

Market Ecology, Pareto Wealth Distribution and Leptokurtic Returns in Microscopic Simulation of the LLS Stock Market Model

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Abstract:

The LLS stock market model is a model of heterogeneous quasi-rational investors operating in a complex environment about which they have incomplete information. We review the main features of this model and several of its extensions. We study the effects of investor heterogeneity and show that predation, competition, or symbiosis may occur between different investor populations. The dynamics of the LLS model lead to the empirically observed Pareto wealth distribution. Many properties observed in actual markets appear as natural consequences of the LLS dynamics: truncated Levy distribution of short-term returns, excess volatility, a return autocorrelation "U-shape" pattern, and a positive correlation between volume and absolute returns.

1. The LLS Model

LLS is a microscopic representation model of the stock market. Its details and some generalizations of it can be found in [2]. In the present account we introduce the basic LLS ideas and the model main results. We consider below a market with only two investment options: a bond and a stock (see [3] for an extension to a multiple stocks case). The model involves a large number of virtual investors characterized each by a current wealth, portfolio structure, probability expectations and risk taking preferences. These personal characteristics come into play in each investor's decision making process as schematically seen in Fig. 1.

The bond is assumed to be a risk-less asset yielding a return at the end of each time period. The bond is exogenous and investors can buy from it as much as they wish at a given rate.

The stock is a risky asset with overall returns rate $H(t)$ composed of two elements:

- (i). **Capital gain (loss)**: If an investor holds a stock, any rise (fall) in the market price of the stock contributes to an increase (decrease) in the investors' wealth.
- (ii). **Dividends**: The company earns income and distributes dividends.

Each investor i is confronted with a decision where the outcome is uncertain: which is the optimal fraction $X(i)$ of his/her wealth to invest in stock? According to the standard theory of investment each investor is characterized by a utility function (of its wealth) $U(W)$ that reflects his/her personal risk taking preference (here we take for simplicity $U(W) = \ln W$ see [1] for a prospect theory extension). The optimal $X(i)$ is the one that maximizes the expected value of his/her $U(W)$ (we take into account all the unknown factors influencing decision-making (such as liquidity constraints or deviations from rationality) by adding a small random variable (or "noise") to the optimal proportion $X(i)$).

The expected value of $U(W)$ depends of course on the expected probabilities for the various values of H to be realized in the future. In LLS the investors expectations for the future H 's are based on extrapolating the past values. More precisely each investor recalls the last k returns on the stock and expects that each of them may take place again with equal probability. The extrapolation range k differs between various investors and it will be in the sequel of this paper the main parameter inducing market inhomogeneity.

At fixed time intervals each investor revises the composition of its portfolio and decides for a new market order. The aggregate of these orders determines the new stock price by the market clearance condition as explained below. Once each investor decides on the proportion of his/her wealth $X(i)$ that (s)he wishes to hold in stocks, one can derive the number of stocks $N(i, p_h)$ it wishes to hold corresponding to each hypothetical stock price p_h . Since the total number of shares in the market N , is fixed there is a particular value of the price p for which the sum of the $N(i, p)$ equals N . This value p is the new market equilibrium price. Upon updating accordingly the traders' portfolios, wealth and list of last k returns, one is ready for the next market iteration. This process is repeated for each time step, and the market prices are recorded throughout the run.

Figure 1: The Flow Chart of the LLS market framework

2. Crashes, Booms and Cycles

The LLS model provides already at the level of a quite homogenous traders population a convincing description of the emergence of cycles of booms and crashes in the stock markets. In a market with one species of investors all having a homogenous memory (extrapolation) range spanning the last k returns of the stock, the stock price alternates regularly between two very different price levels. The explanation for this behavior is as follows:

Assume that the rate of return $H(t)$ on the stock at a time t is higher than the oldest remembered return ($H(t - k)$). The addition of $H(t)$ and the elimination of $H(t - k)$ creates then a new distribution of past returns that is better than the previous one. Since the LLS extrapolating investors use the past k returns to estimate the distribution of the next period's return, they will be lead to be more optimistic and increase their investments in the stock. This, in turn, will cause the stock price to rise, which will generate an even higher return.

This positive feedback loop stops only when investors reach the maximum investment proportion (i.e. $X(i) = 100\%$: we do not allow borrowing or short selling), and can no longer increase their investment proportion in the stock. The dividend contribution to the returns is small compared with this high price at this stage. In the absence of noise the returns on the stock at this plateau converge to a constant growth rate which is just slightly higher than the riskless interest rate (see [4]). In other words, in the absence of noise the price remains almost constant, growing only because of the interest paid on the bond (more money entering the system and being invested in the stock).

When there is some noise in the system the price fluctuates a little around the asymptotic high level, because of the small random fluctuations in the investment proportions. These fluctuations generate some negative returns (on a downward fluctuation) and some high returns (when the price goes back up). One might suspect that a large downward

fluctuation might trigger a reverse positive feedback effect, trader expectations will lower, investment proportions will decrease, the price will drop, generating further negative returns and so on: a crash. This can happen during the "plateau" period but only after the previous sharp price boom which generated an extremely high return, is forgotten. And, indeed, this is exactly what happens. Since it takes k steps to forget the boom, the high price plateaus are a bit longer than the extrapolation span (k days to forget the boom $+O(1)$ more days until a large enough negative fluctuation occurs).

