Seminar 7 — FIR Filters & Frequency Response

Professor-Style Theory Summary & Cheat Sheet

1. FIR Systems and Convolution

Theory

A finite impulse response (FIR) LTI system of order M is defined by:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k], \qquad h[n] = b_n.$$

Impulse response is finite-length, causal if $h[n] = 0$ for $n < 0$.

FIR = no feedback, always BIBO stable.

Method

Computing the output:

$$y[n] = \sum_{k=0}^{M} b_k x[n-k].$$

1. Identify b_k from the given expression or $H(e^{j\omega})$.
2. Use direct convolution for time-domain signals.

Example / Result

If $H(e^{j\omega}) = 1 - e^{-j\omega}$ then $h[n] = \delta[n] - \delta[n-1]$ and

$$y[n] = x[n] - x[n-1].$$

2. Frequency Response of FIR Filters

Theory

The frequency response is

$$H(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k}.$$

A complex exponential $e^{j\omega_0 n}$ is an eigenfunction:

$$y[n] = H(e^{j\omega_0})e^{j\omega_0 n}.$$

Magnitude = gain, phase = shift.

Method

To compute $H(e^{j\omega})$:

1. Write the FIR expansion $\sum b_k e^{-j\omega k}$.
2. Factor exponentials to express in magnitude–phase form.
3. For cascade systems: multiply responses $H = H_1 H_2 H_3$.

Example / Result

For $y[n] = x[n] - \sqrt{3}\,x[n-1] + x[n-2]$,

$$H(e^{j\omega}) = 1 - \sqrt{3}\,e^{-j\omega} + e^{-j2\omega} = e^{-j\omega}(2\cos\omega - \sqrt{3}).$$

3. Zeros of FIR Systems

Theory

Zeros of $H(e^{j\omega})$ occur when the polynomial

$$\sum_{k=0}^{M} b_k e^{-j\omega k} = 0.$$

A sinusoid at frequency ω_0 is fully cancelled if

$$H(e^{j\omega_0}) = 0.$$

Zeros create notches in frequency.

Method

Finding zeros:

1. Rewrite $H(e^{j\omega})$ as $e^{-jM\omega/2}P(\omega)$.
2. Solve $P(\omega) = 0$ using trigonometric identities.
3. For cascaded systems: each factor yields its own zeros.

Example / Result

If $H(e^{j\omega}) = e^{-j\omega}(2\cos\omega - \sqrt{3})$ then

$$H(e^{j\omega}) = 0 \iff 2\cos\omega = \sqrt{3} \iff \omega = \pm\tfrac{\pi}{6}.$$

4. Cascade of FIR Filters

Theory

For FIR systems in cascade:

$$H(e^{j\omega}) = \prod_i H_i(e^{j\omega}), \qquad h[n] = h_1[n] * h_2[n] * \cdots .$$

Cascading adds delays, multiplies spectra.

Method

To find the equivalent difference equation:

1. Compute each $H_i(e^{j\omega})$.
2. Multiply: $H = \prod H_i$.
3. Expand into $\sum b_k e^{-j\omega k}$.
4. Read off b_k for the final FIR.

Example / Result

Cascading $(1 - e^{-j\omega})(1 - e^{-j2\omega})(1 - e^{-j3\omega})$ yields

$$H(e^{j\omega}) = 1 - e^{-j\omega} - e^{-j2\omega} + e^{-j4\omega} + e^{-j5\omega} - e^{-j6\omega}.$$

5. Sinusoidal Response

Theory

For input $x[n] = Ae^{j(\omega_0 n + \phi)}$,

$$y[n] = A|H(e^{j\omega_0})|e^{j(\omega_0 n + \phi + \angle H(e^{j\omega_0}))}.$$

Frequency does not change; only amplitude and phase do.

FIR acts as frequency-dependent scaling.

Method

To express output as $B\cos(\omega_0 n + \varphi)$:

1. Compute $H(e^{j\omega_0})$.
2. Let $B = A|H(e^{j\omega_0})|$.
3. Set $\varphi = \phi + \angle H(e^{j\omega_0}) \pm \frac{\pi}{2}$ depending on sine/cosine form.

Example / Result

If $x[n] = \sin(\omega_0 n)$ and $H(e^{j\omega_0}) = e^{-j\omega_0}(2\cos\omega_0 - \sqrt{3})$,

$$y[n] = (2\cos\omega_0 - \sqrt{3})\cos(\omega_0 n - \omega_0 - \tfrac{\pi}{2}).$$

6. Moving-Average (Rectangular) FIR Filters

Theory

An L-point moving average:

$$h[n] = \frac{1}{L}u[n] - \frac{1}{L}u[n-L], \qquad H(e^{j\omega}) = \frac{1}{L}\sum_{k=0}^{L-1} e^{-j\omega k}.$$

Prototype low-pass; linear phase.

Method

Standard simplification:

$$H(e^{j\omega}) = \frac{1}{L}\, e^{-j\omega\frac{L-1}{2}} \frac{\sin(L\omega/2)}{\sin(\omega/2)}.$$

Steps:

1. Use geometric series: $\sum e^{-j\omega k}$.
2. Factor exponentials to isolate linear phase.
3. Identify zeros at $\omega = \frac{2\pi m}{L}$.

Example / Result

Magnitude:

$$|H(e^{j\omega})| = \frac{1}{L}\left|\frac{\sin(L\omega/2)}{\sin(\omega/2)}\right|.$$

Phase:

$$\angle H(e^{j\omega}) = -\frac{L-1}{2}\omega \quad \text{(plus sign flips at zeros)}.$$

7. Exam Strategy

Method

1. Identify FIR order and coefficients.
2. Compute $H(e^{j\omega})$ in exponential form.
3. Factor into magnitude–phase when useful.
4. Locate zeros: solve $H(e^{j\omega_0}) = 0$.
5. Evaluate system response to sinusoids via $H(e^{j\omega_0})$.
6. For cascades: multiply responses before expanding.