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Engineering Electromagnetics using Geometric Algebra

Contents

1

Physical and mathematical preliminaries

I Motivation

There are already many excellent books dealing with Electromagnetic theory [1]—– [9] to cite just a few and apologizing for not citing many others of great value. However, Electromagnetic engineering is typically presented in terms of Gibbs–Heaviside vector formalism which has been developed at the beginning of 1900 [10]. What we generally denote as Maxwell's equations (the four equations with divergence and curl) have never been written by Maxwell [11]. By far and large all the most relevant books on Electromagnetic theory use the four Maxwell's equations based on the Gibbs–Heaviside formalism.

However, the scientific progress in the last century has proposed some considerable novelties. Around 1928 Wolfgang Pauli has introduced the representation of a vector in terms of a two by two matrix, thus allowing novel possible operations as e.g. taking the inverse of a vector. In the same years, completely independently, Paul A. M. Dirac has introduced the Dirac matrices in order to express the Dirac equation, [12, 13]. While these contributions have been made for very different physics subjects' their application as a methodology in Electromagnetism can be very useful. In addition, in the last century the contributions of Hamilton, Grassmann, Clifford and others have been clarified and better understood [14]. In Table 1.1 an approximate historical development of the various vector formalisms is given.

In fact, it has been recognized that both Pauli and Dirac algebras can be related to the Clifford algebra [15], more recently referred to also as Geometric Algebra (GA). The description of the space in terms of GA is considerably more satisfying than traditional vector analysis. In fact, in vector analysis, there are only scalars (points) and vectors. In addition, vector analysis do not represent an algebra and therefore a pletora of different rules have to be followed for performing

various tasks. Conventional vector analysis only holds true in the three–dimensional space, while GA can be used in whatever number of dimensions. In particular, in the three–dimensional space, according to GA, we need to represent a scalar (which is a point), a vector (oriented line), a bivector (oriented surface) and a pseudo–scalar (a volume element). It is possible to introduce a multivector which is the sum of a scalar, a vector, a bivector and a pseudo–scalar. It is possible to multiply multivectors as simply as we can multiply two by two matrices. Such multivector can be represented either by a two by two Pauli matrix or in terms of a Dirac quadrivector. Maxwell equations can be grouped into a single equation, which has a very simple form (and, noticeably, has the same form also for potentials), and which presents considerable similarities with the Dirac equation. In addition, the algebraic expressions allows noticeable simplifications. In recent years, the limitations due to the use of Gibbs–Heaviside formalism have been recognized; as an example, differential forms have been adopted in the excellent book [16] and in the very interesting paper [17], for providing a better understanding of the underlying physical phenomenology.

In recent years, the interest on the application of geometric algebra has grown significantly and many excellent books have appeared [18]— [23]. However, none of these books is suitable for teaching a basic course of Electromagnetic Fields at engineering.

A book edited by IEEE and entitled “Understanding Geometric Algebra for Electromagnetic Theory”, [24], has appeared in 2011 applying GA concepts to electromagnetic theory. Another excellent book namely “Application of Geometric Algebra to Electromagnetic Scattering” by A. Seagar [25] has been published, showing the potentiality, mainly at the research level, of using a GA approach. Recently, a seminal paper [26] is appeared on the IEEE proceedings and this paper is an excellent introduction to GA for Electromagnetic engineers.

However, often GA has been employed in electromagnetic (EM) by considering a point of view not suitable for engineers [27], [28]. For engineering purposes it is important to deal with guided wave propagations and antennas and not so much with Lorentz transformations.

In addition, while Clifford or GA algebra are very nice for theoretical framework, in practical engineering computations the use of Pauli and Dirac matrices provides a significant advantage. They are very suitable for introduction in a Computer Algebra System (CAS) and tedious computations can be completely avoided. As a matter of fact, with present day CAS it is feasible

to develop the required vector algebra entirely at computer level. In this book we have reported several listings illustrating the various procedures.

