a question of implementation. In their way, both of these issues are dependent on the implementation. Finite size effects are inherent to a specific model and thus an efficient, fast simulator is required for simulations allowing for multiple simulations sufficiently large numbers of agents (up to 10^6 agents and more), in acceptable time, in order to thoroughly test a model for existence of finite size effects. Contrary to this, the choice of pseudo random number generators is not directly dictated by the model as all models assume the random numbers to be truly random. Hence, the choice of a high-quality pseudo random number generator is left to the implementation alone. We perform our simulations with the recently introduced SABCEMM simulator [63]. This simulator is implemented in C++, is designed for large scale simulations and supports several well tested pseudo random number generators.

The further aspects are rather a question of modeling. Difference equations are able to model a time evolution of quantities with a fixed time lag. In contrast, a time continuous model, e.g. an ordinary differential equation or stochastic differential equation, may be discretized using various time steps and is thus able to model the same quantity using different time scales. Any difference equations can be interpreted as a scaled explicit Euler discretization of a time continuous differential equation. Starting from a difference equation, the time continuous formulation is not unique. Furthermore, the advantage of a time continuous formulation is that it may be used to explain instabilities on the level of the difference equations and thus to guide the choice of appropriate discretization schemes. Finally, we emphasize that a time continuous dynamical system may be translated to mesoscopic descriptions modeled using partial differential equations [52]. This limit process leading from microscopic dynamics to a mesoscopic description is at the heart of kinetic theory which has been successfully applied to several ABCEM models in the past [61, 64, 49]. Moreover, we would like to stress that not only the discretization scheme applied to a differential equations but any numerical solver used, e.g. a root finding algorithm itself along equipped with a particular stopping criterion, may have a huge impact on the model behavior.

In this study we exemplarily discuss these numerical and mathematical aspects using the example of the Levy-Levy-Solomon model [42] and Franke-Westerhoff model [26]. The Levy-Levy-Solomon model is one of the most influential ABCEM models and an early example of an ABCEM model in general [54]. The Levy-Levy-Solomon model considers the wealth evolution of agents and the stock price evolution. Furthermore, each agent has to decide in each time step on the optimal asset allocation between the stocks and the asset class bonds. The stock price is fixed in each time step by the clearance mechanism that perfectly matches supply and demand and can be consequently seen as a rational market. The authors claimed that their model is able to reproduce several stylized facts form financial markets such as fat-tails in asset returns. In fact, it has been documented in [68] that the stock price returns are normally distributed and that the model exhibits finite-size effects. In comparison to the Levy-Levy-Solomon model, the Franke-Westerhoff model is rather recent and has been first introduced in 2009 [26]. The model tracks the time evolution of two agent groups and not of an individual agent as in the Levy-Levy-Solomon model. The stock price is modeled as a stochastic difference equation and is thus modeled as a disequilibrium model. The Franke-Westerhoff model is fully described by a system of three difference equations. It has been documented that the Franke-Westerhoff model is able to reproduce several stylized facts. The reason to choose the Franke-Westerhoff model in this study is on the one hand the simplicity of this model and on the other hand the prototypical nature of this model especially w.r.t. the stock price update mechanism. We have selected the Levy-Levy-Solomon model in this work not only because of the popularity and the existence of finite-size effects but as well in virtue of the pricing mechanism which is modeled as a rational market.

The outline of the paper is as follows. In the next section, we provide a short introduction to the SABCEMM simulator [63] and present the supported pseudo random number generators. Secondly, we discuss the connection of difference equations and time continuous differential equations in the context of a disequilibrium financial market models. In section three, we present all four numerical issues by means of simulation results of the Levy-Levy-Solomon model or Franke-Westerhoff model. Using the example of the Levy-Levy-Solomon model, we discuss finite-size effects and the impact of bad pseudo random number generators. In addition, we present continuous formulations and continuous limits of both models. We specifically discuss the impact of different time scalings in the Levy-Levy-Solomon model. Furthermore, we show that a naive numerical discretization of the time continuous Franke-Westerhoff model leads to blow ups. Therefore, we introduce a semi-implicit discretization of the time continuous Franke-Westerhoff model and show that the qualitative output coincides with the original model. Furthermore, we demonstrate that the stopping criterion in the root finding algorithms of the clearance mechanism in the Levy-Levy-Solomon model has a huge impact on the model output. We finish this with a short conclusion.

2 Methodology

The four different numerical aspects discussed in this work can be classified in two groups: those concerning to implementation and those concerning to numerics. In the first group, we consider the aspects finite size effects and random numbers. Both aspects can be regarded as a question of implementation. Finite size effects can only be observed using a sufficiently large number of agents – requiring a suitable simulation tool – but cannot be avoided by the mere choice of the simulation tool. Thus, a sensible choice of simulation tool has to be made. For our simulations we employ the recently introduced SABCEMM simulator [63] which is well suited for large-scale simulations and supports several reliable random number generators. In the next paragraph we will provide a short introduction to the SABCEMM simulator.
In the second group of numerical issues we classify the numerical method and the continuum limit. These aspects cannot be resolved by the correct choice of simulator since it is rather deeply connected to each specific model and the corresponding application of the ABCEM model. More precisely we introduce the connections of ABCEM models to time continuous dynamical systems. Especially we aim to explain the advantages of time continuous models in comparison to difference equations.

2.1 The Simulator SABCEMM

The SABCEMM simulator has been recently introduced and published under an open source license by the authors [63, 60]. The tool is designed for large-scale simulations for ABCEM models and is able to simulate models with several million agents as shown in [5]. SABCEMM is implemented in C++ and employs an object oriented design. This enables the user to create new models by plugging together novel and existing agent and market designs as easily as plugging together pieces of a puzzle. Furthermore, the user is able to choose between different well tested pseudo random number generators. The distinct features of SABCEMM in comparison to other simulation tools such as the JAMEL or JASA simulators [1] are:

1. **Generality**: It is built on the basis of a generalized ABCEM model, suitable for implementation of a very wide spectrum of ABCEM models. Thus, the simulator supports ABCEM models formulated as difference equations as well as time continuous dynamical system.

2. **Recombination**: It allows recombination of the building blocks of different ABCEM models via configuration files for evaluation of novel ABCEM models.

3. **Comparability**: It provides a common foundation, including random number generators, for fair comparisons of different ABCEM models.

4. **Extensibility**: Implementation of additional ABCEM models is facilitated by object-oriented design.

5. **Efficiency**: Suitability for simulations with a large number ($> 10^6$) of agents, allowing for testing for finite size effects.

In the following, we present the main conceptional ideas behind the simulator and present in detail the available pseudo random number generators.

SABCEMM is well suited for any economic market model which consists of at least one *agent* and one *market mechanism*. As an agent we understand an investor who has a supply of or demand for a certain good or asset, which is traded at the market. Thus, a market mechanism has to determine the price by the demand and supply of all market participants. More precisely, we differentiate between the so called *price adjustment process* and the *excess demand calculator*. The latter one aggregates the supply and demand of all market participants (agents) to one quantity, the aggregated excess demand. The former one is rather the method of how the market price is fixed based on this excess demand. In the SABCEMM simulator the aggregated excess demand ED of an ABCEM model of N agents is defined as the mean value of agents' excess demands $ed_i \in \mathbb{R}$.

$$ED(t_k) := \frac{1}{N} \sum_{i=1}^{N} ed_i(t_k).$$

For an economic definition of aggregated excess demand we refer to [50, 20, 56]. In the following we present examples of agent's excess demand. In fact many ABCEMM models such as [11, 6, 34, 33, 45, 26] only consider two agents.

Example 1. *Frequently used financial agents in ABCEM models are fundamentalist and chartists [46, 33]. A fundamental agent believes that to every stock or good there is a fair market price (or fundamental value) $P^F > 0$ and that the market price will converge to this fundamental value. Hence, a fundamentalist wants to sell stocks if the stock price is above the fundamental value and he buys stocks if the market price is below the fundamental value. Hence the agent's demand can be calculated as follows*

$$ed^F(t_k) := a\ (P^F(t_k) - P(t_k)),$$

for a positive weight $a > 0$ and a logarithmic stock price $P(t_k) \in \mathbb{R}$ with the discretized time $t_k := t_0 + k\ \Delta t,\ \ \Delta t > 0\ k \in \mathbb{N}$. Here, $t_0 > 0$ denotes the initial time. A chartist assumes that the future stock return is best approximated by extrapolating past returns. One simple example of such an investor is

$$ed^C(t_k) := b\ (P(t_k) - P(t_{k-1})),$$

for a positive weight $b > 0$. Examples of such agents can be found in the models [11, 6, 34, 33, 45, 26].

In the previous example we presented two possible excess demands of two financial agents. In the following we provide an example of how these excess demands can be aggregated into the aggregated excess demand.

Example 2. *A simple example is the Franke-Westerhoff model [26, 27, 28] where the population of agents is fully described by two agents ($N = 2$), a chartist and a fundamentalist. Hence, the excess demand is defined as*

$$ED^{FW}(t_k) = \frac{1}{2}\left(ed_C^{FW}(t_k) + ed_F^{FW}(t_k)\right).$$

In comparison to the general chartist and fundamental demand defined in example 1 the Franke-Westerhoff models considers time dependent weights a, b and additionally there is a random variable added. For details we refer to the appendix A.2.

Then the price adjustment process uses the aggregated excess demand in order to fix the price. The price adjustment process is often given by an iterative update rule and can be interpreted as a discretized ODE [63]. Such a model can be interpreted as an irrational market i.e. the price is not defined as the argument of the root of the aggregated excess demand. Further details on the numerical interpretation of such price adjustment processes will be provided in the next section. As alternative one may fix the price such that the aggregated excess demand is zero in each time step, which corresponds to a rational market. For a rigorous definition of the meta-model which is the skeletal structur of the SABCEMM simulator we refer to [63]. In order to illustrate the presented ideas we refer to figure 1. The only aspect

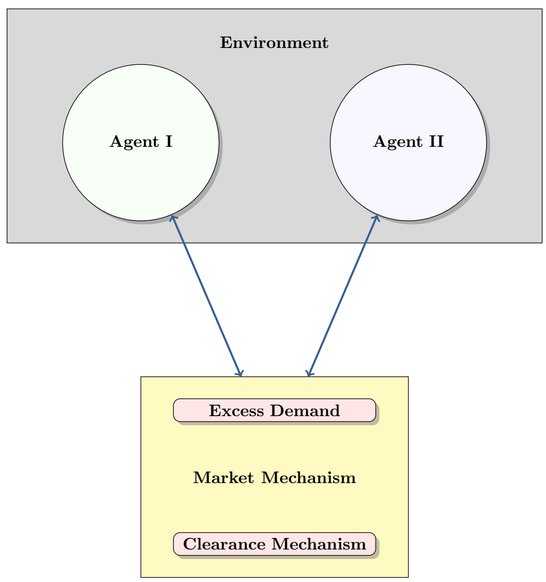


Figure 1: Schematic picture of the abstract ABCEMM model [64].

of our generalized ABCEM model we did not yet discussed is the concept of an *environment*. This concept has been first introduced by the authors [64] and needs to be explained. An environment subsumes any additional coupling, besides of the coupling via the stock price, between the agents. The most famous example possibly is herding, which is frequently used in ABCEM models. We emphasize that such an environment is not mandatory in order to use SABCEMM. For details regarding the definition of an environment we refer to [63]. After having discussed the conceptional ideas behind SABCEMM, we briefly present the possible choices of pseudo random number generators in SABCEMM.