The crash generates a disastrous return and, until it is forgotten, investment proportions and hence the price remains very low. When the price is low, the dividend becomes significant and the returns on the stock are relatively high (compared with the bond). Once the crash has been forgotten, all the returns that are remembered are therefore high, and the price jumps back up. Thus, the low price plateaus are k steps long. This completes one cycle, which is repeated throughout the run. This (quasi-)periodicity is best viewed in the Fourier transform of the price time evolution (Fig. 2) as a series of narrow peaks around the frequency $2k + O(1)$ and its harmonics (note however that the dynamics is not perfectly periodic and therefore in spite of its simplicity, according to some mathematical criteria it may fall into the "complex" category. In the present paper we reserve however the term "complex" for dynamics that are truly complicated to the degree that they do not admit simple verbal or mathematical description or understanding).

The homogenous stock market described above exhibits booms and crashes. However, the homogeneity of investors leads to unrealistic periodicity. As shown below, when there is more than one investor species the dynamics becomes much more complex and realistic.

Figure 2 : The Fourier transform of the price in a market with one species with extrapolation range $k = 10$. The market contained 10000 traders that had initially equal wealth invested half in stock and half in bonds.

3. Realistic Features in LLS with Many Species

Our numerical experiments within the LLS framework have found that already a small number of trader species (characterized by different extrapolation ranges k) leads qualitatively to many of the empirically observed market phenomena.

In reality, we would expect not just a few trader types, but rather an entire spectrum of investors. When the full spectrum of different trader species (fundamentalists and various other types - see [1] for the detailed operational definition) is considered it turns out that "more is different" [5]: the price dynamics becomes realistic: booms and crashes are not periodic or predictable, and they are also less frequent and dramatic. At the same time, we still obtain many of the usual market anomalies described by the experimental studies (however in the limit of infinite times or infinite number of investors, the dynamics may revert to predictable patterns [3]).

We list below a few such realistic features:

Return Autocorrelations: Momentum and Mean-Reversion

In the heterogeneous population LLS model trends are generated by the same positive feedback mechanism that generated cycles in the homogeneous case (section 2): high (low) returns tend to make the extrapolating investors more (less) aggressive, this generates more high (low) returns, etc.

The difference between the two cases is that in the heterogeneous case there is a very complicated interaction between all the different investor species and as a result there are no distinct regular cycles but rather, smoother and more irregular trends. There is no single cycle length - the dynamics is a combination of many different cycles corresponding to the many extrapolation ranges k . This makes the autocorrelation pattern also smoother and more continuous. The return autocorrelations in the heterogeneous LLS model conform to the empirical findings: In the short run the autocorrelation is positive - this is the empirically documented phenomenon known as momentum: high returns during a trading quarter tend to be followed by more high returns in the following months, (and low returns tend to be followed by more low returns). In the longer run the autocorrelation is negative (after a few years of boom, one usually experiences a few "dry" years), which is known as mean-reversion. For even longer lags the autocorrelation eventually tends to zero [1]. The short run momentum, longer run mean-reversion, and eventual diminishing autocorrelation creates the general "U-shape" that is found in empirical studies [7].

Excess Volatility

In markets with a large fundamentalist population (see [1] for their detailed operative definition in the LLS model), the price level is generally determined by the fundamental value of the stock. However, the market extrapolating investors occasionally induce temporary departures of the price away from the fundamental value. These temporary departures from the fundamental value make the price more volatile than the fundamental value.

Following Shiller's [8] methodology we measured the standard deviations of the detrended price and fundamental value. Averaging over 100 independent simulations we found [1] respectively 27.1 and 19.2, which is an excess volatility of 41%.

Heavy Trading Volume

In an LLS market with both fundamentalists and market extrapolating investors (over various k ranges), shares change hands continuously between the various groups:

When a "boom" starts, the extrapolating investors observe higher ex-post returns and become more optimistic, while the fundamentalists view the stock as becoming overpriced and become more pessimistic. Thus, at this stage the market extrapolators buy most of the shares from the fundamentalists.

When the stock crashes, the opposite is true: the extrapolators are very pessimistic, but the fundamentalists buy the stock once it falls below the fundamental value. Thus, there is substantial trading volume in this market. The average trading volume in a typical LLS simulation was about 1,000 shares per period, or about 10% of the total outstanding shares.

Volume is Positively Correlated with Absolute Returns

The typical scenario in an LLS run is that when a positive trend is induced by the extrapolating investors, the opinions of the fundamentalists and the extrapolating investors change in opposite directions:

- The extrapolating investors see a trend of rising prices as a positive indication about the future return distribution, while
- The fundamentalists believe that the higher the price level is (the more overpriced the stock is), the harder it will eventually fall.

The exact opposite holds for a trend of falling prices. Thus, price trends are typically interpreted differently by the two investor types, and therefore induce heavy trading volume. The more pronounced the trend (large price changes), the more likely it is to lead to heavy volume.

In order to verify this relationship quantitatively we regressed volume $V(t)$ on the absolute rates of return $r(t)$ for 100 independent simulations. We run the regressions:

$$V(t) = a + b|r(t)| + \text{random}(t)$$

We found an average value of 870 for b with an average t-value of 5.0. Similar results were obtained for time lagged-returns.

4. Predation, Competition and Symbiosis between Trader Species

In section 2 it was explained that a homogenous population of traders that extrapolate the last k returns leads to cycles of booms and crashes of period $2k + O(1)$. When there are two species with extrapolation ranges k_1 and respectively k_2 , we observe sharp irregular transitions between eras where one species dominates (cycles of period $2k_1 + O(1)$) and market eras where the other species dominates (cycles of period $2k_2 + O(1)$). When the number of trader species is three, there are dramatic qualitative changes: generically, the dynamics becomes complex. We show that complexity is an intrinsic property of the stock market. This suggests an alternative explanation to the widely accepted but empirically questionable random walk hypothesis. We discuss below some of the market ecologies possible with only two species of traders. Of course the picture becomes more complex later, when 3 or more species are introduced.