II Basic concepts

II.1 Units and Dimensions

The measurement of a quantity involves a *unit* and a *number*. The unit is a reference amount of the quantity, and the number expresses the ratio of the magnitude of that quantity to the magnitude of the unit. When the unit of a quantity is assigned an arbitrary value, that unit is called a *fundamental unit* and the quantity a *fundamental quantity*. As an example, in order to measure a distance d we refer to the fundamental quantity of the meter and then express how many meters are in d.

The International Systems of Units (SI) is the normalized system currently used in science and engineering for expressing the measurements unit of physical quantities. The *fundamental quantities* are listed in the Table 1.2. All the secondary quantities can be expressed in terms of the fundamental ones.

In order to express to quantities between 10^{-18} and 10^{18} the conventional prefixes reported in Table 1.3 are often used. Note the use of capital letters.

In electromagnetic engineering both scalar and vector quantities are used. Scalars are denoted in italic, as an example the resistance is denoted by R (what is the dimension of the resistance? see Resistance dimension.)

Vector quantities are typically denoted in bold; e.g. the electric field is denoted by $\mathbf{E}$. When hand writing it is common practice to use the notations $\bar{E}$ or $\underline{\mathrm{E}}$. Versors, i.e. unit vectors, are denoted either by $\hat{\mathbf{x}}$ or by adding a subscript $\mathbf{x}_0$. As an example an electric field in the x direction can be denoted as $\mathbf{E} = \mathbf{x}_0 E$, where E is the module and $\mathbf{x}_0$ is the versor specifying the direction. Often, also the notation $\mathbf{u}_x$ may be found.

History of Electromagnetic Formalisms

Year	Nonrelativistic	Relativistic
1844	*Exterior Algebra (Grassman)*	
1853	*Quaternions (Hamilton)*	
1861	Scalar Components (Maxwell)	
1878	*Geometric Algebra (Clifford)*	
1881	Quaternions (Maxwell)	
1892	**3-Vectors** (Gibbs & Heaviside)	
1899	*Differential Forms (Cartan)*	
1901	Complex 3-Vectors (Riemann)	
1905		*4-Vectors with Imaginary Time (Poincaré)*
1907	Complex 3-Vectors (Silberstein)	
1908		*4-Vectors (Minkowski)*
1910		Complex 6-Vectors (Sommerfeld)
1911		Exterior Algebra (Wilson & Lewis)
1916		Tensor Scalar Components (Einstein)
1918		Differential Forms (Weyl)
1928		Pauli and Dirac Matrices
1966	Spacetime Algebra (Hestenes)	

Table 1.1. A brief history of the development of mathematical formalisms for representing the electromagnetic theory, showing the purely mathematical developments in italic font and their use in electromagnetism unitalicized. The bolded formalism is the one most commonly used today in Electromagnetic engineering: the 3-Vectors of Gibbs and Heaviside.

Table 1.2. Fundamental quantities of the International System

Dimension	Unit	Symbol
length	metre	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
temperature	kelvin	K
quantity	mole	mol
luminous intensity	candela	cd

Table 1.3. Standard prefixes for the SI units of measure

Prefixes	Symbol	Factor
exa	E	10^{18}
peta	P	10^{15}
tera	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}
atto	a	10^{-18}

II.2 Coulomb's law

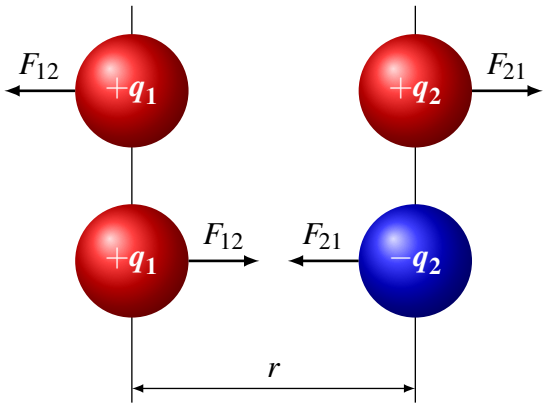
Two charged points with charges of the same sign are repulsing, while points with charges of opposite signs are attracting with force proportional to quantities of their charges and inverse proportional to square of distance between them:

$$F = k\frac{Q_1 Q_2}{r^2}. \tag{1.1}$$

with

$$k = \frac{1}{4\pi\varepsilon_0} = \frac{c^2\mu_0}{4\pi}. \tag{1.2}$$

Name	Symbol	Value	Unit
Number π	π	3.14159265358979323846	
Number e	e	2.71828182845904523536	
Euler's constant	$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \ln(n)\right)$	$= 0.5772156649$	
Elementary charge	e	$1.60217733 \cdot 10^{-19}$	C
Gravitational constant	G, κ	$6.67259 \cdot 10^{-11}$	$m^3kg^{-1}s^{-2}$
Speed of light in vacuum	c	$2.99792458 \cdot 10^{8}$	m/s (def)
Permittivity of the vacuum	ε_0	$8.854187 \cdot 10^{-12}$	F/m
Permeability of the vacuum	μ_0	$4\pi \cdot 10^{-7}$	H/m
Electron mass	m_e	$9.1093897 \cdot 10^{-31}$	kg
Proton mass	m_p	$1.6726231 \cdot 10^{-27}$	kg
Neutron mass	m_n	$1.674954 \cdot 10^{-27}$	kg
Diameter of the Sun	$D_\odot$	$1392 \cdot 10^{6}$	m
Mass of the Sun	$M_\odot$	$1.989 \cdot 10^{30}$	kg
Rotational period of the Sun	$T_\odot$	25.38	days
Radius of Earth	R_A	$6.378 \cdot 10^{6}$	m
Mass of Earth	M_A	$5.976 \cdot 10^{24}$	kg
Rotational period of Earth	T_A	23.96	hours
Earth orbital period	Tropical year	365.24219879	days
Astronomical unit	AU	$1.4959787066 \cdot 10^{11}$	m
Light year	lj	$9.4605 \cdot 10^{15}$	m



$$F_{12} = F_{21} = k\frac{q_1 q_2}{r^2}$$

Fig. 1.1. Figure illustrating the force between two electrical charges.

Force F defined by the relationship (1.1) is quite strong. However, in everyday life it does not reveal itself. This is due to the *screening*. The numbers of positive and negative charges in nature are exactly balanced. Atoms and molecules, which constitute all observable matter around us,

have the same amount of positive and negative charges. Therefore they are electrically neutral in whole. Force (1.1) reveals itself in form of chemical links only when atoms are pulled together. Electric current arises as a result of motion of charged points. This occurs in metallic conductor, which usually have lengthy form (form of wire). Current in such conductor is determined by the *amount of charge passing through it within the unit of time*. Let's consider straight conducting rod of the length l. Current in it leads to misbalance of charges in its ends. Charges of definite sign move to one end of the rod, while lack of these charges in the other end of the rod is detected as the charge of opposite sign. Then Coulomb force arises that tends to recover balance of charges in electrically neutral rod. This means that in such rod current could not flow in constant direction during long time.

Another situation we have with conductor of the form of ring or circuit. Here current does not break the balance of charges. Direct current can flow in it during unlimitedly long time. Circular conductor itself thereby remains electrically neutral and no Coulomb forces arise.

In spite of absence of Coulomb forces, in experiments the interaction of two circular conductors with currents was detected. This interaction has other nature, it is not due to electrical, but due to magnetic forces.

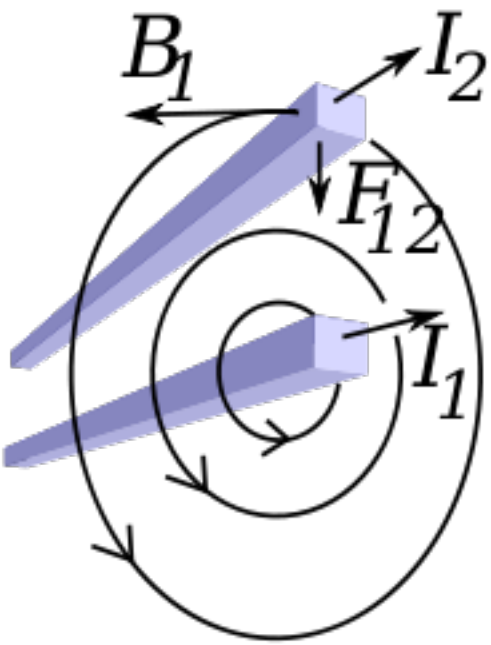


Fig. 1.2. Force between two filaments with currents (taken from internet)

