Energy and Power of Electromagnetic Field

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Electromagnetic Energy

Electric and Magnetic energy

The field concept is based upon the hypothesis that the electromagnetic energy is distributed over the space. We introduce the electric energy density

$$w_e(\mathbf{r}, t) = \frac{\varepsilon}{2}\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \tag{1}$$

and the magnetic energy density

$$w_m(\mathbf{r}, t) = \frac{\mu}{2}\mathbf{H}(\mathbf{r}, t) \cdot \mathbf{H}(\mathbf{r}, t) \tag{2}$$

Field multivectors and energy and power

We use the field multivector$\mathcal{F}$ and the quantity $\mathcal{F}^\dagger$, i.e. the *reverse* of $\mathcal{F}$. The reverse is obtained by taking $-i$ instead of i, thus making the conjugate w.r.t. to i.

$$\begin{aligned} \mathcal{F} &= \mathbf{E} + i\,\eta\mathbf{H} \\ \mathcal{F}^\dagger &= \mathbf{E} - i\,\eta\mathbf{H}\,. \end{aligned} \tag{3}$$

By using the rules of geometric algebra we can compute

$$U = \frac{1}{2}\epsilon\,\mathcal{F}\mathcal{F}^\dagger \quad = \quad \frac{1}{2}\epsilon\,\mathbf{E}^2 + \frac{1}{2}\mu\,\mathbf{H}^2 - i\,\eta\epsilon\mathbf{E}\wedge\mathbf{H}\,. \tag{4}$$

By noting that

$$\mathbf{S} = -i\,\mathbf{E}\wedge\mathbf{H} = \mathbf{E}\times\mathbf{H} \tag{5}$$

Total Energy

We can rewrite U as

$$U = w_e + w_m + \eta \, \epsilon \, \mathbf{S} = \mathcal{E} + \frac{1}{v}\mathbf{S} \tag{6}$$

where we have introduced the total energy $\mathcal{E}$defined as:

$$\mathcal{E} = w_e + w_m \, . \tag{7}$$

Considering the product

$$V = \frac{1}{2}\epsilon \, \mathcal{F}^{\dagger}\mathcal{F} \quad = \quad \mathcal{E} - \frac{1}{v}\mathbf{S} \, . \tag{8}$$

it is possible to write

$$\begin{aligned} \mathcal{E} &= \frac{1}{2}\left(U + V\right) \\ \mathbf{S} &= \frac{v}{2}\left(U - V\right) \end{aligned} \tag{9}$$

Poynting's theorem

Poynting's theorem

By scalar multiplication of Ampére's law with $-\mathbf{E}$ and Faraday's law with $\mathbf{H}$, we obtain

$$\begin{aligned} \nabla \times \mathbf{H} &= \tfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad \Big| \quad \cdot (-\mathbf{E}) \,, \\ \nabla \times \mathbf{E} &= -\tfrac{\partial \mathbf{B}}{\partial t} \quad \Big| \quad \cdot \mathbf{H} \end{aligned} \tag{10}$$

and, by summing, yields:

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \cdot \mathbf{J}. \tag{11}$$

Using the relation

$$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot \nabla \times \mathbf{U} - \mathbf{U} \cdot \nabla \times \mathbf{V}, \tag{12}$$

We transform the left side of (11) and obtain the differential form of *Poynting's theorem*

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \frac{\partial}{\partial t}\left(\frac{\mu}{2}\mathbf{H} \cdot \mathbf{H} + \frac{\varepsilon}{2}\mathbf{E} \cdot \mathbf{E}\right) + \sigma \mathbf{E} \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{J}_0 \,. \tag{13}$$

On the right side of (13), we have the time derivative of the electric and magnetic energy densities corresponding to (1) and (2). The third term is the power loss density

$$p_v(\mathbf{r}, t) = \sigma(\mathbf{r})\mathbf{E} \cdot \mathbf{E}. \tag{14}$$

Due to the impressed current density $\mathbf{J}_0$, a power

$$p_0(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{J}_0(\mathbf{r}, t) \tag{15}$$

is added to the electromagnetic field per unit of volume.

Poynting vector: time domain

Potentials in spinor form in time–domain

Introducing the *Poynting's* vector

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) \tag{16}$$

allows to write down Poynting's Theorem in the following form:

$$\nabla \cdot \mathbf{S} = -\frac{\partial w_m}{\partial t} - \frac{\partial w_e}{\partial t} - p_v + p_0. \tag{17}$$

Integrating (17) over a volume V and transforming the integral over S into a surface integral over the boundary ∂V, we obtain the integral form of Poynting's Theorem:

$$\oint_{\partial V} \mathbf{S} \cdot \mathrm{d}\mathbf{A} = \int_V p_0 \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t} \int_V w_m \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t} \int_V w_e \mathrm{d}V - \int_V p_v \mathrm{d}V\,. \tag{18}$$

Poynting theorem discussion

The first term on the right side of equation (18) describes the power added into the volume V via impressed currents. The second and the third term, respectively, describe time variation of the magnetic and electric energy stored in the volume. The last term describes the conductive losses occurring inside the volume V. The right side of the equation comprises the total electromagnetic power generated within the volume V minus the power losses in the volume minus the increase of electric and magnetic power stored in the volume. This net power must be equal to the power, which is flowing out from the volume V through the boundary ∂V. Therefore we may interpret the surface integral over the pointing vector on the left side of (18) as the total power flowing from inside the volume V to the outside. Since this is valid for an arbitrary choice of volume V, it follows that the Poynting's vector describes the energy flowing by unit of time through an unit area oriented perpendicular to S.