Market Ecologies with Two Trader Species

When there are two trader species with different extrapolation spans it turns out that the nature of the dynamics is determined by the ratio of the extrapolation spans of the two species. In [9] we performed a qualitative theoretical analysis of this phenomenon and supported it by microscopic simulations. We showed that in market eras in which one species (of extrapolation range k_0) dictates the dynamics (i.e. boom-crash cycles have periods of length $2k_0 + O(1)$) the second species (with extrapolation range k) has generically the following performance:

A : If $k_0 < k < 2k_0$ then k is performing very poorly (looses money)

B : If $2nk_0 < k < (2n + 1)k_0$ (with n natural number), then k is doing relatively well

C : If $(2n + 1)k_0 < k < 2nk_0, n > 1$, then k does better than in A but worse than in B

D : If $k < k_0$, then k is doing well .

These facts turned out sufficient to understand the main 3 cases that a 2-species ecology can display:

Case 1: predator - prey dynamics

If one considers one species with an extrapolation range $k_1 = 10$ and a second species with an extrapolation range $k_2 = 14$ it turns out that the resulting ecology dynamics is a predator-prey one. In fact the LLS market dynamics leads in this case to the extinction (total impoverishment) of the k_1 species: after some time the entire wealth on the market belongs to the species $k_1 = 10$ (Fig. 3). As a consequence, the market price presents clear cycles of booms and crashes of periodicity clustered around $24 = 2 * 10 + O(1)$.

This is easily understood since according to the property A above the $k_2 = 14$ population is performing poorly when the k_1 dictates the market periodicity while the population 10 is performing well according D in the hypothetical periods when $k_2 = 14$ dictates the market periodicity. This is only an example of a large class of parameters that lead to predator-prey systems and which may result in the total extinction of one of the species.

Figure 3 : Fraction of the wealth that the species $k_1 = 10$ possesses in Case 1. The traders in the market belonged to 2 species consisting each of 5000 traders. Each trader owed at the beginning 5000 dollars in cache and 5000 shares (worth each 1.4 dollars).

Case 2: competitive species

If one chooses $k_1 = 10$, $k_2 = 26$, the species with extrapolation range 26 gains during the periods when the species $k_1 = 10$ dominates (property B) but species $k_1 = 10$ gains when the species $k_2 = 26$ dominates (property D). It is therefore reasonable that one species can not dominate the other indefinitely. Indeed, a look at the fraction of the wealth held by the species with extrapolation range $k_1 = 10$ reveals alternating eras of dominance (Figure 4). This is also reflected in the alternance between price cycles (~ 56) corresponding to $k_2 = 26$ and price cycles (~ 24) corresponding to $k_1 = 10$. Clearly this alternance between the 2 species corresponds to a classical competitive ecology, in which two competing species take turns in dominating the ecology. Note however that most of the time it is the population k_2 which dominates the wealth. This seems to be a generic tendency in the long runs limit.

Figure 4: Fraction of the wealth that the species $k_1 = 10$ possesses in Case 2. The initial conditions were similar to Figure 3.

Case 3 symbiotic species

In the case $k_1 = 10$, $k_2 = 36$, similarly to the 10 – 26 market, the investors with extrapolation range $k_2 = 36$ are doing better than those with extrapolation range $k_1 = 10$ when $k_1 = 10$ dictates the dynamics (cf. property C). On the other hand $k_1 = 10$ are doing better when the species $k_2 = 36$ dictates the dynamics (cf. D). Hence, we may speculate that again we will find alternating eras of dominance. Figure 5 shows that this is not the case. The difference between this case and the 10 – 26 case is that here the market remains stuck in a "metastable" state: the extrapolation range 36 population never gains enough wealth to dictate long cycles. Thus, the system remains in a state of *symbiosis* throughout the run: the price cycles correspond to the short species extrapolation range span $k_1 = 10$ while 70 – 80% of the wealth stays with the long extrapolation span species k_2 .

For very long k_2 extrapolation ranges, the share of the total wealth detained by k_2 can be even larger (approaching unity).

Figure 5 -Fraction of the wealth that the species $k_1 = 10$ possesses in case 3.

In conclusion [9] has uncovered a quite lively ecology of the traders populations in the LLS model and

"observed phenomena ranging from complete dominance of one population to alternating eras of domination and to symbiosis. . . . Our results suggest that complexity is an intrinsic property of the stock market. The dynamic and complex behavior of the market need not be explained as an effect of external random information. It is a natural property of the market, emerging from the strong nonlinear interaction between the different investor subgroups of the market . . ."

The main source of endogenous dynamics in the LLS model turns out to be the feedback between the market price fluctuations and the wealth of the investors belonging to various species:

- On one hand the wealth of the investors determines their influence on the price changes (at the short range): e.g. the richest determine the periodicity of the boom-crash cycles.

- On the other hand, the variations in the price determine changes in the distribution of wealth, which iterated over longer time intervals, result in changes in the market price cycle periodicity regime.

