Johnson–Cook VUMAT for Plasticity, Damage, and Thermal Softening

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Overview

A VUMAT was developed to implement the Johnson–Cook constitutive formulation. A VUMAT is a user-defined material subroutine in Abaqus/Explicit [1] that specifies the stress update algorithm and governs the material behaviour at each integration point. It allows controlling the constitutive response, including the calculation of stresses, internal variables, and state-dependent properties, such as damage evolution.

The VUMAT was written to incorporate the Johnson–Cook constitutive model with strain-rate sensitivity, thermal softening, damage initiation and evolution, and element deletion. The implementation also includes the Taylor–Quinney effect to estimate the temperature rise resulting from plastic deformation.

Stress Update

In finite element analysis, the governing equation of motion is derived by applying the principle of virtual work and discretising the domain spatially. The resulting semi-discrete dynamic equilibrium equation for element e is:

$$\mathbf{M}^{(e)}\ddot{\mathbf{U}}^{(e)} + \mathbf{C}^{(e)}\dot{\mathbf{U}}^{(e)} + \mathbf{R}_{\text{int}}^{(e)} = \mathbf{R}_{\text{ext}}^{(e)} \tag{1}$$

where $\mathbf{M}^{(e)}$ is the element mass matrix, $\mathbf{C}^{(e)}$ is the damping matrix, and $\mathbf{U}^{(e)}$ is the nodal displacement vector. $\mathbf{R}_{\text{int}}^{(e)}$ and $\mathbf{R}_{\text{ext}}^{(e)}$ represent the internal and external force vectors, respectively, arising from the spatial integration of stress, body forces, and surface tractions. The VUMAT subroutine is called at each integration point within each element, and it updates the stress and state variables for a specific increment of time. In the context of a dynamic explicit formulation, the VUMAT is used in the calculation of internal forces, which for an element are represented by $\mathbf{R}_{\text{int}}^{(e)}$ in Equation (1). Abaqus/Explicit updates the nodal displacements using the central difference time integration scheme:

$$\mathbf{U}^{n+1} = \mathbf{U}^{n} + \delta t \dot{\mathbf{U}}^{n+1/2} \tag{2}$$

Once the updated displacement vector $\mathbf{U}^{n+1}$ is obtained, the strain increment $\Delta\boldsymbol{\varepsilon}$ is computed from the displacement difference over the time step:

$$\Delta\boldsymbol{\varepsilon} = \mathbf{B}\left(\mathbf{U}^{n+1} - \mathbf{U}^{n}\right) = \mathbf{B}\Delta\mathbf{U} \tag{3}$$

Here, $\mathbf{B}$ is the strain-displacement matrix, and $\Delta\mathbf{U}$ represents the displacement increment. This strain increment is passed by reference to the VUMAT. To compute the stress tensor based on the

Johnson–Cook formulation, the following nonlinear equation must be solved:

$$f = \sigma_{\text{eq}} - \sigma_{\text{JC}} \tag{4}$$

where f is the yield function and yielding occurs when $f = 0$. If $f < 0$, the material is in the elastic regime. Expanding this equation using the Johnson–Cook plasticity and the von Mises equivalent stress [2] for $f = 0$ yields:

$$\sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'} = \left[A + B(\varepsilon^{pl} + \Delta\varepsilon_{\text{eq}}^{pl})^n\right]\left[1 + C\ln\left(\frac{\Delta\varepsilon_{\text{eq}}^{pl}/\Delta t}{\dot{\varepsilon}_0}\right)\right][1 - (T^*)^m] \tag{5}$$

where $\boldsymbol{\sigma}'$ is the deviatoric stress tensor. Equation (5) enforces consistency: the stress state must lie on the yield surface. However, the equation cannot be solved analytically since the stress tensor components are unknown, resulting in six unknows and only one equation. This challenge is addressed by recognising that the direction of plastic flow is governed by the associative flow rule, which dictates that the plastic strain increment is normal to the yield surface. This can be written in terms of the yield function, f, as:

$$d\boldsymbol{\varepsilon}^p = d\lambda\frac{\partial f}{\partial \boldsymbol{\sigma}} \tag{6}$$

where $d\boldsymbol{\varepsilon}^p$ is a 6-dimensional vector when expressed in Voigt notation that represents the increment of strain, $d\lambda$ is a scalar, the plastic multiplier, and $\partial f/\partial\boldsymbol{\sigma}$ is the direction of plastic strain increment. Using the von Mises equivalent stress, Equation (6) may be rewritten as follows:

$$d\boldsymbol{\varepsilon}^p = \frac{3}{2}d\varepsilon_{\text{eq}}^{pl}\frac{\boldsymbol{\sigma}'}{\sigma_{\text{eq}}} \tag{7}$$

