

## Problem Sheet 6.2

a)  $\text{succ } N_m \xrightarrow{\beta}^* N_{m+1}$

$$\begin{aligned} \text{succ } N_m &= (\lambda n. \lambda f. \lambda x. f(nfx)) N_m =_{\beta} \lambda f. \lambda x. f(N_m f x) = \\ &= \lambda f. \lambda x. f((\lambda f. \lambda x. N'_m) f x) =_{\beta} \lambda f. \lambda x. f(N'_m [f/f] [x/x]) = \\ &= \lambda f. \lambda x. f(N'_m) = \lambda f. \lambda x. N'_{m+1} = N_{m+1} \end{aligned}$$

b)  $N'_m[N'_k/x] = N'_{m+k}$

*Base case:  $m = 0$*

$$N'_0[N'_k/x] = x[N'_k/x] = N'_{0+k}$$

*Inductive case*

$$N'_{m+1}[N'_k/x] = f N'_m[N'_k/x] = f N'_{m+k} = N'_{m+k+1}$$

c)  $\text{add } N_m N_k \xrightarrow{\beta}^* N_{m+k}$

$$\begin{aligned} \lambda m. \lambda n. \lambda f. \lambda x. mf(nfx) N_m N_k &=_{\beta} \lambda f. \lambda x. N_m f(N_k f x) = \\ &= \lambda f. \lambda x. N_m f((\lambda f. \lambda x. N'_k) f x) =_{\beta} \lambda f. \lambda x. N_m f N'_k = \\ &= \lambda f. \lambda x. (\lambda f. \lambda x. N'_m) f N'_k =_{\beta} \lambda f. \lambda x. N'_{m+k} = N_{m+k} \end{aligned}$$

d)  $N'_m[\lambda x. N'_k/x] \xrightarrow{\beta}^* N'_{m \times k}$

*Base case:  $m = 0$*

$$N'_0[\lambda x. N'_k/x] = x[\lambda x. N'_k/x] = x = N'_{0 \times k}$$

*Inductive case*

$$\begin{aligned} N'_{m+1}[\lambda x. N'_k/x] &= f N'_m[\lambda x. N'_k/x] = \lambda x. N'_k N'_m[\lambda x. N'_k/x] = \\ &= \lambda x. N'_k N'_{m \times k} =_{\beta} N'_k[N'_{m \times k}/x] = N'_{k+(m \times k)} = N'_{k(1+m)} \end{aligned}$$

e)  $\text{mul } N_m N_k \xrightarrow{\beta}^* N_{m \times k}$

$$\begin{aligned} (\lambda m. \lambda n. \lambda f. \lambda x. m(nfx)) N_m N_k &=_{\beta} \lambda f. \lambda x. N_m(N_k f)x = \\ &= \lambda f. \lambda x. N_m((\lambda f. \lambda x. N'_k) f)x =_{\beta} \lambda f. \lambda x. N_m(\lambda x. N'_k)x = \\ &= \lambda f. \lambda x. (\lambda f. \lambda x. N'_m)(\lambda x. N'_k)x =_{\beta} \lambda f. \lambda x. N'_m[\lambda x. N'_k/f] = \\ &= \lambda f. \lambda x. N'_{m \times k} = N_{m \times k} \end{aligned}$$