Problem Sheet 5.2

Exercise 1

a) $Y \triangleq \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$

Let $W = (\lambda x.f(xx))$

$\lambda f.(\lambda x.f(xx))W =_\beta \lambda f.f(WW)$

Since $WW =_\beta f.(WW)$

$\lambda f.(\lambda x.f(xx))W =_\beta \lambda f.f(WW) =_\beta \lambda f.f(f(WW)) =_\beta \lambda f.f\left(f(f(WW))\right)$

b) YF can have a normal form only if F ignores or discards the first argument. This happens when the argument is ignored. Otherwise, YF will keep unfolding indefinitely.

c) The fixed point of $\lambda x.\lambda y.y$ is $\lambda y.y$. The application of the lambda term to any other term always results in $\lambda y.y$ since x is ignored.

d) $Y(\lambda x.\lambda y.y) \triangleq \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))(\lambda x.\lambda y.y) =_\alpha$

$\lambda f.(\lambda z.f(zz))(\lambda z.f(zz))(\lambda x.\lambda y.y) =_\beta (\lambda z.(\lambda x.\lambda y.y)(zz))(\lambda z.(\lambda x.\lambda y.y)(zz)) =_\beta$

$(\lambda z.\lambda y.y)(\lambda z.\lambda y.y) =_\beta \lambda y.y$

Exercise 2

a) $add = \lambda m.\lambda n.ifthen(isZero\ n)m\left(succ(add\ m\ pred(n))\right)$

b) $add = \left(\lambda a.\lambda m.\lambda n.ifthen(isZero\ n)m\left(succ(a\ m\ pred(n))\right)\right)add$

$F_{add} = \lambda a.\lambda m.\lambda n.ifthen(isZero\ n)m\left(succ(a\ m\ pred(n))\right)$

$add = F_{add}\ add$

The function add is a fixed point of the lambda term F_{add}.

$F_{add}(YF_{add}) = YF_{add}$

$add = YF_{add} = Y\left(\lambda a.\lambda m.\lambda n.ifthen(isZero\ n)m\left(succ(a\ m\ pred(n))\right)\right)$