The entire cycle of rise and fall of a given species can be schematically described as: → The species has by chance a (momentary) winning strategy → Investors belonging to the species gain wealth → Overall wealth of the investors belonging to the species increases → Bids of investors belonging to the species become large → Investor bids influence the market price adversely (self-defeating) → Trading of investors belonging to the species becomes inefficient → Investors lose money → Investors belonging to the species become poor → Species wealth and market relevance vanish → Other species with different strategies become winners → Cycle re-starts (with the new winning strategies).

A few comments are in order:

1. the concept of efficient strategy is only a temporary one as it depends crucially of the state of the market: by its very efficiency at a certain moment, a strategy prepares the seeds of its failure in the future.

2. the biological and cognitive analogies are useful but their limits should be understood:

- in biology, the species selection mechanism is based on the disappearance of the inefficient individuals.

- in the learning adaptive agents' case, the individuals discard losing strategies for new ones.

In the LLS market framework, while it is possible to include the above effects, they are not necessary: the strategies selection takes place automatically by their carriers (traders belonging to the species) losing or gaining: for the market to be efficient, no *a priori* intelligence nor explicit criteria for the evaluation and comparison of market performance are required: just the natural (Adam Smith's "invisible hand") market mechanisms.

3. While the adverse influence on the market price implied by the large orders coming from rich agents' leads automatically to inefficiency in their operations (except for rich agents which follow a buy-and-hold strategy and therefore do not influence (adversely) the market), the mere lack of market influence due to poverty does not guarantee a winning strategy. It is necessary therefore that there are enough strategies and enough agents in the market for insuring its efficiency.

Three Investor Species

One might suspect that the three species dynamics is a natural extension of the two species dynamics. Instead of alternating between two cycle lengths the system may just alternate between the three possible states of dominance. Figures 6-7 shows that this is not at all the case. These figures depict a typical part of the dynamics of a three species market, with extrapolation spans of 10, 141 and 256 respectively. With the introduction of a third species the system has underwent a qualitative change: there is no specific cycle length describing the time series. Instead, we see a mixture of different time scales: the system has become complex. Prediction becomes very difficult, and in this sense the market is much more realistic. Figure 6 shows the power struggle between the three species while the Figure 7 depicts the Fourier transform of the price evolution during this run.

Although the dynamics is complex, it is clear from Figures 6 and 7 that there is an underlying structure, which perhaps may be analyzed by the properties A, B, C, D and their generalizations. For instance it would appear from the Figure 6 that 141 and 256 take turns in dominating while 10 has a chance to a non-vanishing wealth share only occasionally in the transition intervals between 141 and 256 dominated eras. The dynamics generated by only three investor species can be extremely complex, even without any external random influences.

Figure 6: The species wealths in a market with 3 species of extrapolation ranges of respectively 10, 141 and 256 days. Initially the 3 species possessed equal wealth distributed equally between stock and bond. Each species consisted of 1000 traders.

Figure 7: Fourier transform of the stock price time evolution in the market described in Fig 6.

5. Generalized Lotka-Volterra models for markets with multiple species

Inspired by the above facts we devised an effective dynamics that stylized the features uncovered in the LLS model and extended them to a more generic framework. Instead of following in detail the way the market price influence each species and individual i , we assumed that this influence can be represented through multiplying their wealth $w_i(t)$ by stochastic multiplicative factors $\lambda_i(t)$. This is natural in the LLS model in which the investments of the individuals (and consequently their returns) are fractions of their wealth (as implied by the constant relative risk aversion utility functions). The stochastic proportionality between personal returns and personal wealth is consistent with the real data that show that the (annual) individual income distribution is proportional to the individual wealth distribution [10].

We proposed [11] therefore a model including the above stochastic autocatalytic properties of the capital as well as the cooperative, diffusive and competitive/ predatory interactions between the species. The resulting model turned out to be a straightforward generalization [11] of the Lotka-Volterra system (discrete logistic equation) well known previously in population biology:

$$w_i(t+1) = \lambda_i(t)w_i(t) + \sum_k a_k w_k(t) - w_i \sum_k b_k w_k(t)$$

Where the sum is over all the N traders participating in the market. There are a few other (mutually non-exclusive) possible interpretations for w_i in addition to the individual wealth: the wealth associated with a particular investing strategy, the capitalization associated with a particular company/industry, or the number of investors following a common trend (herd).

Particularly interesting cases were studied subsequently:

1) The linear case where the total wealth diverges to larger and larger values (inflation, production):

$$w_i(t+1) = \lambda_i(t)w_i(t) + \sum_k a_k w_k(t)$$

2) The case in which the binary interactions between individuals are expressible in terms of interactions with the total wealth $W = \sum_k w_k$:

$$w_i(t+1) = \lambda_i(t)w_i(t) + aW(t) - bw_iW(t)$$

3) The case in which individual wealth is bounded from below by a certain fraction c of the average wealth $\bar{w} = W/N$ [12]:

$$w_i(t+1) = \lambda_i(t)w_i(t)$$

except if

$$\lambda_i(t)w_i(t) < c\bar{w}$$

when

$$w_i(t+1) = c\bar{w}$$

4) The case of the random multiplicative wealth dynamics

$$w_i(t+1) = \lambda_i(t)w_i(t)$$

with variable number of traders [13]:

- traders which fall below a certain fixed minimal wealth w_{min} drop from the market
- a number of traders proportional to the total wealth increase:

$$\Delta N = c(W(t+1) - W(t))/w_{min}$$

join the market at each time step i.e. the average wealth remains constant in this process:

$$\bar{w} = w_{min}/c$$

.