The magnitude of magnetic forces depends essentially on the shape and mutual arrangement of circular conductors. In order to reveal quantitative characteristics for magnetic forces it is convenient to simplify the geometry of conductors. To this end, they are deformed so that each possesses straight rod-shaped part of sufficiently large length l. These rod–shaped parts are arranged parallel to each other with the distance r between them. In the limit, when l is much larger than r, this configuration of conductors can be treated as a pair of infinitely long parallel conductors.

In experiments it was found that such conductors do interact according to the following law.

II.3 Ampere law

Force of interaction of two infinite parallel conductors with currents per unit length of them is proportional to the values of currents in them and inverse proportional to the distance between them:

$$\frac{F}{l} = 2k_A \frac{I_1 I_2}{r}. \tag{1.3}$$

with

$$k_A = \frac{\mu_0}{4\pi} \tag{1.4}$$

Two co-directed currents attract each other, while opposite directed currents repulse each other. In SI measure unit of current 1 A (one *ampere*) is a basic unit. It is determined such that formula is written as

$$\frac{F}{l} = \frac{2\,\mu_0}{4\pi} \frac{I_1 I_2}{r}. \tag{1.5}$$

Here $\pi = 3.14\ldots$ is exact (though it is irrational) mathematical constant with no measure unit. Constant μ_0 is called *magnetic susceptibility* of vacuum. It has the measure unit:

$$\mu_0 = 4\pi \cdot 10^{-7} N \cdot A^{-2}. \tag{1.6}$$

But, in contrast to constant c, the speed of light, it is an exact constant. Its value should not be determined experimentally. One could choose it to be equal to unity, but the above value for this constant was chosen by convention when SI system was established. Due to this value of constant μ_0 current of 1 *ampere* appears to be in that range of currents, that really appear in industrial and household devices. Coefficient 4π in denominator (1.5) is used in order to

simplify some other formulas, which are more often used for engineering calculations in electric technology. Being basic unit in SI, unit of current *ampere* is used for defining unit of charge of 1 *coulomb* as $1\,\mathrm{C} : 1C = 1\,\mathrm{A} \cdot 1\,\mathrm{s}$. Then coefficient of proportionality in Coulomb law (1.1) appears to be not equal to unity. In SI Coulomb law is written as

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1 Q_2}{r^2}. \tag{1.7}$$

Constant ε_0 is called dielectric permittivity of vacuum. In contrast to constant μ_0, this is physical constant determined experimentally:

$$\varepsilon_0 \approx 8.85 \cdot 10^{-12}. \tag{1.8}$$

Constants μ_0, ε_0, and c are related to each other by the following equality:

$$c = \frac{1}{\sqrt{\varepsilon_0\,\mu_0}} \approx 2.998 \cdot 10^8\ m/sec. \tag{1.9}$$

The SI better suits for engineering calculations and is nowadays widely used.

Comparing Coulomb law and Ampere law we see that electrical and magnetic forces reveal themselves in quite different way. However, they have common origin: they both are due to electric charges. Below we shall see that their relation is much more close. Therefore theories of electricity and magnetism are usually united into one theory of electromagnetic phenomena. Theory of electromagnetism is a theory with one measurable constant: this is the light velocity c. Classical mechanics (without Newton's theory of gravitation) has no measurable constants. Newton's theory of gravitation has the constant G given in Table II.1.

This theory is based on Newton's fourth law formulated as follows.

II.4 Universal law of gravitation

Two point masses attract each other with the force proportional to their masses and inverse proportional to the square of distance between them. Universal law of gravitation is given by the formula

$$F = G\frac{M_1 M_2}{r^2} \tag{1.10}$$

in SI system.

According to modern notion of nature classical mechanics and Newton's theory of gravitation are approximate theories. Currently they are replaced by special theory of relativity and general theory of relativity. Historically they appeared as a result of development of the theory of electromagnetism. Below we keep this historical sequence in explaining all three theories.