In SABCEMM we provide two pseudo random number generators (RNG) based on the widely popular Mersenne Twister family of RNGs. First, we include the `MT19937` (64-bit) Mersenne Twister, introduced into the C++ standard library with the C++11 standard. `MT19937` provides an extremely long period of $2^{19937} - 1 \approx 10^{6001.6}$ and particularly good equidistribution (623 dimensions of approximate equidistribution [4]). Secondly, we support the `MT2203` (64-bit) multi-stream Mersenne twister implemented within the Intel Math Kernel Library (MKL)[1] providing a period of $2^{2203} - 1 \approx 10^{663.2}$, very good equidistribution (68 dimensions of approximate equidistribution [4]) and 6024 parallel streams. Note that `MT2203` is included as it outperforms the `MT19937` implementation within the C++11 standard library by far when combined with pooling of generated random numbers. For further details on pooling of random numbers within SABCEMM, the interested reader is refered to [63]. Since `MT19937` and `MT2203` fail only two and four tests the BigCrush battery of statistical tests, respectively [41], we accept these highly popular RNGs as high-quality RNGs. In comparison, in section 3.2, we show that the low-quality RNG `RANDU` [36], failing already 14 tests within the SmallCrush and 125 tests within the Crush battieries [41], is of insufficient quality for application within ABCEM models. RNGs within SABCEMM are seed from the `/dev/urandom` device on Linux/Unix systems. If the device is not available the current time stamp[2] is used.

2.2 Mathematical Perspective on ABCEM Models

As mentioned before, ABCEM models may be interpreted as discretized dynamical systems. Many models in literature neglect the time dependence respectively normalize the time step to one [26, 30, 34]. Then in fact, many models are rather formulated by difference equations. In this section, we lay out the connection between difference equations and discretized differential equations. Furthermore, we introduce a rather general model for an irrational market which is broadly used in literature. Finally, we give a short outlook on possible numerical discretization strategies and discuss the advantage of discretized differential equations in comparison to difference equations.

In genaral one might refer to a market as irrational market if the fixed stock price in each time step does not clear all buy or sell orders. This corresponds to the situation that the aggregated excess demand is non-zero. An early example of an irrational market or disequilibrium model is the Beja and Goldman model [6]. In fact, the Beja and Goldman model can be derived from a rational market model where supply equals demand. Mathematically, such a differential model can be seen as a relaxation of the algebraic demand supply relation. For detailed discussion of the Beja and Goldman model we refer to [6, 63]. In order to be more explicit we give an example of the price adjustment rule present in many models e.g. [18, 2, 45, 9]

$$S(t_{k+1}) = S(t_k) + \Delta t \; ED(t_k). \tag{1}$$

Often the time step Δt is normalized to one such that we are faced with a difference equation. The previous update rule (1) can be interpreted as an explicit Euler discretization of the ODE

$$\frac{d}{dt} S = ED,$$

provided that the aggregated excess demand is deterministic. In the case of the Franke-Westerhoff model [28], the aggregated excess demand is stochastic and thus this pricing

[1] `https://software.intel.com/en-us/mkl` (may require extra license)

[2] `std::chrono::high_resolution_clock::now()`

mechanism cannot be seen as an approximation of an ODE. The pricing rule of the Franke-Westerhoff model can be interpreted as discretized stochastic differential equation (SDE). For details on this specific case we refer to appendix A.2. A very general model of an irrational market has been introduced by Trimborn et al. in [63]. It is given by the following SDE

$$dS = F(S, ED)\ dt + G(S, ED)\ dW, \tag{2}$$

with Wiener process W and arbitrary functions F and G. Notice, that (1) is a special case of the model (2). We use the usual notation for Itô stochastic differential equations. Many market mechanism of ABCEM models are special cases of model (2), for example the models presented in [18, 2, 45, 9, 10, 11, 12, 8, 67, 3, 30, 57, 37, 51, 7, 13, 15, 16, 14, 21, 25, 47, 48, 19]. The simplest discretization of such an SDE (3) is the Euler-Maruyama method. Applying the Euler-Maruyama method to equation (2), we obtain:

$$S(t + \Delta t) = S(t) + \Delta t\ F(S(t), ED(t)) + \sqrt{\Delta t}\ G(S(t), ED(t))\ \eta, \quad \eta \sim \mathcal{N}(0, 1) \tag{3}$$

for a fixed time $t > 0$ and time step $\Delta t > 0$. In the case of a fully deterministic model, the numerical scheme (3) is identical to the standard Euler method.
From a mathematical perspective, we stress that more sophisticated numerical methods for equation (2) exist, which may improve the quality of approximation remarkably [29, 65]. In particular for the case of stiff SDEs or ODEs, one should use implicit solvers to prevent stability problems.

In the ABCEM literature, most models rely on the explicit Euler (in case of deterministic dynamics) or Euler-Maruyama (in case of stochastic dynamics) discretizations. Often, the numerical approximation is rescaled and fixed such that the time step is set to one. Hence, in ABCEM literature, we are rather faced with difference equations of the following type

$$S_{k+1} = S_k + \bar{F}(S_k, ED_k) + \bar{G}(S_k, ED_k)\ \eta, \tag{4}$$

than with differential equations. The model (4) represents a discretized version of the model (2) with discretizations $\bar{F}, \bar{G}$ of functions F, G. Here $k \in \mathbb{N}$ is the index of the discretized time steps $S_k = S(t + k\ \Delta t)$ for a fixed initial time t and time step $\Delta t > 0$. Finally, we would like to stress that a time continuous modes is not only advantageous from a numerical perspective but enables the user to simulate the model on differently coarse time levels by only adapting the time step in the numerical scheme.

3 Numerical Aspects

In this section we discuss the robust formulation of ABCEM models with help of the Levy-Levy-Solomon model and the Franke-Westerhoff model. As pointed out in the introduction are both models structurally very different. The LLS model considers N agents whereas the Franke-Westerhoff model can be regarded as a two agent model.
We discuss the questions of implementation first, namely finite-size effects and the impact of a bad pseudo random number generator on the simulation output. This is performed exemplarily with the LLS model. Then we introduce possible continuous formulations of the LLS and FW model. Furthermore, we show the impact of different continuous formulations on the wealth evolution in the LLS model. Finally, we discuss the impact of different numerical methods on the model behavior of the FW model. In addition, we show that the termination condition in the roots finding algorithm of the LLS model has a huge impact on the model output.

All presented results have been generated with the SABCEMM simulator which is freely available on GitHub [60]. For the used pseudo random number generators used in obtaining the results, please confer to table 4. Furthermore, the simulation data is published [62] such that the reader can reproduce the presented results.

3.1 Finite Size Effects

Finite size effects in ABCEM models have been documented by several authors [22, 32], in particular for the LLS model [68, 40]. We first show the finite size effects and secondly discover the reasons for that behavior.

In our simulations, we observe two different effects caused by different number of agents. First, we obtain that the tail behavior of the wealth distribution changes for increasing numbers of agents. Secondly, we find that dependent on the number of agents, different agent types (differing w.r.t. their memory mechanism) dominate the market. A population is considered dominant if it attains maximum wealth. Levy at al. [44] claimed that the investor group with a higher memory always dominates a group with a smaller memory. We have conducted simulations with 99 agents (33 short memory, 33 medium memory, 33 large memory, for details we refer to appendix A.3) and with 999 (333 short memory, 333 medium memory, 333 long memory, detail in appendix A.3) agents, respectively. The aggregated wealth evolution of the three agent groups (figure 2) shows that the group ranking changes for different number of agents. Furthermore, the quantile-quantile plots in figure 2 clearly indicate the change in tail behavior for different number of agents. Thus, we can conclude that the qualitative output of the model changes with respect to the number of agents. We highlight that this is an utmost undesirable model characteristic.

We aim to understand why changes of the agent number lead to changes in the model behavior. Before discussing the appearance of finite size effects, we need to understand the origins of the *usual* oscillatory model behavior. Studies by Levy, Levy and Solomon [43], Zschischang and Lux [68], and the authors of [5] emphasize the importance of the white noise on the characteristic oscillatory model behavior. In fact, in the zero noise case the oscillatory model characteristic completely vanishes.

As shown in [5] the wave period of the wealth evolution heavily depends on the noise level. Clearly, in the case of no noise we have no oscillatory behavior since the price is in equilibrium, which is computed by the clearance mechanism:

$$n = \sum_{i=1}^{N} n_i(t) = \sum_{i=1}^{N} \frac{w_i(t)}{S(t)} \gamma_i(t).$$

In the noisy case one adds to every optimal γ_i^* a noise term: $\gamma_i = \gamma_i^* + \epsilon_i$, where ϵ is a truncated Gaussian random variable. Then finally one can update the final stock price by an explicit computation since the investment fraction γ_i is constant. For details we refer to (A.2). In fact the stock price is computed as a quantity proportional to

$$S(t) \propto \frac{1}{N} \sum_{i=1}^{N} (\gamma_i^*(t) + \epsilon_i),$$

since the number of stocks n scales with the number of agents. In order to quantify the influence of the noise we define the difference of investment fractions before and after noise application:

$$d_\gamma^N(t) := \Big| \frac{1}{N} \sum_{i=1}^{N} (\gamma_i^*(t) + \epsilon_i) - \frac{1}{N} \sum_{i=1}^{N} \gamma_i^*(t) \Big|.$$

Notice that the difference is not additive due to the additional cutoff of the investment fraction which ensures $\gamma_i \in [0.01, 0.99]$. Figure 3 depicts average difference of investment fractions d_γ^N and the stock price for different number of agents. The simulation results clearly indicate that the variance of d_γ^N decreases for increasing number of agents. Especially, we are able to deduce that large deviations of the mean of the d_γ^N are needed in order to obtain the typical oscillatory behavior. Hence, we can conclude that the increasing number of agents reduces the variance of fluctuations caused by additional noising in the Levy-Levy-Solomon model and heavily influences the stock price behavior. Therefore we claim that this scaling behavior with respect to the number of agents causes the finite size effects in the Levy-Levy-Solomon model.

3.2 Pseudo Random Numbers

Generally, as discussed in [63], ABCEM models require the generation of large numbers, commonly in the order of several million, of high-quality pseudo random numbers. Furthermore, it is well documented that many ABCEM models are sensitive with respect to the precise noise level [5], i.e. the variance of the pseudo random numbers used. Thus, the question whether low-quality pseudo random numbers influence the qualitative results of ABCEM models is legit. We answer this question by examplarily applying the well-known linear congruential pseudo randon number generator `RANDU`, known for poor performance when used for the generation many pseudo random numbers [31, 39], to the LLS model. We compare an implementation of `RANDU` on a processor implementing the ARMv7 32 bit architecture to the C++11 standard pseudo random number generator on processors implementing the Intel 64 (also named AMD64 and x86_64) 64 bit architecture. Figure 4 reveals that the change of pseudo random number generator drastically changes the qualitative model output. Thus, we may conclude that models sensitive to the choice of random variable, generate different model outputs with respect to different pseudo random number generators. Therefore we emphasize that we explicitly do not recommend using the `RANDU` generator. Due to this, the published version of SABCEMM does not support the `RANDU` generator as we have only used this generator for this special test.

3.3 Continuum Limit

In this section, we introduce time continuous versions of the LLS and Franke-Westerhoff model. As usual in the ABCEM literature, the models are originally formulated as difference equations. Obviously, such a continuum limit is not uniquely defined and, in the case of the LLS model, we discuss several different time discretizations in the agent dynamic. Thus, we focus on the impact of different time discretizations in the LLS model and derive the continuum limit of the Franke-Weserhoff model which exhibits stability problems in the case of an explicit Euler discretization. A detailed discussion of the continuum version of the LLS and Franke-Westerhoff model can be found in the appendix A.2.

The LLS Model In the following, we introduce a time scale respectively the time step $\Delta t > 0$ in order to perform the continuum limit in a second step. The time discretized version of the wealth evolution is given by

$$w(t+\Delta t) = w(t) + \Delta t \left[(1-\gamma(t))\ r + \gamma(t)\ \frac{\frac{S(t+\Delta t)-S(t)}{\Delta t} + D(t)}{S(t)} \right]\ w(t). \tag{5}$$

For a proper definition of all parameters and functions we refer to the appendix A.2. Notice that the bond return r and the stock return $\frac{\frac{S(t)-S(t-\Delta t)}{\Delta t}+D(t)}{S(t-\Delta t)}$ are rates and thus scale in time. Equation (5) represents an explicit Euler discretization of the ODE:

$$\frac{d}{dt} w(t) = \left[(1-\gamma(t))\ r + \gamma(t)\ \frac{\frac{d}{dt}S(t) + D(t)}{S(t)} \right]\ w(t).$$

In order to study the time continuous version of the model, we need to properly define the time scaling of the investor. We want to emphasize that several reasonable time scales exist. First, we study the case when the memory variable m_i scales with time, which means: $\bar{m}_i := \frac{m_i}{\Delta t}$. The results for different time steps using an explicit Euler discretization can be seen in figure 5.