One assumes that each of the new traders brings an initial investment equal to w_{min} which means that the total amount of added wealth is

$$\Delta N w_{min} = c(W(t+1) - W(t))$$

i.e. a fraction c of the total wealth increase $(W(t+1) - W(t))$.

In most of these systems were assumed asynchronous: at each time step, only one (randomly chosen) w_i was updated. Very striking generic results can be obtained with all these models in certain relevant regimes. We limit ourselves below to the universal scaling properties (power laws).

6. Pareto Law in LLS and Lotka-Volterra models

The efficient market hypothesis and the Pareto law are some of the most striking and basic concepts in economic thinking. It is therefore very important that our models above succeed to connect them in a very essential way.

Let us discuss this in more detail. More than a hundred years ago, Pareto [14] discovered that the number of individuals with wealth (or incomes) with a certain value w is proportional to $w^{-1-\alpha}$. This later became known as the Pareto Law. The LLS model treats the individual investor wealth as a crucial quantity, and it views its feedback relation to the market dynamics as the main source driving the endogenous dynamics of the market.

It turns out that in the conditions in which the participants in the market do not have a systematic advantage one over the other (which is in fact expected in an efficient market), a dynamics of the LLS type leads always to a Pareto law. The actual value of the exponent α depends on the particular parameters used in the model. Mainly, as explained below α is influenced by the social security policy. If one does not allow any individual to become poorer than a certain fraction c of the current average wealth then, for a wide range of conditions $\alpha = 1/(1 - c)$. This is confirmed in Figure 8 which plots the wealth distribution in the LLS model with $k = 3$ and $c = 0.2$ (and $U(W) = \ln W$).

Figure 8: The wealth distribution of the investors in an LLS model with a poverty line of $c = 20\%$ of the average wealth. On a double logarithmic scale one obtains a straight line with slope 2.2 corresponding to an α of 1.2.

The market consisted of 10000 traders and the measurement was performed as a "snapshot" after 1 000 000 "thermalization" market steps. Initially all the traders had equal wealth (\$1000) equally distributed between bond and stock.

In fact it has been proven [11-13] theoretically that any of the effective dynamics of the type 1-4 with λ_i distribution independent on i or w_i leads always to a power law Pareto distribution.

In a wide range of models, the generic rule is that

$$\alpha = 1/(1-c)$$

where c is essentially the market global impact factor [13]:

$c = \text{exogenous new capital ADDED to the market} / \text{increase in stock capitalization due to market price increase}$

Since the increase in the capitalization is the increase in wealth that the owners incur upon their investment of new capital, the ratio can be also interpreted as the long range market return factor

$$c = 1/(\text{long range market return factor}).$$

Let us explain in short how such results were obtained [15]. The crucial observation is that for large w_i values, the non-stationary multiplicative system of interacting w_i 's is

formally equivalent to a statistical mechanical (additive) system in thermal equilibrium when expressed in terms of the variables $u_i(t) = \ln w_i/\bar{w}(t)$. For instance the system 3) is mapped into a system of particles diffusing in an energy potential field u with a ground level $u_0 = \ln c$. In thermal equilibrium, all such systems (independent on the details of the interactions between their particles) have an universal probability distribution discovered by Boltzmann more than 100 years ago:

$$P(u) \sim \exp(-\alpha u)$$

When re-expressed in terms of the original w_i variables this gives a Pareto power law distribution:

$$P(w) \sim w^{-1-\alpha}$$

The exponent α can be estimated from the integrals representing the total wealth and the total number of traders [12]. For instance, in the models 3-4, in the limit of $N \rightarrow \infty$, the result is:

$$\alpha = 1/(1 - w_{min}/\bar{w})$$

i.e.

$$\alpha = 1/(1 - c)$$

Thus, the Pareto law is the exact analogue of the Boltzmann law for stochastic systems that are multiplicative rather than additive.

For finite N the α is given by the implicit transcendental equation [16]:

$$N = [((1 - (N/c)^\alpha)/\alpha)]/[((1 - (N/c)^{\alpha-1})/(\alpha - 1))]$$

which for $N \ll e^{1/c}$ gives approximately:

$$\alpha \sim \ln N/(\ln(N/c)) < 1$$

which incidentally means that in this regime all the wealth belongs to only a few individuals.

In the system 1 defined above, in the appropriate thermodynamic limit The analog result is:

$$\alpha = 1/(1 - c)$$

with

$$c \sim 2a/(<\lambda> + \sigma^2/2)$$

where σ is the standard deviation of λ

And in the case 2:

$$c \sim 2a/(\sigma^2 + a^2)$$

independent on b and $<\lambda>$.

7. Market Efficiency, Pareto Law and Thermal Equilibrium

The formal equivalence between the non-stationary systems of interacting w_i 's and the equilibrium statistical mechanics systems governed by the universal Boltzmann distribution

has far reaching implications: it relates the Pareto distribution to the efficient market hypothesis: In order to obtain a Pareto power law wealth distribution it is necessary and sufficient that the returns of all the strategies practiced in the market are stochastically the same, i.e. there are no investors that can obtain "abnormal" returns.

Therefore, the presence of a Pareto wealth distribution is a measure of the market efficiency in analogy to the Boltzmann distribution whose presence is a measure to thermal equilibrium. Indeed physical systems which are **not** in thermal equilibrium (e.g. are forced by some external field - say by laser pumping) do **not** fulfill the Boltzmann law. Similarly, markets that are not efficient (e.g. when some groups of investors make systematically more profit than others) do not yield power laws (see Fig 9). Optimal market and power laws are the short time and long time faces of the same medal/phenomenon.

