Boundary conditions

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Transverse Electric and Magnetic field

In order to obtain a unique solution of the Maxwell field equations, one must impose appropriate boundary, radiation and edge conditions.

We shall deal here only with the boundary conditions arising at the interface between two different media.

Consider a regular surface S of a medium discontinuity where the subscripts 1 and 2 distinguish quantities in regions 1 and 2, respectively. From Maxwell's equations, as a consequence of a limiting process, one obtains the following conditions:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}\,, \tag{1a}$$

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = -\mathbf{M}\,, \tag{1b}$$

where $\mathbf{J}$ and $\mathbf{M}$ are, respectively, the electric and magnetic surface current density distributions at the interface.

Normal components

Similarly for a small volume at the interface, a limiting process yields,

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = -\rho_m \,, \tag{2a}$$

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_e \,, \tag{2b}$$

where ρ_e and ρ_m are, respectively, the electric and magnetic surface charge density distributions on the interface.

If neither medium is perfectly conducting, the tangential component of the fields **E** and **H** are continuous while their normal components undergo a jump due to the discontinuity in the permittivity and permeability.

Perfect electric conductor

When medium 1 is a perfect electric conductor, the field inside the medium vanishes everywhere and induced electric charges and currents exist on the surface. In this case we have:

$$\mathbf{n} \times \mathbf{H}_2 = \mathbf{J}\,, \tag{3a}$$

$$\mathbf{n} \times \mathbf{E}_2 = 0\,, \tag{3b}$$

$$\mathbf{n} \cdot \mathbf{B}_2 = 0\,, \tag{3c}$$

$$\mathbf{n} \cdot \mathbf{D}_2 = \rho_e\,, \tag{3d}$$

which states the vanishing, at the metal surface, of the tangential components of **E** and of the normal component of **H**.

Equivalent magnetic currents

In certain cases, it is convenient to include fields generated from equivalent magnetic currents.

Accordingly, the field generated by a magnetic current distribution in the immediate vicinity of a perfectly (electrically) conducting surface is given by

$$\mathbf{n} \times \mathbf{E}_2 \quad = \quad -\mathbf{M}\,. \tag{4}$$

Boundary conditions on multivectors

Boundary conditions for field quadrivectors

When considering the normal **n** directed along z, the boundary conditions can also be stated in the following way

$$\begin{pmatrix} \epsilon_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{1z} \\ i\,E_{1y} + E_{1x} \\ H_{1z} \\ H_{1x} + iH_{1y} \end{pmatrix} = \begin{pmatrix} \epsilon_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{2z} \\ i\,E_{2y} + E_{2x} \\ H_{2z} \\ H_{2x} + iH_{2y} \end{pmatrix} \tag{5}$$

where we have assumed that no superficial currents or charges are present. It is immediate to see that

$$\begin{pmatrix} E_{2z} \\ i\,E_{2y} + E_{2x} \\ H_{2z} \\ H_{2x} + iH_{2y} \end{pmatrix} = \begin{pmatrix} \frac{\epsilon_1}{\epsilon_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\mu_1}{\mu_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{1z} \\ i\,E_{1y} + E_{1x} \\ H_{1z} \\ H_{1x} + iH_{1y} \end{pmatrix} \tag{6}$$

also holds.

Relating field multivectors

However, we recognize that

$$\begin{pmatrix} E_{1z} \\ i\,E_{1y}+E_{1x} \\ H_{1z} \\ H_{1x}+iH_{1y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{i\,\eta_1} & 0 \\ 0 & 0 & 0 & \frac{1}{i\,\eta_1} \end{pmatrix} \begin{pmatrix} E_{1z} \\ i\,E_{1y}+E_{1x} \\ \eta_1\,i\,H_z \\ \eta_1\,(i\,H_x - H_y) \end{pmatrix} \tag{7}$$

and

$$\bar{F}_2 = \begin{pmatrix} E_{2z} \\ i\,E_{2y}+E_{2x} \\ \eta_2\,i\,H_{2z} \\ \eta_2\,(i\,H_{2x} - H_{2y}) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i\,\eta_2 & 0 \\ 0 & 0 & 0 & i\,\eta_2 \end{pmatrix} \begin{pmatrix} E_{2z} \\ i\,E_{2y}+E_{2x} \\ H_{2z} \\ H_{2x}+iH_{2y} \end{pmatrix}. \tag{8}$$

Relating the field multivectors in two regions of space

We can relate the field multivector in the two regions of space as

$$\bar{F}_2 = \begin{pmatrix} \frac{\epsilon_1}{\epsilon_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\eta_2 \mu_1}{\eta_1 \mu_2} & 0 \\ 0 & 0 & 0 & \frac{\eta_2}{\eta_1} \end{pmatrix} \bar{F}_1 \tag{9}$$

and by noting that

$$\frac{\eta_2 \mu_1}{\eta_1 \mu_2} = \frac{v_2}{v_1} \tag{10}$$

we obtain the form

$$\bar{F}_2 = \begin{pmatrix} \frac{\epsilon_1}{\epsilon_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{v_2}{v_1} & 0 \\ 0 & 0 & 0 & \frac{\eta_2}{\eta_1} \end{pmatrix} \bar{F}_1 \, . \tag{11}$$