As pointed out in [5],approximately 90% of the optimal investment decisions γ_i in the original model are located at the boundaries of the interval in $[0.01, 0.99]$. Interestingly, in the case of the previously introduced time scaling of the memory variable, this model characteristic changes. For sufficiently small Δt the optimal investment decisions (γ_i) are all located in $(0.01, 0.99)$ and not at the boundaries. This can be explained by the very small optimization horizon and the smoothing effect of a large return history. For $\Delta t = 0.1$ the percentage of extreme decisions reduces to 72% and for $\Delta t = 0.01$ all optimal investment decisions are located in the interior. Note that these statements are based on the average of the results of over 100 runs.

Alternatively, we may assume that the investor's memory does not scale with time, i.e. the number of time steps which corresponds to the agents' memory is always constant. Simulations show that using a non-scaling memory, i.e. with a memory of a fixed number of time steps, oscillating prices for an explicit Euler discretization can be observed for all chosen timesteps (see fig. 6). Averaging over 100 runs also indicates that the percentage of extreme decisions remain approximately around 90% for any chosen time discretization. The possibility to study further scales of the LLS model is left for future research.

The Franke-Westerhoff Model As before for the LLS model, one may rescale the Franke-Westerhoff Model to time steps $\Delta t > 0$ in the agents' dynamics in the following way, as first step towards a time-continuous model:

$$\begin{aligned} n^f(t+\Delta t) &= n^f(t) + \Delta t\ n^c(t)\pi^{cf}(a(t)) - \Delta t\ n^f(t)\pi^{fc}(a(t)) \\ n^c(t+\Delta t) &= n^c(t) + \Delta t\ n^f(t)\pi^{fc}(a(t)) - \Delta t\ n^c(t)\pi^{cf}(a(t)) \end{aligned} \tag{6}$$

Thus, the limit equations of the previous dynamics are given by the following ODEs.

$$\begin{aligned} \frac{d}{dt}n^f(t) &= n^c(t)\pi^{cf}(a(t)) - n^f(t)\pi^{fc}(a(t)) \\ \frac{d}{dt}n^c(t) &= n^f(t)\pi^{fc}(a(t)) - n^c(t)\pi^{cf}(a(t)) \end{aligned} \tag{7}$$

For the stock price dynamics we introduce the following time scaling

$$P(t) = P(t-\Delta t) + \mu\ \Delta t\ ED^{FW}(t) + \sqrt{\Delta t}\ \mu\ (\sigma_f + \sigma_c)\ \eta, \quad \eta \sim \mathcal{N}(0,1). \tag{8}$$

For a detailed definition of ED^{FW} we refer to appendix A.2. Equation (8) represents an Euler-Maruyama discretization of the stochastic differential equation (SDE):

$$dP = \mu\ ED^{FW}(t)\ dt + \mu(\sigma_f + \sigma_c)\ dW$$

The SDE is interpreted in the Itô sense and the usual notation for SDEs is employed. Hence, the time continuous Franke-Westerhoff model reads:

$$\begin{aligned} dP(t) &= \mu\ ED^{FW}(t)\ dt + \mu(\sigma_f + \sigma_c)\ dW \\ \frac{d}{dt}n^f(t) &= n^c(t)\pi^{cf}(a(t)) - n^f(t)\pi^{fc}(a(t)) \\ \frac{d}{dt}n^c(t) &= n^f(t)\pi^{fc}(a(t)) - n^c(t)\pi^{cf}(a(t)) \end{aligned} \tag{9}$$

Clearly, the equations (8)- (6) can be interpreted as an explicit Euler discretization of the ODE-SDE system. For a detailed introduction to the Franke-Westerhoff model, we refer to the appendix (A.2). In order to clarify the validity of the derived continuum limit, we perform numerical tests.

We first run the model with the parameters defined in [28] and choose the time step to Δt to be $\Delta t = 1$. The qualitative results are identical to the original model (see fig. 7). If we change the noise level of the fundamentalist to $\sigma_f = 1.15$, we obtain a blow up of the dynamics. By blow up we mean that the numerical solution of our equation tends to infinity at finite time. This is an undesirable model characteristic since a minor change in the model parameters has led to an unfeasible model.
This is expected as the only difference is the missing additional constraint (10) for the switching probabilities (cf. Remark 1 in appendix A.2).
If we change the noise level of the fundamentalist we obtain a blow up of the dynamics (see fig. 8). This is a undesirable model characteristic since a minor change in the model parameters has led to an unfeasible model.
One might expect that the large time step $\Delta t = 1$ may be the reason for the numerical instability. The fig. 9 reveals that we still obtain a blow up even for the time step $\Delta t = 0.1$.

The reason for the blow up are visible in fig. 10. The values for $n^f(t)$ and $n^c(t)$ leave the interval of $[0,1]$ while preserving the relation $n^f(t) + n^c(t) = 1$. Subsequently this leads to a failure in the price calculation.
It appears Franke and Westerhoff have been aware of this model behavior since they have stated the following additional constraint in [27, 28]

$$\begin{aligned} \pi^{cf}(a(t_{k-1})) &= \min\{1, \nu \exp(a(t_{k-1}))\}, \\ \pi^{fc}(a(t_{k-1}) &= \min\{1, \nu \exp(-a(t_{k-1}))\}, \end{aligned} \tag{10}$$

to their original model [26] introduced in 2009. This additional constraint clearly guarantees the bounds $[0,1]$ of the fractions of chartists and fundamentalists n^f, n^c. Thus, this additional constraint prevents the dynamics from blowing up. We will show in the next section 3.4 that this constraint can be rendered redundant by applying a superior numerical solver.

3.4 Numerical Solver

In this section we discuss the impact of different numerical solvers on the model output in the LLS and Franke-Westerhoff model. More precisely, we study the impact of the stopping criterion applied in the root finding algorithm of the LLS model. This is employed in the clearance mechanism which fixes the stock price in each time step. Secondly, we introduce a semi-implicit discretization of the continuous Franke-Westerhoff model derived in the previous chapter. This novel scheme is stable for arbitrary time steps and any reasonable parameter sets.

LLS Model The clearance mechanism of the LLS model

$$n = \sum_{k=1}^{N} \frac{\gamma_k(t)\ w_k(t)}{S(t)}$$

clearly models an equilibrium market where supply equals demand. In order to compute the equilibrium price in each time step,the clearance condition is solved numerically using a root finding algorithm. The crucial parameter is the chosen stopping criterion, defined as $\left| \sum_{k=1}^{N} \frac{\gamma_k(t)\ w_k(t)}{S^k} - n \right| \leq \xi,\ \xi > 0$. Thus, the algorithm terminates if the clearance mechanism for the stock price S^k is satisfied with respect to precision ξ. Note that the original publications [42, 43] do not report values or references for the tolerance used.
The choice of stopping criterion is also important economically. For relatively small values of ξ, the market price clearly is forced close to equilibrium resulting in an approximately rational market. For larger ξ, the market may become irrational at times. Note that the market is not inherently irrational for larger tolerances - most of the time, we observe low excess demand despite high tolerances. It is only around crashes that we find high excess demands that actually exhausts the tolerance. From a computational perspective, it is desirable to increase ξ as much as possible to lower computational cost. Figures 11,12, 13 (200 agents) demonstrate that the choice of stopping criterion ξ may strongly impacts the model behavior.

In figure 11, we present the results with tolerances $\xi = 0.05$ and $\xi = 0.1$. There are only subtle differences in return distribution and price trend. In figure 12, simulation results for tolerances $\xi = 0.1$, $\xi = 0.5$, $\xi = 0.75$, and $\xi = 1$ are compared with respect to average investment proportions. There are moderate differences in the plots for $\xi \leq 0.75$. Notably, these results feature peaks at the same times and the trends look generally alike. For $\xi = 1$ however, average investment proportion is overall very low and it never exceeds 0.1. Hence, $\xi = 1$ is found to represent to much of a relaxation of the rational market assumption. Figure 13 presents the autocorrelation of logarithmic stock returns and raw stock returns for multiple tolerance levels. Notably, the highest autocorrelation can be observed for $\xi = 0.25$, $\xi = 0.5$, and $\xi = 0.75$. From qq-plots of logarithmic stock return (see figure 14) one can conclude that increased tolerances result in heavier tails. From an economic point of view, a non-negligible tolerance represents a relaxation of the rational market hypothesis and therefore results in an irrational market. This raises the question whether increasing tolerance, and by this more irrational markets, generally result in heavier tails.

Franke-Westerhoff Model In the previous section, we have shown numerically that the blow up of the dynamics is caused by the violations of the bounds $[0, 1]$ of the agents' fractions. In order to clarify, if these violations are caused by the numerical scheme or by the continuous dynamics itself we test the invariance property of our ODE model. We formulate the following proposition:

Proposition 1. *Any solution of the SDE-ODE system* (9) *remains in the set* $V := \mathbb{R} \times U$ *with* $U := \{(x_1, x_2)^T \in \mathbb{R} | x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$.

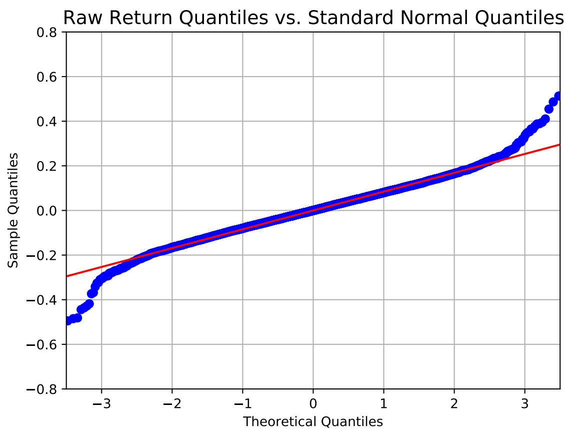
For the proof we refer to appendix A.1. Proposition 1 shows that the observed blow ups are introduced by the discretization of the SDE-ODE system (9). In order to avoid the instabilities we introduce the following semi-implicit Euler discretization:

$$\begin{aligned}
P(t + \Delta t) &= P(t) + \Delta t\ \mu\ ED^{FW}(t) + \sqrt{\Delta t}\ \mu(\sigma_f + \sigma_c)\ \eta, \quad \eta \sim \mathcal{N}(0, 1) \\
n^f(t + \Delta t) &= n^f(t) + \Delta t\ [n^c(t + \Delta t)\pi^{cf}(a(t)) - n^f(t + \Delta t)\pi^{fc}(a(t))] \\
n^c(t + \Delta t) &= n^c(t) + \Delta t\ [n^f(t + \Delta t)\pi^{fc}(a(t)) - n^c(t + \Delta t)\pi^{cf}(a(t))]
\end{aligned} \tag{11}$$

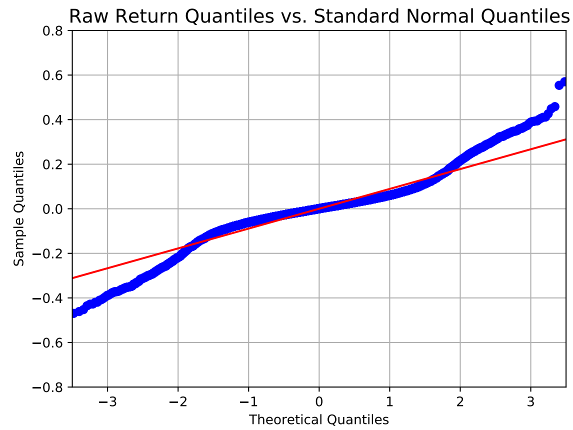
In appendix A.2, we show that the semi-implicit scheme (11) can be rewritten in explicit form. Due to this, the computational cost of the explicit and the semi-implicit schemes are comparable. In addition, the semi-implicit scheme (11) preserves the invariant properties of system (9).

