Sinusoids & Exponentials — Seminar 1 & 2

Professor-Style Theory Summary & Cheat Sheet

1. Continuous-Time Sinusoids

Theory

A real sinusoid is defined as

$$x(t) = A\cos(\omega_0 t + \varphi),$$

with:

$$A > 0 \text{ (amplitude)}, \quad \omega_0 > 0 \text{ (angular frequency [rad/s])}, \quad t \in \mathbb{R}, \quad \varphi \in [0, 2\pi).$$

Fundamental period:

$$T_0 = \frac{2\pi}{\omega_0}, \qquad x(t + T_0) = x(t).$$

Range:

$$-A \leq x(t) \leq A.$$

Intuition: amplitude sets vertical scale; ω_0 sets speed; φ shifts horizontally.

Method

To determine sinusoid parameters from a plot:

1. Amplitude A: half the peak-to-peak value.
2. Period T_0: distance between successive peaks.
3. Angular frequency: $\omega_0 = 2\pi/T_0$.
4. Phase φ: solve $x(0) = A\cos(\varphi)$ or use time shift of max.
5. Time shift relation:

$$\cos(\omega_0 t + \varphi) = \cos(\omega_0(t - t_1)) \Rightarrow \varphi = -\omega_0 t_1.$$

Example / Result

From the sinusoid in Seminar 1 (Solutions p.2):

$$A = 9, \quad \omega_0 = 2\pi\frac{20}{9}, \quad \varphi = -2\pi\frac{1}{9}.$$

2. Sketching Sinusoids and Phase Shifts

Theory

A horizontal shift of $x(t) = A\cos(\omega_0 t + \varphi)$ corresponds to

$$x(t) = A\cos(\omega_0(t - t_1)), \qquad \varphi = -\omega_0 t_1.$$

Positive t_1 shifts the waveform right; negative shifts it left.

Intuition: cosine achieves its max when argument $= 0$ modulo 2π.

Method

To plot shifted cosines:

1. Identify base period $T_0 = 1/f_0$ or $2\pi/\omega_0$.
2. Compute shift $t_1 = -\varphi/\omega_0$.
3. Translate the standard cosine plot by t_1.
4. Mark maxima at $t = t_1 + kT_0$.

Example / Result

Seminar 1: For $\omega_0 = \pi/5$ and $\varphi = -\pi/3$,

$$t_1 = -\frac{-\pi/3}{\pi/5} = \frac{5}{3}.$$

3. Euler Representation and Phasors

Theory

Euler identities:

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.$$

A phasor is the complex constant

$$X = Ae^{j\varphi}, \qquad x(t) = \Re\{Xe^{j\omega_0 t}\}.$$

Intuition: sinusoid = rotating phasor projected onto real axis.

Method

Sum of sinusoids of same frequency:

1. Convert each into phasor $X_k = A_k e^{j\varphi_k}$.
2. Add phasors algebraically: $X = \sum_k X_k$.
3. Resulting sinusoid:

$$x(t) = |X|\cos(\omega_0 t + \arg X).$$

Example / Result

Seminar 2, Ex. 9:

$$2\cos(200\pi t + \tfrac{\pi}{3}) + 2\cos(200\pi t - \tfrac{3\pi}{4})$$

Phasors:

$$X_1 = 2e^{j\pi/3}, \quad X_2 = 2e^{-j3\pi/4}, \quad X = X_1 + X_2 = 5.536\,e^{j0.2747}.$$

Thus,

$$x(t) = 5.536\cos(200\pi t + 0.2747).$$

4. Trigonometric Identities from Euler

Theory

Using Euler,

$$\cos(\theta_1 \pm \theta_2) = \cos\theta_1 \cos\theta_2 \mp \sin\theta_1 \sin\theta_2.$$

Intuition: product of phasors encodes angle sums.

Method

Procedure:

1. Replace cosines/sines using Euler expressions.
2. Multiply exponentials: $e^{j\theta_1} e^{j\theta_2} = e^{j(\theta_1+\theta_2)}$.
3. Collect terms and recover cosine/sine via inverse Euler.

