Spectral Representation — Seminar 3 & 4

Professor-Style Theory Summary & Cheat Sheet

1. Complex Exponentials and Euler Identities

Theory

A complex exponential in continuous time:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t).$$

Euler identities:

$$\cos(\omega t+\varphi) = \frac{e^{j(\omega t+\varphi)} + e^{-j(\omega t+\varphi)}}{2}, \qquad \sin(\omega t+\varphi) = \frac{e^{j(\omega t+\varphi)} - e^{-j(\omega t+\varphi)}}{2j}.$$

Intuition: cos/sin are built from a positive and negative rotating phasor.

Method

Rewrite a real sinusoid as sum of complex exponentials:

1. Start from $x(t) = A\cos(\omega t+\varphi)$ or $A\sin(\omega t+\varphi)$.
2. Apply Euler:
$$A\cos(\omega t+\varphi) = \frac{A}{2}e^{j(\omega t+\varphi)} + \frac{A}{2}e^{-j(\omega t+\varphi)}.$$
3. Identify:
$$X(+\omega) = \frac{A}{2}e^{j\varphi}, \quad X(-\omega) = \frac{A}{2}e^{-j\varphi}.$$

Example / Result

For

$$x(t) = 10\cos(800\pi t + \tfrac{\pi}{4}),$$

we get:

$$x(t) = 5e^{j\pi/4}e^{j800\pi t} + 5e^{-j\pi/4}e^{-j800\pi t}.$$

Complex amplitudes:

$$X(800\pi) = 5e^{j\pi/4}, \quad X(-800\pi) = 5e^{-j\pi/4}.$$

2. Sinusoids and Two-Sided Line Spectra

Theory

A real sinusoid $A\cos(\omega t+\varphi)$ has a two-sided discrete spectrum with lines at $\omega = \pm\omega_0$:

$$X(\omega) = \frac{A}{2}e^{j\varphi}\,\delta(\omega-\omega_0) + \frac{A}{2}e^{-j\varphi}\,\delta(\omega+\omega_0).$$

Magnitude: $|X(\pm\omega_0)| = A/2$. Phase: $\angle X(+\omega_0) = \varphi$, $\angle X(-\omega_0) = -\varphi$.

Intuition: each cosine $\Rightarrow$ two symmetric spectral lines.

Method

Given $x(t)$ as sum of cosines:

$$x(t) = \sum_i A_i \cos(\omega_i t + \varphi_i),$$

to draw its spectrum:

1. For each term, compute ω_i and $A_i/2$.
2. Plot impulses in $|X(\omega)|$ at $\omega = \pm\omega_i$ with height $A_i/2$.
3. Plot phases in $\angle X(\omega)$: φ_i at $+\omega_i$, $-\varphi_i$ at $-\omega_i$.
4. DC term A_{DC} appears as a single line at $\omega = 0$ with amplitude A_{DC}.

Example / Result

For

$$x(t) = 10\cos(800\pi t + \tfrac{\pi}{4}) + 7\cos(1200\pi t - \tfrac{\pi}{3}) - 3\cos(1600\pi t),$$

angular frequencies:

$$\omega_1 = 800\pi, \quad \omega_2 = 1200\pi, \quad \omega_3 = 1600\pi.$$

Line spectrum:

$$\begin{aligned} X(\pm 800\pi) &= 5e^{\pm j\pi/4}, \\ X(\pm 1200\pi) &= \tfrac{7}{2}e^{\mp j\pi/3}, \\ X(\pm 1600\pi) &= \tfrac{3}{2}e^{\pm j\pi}. \end{aligned}$$

3. Periodicity of Multi-Sinusoid Signals

Theory

A sum of sinusoids

$$x(t) = \sum_i A_i \cos(\omega_i t + \varphi_i)$$

is periodic if all frequencies are integer multiples of a fundamental frequency f_0:

$$f_i = \frac{\omega_i}{2\pi} = n_i f_0, \quad n_i \in \mathbb{Z}.$$

Then the fundamental period is:

$$T_0 = \frac{1}{f_0}.$$

Intuition: a common period exists only if all frequencies share a common base.

Method

To determine periodicity and T_0:

1. Convert each angular frequency: $f_i = \omega_i/(2\pi)$.
2. Compute $f_0 = \gcd(f_1, f_2, \dots)$ (using integer ratios given in the problem).
3. If $f_0 > 0$, then $x(t)$ is periodic with $T_0 = 1/f_0$.
4. If no finite f_0 exists, $x(t)$ is aperiodic.