Proposition 2. *For all* $\Delta t > 0$ *and correct initial conditions* $(P(t_0), n^f(t_0), n^c(t_0)) \in V$ *the numerical solution* $(P(t_0 + k\ \Delta t), n^f(t_0 + k\ \Delta t), n^c(t_0 + k\ \Delta t)) \in V,\ k = 0, 1, ..., N$ *defined by the scheme in* (11) *remains in the set* V *for any number* $N \in \mathbb{N}$.

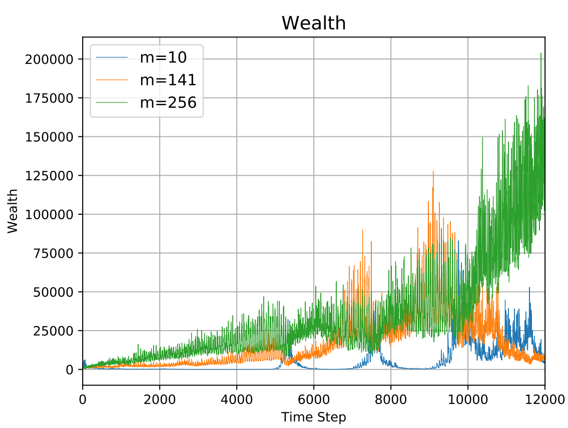
Hence, we have shown that using a proper numerical discretization the invariance property of the ODE system is satisfied for any time step $\Delta t > 0$ and arbitrary choices of constants. This is a huge advantage in comparison of the original model, formulated as a system of difference equations. In particular, the semi-implicit discretization renders the additional constraints (10) as introduced by Franke-Westerhoff in [27] redundant. We conclude with a numerical example of the semi-implicit discretization (11).



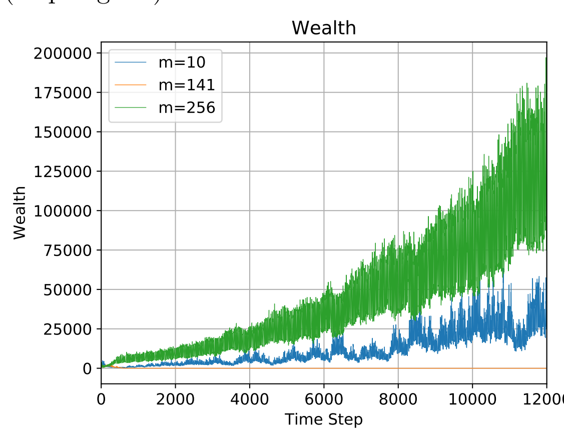
(a) QQ-plot of returns with $N = 99$. Heavy tails are visible.



(b) QQ-plot of returns with $N = 999$ agents. Heavy tails are visible but changed their shape (cmp. fig. 2a).

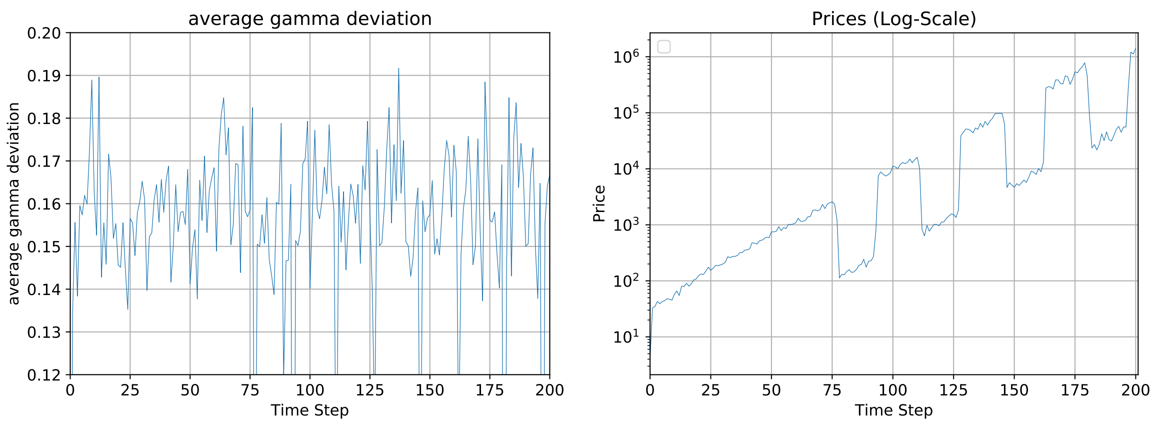


(c) Aggregated wealth of each agent group with $N = 33$ agents.

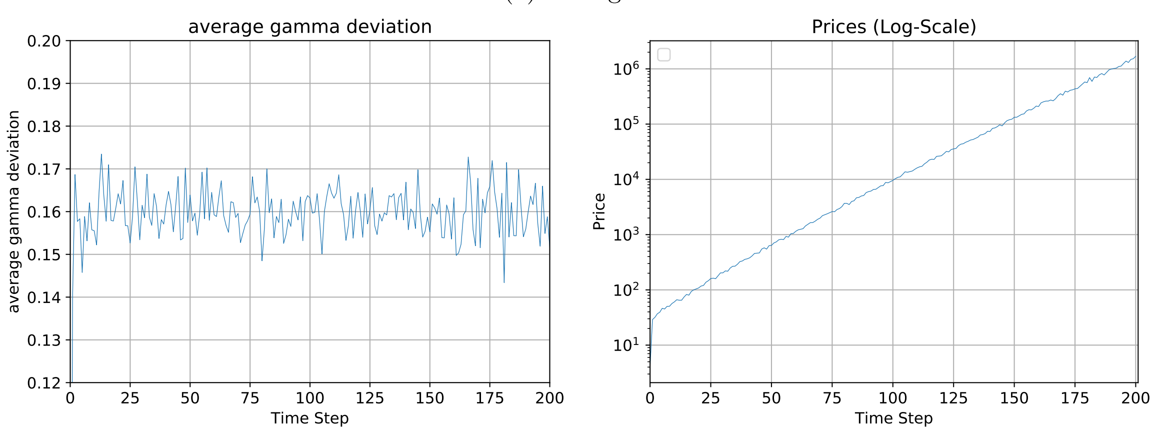


(d) Aggregated wealth of each agent group with $N = 333$ agents. The group ranking has changed.

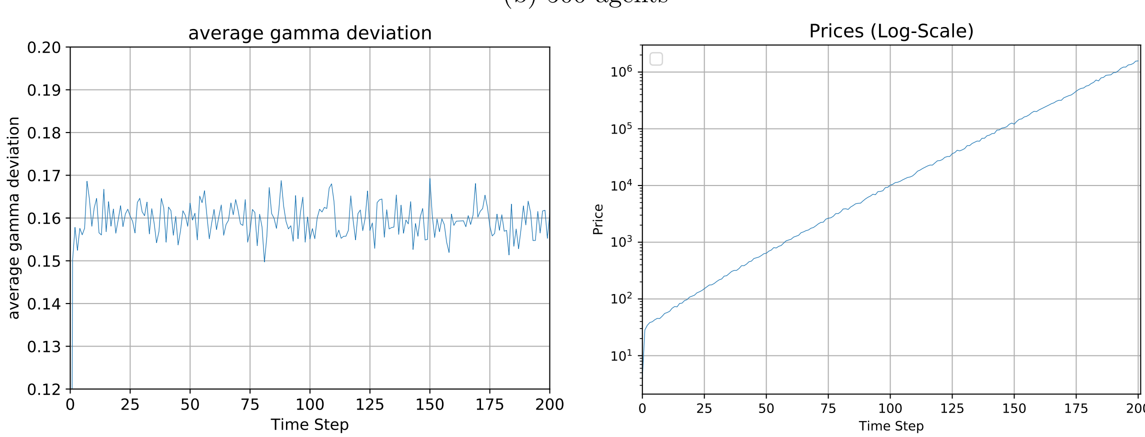
Figure 2: Simulation of the LLS model with 99 EMB agents (left) and 999 EMB agents (right). The agents are divided in three groups with different memory sizes. Figure 2a and 2b reveal a change in the tail behavior, while fig. 2a and 2b shows a change in the group ranking. Parameters as in table 2. For colored plots, please refer to the online version.



(a) 200 agents

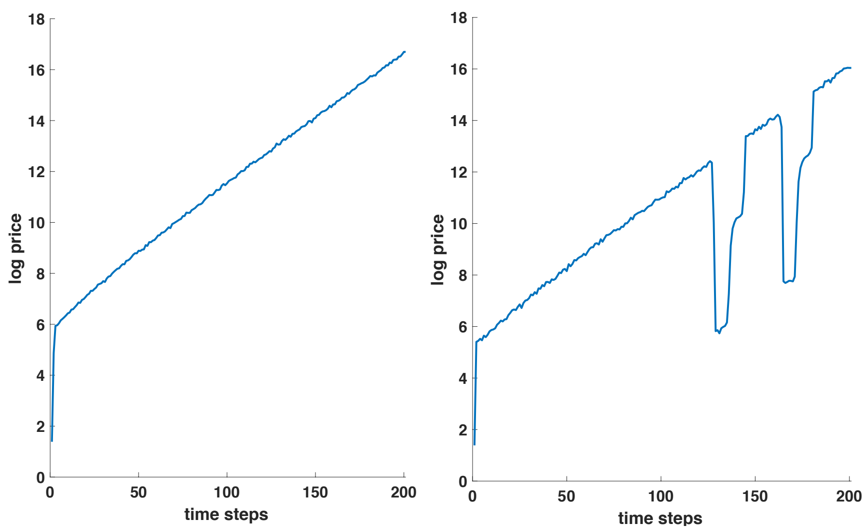


(b) 500 agents



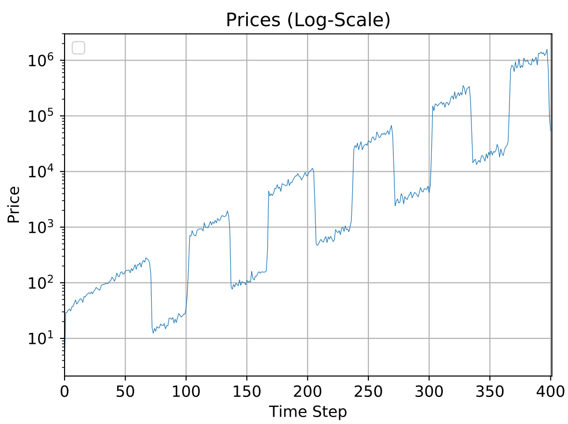
(c) 1000 agents

Figure 3: Prices and gamma differences d_γ^N for different agent counts.

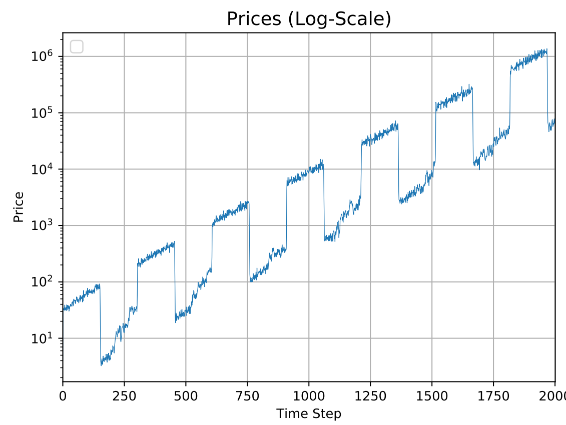


(a) C++ MT19937 RNG (64 bit) (b) `RANDU` generator (32 bit)

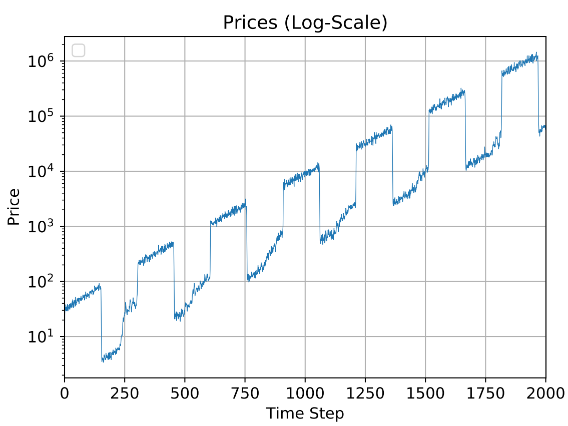
Figure 4: Simulations of the LLS model conducted with different pseudo random number generators. Parameters as in table 1 with $\sigma_\gamma = 0.01$.