This analogy is consistent with the interpretation of market efficiency as analog to the Second law of Thermodynamics:

- one can extract energy (only) from systems that are not in thermal equilibrium
- one can extract wealth (only) from markets that are not efficient.

- by extracting energy from a non-equilibrium thermal system one gets it closer to an equilibrium one.
- by extracting wealth from a non-efficient market one brings it closer to an efficient one

- in the process of approaching thermal equilibrium, one also approaches the Boltzmann energy distribution
- in the process of approaching the efficient market one also approaches the Pareto wealth distribution.

- by having additional knowledge on a thermodynamic system state one can extract additional energy (e.g. Maxwell demons gedanken experiment)
- by having additional knowledge on a financial system one can extract additional wealth.

This double analogy

thermodynamic equilibrium \sim efficient market

Boltzmann law \sim Pareto law

holds in the details of their microscopic origins:

- the convergence to statistical mechanics equilibrium depends on the balance of the probability flow entering and exiting each energy level. This is usually insured microscopically by the fact that the a priori probability for a molecule to gain or loose an energy quanta in a collision is the same for any energy level with the exception of the collisions including molecules in the ground state which can only receive (but not give) energy.
- in the stochastic models 1-4, the convergence of the wealth to the power-law is insured by the balance of flow of investors from one level of $[\log(\text{relative wealth})]$ to another. At the individual level, this is enforced by all the individuals having the same (relative) returns probability distribution (except for the individuals possessing the lowest

allowed wealth). If this condition is not fulfilled, one does not get a wealth distribution power law.

These facts should guide us in the practical runs in establishing which combinations of strategies (or the strategy selection strategies) are producing a realistic market "in the Pareto sense". In Figure 9 one sees the wealth distribution in a model in which there are 2 trader species with slightly different distributions of λ . One sees that even a small violation of the λ uniformity leads to significant departures from the Pareto law which are inconsistent with the historical experimental facts. The absence of such departures in real life is a strong indication of the market efficiency in the weak stochastic sense (that all investors have stochastically the same relative returns distribution).

Figure 9: Wealth distribution for 2 investor species with different return distributions. Model 3 was used with a lower wealth bound of $c = 20\%$.

λ is randomly drawn. For the first species λ is 1.10 or 0.95 with equal probability. For the second "more talented" species λ is 1.11 or 0.96 with equal probability. The 2 species were each composed of 10000 traders with initially equal wealth (1000 dollars each).

The measurement of the wealth distribution was performed after a "thermalization period" of 100 000 wealth updatings.

8. Leptokurtic Market Returns in LLS

It has been long known that the distribution of stock returns is leptokurtic or "fat-tailed". Furthermore, a specific functional form has been suggested for the short-term return distribution (at least in a certain finite range) - the Levy distribution [17]. This feature is present in the LLS model, and is directly related to the Pareto distribution of wealth.

The central limit theorem insures that in a wide range of conditions the distance reached by a random walk of t steps of average squared size s^2 is a Gaussian with standard deviation $s\sqrt{t}$. Suppose that at time $t = 0$ one has N positive numbers $w_i(0)$; $i = 1, \dots, N$ of order 1 and sum $W(0)$. Suppose that at each time step one of the numbers varies (increases or decreases) by a fraction $s_i(t) \ll 1$ extracted from a random distribution with average squared s^2 (and 0 mean). What will be the probability distribution of the sum $W(t)$ after t steps? According to the central limit theorem this would be the Gaussian

$$P(W, t) = 1/(\sqrt{2\pi ts^2})e^{-(W(t)-W(0))^2/2ts^2}$$

since it consists of t steps of average squared size s^2 .

If one interprets $w_i(t)$ as the value of the stocks owned by the trader i at time t , then $W(t) = \sum_i w_i$ is the total market value of the stock and therefore $(W(t) - W(0))/W(0)$ is the relative stock return for the time interval t .

One sees that if the central limit theorem would hold, one would predict a Gaussian stock returns distribution. This is in fact the case for real stocks and time intervals longer than a few weeks. For significantly shorter times t however, the distribution of returns is very different from a Gaussian. Even though the exact shape of the returns distribution is not yet established experimentally, it is generally agreed that in certain ranges (typically "in the tails" - i.e. for large w_i values) it fits better a power law rather than a Gaussian.

Such a situation can in principle be explained by the following scenario:

Suppose that at time $t = 0$ one has an arbitrarily large number of positive numbers $w_i(0)$. Suppose moreover that the probability distribution for the sizes of $w_i(0)$ is

$$P(w) \sim w^{-1-\alpha}$$

Suppose that at each time step one of the w_i 's varies (increases or decreases) by a fraction $s_i(t) \ll 1$ of average squared size s^2 . What will be the probability distribution of the variation of the w_i 's sum $W(t) - W(0)$ after t steps?

One is tempted to think that the correct answer is given by

$$P(W, t) = 1/(\sqrt{2\pi ts^2})e^{-(W(t)-W(0))^2/2ts^2}$$

for some s . However this is wrong. Indeed, assuming such an s exists would imply that the probability for the sum variation $W(t) - W(0)$ to be 10 after a time $t = 1/(2s^2)$ is:

$$P(W(t) = W(0) + 10, t = 1/(2s^2)) \sim e^{-10^2} \sim 10^{-32}$$

while in reality a lower bound for the probability of getting $W(t) - W(0) = 10$ in just one step it is obviously that given by

$$P(w) \sim w^{-1-\alpha}$$

I.e. $P(W(t) = W(0) + 10, t = 1/(2s^2))$ is at least of order $10^{-1-\alpha}$ which for $\alpha < 2$ means it is larger than 10^{-3} !

