Problem Sheet 7.2

Exercise 1

a) $\lambda x^{o}. x$

The type is $o \to o$ and the derivation is:

$$\frac{x: o \vdash x: o}{\vdash \lambda x^{o}. x: o \to o}$$

b) $\lambda x^{o}. \lambda y^{o}. \lambda z^{o\to o}. zx$

The type is $o \to o \to (o \to o) \to o$. Considering $\Gamma = x: o, y: o, z: o \to o$, the typing derivation is:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\overline{\Gamma \vdash z: o \to o} \quad \overline{\Gamma \vdash x: o}}{\Gamma \vdash zx: o}}{x: o, y: o, z: o \to o \vdash zx: o}}{x: o, y: o \vdash \lambda z^{o\to o}. zx: (o \to o) \to o}}{x: o \vdash \lambda x^{o}. \lambda y^{o}. \lambda z^{o\to o}. zx: o \to (o \to o) \to o}}{\vdash \lambda x^{o}. \lambda y^{o}. \lambda z^{o\to o}. zx: o \to o \to (o \to o) \to o}$$

c) $\lambda f^{(o\to o)\to o}. \lambda g^{o\to o\to o}. f(\lambda x^{o}. gxx)$

The type is $\big((o \to o) \to o\big) \to (o \to o \to o) \to o$. Considering $\Gamma = f: (o \to o) \to o, g: o \to o \to o$, the typing derivation is:

$$\cfrac{\cfrac{\cfrac{\cfrac{\overline{\Gamma \vdash f: (o \to o) \to o} \quad \cfrac{\cfrac{\overline{\Gamma, x: o \vdash g: o \to o \to o} \quad \overline{\Gamma, x: o \vdash x: o}}{\Gamma, x: o \vdash gx: o \to o} \quad \overline{\Gamma, x: o \vdash x: o}}{\Gamma \vdash \lambda x^{o}. gxx: o \to o}}{\Gamma \vdash f(\lambda x^{o}. gxx): o}}{f: \big((o \to o) \to o\big), g: (o \to o \to o) \vdash f(\lambda x^{o}. gxx): o}}{\cfrac{f: \big((o \to o) \to o\big) \vdash \lambda g^{o\to o\to o}. f(\lambda x^{o}. gxx): (o \to o \to o) \to o}{\vdash \lambda f^{(o\to o)\to o}. \lambda g^{o\to o\to o}. f(\lambda x^{o}. gxx): \big((o \to o) \to o\big) \to (o \to o \to o) \to o}}$$

Exercise 2

The term $(\lambda x. x)(\lambda y. y)$ can be typed. According to the derivation rule for applications:

$$\frac{\Gamma \vdash M: \tau \to \sigma \quad \Gamma \vdash N: \tau}{\Gamma \vdash MN: \sigma}$$

The argument of M must have the same type as N. Therefore, if M is of type $\tau \to \tau$ the type of N must be τ. Expressions like $\backslash x. xx$ cannot be typed since it would lead to a contradiction where x must be of type $\tau \to \sigma$ and τ at the same time.

Exercise 3

$N = (o \to o) \to o \to o$

a) Considering $\Gamma = n\colon N, f\colon o \to o, x\colon o$, the typing derivation for the successor operator is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\theta = o,\ \eta = o}{\Gamma \vdash f\colon \theta \to \eta} \quad \dfrac{\dfrac{\dfrac{\tau = o \to o, \omega = o, \theta = o}{\Gamma \vdash n\colon \tau \to (\omega \to \theta)} \quad \dfrac{\tau = o \to o}{\Gamma \vdash f\colon \tau}}{\Gamma \vdash nf\colon \omega \to \theta} \quad \dfrac{\omega\colon o}{\Gamma \vdash x\colon \omega}}{\Gamma \vdash nfx\colon \theta}}{n\colon N, f\colon o \to o, x\colon o \vdash f(nfx)\colon \eta}}{n\colon N, f\colon o \to o \vdash \lambda x^o.f(nfx)\colon \alpha = \epsilon = o \to \eta}}{n\colon N \vdash \lambda f^{o \to o}.\lambda x^o.f(nfx)\colon \alpha = \gamma\colon (o \to o) \to \epsilon}}{\vdash \lambda n^N.\lambda f^{o \to o}.\lambda x^o.f(nfx)\colon \alpha = N \to \gamma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f\colon o \to o} \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash n\colon \tau \to N} \quad \dfrac{}{\Gamma \vdash f\colon o \to o}}{\Gamma \vdash nf\colon o \to o} \quad \dfrac{}{\Gamma \vdash x\colon o}}{\Gamma \vdash nfx\colon o}}{n\colon N, f\colon o \to o, x\colon o \vdash f(nfx)\colon o}}{n\colon N, f\colon o \to o \vdash \lambda x^o.f(nfx)\colon o \to o}}{n\colon N \vdash \lambda f^{o \to o}.\lambda x^o.f(nfx)\colon (o \to o) \to o \to o}}{\vdash \lambda n^N.\lambda f^{o \to o}.\lambda x^o.f(nfx)\colon N \to N}$$

