A new proof of Chen's theorem for Markoff graphs

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Source

William Chen, Nonabelian level structures, Nielsen equivalence, and Markoff triples, 2021.

Daniel E. Martin, A new proof of Chen's theorem for Markoff graphs, 2025.

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The two papers, William Chen's "Nonabelian level structures, Nielsen equivalence, and Markoff triples" (2021/2024) and Daniel E. Martin's "A new proof of Chen's theorem for Markoff graphs" (2025), are distinct in their primary contributions, scope, and methodology, although Martin's paper is a **direct follow-up and provides an alternative proof for a specific result from Chen's work**.

Here are the key differences:

- **Primary Contribution and Relationship:**
  - **Chen's paper** (published online in 2021 and in Annals of Mathematics in 2024) establishes a **broad theoretical framework** based on congruences on the degree of maps from components of Hurwitz spaces of covers of elliptic curves to the moduli stack of elliptic curves. This framework is then applied to various problems, including the Markoff equation. Crucially, it provides a **resolution to the conjecture of Bourgain, Gamburd, and Sarnak (originally by Baragar in 1991) for all but finitely many primes p, implying strong approximation for Markoff triples**.
  - **Martin's paper** (published online in 2025) acknowledges Chen's significant contribution and specifically provides an **alternative proof for a "special case/corollary" of Chen's main theorem**. This particular result is that "the size of any connected component of the Markoff mod p graph is divisible by p".
- **Scope and Generality:**
  - **Chen's paper** is **more general and foundational**, introducing concepts like nonabelian level structures, Nielsen equivalence for generating pairs of finite groups, and detailed analysis of various Hurwitz spaces and moduli stacks. It applies its congruences not just to the

Markoff equation but also to generalized Markoff equations and the connectedness of Hurwitz stacks.
  - **Martin's paper** has a **narrower, more focused scope**, concentrating solely on the Markoff equation (x^2 + y^2 + z^2 = 3x1x2x3) and its associated Markoff graph Gp over Fp. Its goal is to provide a simpler or different pathway to a specific, albeit crucial, divisibility property related to this equation.
- **Methodology of Proof:**
  - **Chen's proof** is deeply rooted in **algebraic geometry and combinatorial group theory**. It involves sophisticated concepts such as the *Higman invariant*, analysis of *sheaves of relative differentials and dualizing sheaves*, *deformation theory of algebraic stacks*, and detailed calculations involving *character varieties of SL2-representations*. The core argument establishes congruences on the *degree of the "forgetful map"* from Hurwitz spaces, which translates to divisibility properties of orbit cardinalities.
  - **Martin's alternative proof** employs a **more arithmetic and direct combinatorial approach**. It introduces "Penner coordinates" to transform the Markoff equation and then uses summations over the vertices of a connected component of the Markoff graph, leveraging properties of Vieta involutions and modular arithmetic to show the divisibility by p.
- **Chronological Context:**
  - **Chen's paper** was initially posted on arXiv in November 2020 and updated in June 2021, with its final version published in *Annals of Mathematics* in 2024. It represented a **breakthrough** in the field.
  - **Martin's paper** was published online in June 2025, building directly on Chen's seminal work and offering a **new perspective on a proven result**. Martin explicitly credits Chen for the original theorem and acknowledges the collaborative discussions that contributed to his new proof.

In essence, Chen's paper provides the overarching theory and the initial proof of the divisibility theorem (Theorem 1.2.5 / 5.5.4 in Chen, Theorem 1.4 in Martin), which was critical in resolving a long-standing conjecture. Martin's paper then offers an entirely different, arguably more elementary, method to arrive at that specific divisibility result for the Markoff equation.