This coarse estimations highlights the difference between the Gaussian distributions and the distributions generated by random walks with power distributed step sizes (called Levy distributions [18,17]): the presence of w_i 's of arbitrary size implied by a power law distribution insures that the large returns distribution is dominated by the power law of the individual step sizes rather than the combinatorics of the multiple events characterizing the Gaussian system.

One sees now that the systems 1-4 (and consequently LLS) are exactly of the type one needs to explain returns distribution power tails:

- on one hand according to section 6, the models 1-4 (and consequently LLS) insure a power distribution of w_i 's.

- on the other hand, in the models 1-4 the variation of the stock index $W(t)$ is the sum of the variations of the individual $w_i(t)$'s.

- these variations $w_i(t+1) - w_i(t)$ are stochastic fractions $s_i(t) = \lambda_i(t) - 1$ of w_i as above (the fact that $\lambda_i(t) - 1$ has not 0 mean is taken care by working actually with $u_i = \ln(w_i/\bar{w})$).

Therefore, according to the argument above, the effective models 1-4, which reflect the stochastic proportionality in LLS between individual wealth, individual investments and individual gains/losses predict that the price fluctuations in the LLS model will obey a Levy distribution (and in particular fit a power in some range of the "tail").

There is a proviso for this argument to hold: the number of individual terms N has to be larger than the number of time steps t . Otherwise the finite size of the sample of

w_i 's will show up in the absence of sizes w_i larger than a certain value. In fact for $t \gg N$ one recovers (slowly) the Gaussian distribution.

In the LLS case, if the portfolio updatings are performed simultaneously by all the investors, the unit time step corresponds already to a time $t = N$. In order to verify the (truncated) Levy distribution and the power "tail" predictions, one has to look at the dynamics at a finer time scale. We therefore performed [12] LLS runs in which at each time step only one trader i reconsiders its portfolio investment proportion $X(i)$. In such conditions, one expects to obtain a distribution which fits in a significant range a power law (up to large w_i values where the finite N effects become important).

This is in fact confirmed by the numerical experiments. While for the global updating steps one gets a Gaussian distribution, for the trader-by-trader procedure one obtains a truncated Levy distribution (Fig. 10).

Figure 10: The returns distribution in the LLS model in which only one trader re-evaluates his/her portfolio per unit time. $c = 0.2$, $k = 3$, $U = \ln W$.

The market contained 10 000 traders with initially equal wealth and portfolio composition (half in stock and half in bonds).

The number of market returns in intervals of 0.001 were measured during 5 000 000 market steps (after an initial 1 000 000 equilibration period).

Note that in the central region of the short time returns (before the cut-off becomes relevant) the Levy distribution is characterized by an α equal to the exponent α of the traders' wealth distribution.

As explained in Section 6, in certain conditions (e.g. model 4) one can interpret α as $\alpha = 1/(1 - 1/(\text{long term market return factor}))$.

Therefore the analysis above relates the **stochastic distribution of the short term returns** to the **value of the long term returns** via the **exponent of the Pareto power law of individual incomes/wealths**.

Moreover the **long term returns** are related (e.g. model 3) via the value of the Pareto exponent α to the ratio (\bar{w}/w_{min}) between the average wealth/income and the **lowest admissible wealth/income**: the value $\alpha \sim 1.4$ implies (cf. models 3-4) for both these quantities values of the order of

$$1/c = \alpha/(\alpha - 1) \sim 3.5.$$

Speculatively, one may try to use the above relation in order to explain the stability of the Pareto constant α over the past century (and over the various countries and economies).

Indeed one may relate the implied value 3.5 for both \bar{w}/w_{min} and the long term market return to some basic biological invariant which is the average number of dependents / offsprings humans have:

- if w_{min} is the minimal amount necessary to keep alive one person in a certain society (cost of life), then the average income \bar{w} will have to equal roughly w_{min} times the number of dependents the average household head has to support.

- at the social level, the total effort/wealth that one generation invests in the economy has to result in an economical growth capable to support a population larger by a factor equal to the average number of descendents.