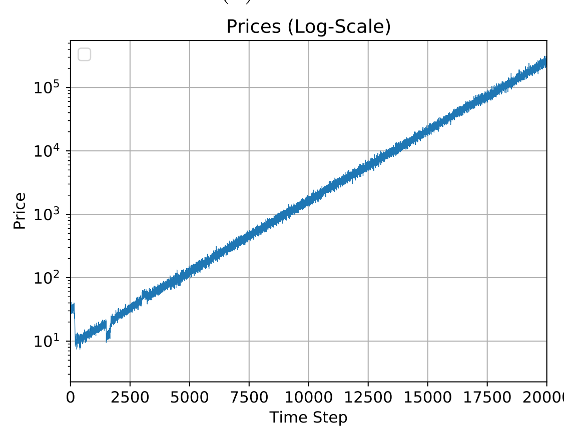
(a) $\Delta t = 0.5$



(b) $\Delta t = 0.1$

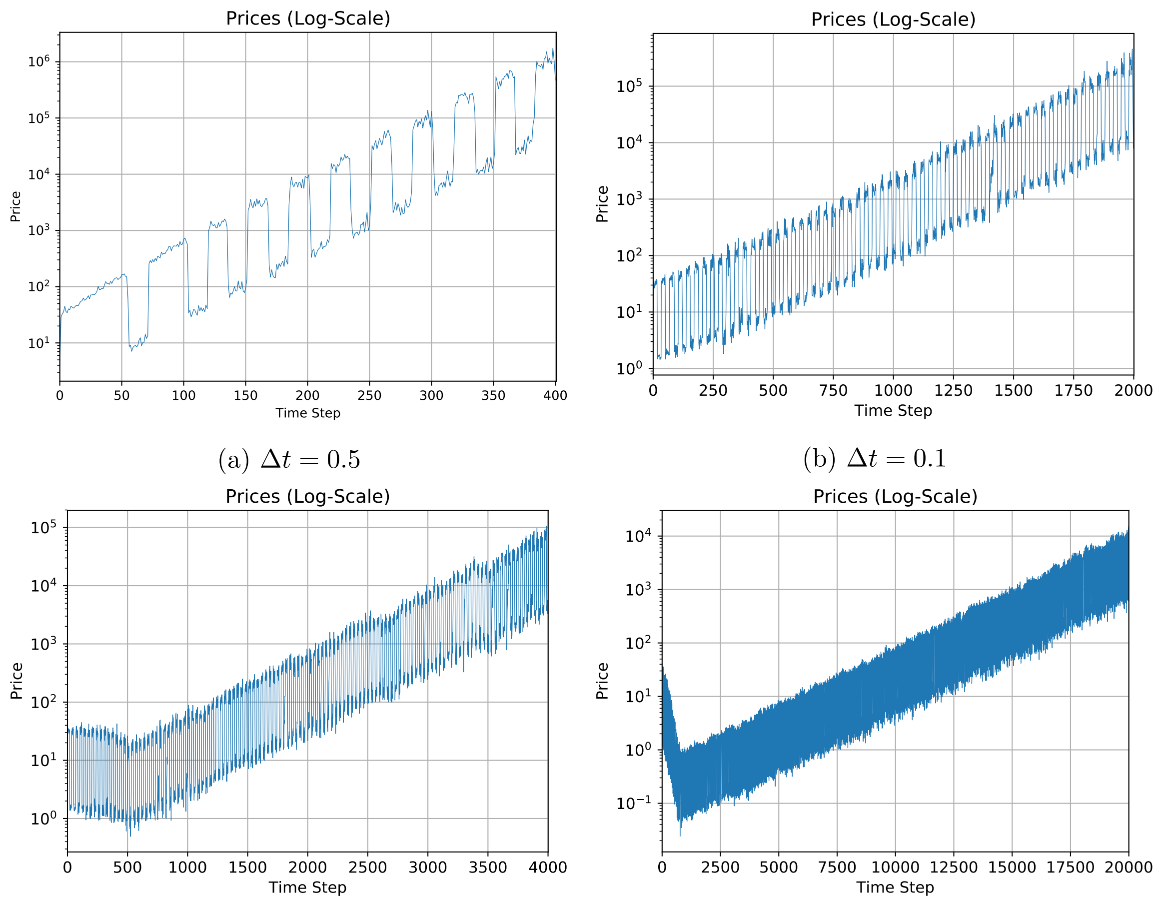


(c) $\Delta t = 0.05$



(d) $\Delta t = 0.01$

Figure 5: Simulations of time continuous LLS model with scaled memory variable and different time discretizations. Further parameters as defined in table 1 with $\sigma_\gamma = 0.2$.



(a) $\Delta t = 0.5$

(b) $\Delta t = 0.1$

(c) $\Delta t = 0.05$

(d) $\Delta t = 0.01$

Figure 6: Simulations of time continuous LLS model with fixed memory variable and different time discretizations. Further parameters as defined in table 1 with $\sigma_\gamma = 0.2$.

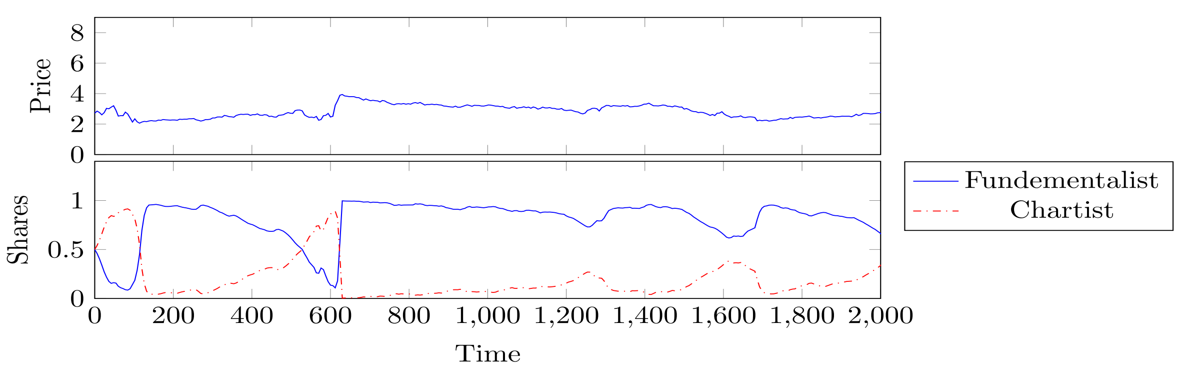


Figure 7: Franke-Westerhoff model with explicit Euler discretization. Parameters as in table table 3.

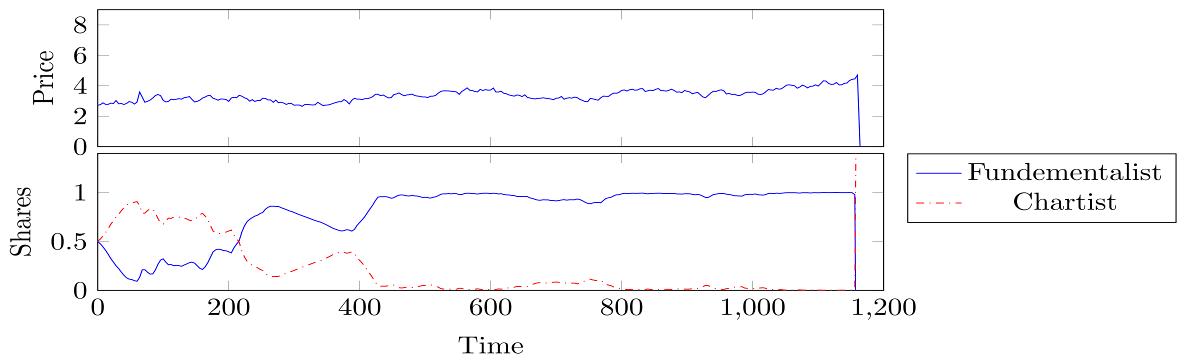


Figure 8: Blow up in the dynamics of the Franke-Westerhoff model with explicit Euler discretization. Parameters as in table table 3 with $\sigma_f = 1.15$.

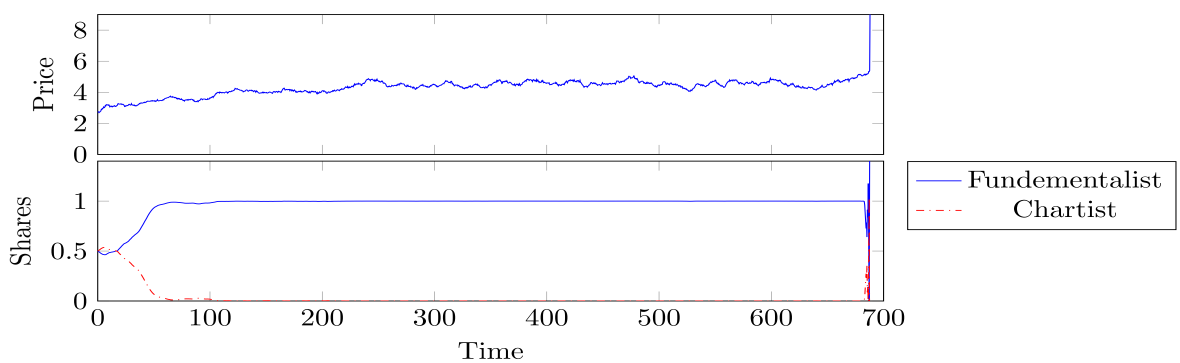


Figure 9: Blow up of the dynamics of the Franke-Westerhoff model with explicit Euler discretization. Parameters as in table table 3 with $\sigma_f = 1.15$ and $\Delta t = 0.1$.