b) $N \to N \to N$

c) $add \triangleq \lambda m^N.\lambda n^N \lambda f^{o \to o}.\lambda x^o.mf(nfx)$

d) Let $\Gamma = m\colon N, n\colon N, f\colon o \to o, x\colon o$, the typing derivation for the addition term is:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\tau \to (\omega \to \theta) = N}{\Gamma \vdash m\colon \tau \to (\omega \to \theta)} \quad \dfrac{\tau = o \to o}{\Gamma \vdash f\colon \tau}}{\Gamma \vdash mf\colon \omega \to \theta} \quad \dfrac{\dfrac{\dfrac{\mu \to (\Omega \to \omega)\colon N}{\Gamma \vdash n\colon \mu \to (\Omega \to \omega)} \quad \dfrac{\mu = o \to o}{\Gamma \vdash f\colon \mu}}{\Gamma \vdash nf\colon \Omega \to \omega} \quad \dfrac{\Omega = o}{\Gamma \vdash x\colon \Omega}}{\Gamma \vdash nfx\colon \omega}}{\Gamma \vdash mf(nfx)\colon \theta}}{m\colon N, n\colon N, f\colon o \to o \vdash \lambda x^o.mf(nfx)\colon \eta = o \to \theta}}{m\colon N, n\colon N \vdash \lambda f^{o \to o}.\lambda x^o.mf(nfx)\colon \epsilon = (o \to o) \to \eta}}{m\colon N \vdash \lambda n^N.\lambda f^{o \to o}.\lambda x^o.mf(nfx)\colon \alpha = N \to \gamma}}{\vdash \lambda m^N.\lambda n^N.\lambda f^{o \to o}.\lambda x^o.mf(nfx)\colon \alpha = N \to \gamma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash m\colon N} \quad \dfrac{}{\Gamma \vdash f\colon o \to o}}{\Gamma \vdash mf\colon o \to o} \quad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash n\colon N} \quad \dfrac{}{\Gamma \vdash f\colon o \to o}}{\Gamma \vdash nf\colon o \to o} \quad \dfrac{}{\Gamma \vdash x\colon o}}{\Gamma \vdash nfx\colon o}}{\Gamma \vdash mf(nfx)\colon o}}{m\colon N, n\colon N, f\colon o \to o \vdash \lambda x^o.mf(nfx)\colon o \to o}}{m\colon N, n\colon N \vdash \lambda f^{o \to o}.\lambda x^o.mf(nfx)\colon (o \to o) \to o \to o}}{m\colon N \vdash \lambda n^N.\lambda f^{o \to o}.\lambda x^o.mf(nfx)\colon N \to N}}{\vdash \lambda m^N.\lambda n^N.\lambda f^{o \to o}.\lambda x^o.mf(nfx)\colon N \to N \to N}$$

e) No, it cannot be similarly typed. The exponentiation term is:

$$exp \triangleq \lambda m.\lambda n.\lambda f.\lambda x.(nm)fx$$

Since n is applied to m, and n expects a function but the type of m does not match $\alpha \to \alpha$, the expression cannot be similarly typed. The first argument, m, must have a smaller type than n.