This allows to reduce the six unknowns of Equation (5) to solving for a single scalar quantity, $d\varepsilon_{\text{eq}}^{pl}$. The return mapping algorithm assumes a trial stress based on an elastic predictor and iteratively adjusts $d\varepsilon_{\text{eq}}^{pl}$ to satisfy the consistency condition. When advancing from step n to $n+1$ the total strain increment ($\Delta\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{n+1} - \boldsymbol{\varepsilon}^n$) is known, however, it is not known whether it will cause plastic flow. Therefore, for a step n, it is assumed that the entire strain increment is elastic:

$$\boldsymbol{\sigma}^{\text{trial}} = \boldsymbol{\sigma}^{(n)} + \mathbf{C} : \Delta\boldsymbol{\varepsilon} \tag{8}$$

If, for the calculated trial stress, $f \leq 0$, the stress is elastic. Otherwise, a correction must be applied:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{trial}} - \mathbf{C} : \Delta\boldsymbol{\varepsilon} \tag{9}$$

The increment of equivalent plastic strain, $\Delta\varepsilon_{\text{eq}}^{pl}$, and the stress state are accepted once the difference between the equivalent stress and the Johnson–Cook stress is less than a defined tolerance:

$$\begin{aligned}&\sqrt{\frac{3}{2}\left[\left(\boldsymbol{\sigma}^{\text{trial}} - \Delta\varepsilon_{\text{eq}}^{pl}\,\mathbf{C}\,\mathbf{n}\right)' : \left(\boldsymbol{\sigma}^{\text{trial}} - \Delta\varepsilon_{\text{eq}}^{pl}\,\mathbf{C}\,\mathbf{n}\right)'\right]} \\ &\quad - \left[A + B\left(\varepsilon^{pl} + \Delta\varepsilon_{\text{eq}}^{pl}\right)^n\right]\left[1 + C\ln\left(\frac{\Delta\varepsilon_{\text{eq}}^{pl}}{\Delta t\,\dot{\varepsilon}_0}\right)\right][1 - (T^*)^m] < \text{tol}\end{aligned} \tag{10}$$

Damage Model

Damage is implemented using the Johnson–Cook damage model [3]. The evolution of damage is described in the Johnson–Cook model with the strain at fracture (ε^f) for each element using the following equation:

$$\begin{aligned} \varepsilon^f &= \left[D_1 + D_2 e^{D_3 \sigma^*}\right] \left[1 + D_4 \ln(\dot{\varepsilon}^*)\right] \left[1 + D_5 T^*\right] \\ \sigma^* &= \frac{\sigma_m}{\sigma_{\mathrm{VM}}}, \quad \sigma_m = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} \end{aligned} \tag{11}$$

where D_1 to D_5 are material constants that describe the material's susceptibility to damage, and σ^* is the stress triaxiality factor. The strain at fracture is used in the damage initiation parameter ω_D:

$$\omega_D = \int_0^t \frac{d\varepsilon^{pl}}{\varepsilon^f} \tag{12}$$

When the integral in Equation (12) reaches 1, damage initiates. From that point on, the progression of damage is determined by the damage variable D, given by:

$$D = \int_{t_0}^t \frac{l_c \, d\varepsilon^{pl}}{u_f^{pl}} \tag{13}$$

l_c represents the characteristic length of the element, u_f^{pl} is the plastic displacement at failure, and t_0 is the time at which damage initiates. After damage initiates, the nominal Cauchy stress tensor $\boldsymbol{\sigma}$ follows a degradation law that relates it to the effective stress $\boldsymbol{\sigma}^{eff}$, which corresponds to the undamaged material response. This is expressed by the following isotropic damage rule:

$$\boldsymbol{\sigma} = (1 - D)\, \boldsymbol{\sigma}^{eff} \tag{14}$$

This formulation implies a uniform degradation of the material's stress-carrying capacity as damage evolves. The damage variable D increases from 0 (no damage) to 1 (complete failure), and directly scales the stress. As a result, once $D = 1$, the stress vanishes and the material can no longer sustain load. In the VUMAT in this repository, the lack of load-carrying capacity associated with full damage can be used as a criterion for element deletion in the explicit dynamic analysis.