II.5 Concept of near action

Let's consider pair of charged bodies, which are initially fixed, and let's do the following mental experiment with them. When we start moving second body apart from first one, the distance r begins increasing and consequently force of Coulomb interaction will decrease. In this situation we have a natural question: how soon after second body starts moving second body will feel change of Coulomb force of interaction? There are two possible answers to this question:

- immediately;
- with some delay depending on the distance between bodies.

First answer is known as concept of *distant action*. Taking this concept we should take formula (1.1) as absolutely exact formula applicable for charges at rest and for moving charges as well. Second answer is based on the concept of *near action*. According to this concept, each interaction (and electric interaction among others) can be transmitted immediately only to the point of space infinitesimally close to initial one. Transmission of any action to finite distance should be considered as a process of successive transmission from point to point. This process always leads to some finite velocity of transmission for any action. In the framework of the concept of near action Coulomb law is treated as approximate law, which is exact only for the charges at rest that stayed at rest during sufficiently long time so that process of transmission of electric interaction has been terminated. Theory of electromagnetism has measurable constant c (light velocity), which is first pretender for the role of transmission velocity of electric and magnetic interactions. For this reason electromagnetic theory is much more favorable as compared to Newton's theory of gravitation. The value of light velocity is a very large quantity. If we settle an experiment of measuring Coulomb force at the distances of the order of $r \approx 10\ cm$, for the time of transmission of interaction we would get times of the order of $t \approx 3 \cdot 10^{-10}\ sec$. Experimental technique of XIX-th century was unable to detect such a short interval of time. Therefore

the problem of choosing concept could not be solved experimentally. In XIX-th century it was subject for contests. The only argument against the concept of distant action that time, quite likely, was its straightness, its self-completeness, and hence its scarcity. In present time concept of near action is commonly accepted. Now we have the opportunity for testing it experimentally in the scope of electromagnetic phenomena. Let's study this concept more attentively. According to the concept of near action, process of transmitting interaction to far distance exhibits an inertia. Starting at one point, where moving charge is placed, for some time this process exist in hidden form with no influence to both charges. In order to describe this stage of process we need to introduce new concept. This concept is a *field*.

II.6 Field concept

A Field is a material entity able to fill the whole space and able to act upon other material bodies transmitting mutual interaction of them. The number of fields definitely known to scientists is not big. There are only four fundamental fields: *strong field, weak field, electromagnetic field*, and *gravitational field*. Strong and weak fields are very short distance fields, they reveal themselves only in atomic nuclei, in collisions and decay of elementary particles, and in stellar objects of extremely high density, which are called neutron stars. Strong and weak interactions and fields are not considered in this book. There are various terms using the word field: *vector field, tensor field, spinor field, gauge field,* and others. These are mathematical terms reflecting some definite properties of real physical fields.

Let's apply concept of near action to Coulomb law for two charged points. Coulomb force in the framework of this concept can be interpreted as follows: first charge produces electric field around itself, and this field acts upon other charge. Result of such action is detected as a force F applied to second charge. Force is vectorial quantity. Let's denote by $\mathbf{F}$ vector of force and take into account the direction of this vector determined by verbal statement of Coulomb law above. This yields

$$\mathbf{F} = Q_1\, Q_2\, \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{1.11}$$

Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are radius-vectors of points, where charges Q_1 and Q_2 are placed. Let's consider vector $\mathbf{E}$ determined as the ratio $\mathbf{E} = \mathbf{F}/Q_2$. For this vector from formula (1.11) we derive

$$\mathbf{E} = Q_1 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{1.12}$$

Vector $\mathbf{E}$ depends upon the position of first charge and upon its value. It depends also on the position of second charge, but it doesn't depend on the value of second charge. One can take vector $\mathbf{E}$ for quantitative measure of electric field produced by first charge Q_1 at the point $\mathbf{r}_2$, where second charge is placed. Vector $\mathbf{E}$ can be determined by formula (1.12) or it can be measured experimentally. For this purpose one should place test charge q to the point $\mathbf{r}_2$ and one should measure Coulomb force $\mathbf{F}$ acting upon this test charge. Then vector $\mathbf{E}$ is determined by division of $\mathbf{F}$ by the value of test charge q:

$$\mathbf{E} = \mathbf{F}/q. \tag{1.13}$$

II.7 Superposition principle

Now consider more complicated situation. Suppose that charges $Q_1, \ldots, Q_n$ are placed at the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$. They produce electric field around them, and this field acts upon test charge q placed at the point $\mathbf{r}$. This action reveals as a force $\mathbf{F}$ applied to the charge q. Again we can define vector $\mathbf{E}$ of the form (1.13) and take it for the quantitative measure of electric field at the point $\mathbf{r}$. This vector is called *vector of intensity of electric field* or simply *vector of electric field* at that point. Generally speaking, in this case one cannot be a priori sure that vector $\mathbf{E}$ does not depend on the quantity of test charge q. However, there is the following experimental fact. Electric field $\mathbf{E}$ at the point $\mathbf{r}$ produced by a system of point charges $Q_1, \ldots, Q_n$ is a vectorial sum of electric fields that would be produced at this point by each charge $Q_1, \ldots, Q_n$ separately. This is the *Superposition principle*. Superposition principle combined with Coulomb law leads to the following formula for the intensity of electric field produced by a system of point charges at the point $\mathbf{r}$:

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n} Q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{1.14}$$

Using superposition principle, one can pass from point charges to continuously distributed charges. Suppose that the number of point charges tends to infinity: $n \to \infty$. In such limit sum in formula (1.14) is replaced by integral over 3-dimensional space:

$$\mathbf{E}(\mathbf{r}) = \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}'. \tag{1.15}$$

Here $\rho(\mathbf{r}')$ is spatial density of charge at the point $\mathbf{r}'$. This value designates the amount of charge per unit volume. In order to find force acting on test charge q we should invert formula (1.13) As a result we obtain

$$\mathbf{F} = q\,\mathbf{E}(\mathbf{r}). \tag{1.16}$$

Force acting on a charge q in electric field is equal to the product of the quantity of this charge by the vector of intensity of field at the point, where charge is placed. However, charge q also produces electric field. Does it experience the action of its own field? For point charges the answer to this question is negative. This fact should be treated as a supplement to principle of superposition. Total force acting on a system of distributed charges in electric field is determined by the following integral:

$$\mathbf{F} = \int \rho(\mathbf{r})\,\mathbf{E}(\mathbf{r})\,d^3\mathbf{r}. \tag{1.17}$$

Field $\mathbf{E}(\mathbf{r})$ in (1.17) is external field produced by external charges. Field of charges with density $\rho(\mathbf{r})$ is not included into $\mathbf{E}(\mathbf{r})$. Concluding this section, note that formulas (1.14) and (1.15) hold only for charges at rest, which stayed at rest for sufficiently long time so that process of interaction transmitting reached the point of observation $\mathbf{r}$. Fields produced by such systems of charges are called *static fields*, while branch of theory of electromagnetism studying such fields is called *electrostatics*.

Superposition principle is a consequence of the linearity of the equations. If we have a linear operator $\mathscr{L}$ that, with given sources s_1 produces a field f_1 as

$$f_1 = \mathscr{L}(s_1) \tag{1.18}$$

and if we consider other sources s_2 such that

$$f_2 = \mathscr{L}(s_2)\,. \tag{1.19}$$

When they are considered together for linearity we have

$$f_1 + f_2 = \mathscr{L}(s_1 + s_2)\,. \tag{1.20}$$

This is the superposition principle that apply for linear equations. Fortunately, Maxwell equations describing the behavior of the Electromagnetic field are linear ones. Therefore in the study of Electromagnetic fields we can always apply superposition.
As we have seen the field is, in general, a *vector* quantity. It is therefore appropriate to recall the main operations on vectors.