Example / Result

From Exercise 1:

$$f_1 = 400 \text{ Hz}, \quad f_2 = 600 \text{ Hz}, \quad f_3 = 800 \text{ Hz}.$$

$$f_0 = \gcd(400, 600, 800) = 200 \text{ Hz}, \quad T_0 = \frac{1}{200} = 5 \cdot 10^{-3} \text{ s}.$$

Adding a cosine at 500 Hz yields frequencies $(400, 500, 600, 800)$ with $f_0 = \gcd(400, 500, 600, 800) = 100$ Hz, giving $T_0 = 10$ ms.

4. Complex Fourier Series Representation

Theory

A real periodic signal with period T_0 admits the complex Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}, \qquad f_0 = \frac{1}{T_0},$$

with complex coefficients:

$$a_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-j2\pi k f_0 t}\, dt.$$

Intuition: a_k are phasors describing each harmonic kf_0.

Method

To express $x(t)$ in FS form:

1. Identify the fundamental frequency f_0 from the sinusoid frequencies.
2. Rewrite each term as $A\cos(2\pi f t + \varphi)$.
3. Use Euler to obtain terms $C_k e^{j2\pi k f_0 t}$ and $C_{-k} e^{-j2\pi k f_0 t}$.
4. Group coefficients multiplying $e^{j2\pi k f_0 t}$ to read off a_k.

Example / Result

From Exercise 4:

$$x(t) = 10 + 20\cos(2\pi 100 t + \tfrac{\pi}{4}) + 10\sin(2\pi 250 t).$$

Frequencies: 100 Hz and 250 Hz.

$$f_0 = \gcd(100, 250) = 50 \text{ Hz}, \quad T_0 = 0.02 \text{ s}.$$

Non-zero coefficients:

$$\begin{aligned} a_0 &= 10, \\ a_{\pm 2} &= 10\, e^{\pm j\pi/4}, \\ a_{\pm 5} &= 5\, e^{\mp j\pi/2}. \end{aligned}$$

All other a_k are zero.

5. Spectral Symmetry and DC Component

Theory

For a real-valued signal $x(t)$, its complex spectrum satisfies conjugate symmetry:

$$X(-\omega) = X^*(\omega).$$

Thus magnitude is even and phase is odd:

$$|X(-\omega)| = |X(\omega)|, \quad \angle X(-\omega) = -\angle X(\omega).$$

Theory

The DC component is the average value of $x(t)$:

$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt \quad \text{(Fourier transform)}, \qquad a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt \quad \text{(Fourier series)}.$$

DC appears as a single line at $\omega = 0$ and is not split into $\pm$ frequencies.

Intuition: real time signals $\Rightarrow$ symmetric magnitudes; DC is a single average level.

Example / Result

From Exercise 2, a real $x(t)$ defined by symmetric spectral lines:

$$X(+50\text{ Hz}) = 7e^{-j\pi/3}, \quad X(-50\text{ Hz}) = 7e^{+j\pi/3}$$

gives

$$x(t) = 11 + 14\cos(100\pi t - \tfrac{\pi}{3}) + 8\cos(350\pi t - \tfrac{\pi}{2}).$$

The DC line at $f = 0$ has amplitude 11, not halved.

6. Nonlinear Spectral Generation: Powers of Sinusoids

Theory

Nonlinear operations (e.g., squaring, cubing) on sinusoids generate harmonics at integer multiples of the base frequency. For a single sinusoid with angular frequency ω_0:

$$\sin^3(\omega_0 t) \Rightarrow \text{components at } \omega_0,\ 3\omega_0.$$

Intuition: nonlinearities mix and replicate frequencies.

Method

To find the spectrum of powers of a sinusoid:

1. Write $\sin(\omega_0 t)$ using exponentials:

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}.$$

2. Raise to the desired power and expand using binomial identities.
3. Collect terms of the form $Ce^{jm\omega_0 t}$ to identify harmonics at $m\omega_0$.
4. Optionally convert back to sines/cosines with phases.