Example / Result

Seminar 2, Ex. 4:

$$\cos(\theta_1 + \theta_2) = \frac{e^{j\theta_1} + e^{-j\theta_1}}{2} \cdot \frac{e^{j\theta_2} + e^{-j\theta_2}}{2} \Rightarrow \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2.$$

5. Time Shift and Maximum Location

Theory

For $x(t) = A\cos(\omega_0 t + \varphi)$, the maxima occur at

$$\omega_0 t + \varphi = 2\pi k, \qquad t = \frac{-\varphi}{\omega_0} + kT_0.$$

Method

To check maximum at time t_1:

1. Set $t = t_1$ and check whether $\omega_0 t_1 + \varphi \equiv 0 \pmod{2\pi}$.
2. If yes, $x(t_1) = A$ is a maximum.

Example / Result

Seminar 2, Ex. 5:

$$x(t) = A\sin(11\pi t) = A\cos(11\pi t - \tfrac{\pi}{2}).$$

Thus $t_1 = \frac{1}{22}$ gives $\omega_0 t_1 - \pi/2 = 0$.

6. Scaling and Shifting a Sinusoid

Theory

If
$$y(t) = Gx(t - t_1),$$
and $x(t) = A\cos(\omega_0 t + \varphi)$, then
$$y(t) = GA\cos(\omega_0 t + \varphi - \omega_0 t_1).$$

Intuition: time shift modifies phase; scaling modifies amplitude.

Method

To match $y(t) = B\cos(\omega_0 t)$:
1. Set $GA = B$ for amplitude.
2. Solve $\varphi - \omega_0 t_1 = 0$ for t_1.

Example / Result

Seminar 2, Ex. 6:
$$x(t) = 20\cos(80\pi t - 0.4\pi).$$
We want $y(t) = 5\cos(80\pi t)$.
$$G = \frac{5}{20} = \frac{1}{4}, \qquad t_1 = \frac{-(-0.4\pi)}{80\pi} = -\frac{1}{200}.$$

7. Phasor Sum Identity

Theory

For $X_1 = e^{j\alpha}$ and $X_2 = e^{j\beta}$,
$$X_3 = X_1 + X_2 = 2\cos\left(\frac{\alpha - \beta}{2}\right) e^{j(\alpha+\beta)/2}.$$

Intuition: sum of two unit phasors is determined by midpoint angle and half-angle difference.

Example / Result

Seminar 2, Ex. 7: Plotting X_1, X_2, their vector sum has magnitude $2\cos((\alpha - \beta)/2)$ and angle $(\alpha + \beta)/2$.

8. Discrete-Time Complex Exponentials

Theory

A discrete complex exponential:
$$x[n] = Xz_0^n,$$
with phasor $X = Ae^{j\varphi}$ and $z_0 = re^{j\omega_0}$, is a discrete sinusoid when $r = 1$.

Intuition: repeated rotation/scaling in the complex plane.

Method

First difference:

$$y[n] = x[n] - x[n-1]$$

has form $Ae^{j(\omega_0 n+\phi)}$ when

$$A = |1 - z_0^{-1}||X|, \quad \phi = \arg(X(1 - z_0^{-1})).$$

Example / Result

Seminar 2, Ex. 8:

$$x[n] = e^{j(0.4\pi n - 0.5\pi)}, \quad y[n] = x[n] - x[n-1] = Ae^{j(\omega_0 n+\phi)}.$$

Computed:

$$A = 1.37, \qquad \phi = -0.81, \qquad \omega_0 = 0.4\pi.$$

9. Complex-Valued $z(t)$ for Real $x(t)$

Theory

Any real sinusoid can be expressed as

$$x(t) = \Re\{z(t)\}, \qquad z(t) = Xe^{j\omega_0 t}.$$

Intuition: real part picks out projection of rotating phasor.

Example / Result

Seminar 2, Ex. 10:

$$x(t) = 20\cos(300\pi t + \tfrac{\pi}{4}) + 5\sqrt{2}\cos(300\pi t - \pi) + 5\sqrt{2}\cos(300\pi t - \tfrac{\pi}{2}).$$

Phasor sum:

$$X = 10e^{j\pi/4}, \quad z(t) = Xe^{j300\pi t}, \quad x(t) = 10\cos(300\pi t + \tfrac{\pi}{4}).$$

10. Exam Strategy for Sinusoid Problems

Method

1. Identify amplitude, period, frequency, and phase directly from plots or formulas.
2. Convert between cosine and sine using phase shifts.
3. For sums of sinusoids at same ω_0:
   - Rewrite each as a phasor.
   - Add complex phasors.
   - Convert back to single cosine.
4. For discrete exponentials:
   - Identify z_0 and X.
   - For differences, compute $1 - z_0^{-1}$.
   - Extract amplitude and phase from magnitude and angle.
5. Always track angles in radians unless otherwise stated.