Sampling — Seminar 5

Professor-Style Theory Summary & Cheat Sheet

1. Sampling Operator and Invertibility

Theory

Sampling operator: $x[n] = x(nT_s)$ with $f_s = 1/T_s$. Invertibility requires a bandlimited signal with spectral support $|f| \leq f_{\max}$. Nyquist–Shannon condition:

$$f_s > 2f_{\max}.$$

If $f_s < 2f_{\max}$, different continuous signals may yield the same $x[n]$.

Method

To determine invertibility:

1. Identify highest spectral component $f_{\max}$.
2. Check Nyquist: $f_s > 2f_{\max}$.
3. If violated, enumerate aliases $f = \pm f_0 + mf_s$.

Example / Result

For $x(t) = 4 + 4\cos(1000\pi t)\sin(50000\pi t)$, expansion gives frequencies $\{24.5, 25.5\}$ kHz $\Rightarrow f_{\max} = 25.5$ kHz. Nyquist: $f_s > 51$ kHz.

Intuition: two samples per period prevent ambiguity between oscillations.

2. Spectral Expansion of Products

Theory

Trig identity used repeatedly:

$$\sin(a)\cos(b) = \frac{\sin(a+b) + \sin(a-b)}{2}.$$

Frequencies extracted from $\cos(\omega t)$ and $\sin(\omega t)$ via $f = \omega/(2\pi)$.

Method

1. Convert products into sum of sinusoids.
2. Read frequencies directly from ω.
3. Determine $f_{\max}$ from resulting components.

Example / Result

$\cos(50\pi t)\sin(700\pi t) = \frac{1}{2}[\sin(750\pi t) + \sin(650\pi t)] \Rightarrow f = \{375, 325\}$ Hz.

Intuition: product signals create sum/difference frequencies.

3. Aliasing and Equivalence Classes

Theory

Two continuous signals produce the same samples iff their frequencies satisfy

$$f' = \pm f_0 + m f_s, \qquad m \in \mathbb{Z}.$$

The discrete-time normalized frequency is

$$\Omega = 2\pi \frac{f}{f_s}.$$

Aliasing occurs when different f map to identical Ω.

Method

1. Compute base continuous frequency from Ω: $f_0 = \Omega f_s/(2\pi)$.
2. Enumerate aliases: $f = \pm f_0 + m f_s$.
3. Retain those $|f| < f_s$ or within problem constraints.

Example / Result

For $x[n] = 100\cos(0.4\pi n + \pi/4)$ with $f_s = 5$ kHz: $f_0 = (0.4\pi)/(2\pi) \cdot 5000 = 1000$ Hz. Aliases < 5 kHz: $f_1 = 1000$ Hz, $f_2 = 4000$ Hz.

Intuition: discrete-time frequencies "wrap" modulo 2π.

4. Continuous-to-Discrete Frequency Mapping

Theory

Sampling transforms $\cos(2\pi f t + \phi)$ into

$$x[n] = \cos(2\pi f n T_s + \phi) = \cos(\Omega n + \phi), \qquad \Omega = 2\pi \frac{f}{f_s}.$$

Method

1. Replace t by nT_s.
2. Convert $2\pi f T_s$ to normalized Ω.
3. Reduce Ω modulo 2π.

Example / Result

$x(t) = 10 + 18\cos(140\pi t - \frac{2\pi}{3})$ sampled at $f_s = 400$:

$$\Omega = 140\pi T_s = \frac{140\pi}{400} = 0.35\pi.$$

Thus $x[n] = 10 + 18\cos(0.35\pi n - \frac{2\pi}{3})$.

5. Minimum Sampling Rate

Theory

Sampling without aliasing requires
$$f_s > 2f_{\max}.$$
This ensures unique reconstruction and avoids overlapping shifted spectra.

Method

1. Compute all sinusoidal frequencies of $x(t)$.
2. Identify $f_{\max}$.
3. Apply Nyquist inequality.

Example / Result

For $x(t) = \frac{1}{2}[\sin(750\pi t) + \sin(650\pi t)]$: $f_{\max} = 375$ Hz $\Rightarrow f_s > 750$ Hz.

Intuition: prevents spectral replicas from touching.

6. Reconstruction from Spectral Lines

Theory

Given spectral pairs (A_k, f_k, ϕ_k), the signal is

$$x(t) = \sum_k A_k \cos(2\pi f_k t + \phi_k).$$

Hermitian symmetry ensures real-valuedness.

Method

1. Read amplitude/phase from spectrum.
2. Convert angular frequencies ω_k to f_k.
3. Sum all cosine components.

Example / Result

From spectral points in Ex. 4:

$$x(t) = 10 + 18\cos(140\pi t - \tfrac{2\pi}{3}) + 10\cos(350\pi t + \tfrac{\pi}{2}).$$

7. Chirp Signals and Instantaneous Frequency

Theory

For a phase $\theta(t)$, instantaneous frequency is

$$f(t) = \frac{1}{2\pi}\frac{d\theta(t)}{dt}.$$

Sampling constraints force $f(t) \le f_s/2$ (folding).

Method

1. Replace n by tf_s when converting $x[n]$ to $x(t)$.
2. Differentiate $\theta(t)$ to obtain $f(t)$.
3. Clip at Nyquist if required.

Example / Result

For $\theta[n] = \pi 10^{-4} n^2$ and $f_s = 4$ kHz: $x(t) = \cos(1600\pi t^2)$,

$$f(t) = 1600t.$$

Folded spectrum limits $f(t) \leq 2000$ Hz.

Intuition: chirps sweep frequency continuously; sampling restricts usable range.

8. Phasor Interpretation

Theory

Phasor: $z[n] = e^{j\theta[n]}$. If $\theta[n] = \Omega n + \phi$, rotation is uniform with period $N = 2\pi/\Omega$. Nonlinear $\theta[n]$ yields non-periodic rotation (chirp phasor).

Method

1. Compute angle $\theta[n]$.
2. Plot $e^{j\theta[n]}$ on unit circle.
3. Identify periodicity from $\theta[n+N] - \theta[n] = 2\pi$.

Example / Result

$z[n] = e^{j(0.08\pi n - 0.25\pi)}$ has period 25. Chirp $c[n] = e^{j0.1\pi n^2}$ is non-periodic.