Algebraic version of the boundary conditions

In many cases it is possible to assume that $\mu_1 = \mu_2 = \mu$; by introducing the refractive index

$$n_{12} = \sqrt{\epsilon_1/\epsilon_2} \tag{12}$$

we can write the boundary conditions as

$$\bar{F}_2 = \begin{pmatrix} n_{12}^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n_{12} & 0 \\ 0 & 0 & 0 & n_{12} \end{pmatrix} \bar{F}_1 \tag{13}$$

which is an algebraic version of the boundary conditions.

Boundary conditions for waves

Boundary conditions for waves

Equation (13) also allows to recover the condition on the incident and reflected waves previously introduced. In fact, by defining

$$\begin{pmatrix} a'_0 + b'_0 \\ a'_1 + b'_1 \\ a'_0 - b'_0 \\ a'_1 - b'_1 \end{pmatrix} = \begin{pmatrix} E_{1z} \\ i\,E_{1y} + E_{1x} \\ \eta_1\, i\, H_{1z} \\ \eta_1\, (i\,H_{1x} - H_{1y}) \end{pmatrix} = \bar{F}_1 \tag{14}$$

and

$$\begin{pmatrix} a''_0 + b''_0 \\ a''_1 + b''_1 \\ a''_0 - b''_0 \\ a''_1 - b''_1 \end{pmatrix} = \begin{pmatrix} E_{2z} \\ i\,E_{2y} + E_{2x} \\ \eta_2\, i\, H_{2z} \\ \eta_2\, (i\,H_{2x} - H_{2y}) \end{pmatrix} = \bar{F}_2 \tag{15}$$

A first relation between incident and reflected waves

we can obtain the following relationship

$$\begin{pmatrix} a_0'' + b_0'' \\ a_1'' + b_1'' \\ a_0'' - b_0'' \\ a_1'' - b_1'' \end{pmatrix} = \begin{pmatrix} n_{12}^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n_{12} & 0 \\ 0 & 0 & 0 & n_{12} \end{pmatrix} \begin{pmatrix} a_0' + b_0' \\ a_1' + b_1' \\ a_0' - b_0' \\ a_1' - b_1' \end{pmatrix} . \tag{16}$$

By introducing of the symbols

$$r = \frac{n_{12} + 1}{n_{12} - 1} \tag{17}$$

$$t = \frac{2}{n_{12} - 1} . \tag{18}$$

Generalized scattering matrix for fields

We can express the reflected waves in terms of the incident ones. By performing the solution we obtain

$$\begin{pmatrix} b_0' \\ b_1' \\ b_0'' \\ b_1'' \end{pmatrix} = \begin{pmatrix} -r & 0 & \frac{t}{n_{12}} & 0 \\ 0 & r & 0 & -t \\ -n_{12}^2\, t & 0 & r & 0 \\ 0 & n_{12}\, t & 0 & -r \end{pmatrix} \begin{pmatrix} a_0' \\ a_1' \\ a_0'' \\ a_1'' \end{pmatrix} \tag{19}$$

The above equation has the form of a scattering matrix with four ports.

Traveling and reflected waves

In conclusion we have (either in region 1 or region 2)

$$\begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_0 - b_0 \\ a_1 - b_1 \end{pmatrix} = \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\,i\,H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} . \tag{20}$$

which corresponds to:

$$\begin{aligned} a_0 &= \frac{1}{2}\,(i\,\eta\,H_z + E_z) \\ a_1 &= \frac{1}{2}\,(-\eta\,H_y + i\,E_y + i\,\eta\,H_x + E_x) \\ b_0 &= \frac{1}{2}\,(E_z - i\,\eta\,H_z) \\ b_1 &= \frac{1}{2}\,(\eta\,H_y + i\,E_y - i\,\eta\,H_x + E_x)\ . \end{aligned} \tag{21}$$

two sets of two equations which are decoupled

The reason for introducing the wave parameters is that with them we have two sets of two equations which are decoupled. One set represent progressive waves and is expressed by:

$$\begin{pmatrix} \partial_z + \frac{\partial_t}{v} & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & \frac{\partial_t}{v} - \partial_z \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0 \tag{22}$$

while the regressive waves are given by:

$$\begin{pmatrix} \partial_z - \frac{\partial_t}{v} & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & -\partial_z - \frac{\partial_t}{v} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0 \tag{23}$$

At an interface perpendicular to the z direction the wave parameters are mixed according to (19).