Taylor–Quinney formulation

The Taylor–Quinney [4] coefficient was included to represent the fraction of plastic work converted into heat. The full formulation was implemented to maintain consistency with standard Johnson–Cook implementations and to enable a general representation of high strain-rate deformation behaviour.

The Taylor–Quinney expression is given by:

$$\Delta T = \frac{\beta \, \delta W^{pl}}{\rho \, C_p} \tag{15}$$

In this formulation, β denotes the Taylor–Quinney coefficient, which defines the fraction of plastic work converted into heat. The term δW^{pl} represents the incremental plastic work, ρ is the material density, and C_p is the specific heat capacity.

Algorithm 1 Return Mapping Algorithm for Johnson–Cook Plasticity with Damage (VUMAT)

for each integration point i **do**
  if first increment (`SV1` == 0) **then**
    Compute trial stress using Hooke's law
    Initialise state variables (strain, temperature, etc.)
    Store stress in history for damage evaluation
  else
    Retrieve history values (`SV2--SV9`)
    Set $\sigma_{\text{old}} \leftarrow$ previous stress
    Compute trial stress: $\sigma^{\text{trial}} = \sigma^{\text{old}} + \mathbf{C} : \Delta\varepsilon$
    Compute $\sigma^{\text{dev}}, \sigma_{\text{eq}}$ from σ^{trial}
    Set plastic direction $\mathbf{n} = \frac{3}{2}\frac{\sigma^{\text{dev}}}{\sigma_{\text{eq}}}$
    Precompute $\mathbf{Cn}$ for stress corrector
    Initialise $\Delta\varepsilon^{pl}_{\text{eq}} \leftarrow 0$
    Bracket solution for plastic increment (min/max)
    while not converged and iteration limit not exceeded **do**
      $\varepsilon^{pl}_{\text{eq,iter}} \leftarrow \varepsilon^{pl}_{\text{old}} + \Delta\varepsilon^{pl}_{\text{eq}}$
      Compute stress: $\sigma = \sigma^{\text{trial}} - \Delta\varepsilon^{pl}_{\text{eq}} \cdot (\mathbf{Cn})$
      Compute σ_{eq} from updated stress
      Evaluate σ_{JC} using current $\varepsilon^{pl}, T, \dot{\varepsilon}$
      if $|\sigma_{\text{eq}} - \sigma_{\text{JC}}| <$ tolerance **then**
        break
      end if
      Update bisection interval and midpoint
    end while
    Store final stress and updated state variables
    Compute work and plastic work
    Update energy terms
    Evaluate fracture strain ε^f using JC damage model
    Update damage initiation: $\omega_D \leftarrow \omega_D + |\Delta\varepsilon^{pl}/\varepsilon^f|$
    if $\omega_D \geq 1$ **then**
      Cap $\omega_D \leftarrow 1$
      Update damage: $D \leftarrow D + |\Delta\varepsilon^{pl} \cdot l_c/u^p_f|$
      if $D \geq 1$ **then**
        $D \leftarrow 1$
        Apply stress softening: $\boldsymbol{\sigma} \leftarrow (1 - D) \cdot \boldsymbol{\sigma}$
        Flag element for deletion
      end if
    end if
  end if
end for

References

[1] Dassault Systèmes. *Abaqus 2024 Documentation*. https://help.3ds.com/. Dassault Systèmes Simulia Corp. 2024.

[2] Gordon R. Johnson and William H. Cook. "A Constitutive Model and Data for Metals Subjected to Large Strains, High Strain Rates and High Temperatures". In: *Proceedings of the 7th International Symposium on Ballistics*. The Hague, Netherlands, 1983, pp. 541–547.

[3] Gordon R. Johnson and William H. Cook. "Fracture Characteristics of Three Metals Subjected to Various Strains, Strain Rates, Temperatures and Pressures". In: *Engineering Fracture Mechanics* 21.1 (1985), pp. 31–48. DOI: 10.1016/0013-7944(85)90052-9.

[4] G. I. Taylor and H. Quinney. "The latent energy remaining in a metal after cold working". In: *Philosophical Transactions of the Royal Society A* 143 (1934), pp. 307–326.