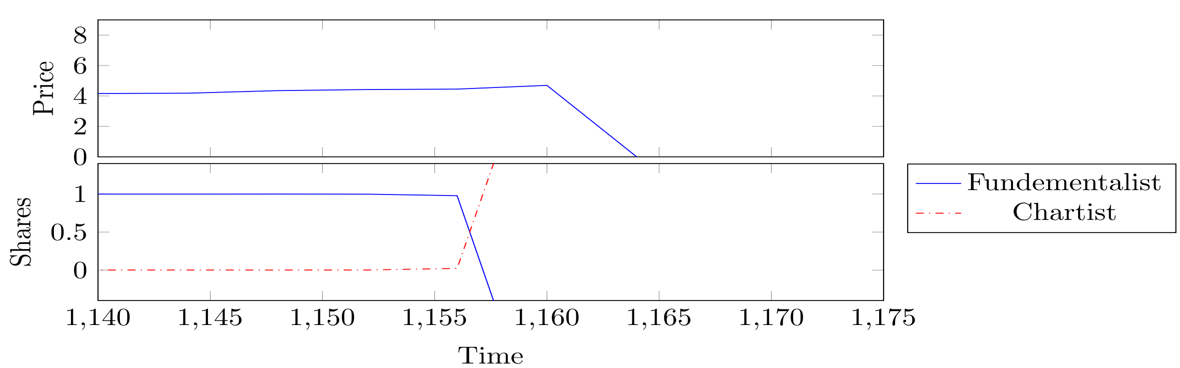
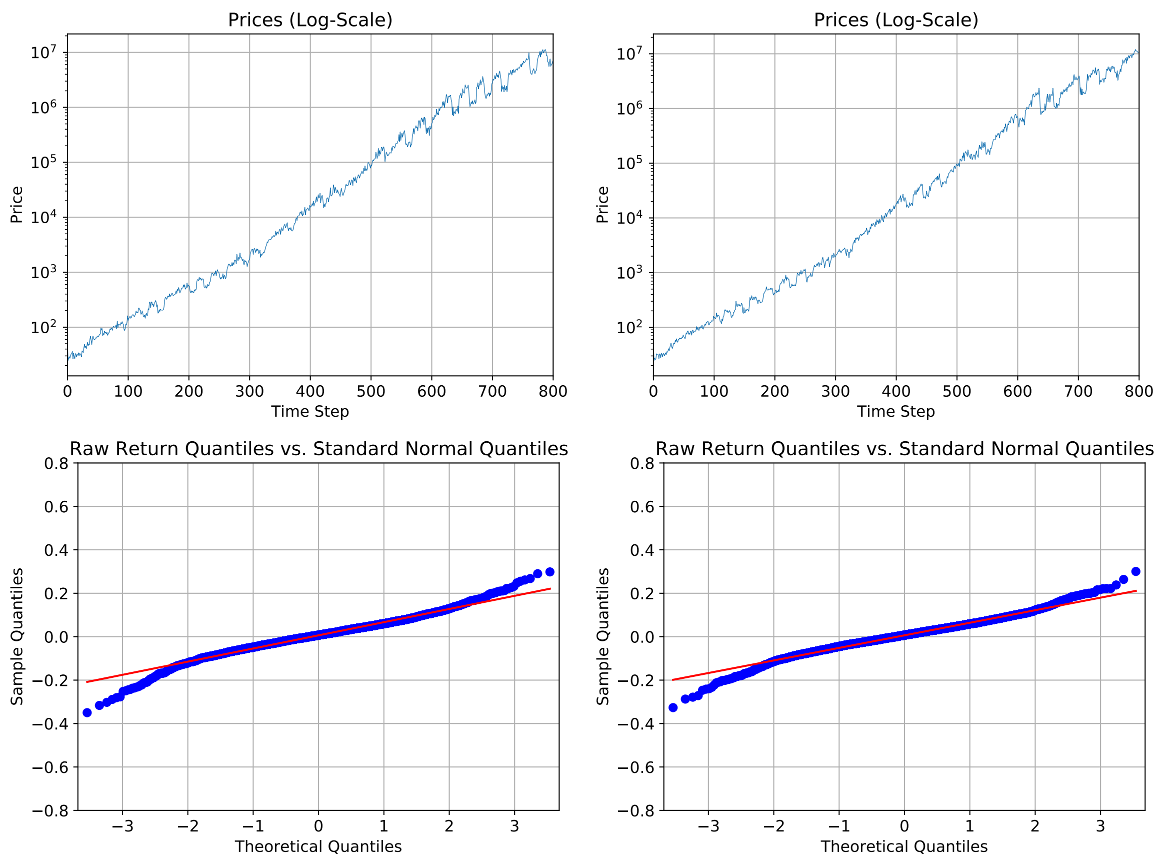


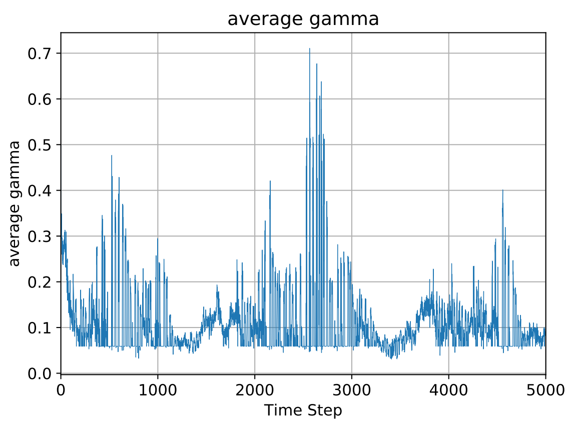
Figure 10: Zoom on the instability of the Franke Westerhoff model with explicit Euler discretization. See fig. 8 for full plot. Parameters as in table table 3 with $\sigma_f = 1.15$.



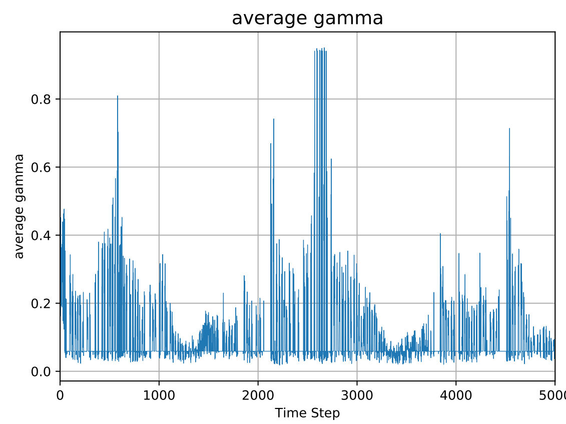
(a) $\xi = 0.1$

(b) $\xi = 0.05$

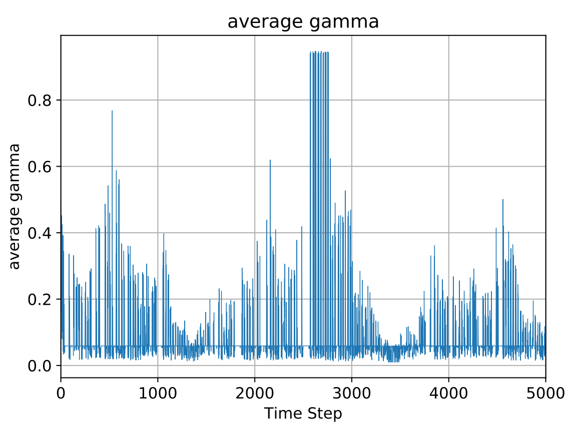
Figure 11: Price trends and return distribution for low tolerances



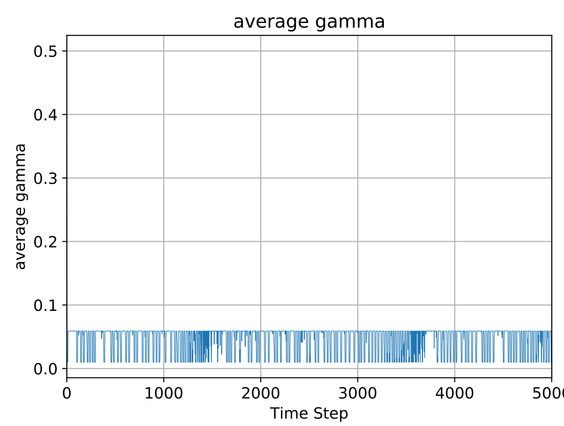
(a) $\xi = 0.1$



(b) $\xi = 0.5$

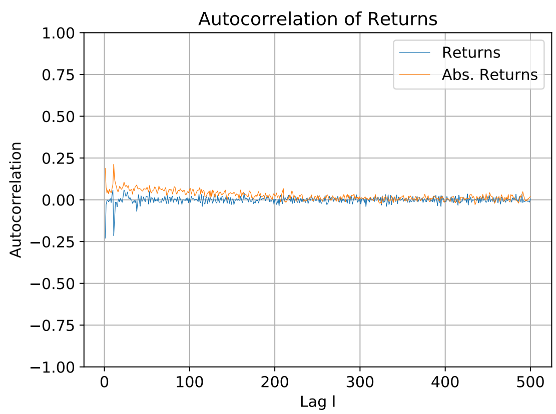


(c) $\xi = 0.75$

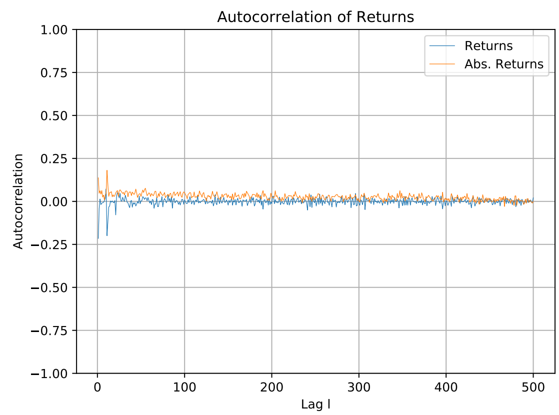


(d) $\xi = 1$

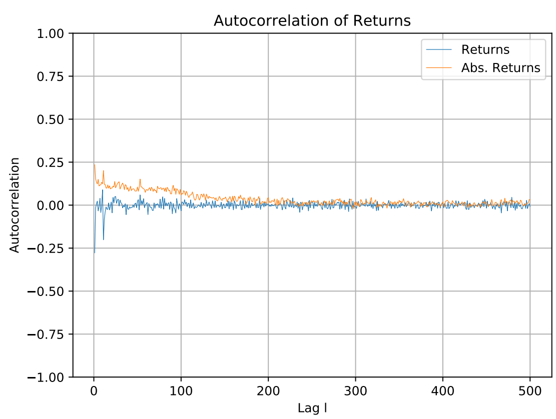
Figure 12: Average investment proportion for various tolerances



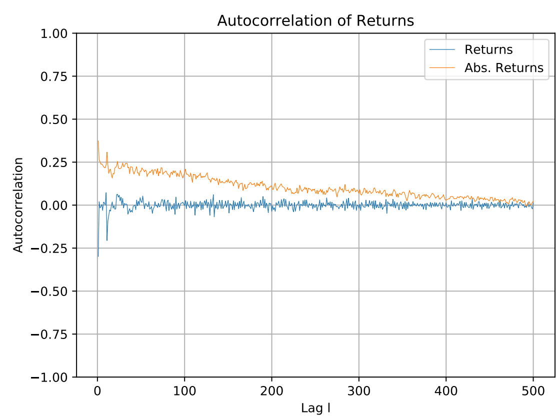
(a) $\xi = 0.05$



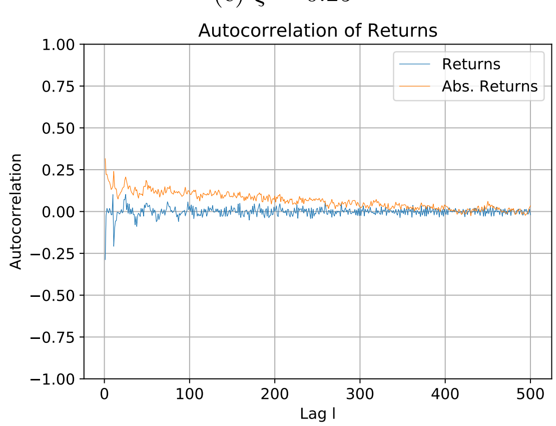
(b) $\xi = 0.1$



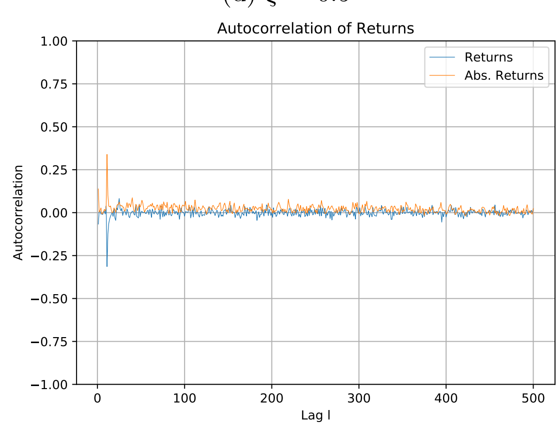
(c) $\xi = 0.25$



(d) $\xi = 0.5$

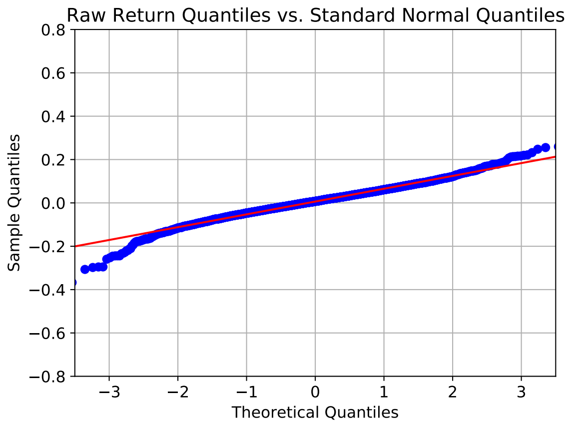


(e) $\xi = 0.75$

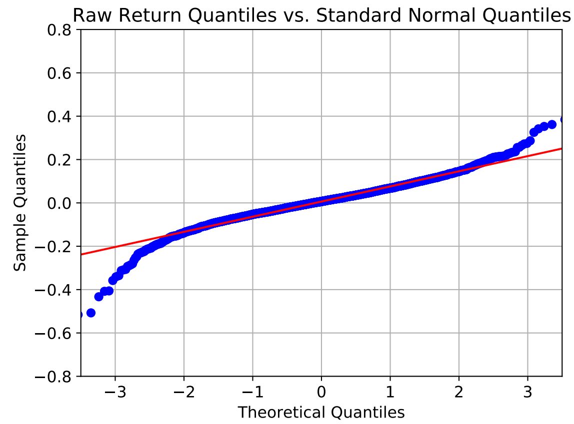


(f) $\xi = 1$

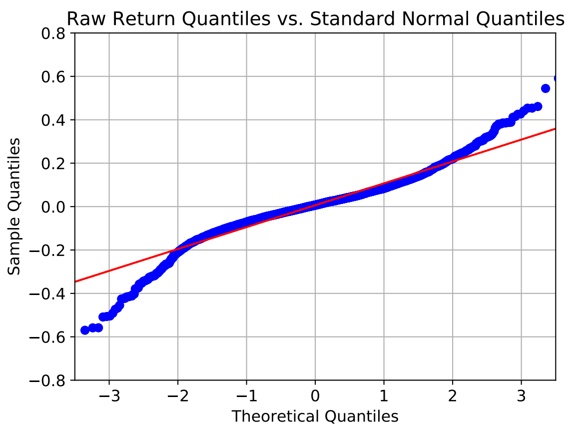
Figure 13: Autocorrelation for various tolerances. For colored plots, please refer to online version.



