Z-Transform — Seminar 8

Professor-Style Theory Summary & Cheat Sheet

1. Z-Transform & ROC

Theory

Bilateral z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

ROC is a ring in the z-plane; it cannot include poles.
Unit circle must lie in ROC to define $H(e^{j\omega})$.

Example / Result

Key pairs used in Seminar 8:

$$\delta[n] \leftrightarrow 1, \qquad a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \ |z| > |a|.$$

2. FIR System Functions

Theory

FIR systems:

$$H(z) = \sum_{k=0}^{M} h[k]z^{-k} \quad \Longleftrightarrow \quad y[n] = \sum_{k=0}^{M} h[k]x[n-k].$$

All poles located at $z = 0$ (multiplicity M). No stability concerns.

Method

To compute $H(z)$:

1. Write $h[n]$ explicitly (from difference form or exercise).
2. Apply z-transform using shifts z^{-k}.
3. Simplify into polynomial in z^{-1}.

Example / Result

Three-point moving average:

$$h[n] = \tfrac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]), \quad H(z) = \tfrac{1}{3}(1 + z^{-1} + z^{-2}).$$

3. Factorization & Conjugate-Pair Expansion

Theory

A factor $(1 - az^{-1})$ gives a zero at $z = a$.
Conjugate pair:

$$(1 - az^{-1})(1 - a^* z^{-1}) = 1 - (a + a^*)z^{-1} + |a|^2 z^{-2}.$$

$$a = re^{j\theta} \Rightarrow a + a^* = 2r\cos\theta.$$

Method

Factor all terms as $(1 - cz^{-1})$ or $(1 \pm z^{-2})$ to identify:

- unit-circle zeros (nulls),
- radial zeros ($r < 1$),
- multiplicities.

Example / Result

$$(1 - 0.9e^{j\pi/3}z^{-1})(1 - 0.9e^{-j\pi/3}z^{-1}) = 1 - 0.9z^{-1} + 0.81z^{-2}.$$

4. Frequency Response $H(e^{j\omega})$

Theory

Evaluate on unit circle:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}}.$$

If $H(e^{j\omega_0}) = 0$, the system cancels a sinusoid of frequency ω_0.

Method

To compute $H(e^{j\omega})$:

1. Substitute $z = e^{j\omega}$, $z^{-1} = e^{-j\omega}$.
2. Factor out $e^{-j\alpha\omega}$ if needed.
3. Reduce expressions with $e^{j\theta} \pm e^{-j\theta}$.

Example / Result

For $H(z) = 1 - z^{-4}$:

$$H(e^{j\omega}) = 1 - e^{-j4\omega} = 2j\sin(2\omega)e^{-j2\omega}.$$

Magnitude:

$$|H(e^{j\omega})| = 2|\sin(2\omega)|.$$

Nulls at $\omega = \frac{k\pi}{2}$.

5. Sinusoidal Steady-State Response

Theory

Input:

$$x[n] = A\cos(\omega_0 n + \phi).$$

Output of an LTI system:

$$y[n] = A|H(e^{j\omega_0})|\cos(\omega_0 n + \phi + \angle H(e^{j\omega_0})).$$

Example / Result

If $H(e^{j\pi/2}) = 0$, all components at $\omega = \pi/2$ vanish.

6. Impulse Response Patterns in Seminar 8

Theory

Impulse response is obtained by inverse z-transform:

$$H(z) = \sum h[k]z^{-k} \quad \Longrightarrow \quad h[n] = \sum h[k]\delta[n-k].$$

FIR structures yield finite support.

Example / Result

$$H(z) = 1 - z^{-4} \quad \Rightarrow \quad h[n] = \delta[n] - \delta[n-4].$$

Example / Result

$$H(z) = \tfrac{1}{3}(1 + z^{-1} + z^{-2}) \Rightarrow h[n] = \tfrac{1}{3}(\delta[n] + \delta[n-1] + \delta[n-2]).$$

7. Cascade Decomposition (H = H_1H_2)

Theory

If $H(z) = H_1(z)H_2(z)$,

$$x[n] \xrightarrow{H_1} w[n] \xrightarrow{H_2} y[n].$$

Choice of H_1 can simplify intermediate signals.

Method

1. Choose $H_1(z)$ producing a simple time-domain operation (e.g. $1 - z^{-4}$ gives $w[n] = x[n] - x[n-4]$).
2. Let $H_2(z) = H(z)/H_1(z)$.
3. Expand $H_2(z)$ into FIR form to write $y[n]$.

Example / Result

If

$$H(z) = (1 - z^{-4})(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1}),$$

choose

$$H_1(z) = 1 - z^{-4}, \quad w[n] = x[n] - x[n-4].$$

Then

$$H_2(z) = 1 - 0.8\sqrt{2}\,z^{-1} + 0.64z^{-2}.$$

Thus

$$y[n] = w[n] - 0.8\sqrt{2}\,w[n-1] + 0.64w[n-2].$$

8. Standard FIR Examples Used in Seminar 8

Example / Result

Three-point average:

$$H(z) = \tfrac{1}{3}(1 + z^{-1} + z^{-2}), \quad |H(e^{j\omega})| = \tfrac{|1+2\cos\omega|}{3}.$$

Example / Result

Two-point difference:

$$H(z) = 1 - z^{-1}, \quad |H(e^{j\omega})| = 2|\sin(\omega/2)|.$$

Example / Result

Four-sample difference:

$$H(z) = 1 - z^{-4}, \quad |H(e^{j\omega})| = 2|\sin(2\omega)|.$$

9. Exam Strategy (Seminar 8)

Method

1. Factor $H(z)$ fully; identify all zeros and their angles.
2. Check if unit circle is in ROC before evaluating $H(e^{j\omega})$.
3. For sinusoidal inputs, evaluate $H(e^{j\omega_0})$.
4. Detect nulling frequencies via unit-circle zeros.
5. For cascades, pick $H_1(z)$ producing simple shifts.
6. Express final outputs in exact sinusoidal/frequency form.