Poyinting vector for harmonic fields

Poyinting vector for harmonic fields

For harmonic electromagnetic fields, the introduction of a complex Poynting's vector is useful. For this we construct

$$\begin{aligned} \nabla \times \mathbf{H}^* &= -j\omega\varepsilon^* \mathbf{E}^* + \mathbf{J}_0^* \quad &| \quad &\cdot (-\mathbf{E})\,, \\ \nabla \times \mathbf{E} &= -j\omega\mu \mathbf{H} \quad &| \quad &\cdot \mathbf{H}^*\,. \end{aligned} \tag{19}$$

Summing both equations, we obtain

$$\nabla \times \mathbf{H}^* \cdot \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}^* = -j\omega(\mu \mid \mathbf{H} \mid)^2 - \varepsilon^* \mid \mathbf{E} \mid^2) - \mathbf{E} \cdot \mathbf{J}_0^*\,. \tag{20}$$

With the relation

$$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot \nabla \times \mathbf{U} - \mathbf{U} \cdot \nabla \times \mathbf{V}\,, \tag{21}$$

we can transform (20) into the differential form of the *complex Poynting's theorem*

$$\mathrm{div} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*) = -2j\omega \left(\frac{\mu}{4} \mid \mathbf{H} \mid^2 - \frac{\varepsilon^*}{4} \mid \mathbf{E} \mid^2 \right) - \frac{1}{2} \mathbf{E} \cdot \mathbf{J}_0^*\,. \tag{22}$$

complex Poynting's vector

We now introduce the *complex Poynting's vector* $\mathbf{T}$:

$$\mathbf{T}(\mathbf{r}) = \frac{1}{2}\left(\mathbf{E}(\mathbf{r}) \times \mathbf{H}^{*}(\mathbf{r})\right). \tag{23}$$

We have to note that $\mathbf{T}$ is not the phasor corresponding to $\mathbf{S}$. Therefore we have used a different character to distinguish between the complex Poynting's vector and the real Poynting's vector. In order to give an interpretation of the complex Poynting's vector $\mathbf{T}$, we compute first the time-dependent Poynting's vector $\mathbf{S}$ for a harmonic electromagnetic field

the time average of the Poynting's vector S

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left\{\mathbf{E}(\mathbf{r})\mathrm{e}^{\mathrm{j}\omega \mathrm{t}}\right\} = \frac{1}{2}\left(\mathbf{E}(\mathbf{r})\mathrm{e}^{\mathrm{j}\omega \mathrm{t}} + \mathbf{E}^*(\mathbf{r})\mathrm{e}^{-\mathrm{j}\omega \mathrm{t}}\right), \tag{24a}$$

$$\mathbf{H}(\mathbf{r}, t) = \mathrm{Re}\left\{\mathbf{H}(\mathbf{r})\mathrm{e}^{\mathrm{j}\omega \mathrm{t}}\right\} = \frac{1}{2}\left(\mathbf{H}(\mathbf{r})\mathrm{e}^{\mathrm{j}\omega \mathrm{t}} + \mathbf{H}^*(\mathbf{r})\mathrm{e}^{-\mathrm{j}\omega \mathrm{t}}\right) \tag{24b}$$

we obtain

$$\mathbf{S}(\mathbf{r}, t) = \frac{1}{2}\,\mathrm{Re}\left\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})\right\} + \frac{1}{2}\,\mathrm{Re}\left\{\mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r})\mathrm{e}^{2\mathrm{j}\omega \mathrm{t}}\right\}. \tag{25}$$

The first term on the right side of (25) is equal to the real part of the complex Poynting's vector **T** after equation (23). This term is independent of time. The second on the right-hand side of (25) oscillates with twice the frequency of the alternating electromagnetic field. The time average of this part vanishes. Therefore the real part of the complex Poynting's vector **T** is the time average of the Poynting's vector **S**.

$$\overline{\mathbf{S}(\mathbf{r}, t)} = \mathrm{Re}\{\mathbf{T}(\mathbf{r})\}. \tag{26}$$

average electric power dissipation density

The real part of the complex Poynting's vector $\mathbf{T}$ denotes the power flowing through an unit area oriented perpendicular to $\mathbf{T}$. We write the time averages of the electric and magnetic energy densities $\overline{w}_e$ and $\overline{w}_m$ as

$$\overline{w}_e = \frac{\varepsilon}{2}\overline{\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)} = \frac{\varepsilon'}{4} \mid \mathbf{E}(\mathbf{r}) \mid^2, \tag{27}$$

$$\overline{w}_m = \frac{\mu}{2}\overline{\mathbf{H}(\mathbf{r},t)\cdot\mathbf{H}(\mathbf{r},t)} = \frac{\mu'}{4} \mid \mathbf{H}(\mathbf{r}) \mid^2 . \tag{28}$$

We have to consider that the quantities ε' and μ' in the complex representation correspond to the quantities ε and μ in the time-dependent formulation.