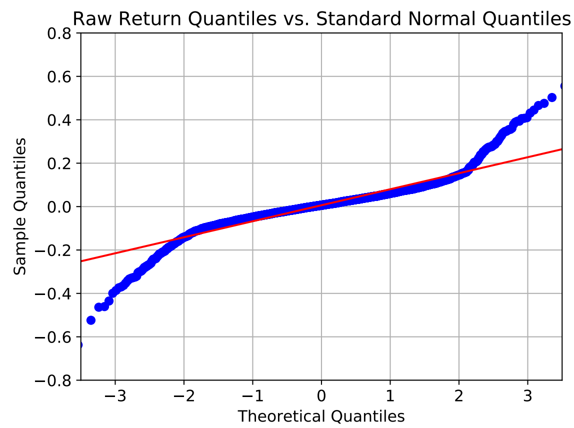
(a) $\xi = 0.1$



(b) $\xi = 0.25$



(c) $\xi = 0.5$



(d) $\xi = 1$

Figure 14: QQ plot of returns for various tolerances

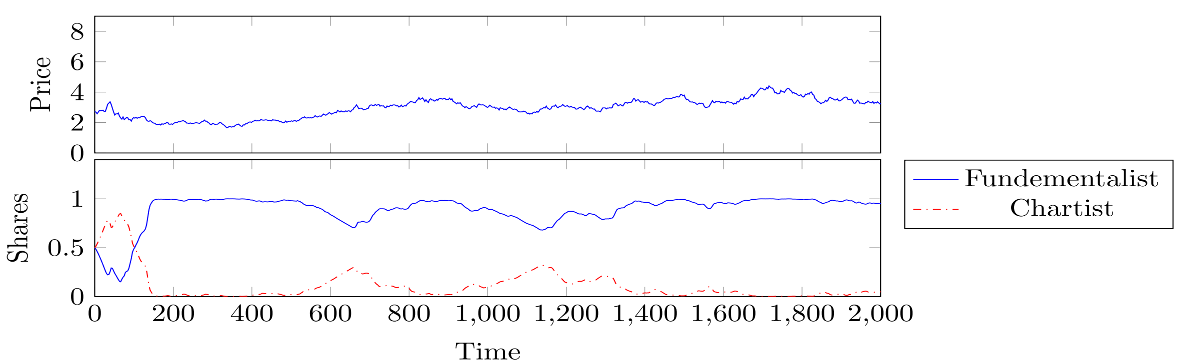


Figure 15: Franke Westerhoff model with semi-implicit discretization. Parameters as in table table 3 with $\sigma_f = 1.15$.

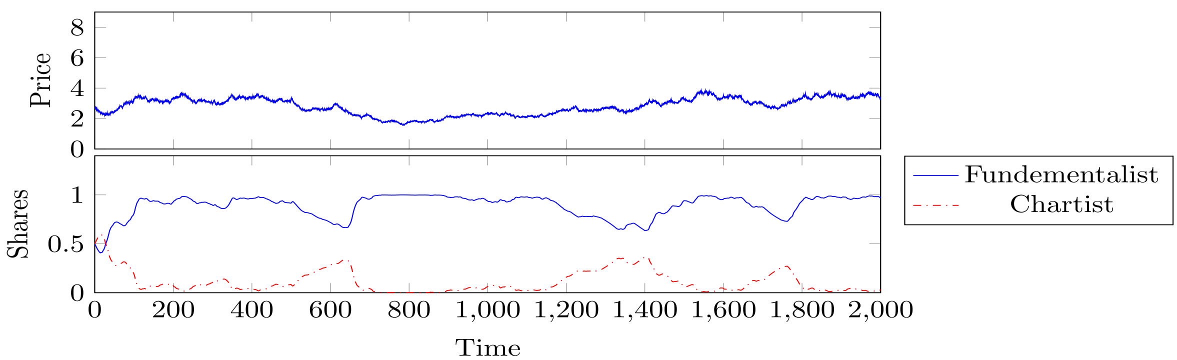


Figure 16: Franke Westerhoff model with semi-implicit discretization. Parameters as in table table 3 with $\sigma_f = 1.15$ and $\Delta t = 0.1$.

4 Conclusion

In this study we have discussed several mathematical and numerical aspects in ABCEM models, namely finite-size effects, the impact of a bad random number generator, the continuous formulation of difference equations and numerical solvers for differential equations. More precisely, we have exemplarily shown finite-size effects in the Levy-Levy-Solomon model. In addition, we have presented that a bad pseudo random number generator heavily influences the qualitative simulation output in the Levy-Levy-Solomon model. These observations lead us to conclude that in Monte Carlo simulations of ABCEM models it is of immanent importance to:

- simulate the model with a large number of agents to exclude finite-size effects; and
- to employ a reliable and well tested pseudo random number generator.

Our SABCEMM simulator may be seen as help to exclude these lacks in implementation by allowing the efficient implementation of ABCEM models allowing more than $> 10^6$ agents and providing access to proven-quality random number generators.

The second class of numerical issues discussed in this manuscript concerned the proper discretization. For this, we first derived the time-continuous formulations of the difference equations used in the LLS and the Franke-Westerhoff model. We further showed that these continuous formulations are not unique. Then, we showed that the numerical instabilities present in the standard Franke-Westerhoff model stem from the explicit Euler discretization and can be alleviated by applying a proper semi-implicit Euler discretization. Finally, we highlighted the impact of the choice of stopping criterion in the root finding algorithm within the clearance mechanism on the qualitative output of the model.

In this manuscript, we showed the immanent importance of the proper choice of numerical discretization on the model behavior. As a consequence, we strongly recommend to model ABCEM models on the continuous level as this allows for studying different time scales and the effects of different time discretizations, some of which may be suitable to overcome additional constraints for stability. Finally, we want to emphasize that a dynamical system of differential equations enables derivation of reduced models with the help of kinetic theory. In such reduced models, the agents' dynamics is characterized by a continuous density function which is defined as a solution of a partial differential equation. The advantage is the possibility to study the long time dynamics of the system and to derive analytical results such as the steady state distribution. Examples of such a modeling approach are [64, 61, 52] and a comprehensive introduction to this modeling tool can be found in [52].

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A Appendix

A.1 Analysis

The proof of the Proposition 1 is given by:

Proof. Since we consider a stochastic Itô integral it is clear that P remains in $\mathbb{R}$. It remains to show that $U := \{(x_1, x_2)^T \in \mathbb{R} | x_1 \geq 0,\ x_2 \geq 0,\ x_1 + x_2 \leq 1\}$ is an invariant set for the equations (7). We define

$$\begin{aligned} f_1(x_1, x_2) &:= x_2 \pi^{cf} - x_1 \pi^{fc}, \\ f_2(x_1, x_2) &:= x_1 \pi^{fc} - x_2 \pi^{cf}, \end{aligned}$$

and show the invariance of $U^+ := \{(x_1, x_2)^T \in \mathbb{R} | x_1 \geq 0,\ x_2 \geq 0\}$. We directly obtain that $f_1(0, x_2) = x_2 \pi^{cf} > 0$ and $f_2(x_1, 0) = x_1 \pi^{fc} > 0$ holds and thus U^+ is positive invariant. Secondly, we define $\phi(x_1, x_2) = x_1 + x_2$ and compute the Lie derivative of f along ϕ.

$$\langle \nabla \phi(x_1, x_2), f(x_1, x_2) \rangle = x_2 \pi^{cf} - x_1 \pi^{fc} + x_1 \pi^{fc} - x_2 \pi^{cf} = 0.$$

Hence, we have shown the positive invariance of the set $\{(x_1, x_2)^T \in \mathbb{R} | \phi(x) \leq 1\}$ and consequently U is an invariant set of our ODE system (7).

□

The proof of the Proposition 2 reads:

Proof. Since the Euler-Maruyama method updating rule is simply a sum of real numbers and the real line is a closed set for any stock price, $P \in \mathbb{R}$ holds. The updating rule of the agents' fraction can be rewritten as follows:

$$\begin{aligned} n^f(t_0 + (k+1)\Delta t) &= \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\ n^c(t_0 + (k+1)\Delta t) &= \frac{n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))}. \end{aligned}$$

Then, we can perform a simple induction. We assume that $0 < n^f(t_0 + k\Delta t) < 1,\ 0 < n^c(t_0 + k\Delta t) < 1$ holds and obtain

$$\begin{aligned} n^f(t_0 + (k+1)\Delta t) &= \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\ &\leq \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t)) + n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\ &= 1, \\ n^c(t_0 + (k+1)\Delta t) &= \frac{n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\ &\leq \frac{n^f(t_0 + k\Delta t) + \Delta t \pi^{cf}(a(t_0 + k\Delta t)) + n^c(t_0 + k\Delta t) + \Delta t \pi^{fc}(a(t_0 + k\ \Delta t))}{1 + \Delta t(\pi^{fc}(a(t_0 + k\ \Delta t)) + \pi^{cf}(a(t_0 + k\ \Delta t)))} \\ &= 1, \end{aligned}$$

since $\pi^{fc}, \pi^{cf} > 0$ holds by definition and we have used that $n^f + n^c = 1$ holds. The previous inequality shows that $n^f(t_0 + (k+1)\Delta t), n^c(t_0 + (k+1)\Delta t)$ remain in the set U for all $k \in \mathbb{N}$. □

xs

A.2 Models

LLS Model We have implemented the model as defined in [42, 43]. As described in section 3.3 we have added the correct time scale to the model. In order to obtain the original model one needs to set $\Delta t = 1$.