Example / Result

From Exercise 3:

$$x(t) = \sin^3(27\pi t).$$

Using

$$\sin^3\theta = \left(\frac{e^{j\theta} - e^{-j\theta}}{2j}\right)^3$$

and expansion:

$$x(t) = \frac{3}{4}\sin(27\pi t) - \frac{1}{4}\sin(81\pi t),$$

with components at $\omega = 27\pi$ and $\omega = 81\pi$. Fundamental:

$$\omega_0 = 27\pi, \quad f_0 = \frac{27}{2}\ \text{Hz}, \quad T_0 = \frac{2}{27}\ \text{s}.$$

7. Single-Tone Amplitude Modulation (AM)

Theory

A single-tone AM signal of the form

$$x(t) = [A_c + m(t)]\cos(\omega_c t)$$

with $m(t)$ containing a single sinusoid at ω_m produces spectral components at:

$$\omega_c \quad \text{(carrier)}, \qquad \omega_c \pm \omega_m \quad \text{(sidebands)}.$$

For a pure sinusoidal modulator $m(t) = M\sin(\omega_m t + \varphi)$ there are exactly two sidebands.

Intuition: multiplication in time shifts spectra and creates sum/difference frequencies.

Method

To find the spectrum of a product $[A + B\sin(\omega_m t + \phi)]\cos(\omega_c t)$:

1. Rewrite $\sin(\cdot)$ and $\cos(\cdot)$ with Euler exponentials.
2. Separate the carrier term $A\cos(\omega_c t)$.
3. Expand the product of exponentials to obtain exponents at $\omega_c \pm \omega_m$.
4. Group conjugate pairs to obtain cosines for carrier and sidebands.

Example / Result

From Exercise 5:

$$x(t) = [12 + 7\sin(\pi t - \tfrac{\pi}{3})]\cos(13\pi t).$$

Modulating frequency: $\omega_m = \pi$, carrier frequency: $\omega_c = 13\pi$. Result:

$$x(t) = \frac{7}{2}\cos\left(12\pi t + \tfrac{5\pi}{6}\right) + 12\cos(13\pi t) + \frac{7}{2}\cos\left(14\pi t - \tfrac{5\pi}{6}\right),$$

with components at ω_c and sidebands $\omega_c \pm \omega_m$.

8. Rectangular Pulse Train Fourier Series

Theory

A periodic pulse train of period T_0 and pulse width $2t_c$:

$$x(t) = \begin{cases} 1, & |t| < t_c \text{ (within one period)}, \\ 0, & t_c < |t| \leq T_0/2, \end{cases} \qquad x(t+T_0) = x(t)$$

admits complex FS coefficients:

$$a_k = \frac{1}{T_0} \int_{-t_c}^{t_c} e^{-j2\pi kt/T_0}\, dt.$$

Intuition: duty cycle controls DC level and spectral envelope.

Method

To compute a_k for the pulse train:

1. Use $a_k = \frac{1}{T_0} \int_{-t_c}^{t_c} e^{-j2\pi kt/T_0}\, dt$.
2. For $k = 0$, integrate directly:
$$a_0 = \frac{1}{T_0} \int_{-t_c}^{t_c} 1\, dt = \frac{2t_c}{T_0}.$$
3. For $k \neq 0$, integrate exponentials:
$$a_k = \frac{1}{T_0} \left[\frac{T_0}{-j2\pi k} e^{-j2\pi kt/T_0} \right]_{-t_c}^{t_c}.$$
4. Simplify using $e^{-j\theta} - e^{j\theta} = -2j\sin\theta$ to obtain a sine form.

Example / Result

From Exercise 6:

$$a_0 = \frac{2t_c}{T_0},$$

and for $k \neq 0$:

$$a_k = \frac{1}{\pi k} \sin\left(\frac{2\pi k t_c}{T_0}\right), \quad k \in \mathbb{Z} \setminus \{0\}.$$

For $t_c = T_0/4$, the duty cycle is $1/2$ and the DC level is $a_0 = 1/2$.

9. Exam Strategy for Spectral Problems

Method

1. Identify all sinusoidal components and their frequencies (or read them from the spectrum).
2. Check periodicity by computing the fundamental frequency f_0 via greatest common divisor of the sinusoid frequencies.
3. For time-domain $\rightarrow$ spectrum:
   - Rewrite using Euler.
   - Group terms at $\pm\omega$ to obtain line magnitudes and phases.
4. For spectrum $\rightarrow$ time-domain:
   - Pair conjugate lines $X(\pm\omega)$.
   - Use $A = 2|X(\omega)|$, $\varphi = \angle X(\omega)$ to form $A\cos(\omega t + \varphi)$.
5. For FS representations:
   - Choose f_0 consistent with all observed frequencies.
   - Express each component as harmonic kf_0 and read off a_k.
6. For modulation and nonlinearities:
   - Use exponentials to find sum/difference and harmonic frequencies.
   - Identify carrier, sidebands, and harmonic structure.