Haskell Notes

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Equivalence Relation

A relation $R \subseteq S \times S$ is an **equivalence relation** whenever, for $s, t, u \in S$:

- R is **reflexive**, i.e., $(s, s) \in R$;
- R is **symmetric**, i.e., if $(s, t) \in R$, then $(t, s) \in R$;
- R is **transitive**, i.e., if $(s, t) \in R$ and $(t, u) \in R$, then $(s, u) \in R$.

This definition applies to various contexts in mathematics and computer science. Equivalence relations are useful in defining partitions of sets and modeling relationships like equality, congruence, or similarity.

Define the Renaming $[y/x]$, **in the most general way:**

- regardless of the free/bound/binding position,
- to be applied only if y does not occur in M.

$$x[y/x] \equiv y$$

$$z[y/x] \equiv z \quad \text{if } x \neq z$$

$$(MN)[y/x] \equiv (M[y/x])(N[y/x])$$

$$(\lambda x.M)[y/x] \equiv \lambda y.(M[y/x])$$

$$(\lambda z.M)[y/x] \equiv \lambda z.(M[y/x]) \quad \text{if } x \neq z$$

α-Equivalence Relation

Define when two lambda-terms are **"the same up to renaming of bound variables"**.

Definition

α-equivalence: The smallest congruence relation on λ terms such that for all terms M and all variables y that do not occur in M:

$$\lambda x.M =_\alpha \lambda y.(M[y/x])$$

Key Properties

- **"Equivalence"**: must satisfy reflexivity, symmetry, and transitivity.
- It must respect the structures of lambda terms and the free/bound occurrences therein:
  - If $M = M'$ and $N = N'$, then: $MN = M'N'$
  - If $M = M'$, then: $\lambda x.M = \lambda x.M'$
  - If $y \notin M$, then: $\lambda x.M = \lambda y.(M'[y/x])$

β-Equivalence Relation

Definition

$\to_\beta$ is the smallest relation on terms such that:

- $(\lambda x.M)N \to_\beta M[N/x]$
- if $M \to_\beta M'$, then $MN \to_\beta M'N$
- if $N \to_\beta N'$, then $MN \to_\beta MN'$
- if $M \to_\beta M'$, then $\lambda x.M \to_\beta \lambda x.M'$

β-Equivalence

β-equivalence is the smallest equivalence relation, denoted by $=_\beta$, between pairs of terms, obtained by taking the reflexive, symmetric, and transitive closure of the relation $\to_\beta$.
section*β-Equivalence and Normal Forms

- A term that has no redexes is in **normal form**.
- Not all terms have a normal form. For example, the term Ω:

$$(\lambda x.xx)(\lambda x.xx)$$

  has a redex, but it β-reduces to itself, without ever terminating.

- Other terms have different computations, all reaching the **normal form**:

$$(\lambda x.x)((\lambda z.zz)(\lambda y.y)) \to_\beta (\lambda z.zz)(\lambda y.y) \to_\beta (\lambda y.y)(\lambda y.y) \to_\beta \lambda y.y$$

$$(\lambda x.x)((\lambda z.zz)(\lambda y.y)) \to_\beta (\lambda x.x)((\lambda y.y)(\lambda y.y)) \to_\beta (\lambda y.y)(\lambda y.y) \to_\beta \lambda y.y$$

$$(\lambda x.x)((\lambda z.zz)(\lambda y.y)) \to_\beta (\lambda x.x)((\lambda y.y)(\lambda y.y)) \to_\beta (\lambda x.x)(\lambda y.y) \to_\beta \lambda y.y$$

... these are different reduction strategies!

Church-Rosser Theorem and Confluence

Assume that this multi-step reduction exists, $h > 1$, so that $M =_\beta M'$:

$$M = M'_0 \to_\beta M'_1 \to_\beta \cdots \to_\beta M'_h = M'$$

Church-Rosser theorem then asserts that there is a common reduct N for M and M' that can be reached (with β-reductions):

$$\begin{tikzcd} M \arrow[rr, leftrightarrow] \arrow[dr, dotted] & & M' \arrow[dl, dotted] \\ & N & \end{tikzcd}$$

Figure 1: Church Rosser Theorem - Confluence Example 1

Key Question

What happens when M' is *a* normal form?
... and when both M' and N are normal forms?
This is about *THE* normal form: if one exists, it is unique.

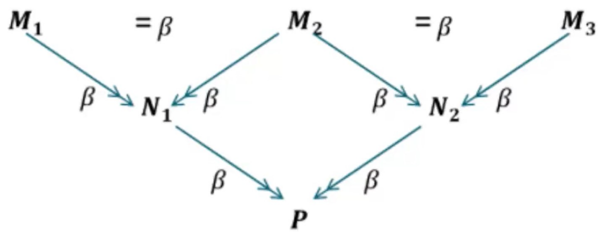

Figure 2: Church Rosser Theorem - Confluence Example 2

Theorem

Church Rosser Theorem: For any two λ-terms M and N, $M =_\beta N$ if and only if there is some λ-term P such that:

$$M \twoheadrightarrow_\beta^* P \quad \text{and} \quad N \xrightarrow{\beta^*} P$$