From equations (2), (23) and (**??**) we obtain the average electric power dissipation density

$$\overline{p}_{ve} = \frac{1}{2}\sigma \mid \mathbf{E}(\mathbf{r}) \mid^2 = \frac{1}{2}\omega\varepsilon'' \mid \mathbf{E}(\mathbf{r}) \mid^2 . \tag{29}$$

The introduction of the complex permittivity μ allows also to consider the magnetic losses. The average power dissipation density is given by

$$\overline{p}_{v} = \frac{w}{2}(\varepsilon'' \mid \mathbf{E}(\mathbf{r}) \mid^2 + \mu'' \mid \mathbf{H}(\mathbf{r}) \mid^2). \tag{30}$$

complex powe

The complex power, which is added to the field due to the impressed current density $\mathbf{J}_0$ is given by

$$p_{s0} = -\frac{1}{2}\mathbf{E} \cdot \mathbf{J}_0^* \tag{31}$$

The real part of p_{s0} equals the time average $\overline{p}_{s0}$ according to equation (17).

$$\overline{p}_0 = \mathrm{Re}\{p_{s0}\}. \tag{32}$$

The proof is similar to the one of (26). After inserting of (23), (27), (28), (30) and (31) into (22), we can write down the complex Poynting's theorem in the following form

$$\operatorname{div} \mathbf{T} = -2j\omega(\overline{w}_m - \overline{w}_e) - \overline{p}_v + p_{s0}\,. \tag{33}$$

Integral form of the complex Poynting's Theorem

By integration over a volume V, we obtain the integral form of the complex Poynting's Theorem

$$\oint_{\partial V} \mathbf{T} \cdot \mathrm{d}\mathbf{A} = \int_{V} p_{s0} \mathrm{d}V - 2j\omega \int_{V} (\overline{w}_m - \overline{w}_e) \mathrm{d}V - \int_{V} \overline{p}_v \mathrm{d}V \,. \tag{34}$$

We consider first the real part of (34).

$$\mathrm{Re}\left\{ \oint_{\partial V} \mathbf{T} \cdot \mathrm{d}\mathbf{A} \right\} = \mathrm{Re}\left\{ \int_{V} p_{s0} \mathrm{d}V \right\} - \int_{V} \overline{p}_v \mathrm{d}V \,. \tag{35}$$

The left side of (35) equals the active power radiated from inside the volume V through the boundary ∂V. On the right side of this equation, the first term denotes the power added via the impressed current density $\mathbf{J}_0$; the second term describes the conductive losses, the dielectric losses and the magnetic losses inside the volume V.

The imaginary part of (34) is

$$\operatorname{Im}\left\{\oint_{\partial V} \mathbf{T} \cdot \mathrm{d}\mathbf{A}\right\} = \operatorname{Im}\left\{\int_{V} p_{s0} \mathrm{d}V\right\} - 2\omega \int_{V} (\overline{w}_m - \overline{w}_e) \mathrm{d}V \,. \tag{36}$$

The first term on the right side gives the reactive power inserted into the volume V via the impressed current density $\mathbf{J}_0$. Let us first consider the case where the second term on the right side is vanishing. In this case we see that the left side of (36) denotes the power radiated from volume V. Since the volume V can be chosen arbitrarily, it follows that the imaginary part of the complex Poynting's vector $\mathbf{T}$ describes the reactive power radiated through an unit area normally oriented to the vector $\mathbf{T}$.

The second term on the right side of (36) contains the product of the double angle of frequency with the difference of the average stored magnetic and electric energies. This term yields no contribution, if the magnetic energy stored in the volume V equals the average electric energy stored in V. The magnetic energy as well as electric energy oscillate with an angular frequency 2ω. The energy is permanently converted between electric energy and magnetic energy. If the averages $\overline{w}_e$ and $\overline{w}_m$ are equal, electric and magnetic energies may be mutually converted completely. In this case the energy oscillates between electric and magnetic field inside the volume V. If the average electric and magnetic energies are not equal, energy as well oscillates between volume V and the space outside V. In this case there is a power flow between V and the outer region. For $\overline{w}_m > \overline{w}_e$ the reactive power flowing into volume V is positive, whereas for $\overline{w}_m < \overline{w}_e$ the reactive power flowing into V is negative.