The model considers $N \in \mathbb{N}$ financial agents who can invest $\gamma_i \in [0.01, 0.99]$, $i = 1, ..., N$ of their wealth $w_i \in \mathbb{R}_{>0}$ in a stocks and have to invest $1 - \gamma_i$ of their wealth in a safe bond with interest rate $r \in (0, 1)$. The investment propensities γ_i are determined by a utility maximization and the wealth dynamic of each agent at time $t \in [0, \infty)$ is given by

$$\begin{aligned} w_i(t) &= w_i(t-\Delta t) \\ &+ \Delta t \left((1-\gamma_i(t-\Delta t))\ r\ w_i(t-\Delta t) + \gamma_i(t-\Delta t)\ w_i(t-\Delta t)\ \underbrace{\frac{\frac{S(t)-S(t-\Delta t)}{\Delta t} + D(t)}{S(t)}}_{=:x(S,t,D)} \right). \end{aligned}$$

The dynamics is driven by a multiplicative dividend process. Given by:

$$D(t) := (1 + \Delta t\ \tilde{z})\ D(t - \Delta t),$$

where $\tilde{z}$ is a uniformly distributed random variable with support $[z_1, z_2]$. The price is fixed by the so called *market clearance condition*, where $n \in \mathbb{N}$ is the fixed number of stocks and $n_i(t)$ the number of stocks of each agent.

$$n = \sum_{i=1}^{N} n_i(t) = \sum_{k=1}^{N} \frac{\gamma_k(t)\ w_k(t)}{S(t)}. \tag{12}$$

The utility maximization is given by

$$\max_{\gamma_i \in [0.01, 0.99]} E[\log(w(t + \Delta t, \gamma_i, S^h))].$$

with

$$\begin{aligned} E[\log(w(t+\Delta t, \gamma_i, S^h))] = \frac{1}{m_i} \sum_{j=1}^{m_i} U_i \Big(& (1-\gamma_i(t)) w_i(t, S^h)\ (1 + r\Delta t) \\ & + \gamma_i(t) w_i(t, S^h) \Big(1 + x\big(S, t - j\Delta t, D\big)\ \Delta t \Big) \Big). \end{aligned}$$

The constant m_i denotes the number of time steps each agent looks back. Thus, the number of time steps m_i and the length of the time step Δt defines the time period each agent extrapolates the past values. The superscript h indicates, that the stock price is uncertain and needs to be fixed by the market clearance condition. Finally, the computed optimal investment proportion gets blurred by a noise term.

$$\gamma_i(t) = \gamma_i^*(t) + \epsilon_i,$$

where ϵ_i is distributed like a truncated normally distributed random variable with standard deviation σ_γ. Finally, we have to update the price after the nosing process. Since the investment fraction is constant we are able to compute the stock price explicitly:

$$S(t) = \frac{\frac{1}{n} \sum\limits_{i=1}^{N} \gamma_i(t) \Big(w_i(t-\Delta t) + \Delta t\ w_i(t-\Delta t) \big(\gamma_i(t-\Delta t) \frac{D(t-\Delta t-) S(t-\Delta t)}{\Delta t\ S(t-\Delta t)} + (1-\gamma_i(t-\Delta t))\ r \big) \Big)}{1 - \frac{1}{n} \sum\limits_{i=1}^{N} \frac{\gamma_i(t)\gamma_i(t-\Delta t) w_i(t-\Delta t)}{S(t-\Delta t)}}.$$

Utility maximization Thanks to the simple utility function and linear dynamics we can compute the optimal investment proportion in the cases where the maximum is reached at the boundaries. The first order necessary condition is given by:

$$f(\gamma_i) := \frac{d}{dt}E[\log(w(t+\Delta t, \gamma_i, S^h))] = \frac{1}{m_i}\sum_{j=1}^{m_i} \frac{\Delta t\ (x\big(S, t-j\Delta t, D\big) - r)}{\Delta t\ (x\big(S, t-j\Delta t, D\big) - r)\ \gamma_i + 1 + \Delta t\ r}.$$

Thus, for $f(0.01) < 0$ we can conclude that $\gamma_i = 0.01$ holds. In the same manner, we get $\gamma_i = 0.99$, if $f(0.01) > 0$ and $f(0.99) > 0$ holds. Hence, solutions in the interior of $[0.01, 0.99]$ can be only expected in the case: $f(0.01) > 0$ and $f(0.99) < 0$. This coincides with the observations in [54].

Franke-Westerhoff model We present the Franke-Westerhoff model as introduced in [26] and considered with minor modifications in [27, 28]. As described in section 3.3 we have added a time scaling to the model. In order to obtain the original model one needs consider the explicit Euler discretization of the agents' shares and has to set $\Delta t = 1$. The Franke-Westerhoff model considers tow types of agents, chartists and fundamentalists. The demand of each agent reads

$$d^f(t) = \phi(P_f(t) - P(t)) + \epsilon_k^f, \quad \phi \in \mathbb{R}^+, \quad \epsilon_k^f \sim \mathcal{N}(0, \sigma_f^2), \tag{13}$$

$$d^c(t) = \chi(P(t) - P(t-\Delta t)) + \epsilon_k^c, \quad \chi \in \mathbb{R}^+, \quad \epsilon_k^c \sim \mathcal{N}(0, \sigma_c^2), \tag{14}$$

where $P(t)$ denotes the logarithmic market price and $P_f(t)$ denotes the fundamental price. The noise terms ϵ_k^f and ϵ_k^c are normally distributed, with zero mean and different standard deviations σ_c^2 and σ_f^2. The second important features are the fractions of the chartist or fundamental population. In that sense the two agents can be seen as representative agents of a population. The fraction of chartists $n^C(t) \in [0,1]$ and the fraction of fundamentalitst $n^F(t) \in [0,1]$ have to fulfill $n^C(t) + n^F(t) = 1$. Hence, the deterministic excess demand can be defined as:

$$ED^{FW}(t) := \frac{1}{2}(ed^f(t) + ed^c(t)) \tag{15}$$

$$\begin{aligned} ed^f(t) &:= 2\ n^f(t)E[d^f(t)] \\ ed^c(t) &:= 2\ n^c(t)E[d^c(t)]. \end{aligned} \tag{16}$$

Here, E denotes the expected value. The pricing equation is then given by the simple rule

$$P(t) = P(t-\Delta t) + \mu\ \Delta t\ ED^{FW}(t) + \sqrt{\Delta t}\ \mu\ (\sigma_f + \sigma_c)\ \eta, \quad \eta \sim \mathcal{N}(0,1). \tag{17}$$

Finally, we need to specify the switching mechanism. This switching mechanism is known as the transition probability approach (TPA) [66, 45]. We consider the so called switching index $a(t) \in \mathbb{R}$ which describes the attractiveness of the fundamental strategy over the chartist strategy. Thus, a positive $a(t)$ reflects an advantage of the fundamental strategy in comparison to the chartist and if $a(t)$ is negative we have the opposite situation. We define the switching probabilities

$$\pi^{cf}(a(t)) := \nu \exp(a(t)) \tag{18}$$

$$\pi^{fc}(a(t) := \nu \exp(-a(t)). \tag{19}$$

where π^{xy} is the probability that an agent with strategy x switches to strategy y. The flexibility parameter $\nu > 0$ is a scaling factor for $a(t)$.

Remark 1. *A minor modification of the Franke-Westerhoff model introduced in 2011 [27, 28] considers the following switching probabilities*

$$\pi^{cf}(a(t)) := \min\left(1, \nu \exp(a(t))\right) \tag{20}$$

$$\pi^{fc}(a(t) := \min\left(1, \nu \exp(-a(t))\right). \tag{21}$$

The previous definition ensures that switching probabilities are restricted to the interval $[0, 1] \subset \mathbb{R}$. *In the original model introduced in 2009 [26] there has been no additional limits as introduced in* (20).

Explicit Euler Discretization Then the explicit Euler discretization of the time evolution of chartist and fundamentalist shares as presented in (9) is given by:

$$n^f(t) = n^f(t - \Delta t) + \Delta t n^c(t)\pi^{cf}(a(t - \Delta t)) - \Delta t n^f(t)\pi^{fc}(a(t - \Delta t))$$
$$n^c(t) = n^c(t - \Delta t) + \Delta t n^f(t - \Delta t)\pi^{fc}(a(t - \Delta t)) - \Delta t n^c(t - \Delta t)\pi^{cf}(a(t - \Delta t))$$

Semi-Implicit Euler Discretization Alternatively one may use the semi-implicit scheme introduced in section 3 for the time evolution of chartist and fundamentalist shares.

$$\begin{aligned} n^f(t) &= \frac{n^f(t - \Delta t) + \Delta t \pi^{cf}(a(t - \Delta t))}{1 + \Delta t(\pi^{fc}(a(t - \Delta t)) + \pi^{cf}(a(t - \Delta t)))} \\ n^c(t) &= \frac{n^c(t - \Delta t) + \Delta t \pi^{fc}(a(t - \Delta t))}{1 + \Delta t(\pi^{fc}(a(t - \Delta t)) + \pi^{cf}(a(t - \Delta t)))} \end{aligned} \tag{22}$$

As shown in section 3.3 as well, this numerical approximation stable and conserves the invariance property of the ODEs.

Finally, we have to specify how the switching index $a(t)$ is calculated. The switching index $a(t)$, encodes how favourable a fundamentalist strategy is over a chartist strategy. The switching index is determined linearly out of the the three principles *predisposition, herding and misalignment.*

$$\alpha(t) = \alpha_p + \alpha_h \left(n^f(t) - n^c(t)\right) + \alpha_m \left(P(t) - P_f\right)^2,$$

where $\alpha_p, \alpha_h, \alpha_m > 0$ are weights respectively scaling factors. The sign $\alpha_p \in \mathbb{R}$ determines the predisposition with respect to a fundamental or chartist strategy. For details regarding the modeling we refer to [26].

A.3 Parameter sets

LLS Model The initialization of the stock return is performed by creating an artificial history of stock returns. The artificial history is modeled as a Gaussian random variable with mean μ_h and standard deviation σ_h. Furthermore, we have to point out that the increments of the dividend is deterministic, if $z_1 = z_2$ holds. We used the C++ standard random number generator for all simulations of the LLS model if not otherwise stated.

Franke-Westerhoff Model

Parameter	Value
N	100
m_i	15
σ_γ	0 or 0.2
r	0.04
$z_1 = z_2$	0.05
Δt	1
time steps	200

(a) Parameters of LLS model.

Variable	Initial Value
μ_h	0.0415
σ_h	0.003
$\gamma(t=0)$	0.4
$w_i(t=0)$	1000
$n_i(t=0)$	100
$S(t=0)$	4
$D(t=0)$	0.2

(b) Initial values of LLS model.

Table 1: Basic setting of the LLS model.

Parameter	Value
N	99
m_i	$10,\ 1 \leqslant i \leqslant 33$ $141,\ 34 \leqslant i \leqslant 66$ $256,\ 67 \leqslant i \leqslant 99$
σ_γ	0.2
r	0.0001
$z_1 = z_2$	0.00015
Δt	1
time steps	$20,000$

(a) Parameters of LLS model.

Variable	Initial Value
μ_h	0.0415
σ_h	0.003
$\gamma_i(t=0)$	0.4
$w_i(t=0)$	1000
$n_i(t=0)$	100
$S(t=0)$	4
$D(t=0)$	0.004

(b) Initial values of LLS model.

Table 2: Setting for the LLS model (3 agent groups).

Parameter	Value
ϕ	0.18
χ	2.3
α_0	-0.161
α_h	1.3
α_m	12.5
σ_f	0.79
σ_c	1.9
ν	0.05
P_f	1
μ	0.01
Δt	1

(a) Parameters

Variable	Initial Value
$P(t = 0)$	1

(b) Initial Values

Table 3: Parameters and initial values for the Franke Westerhoff model.

Simulation	Random Number Generator
Figures 2a to 2d	C++ MT19937 RNG (64 bit)
Figures 3a to 3c	C++ MT19937 RNG (64 bit)
Figure 4a	C++ MT19937 RNG (64 bit)
Figure 4b	`RANDU` generator (32 bit)
Figure 5	C++ MT19937 RNG (64 bit)
Figure 6	C++ MT19937 RNG (64 bit)
Figure 7	IntelMKL MT2203 RNG (64 bit)
Figure 8	IntelMKL MT2203 RNG (64 bit)
Figure 9	IntelMKL MT2203 RNG (64 bit)
Figure 10	IntelMKL MT2203 RNG (64 bit)
Figures 11a to 11b	C++ MT19937 RNG (64 bit)
Figures 12a to 12d	C++ MT19937 RNG (64 bit)
Figures 12a to 12d	C++ MT19937 RNG (64 bit)
Figures 13a to 13f	C++ MT19937 RNG (64 bit)
Figures 14a to 14d	C++ MT19937 RNG (64 bit)
Figure 15	IntelMKL MT2203 RNG (64 bit)
Figure 16	IntelMKL MT2203 RNG (64 bit)

Table 4: Random Generators for the Simulations

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