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References

1. M. Levy, H. Levy, S. Solomon, *Microscopic Simulation of Financial Markets*, Academic Press, New York, 2000.
2. Levy, M., Levy H. and S. Solomon, "A Microscopic Model of the Stock Market : Cycles, Booms, and Crashes," *Economics Letters*, 45, 1994.
- Moss de Oliveira S., H. de Oliveira and D. Stauffer, *Evolution, Money, War and Computers*, B.G. Teubner Stuttgart-Leipzig 1999.
- Solomon, S. [1995], The microscopic representation of complex macroscopic phenomena, *Annual Reviews of Computational Physics II*, 243-294, D. Stauffer (editor), World Scientific 1995.
- Solomon S. Behaviorally realistic simulations of stock markets; Traders with a soul, *Computer Physics Communications* 121-122 (1999) 161
3. Hellthaler T., "The Influence of Investor Number on a Microscopic Market", *Int. J. Mod. Phys.C* 6, 1995
- Kohl R., "The Influence of the Number of Different Stocks on the Levy, Levy Solomon Model," *Int. J. Mod. Phys. C* 8, 1997.
4. Levy, M., Levy H. and S. Solomon, "Simulation of the Stock Market: The Effects of Microscopic Diversity," *Journal de Physique I*, 5, 1995.
5. Anderson P. W., J. Arrow and D. Pines [1988], eds. *The Economy as an Evolving Complex System* (Redwood City, Calif.: Addison-Wesley, 1988);
6. Egenter E., T. Lux and D. Stauffer, "Finite Size Effects in Monte Carlo Simulations of Two Stock Market Models," *Physica A* 268 ,1999.
7. Fama, E., and K. French, "Permanent and Temporary Components of Stock Prices," *Journal of Political Economy*, 96, 1988.
8. Shiller, Robert J., "Do Stock Returns Move too Much to be Justified by Subsequent Changes in Dividends?" *American Economic Review*, 1981.
9. Levy M., Persky N., and Solomon S., "The Complex Dynamics of a Simple Stock Market Model," *International Journal of High Speed Computing*, 8, 1996.
http://www.ge.infn.it/econophysics/papers/solomonpapers/stock_ex_model.ps.gz see also
- Farmer J.D., "Market Force, Ecology and Evolution," e-print adap-org/9812005. See also J.D. Farmer and S. Joshi, "Market Evolution Toward Marginal Efficiency" SFI report 1999.
10. Levy, M., and S. Solomon, "New Evidence for the Power Law Distribution of Wealth", *Physica, A* 242, 1997.
- Levy, M. "Are Rich People Smarter?" UCLA Working Paper 1997.
- Levy, M. "Wealth Inequality and the Distribution of Stock Returns" Hebrew University Working Paper 1999.
11. Solomon S. and M. Levy, "Spontaneous Scaling Emergence in Generic Stochastic Systems," *International Journal of Modern Physics C* , 7(5), 1996.
- S. Solomon, Stochastic Lotka-Volterra systems of competing auto-catalytic agents lead generically to truncated Pareto power wealth distribution, truncated Levy distribution of

- market returns, clustered volatility, booms and crashes, In Computational Finance 97, Eds. A-P. N. Refenes, A.N. Burgess, J.E. Moody, (Kluwer Academic Publishers 1998)
- O. Biham, O. Malcai, M. Levy and S. Solomon, Phys Rev E 58, 1352, (1998).
12. Levy M., S. Solomon, "Power Laws are Logarithmic Boltzmann Laws" International Journal of Modern Physics C , 7 (4) 1996.
13. Aharon Blank and Sorin Solomon cond-mat/0003240 Power Laws and Cities Population
14. V. Pareto, Cours d'Economie Politique, 2 (1897).
15. Sorin Solomon, cond-mat/9901250; Generalized Lotka-Volterra (GLV) Models and Generic Emergence of Scaling Laws in Stock Markets; To appear in the proceedings of the "International Conference on Computer Simulations and the Social Sciences, Paris 2000" Hermes Science Publications.
16. O. Malcai, O. Biham and S. Solomon, Phys. Rev. E, 60, 1299, (1999).
17. B. B. Mandelbrot, Comptes Rendus 232, 1638 (1951)
- H. A. Simon and C. P. Bonini, Amer. Econ. Rev. 607 (1958)
- Mantegna, R. N., "Levy Walks and Enhanced Diffusion in the Milan Stock Exchange." Physica A, 179, 1991.
- Mantegna, R. N., and Stanley, H. E., "Scaling Behavior in the Dynamics of an Economic Index." Nature, 376, 1995.
18. Paul Levy, Theorie de l'Addition des Variables Aleatoires, Gauthier-Villiers, Paris 1937.
- 19 Lotka, A.J., (editor) Elements of Physical Biology, Williams and Wilkins, Baltimore, 1925;
- V. Volterra [1926], Nature, 118, 558.

Figure Captions

Figure 1: The Flow Chart of the LLS market framework

Figure 2 : The Fourier transform of the price in a market with one species with extrapolation range $k = 10$. The market contained 10000 traders that had initially equal wealth invested half in stock and half in bonds.

Figure 3 : Fraction of the wealth that the species $k_1 = 10$ possesses in Case 1. The traders in the market belonged to 2 species consisting each of 5000 traders. Each trader owed at the beginning 5000 dollars in cash and 5000 shares (worth each 1.4 dollars).

Figure 4: Fraction of the wealth that the species $k_1 = 10$ possesses in Case 2. The initial conditions were similar to Figure 3.

Figure 5 -Fraction of the wealth that the species $k_1 = 10$ possesses in case 3.

Figure 6: The species wealths in a market with 3 species of extrapolation ranges of respectively 10, 141 and 256 days. Initially the 3 species possessed equal wealth distributed equally between stock and bond. Each species consisted of 1000 traders.

Figure 7: Fourier transform of the stock price time evolution in the market described in Fig 6.

Figure 8: The wealth distribution of the investors in an LLS model with a poverty line of $c = 20\%$ of the average wealth. On a double logarithmic scale one obtains a straight line with slope 2.2 corresponding to an α of 1.2.

The market consisted of 10000 traders and the measurement was performed as a "snapshot" after 1 000 000 "thermalization" market steps. Initially all the traders had equal wealth (\$1000) equally distributed between bond and stock.

Figure 9: Wealth distribution for 2 investor species with different return distributions. Model 3 was used with a lower wealth bound of $c = 20\%$.

λ is randomly drawn. For the first species λ is 1.10 or 0.95 with equal probability. For the second "more talented" species λ is 1.11 or 0.96 with equal probability. The 2 species were each composed of 10000 traders with initially equal wealth (1000 dollars each).

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