II.8 The frequency spectrum

The electromagnetic frequency spectrum is represented in Figure 1.3 in terms of the wavelength. In this Figure is also shown the portion of spectrum where the transmission characteristics of optical fibers are best utilized. Waveguides used in optical communication systems and in integrated optics are generally operated in this frequency range. Another part of the spectrum intensively used is that corresponding to radio wavelengths, as detailed in Table 1.4 . Waveguides operating at microwave and millimeter waves are of particular interest for their wide range of applications. The frequency band designations employed in the microwave range are reported in Table 1.5.
The wavenumber concept is perhaps more fundamental in electromagnetic wave theory than either of the more popular concepts of wavelength and frequency. The corresponding values of wavenumber k and frequency f are obtained from the wavelength λ by the following relationships

$$f = \frac{c}{\lambda} \quad \lambda = \frac{2\pi}{k} \tag{1.21}$$

where c is the free–space velocity of light,

$$\begin{aligned} c &= \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 2.997925 \times 10^8 \text{m/s} \approx 3 \times 10^8 \text{m/s} \\ \mu_0 &= 4\pi \times 10^{-7} henry/metre \left(\frac{V \cdot s}{A \cdot m}\right) \\ \varepsilon_0 &= \frac{1}{36\pi} \times 10^{-9} farad/metre \left(\frac{A \cdot s}{V \cdot m}\right) \end{aligned} \tag{1.22}$$

Table 1.4. Radio frequency band designations and use.

Frequency	*Wavelength*	*Band designation*	*Typical service*
30-300 Hz 300-3000 Hz	10-1 Mm 1000-100 km	ELF extremely low frequency	
3-30 kHz	100-10 km	VLF very low frequency	Navigation, sonar
30-300 kHz	10-1 km	LF low frequency	Radio beacons, navigational aids
300-3000 kHz	1 km-100 m	MF medium frequency	AM broadcasting, maritime radio, Coast Guard communication, direction finding
3-30 MHz	100-10 m	HF high frequency	Telephone, telegraph, facsimile; shortwave international broadcasting; amateur radio; citizen's band; ship to coast and ship to aircraft communication
30-300 MHz	10-1 m	VHF very high frequency	Television, FM broadcast, air traffic control, police, taxicab mobile radio, navigational aids
300-3000 MHz	1 m-10 cm	UHF ultra high frequency	Television, radioprobes, satellite communication, surveillance radar, navigational aids
3-30 GHz	10-1 cm	SHF super high frequency	Airborne radar, satellite communication, common carrier land mobile communication, microwave links
30-300 GHz	1 cm- 1 mm	EHF extremely high frequency	Experimental, Radar, vehicular anti-collision radar

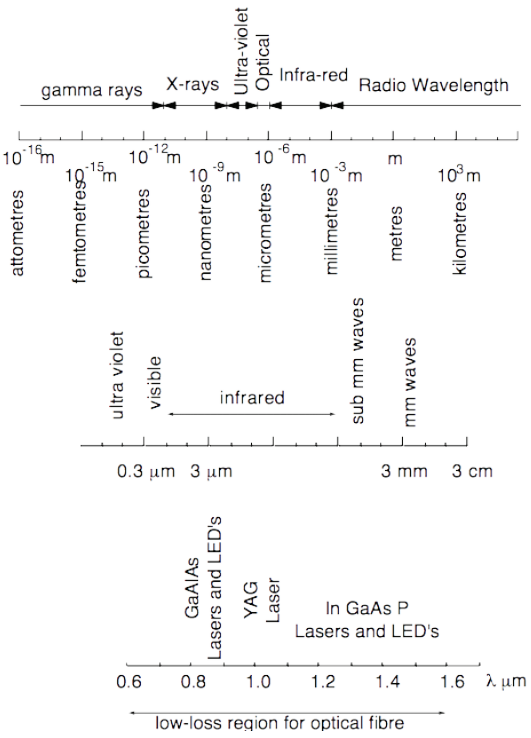


Fig. 1.3. Electromagnetic wave spectrum.

while μ_0 is the magnetic permeability and ε_0 the dielectric permittivity of free–space.

Waveguides may operate up to optical frequencies. For communication purposes, since it is generally desirable to employ as large a bandwidth as possible, it is advantageous to use the higher frequencies now available. Up to the millimetric range this is an added bonus, as circuit dimensions become smaller as frequency increases, thus allowing a saving of space and sometimes a reduction of costs when higher frequencies are used.

Table 1.5. Microwave frequency band designations.

Frequency	Wavelength	IEEE radar band designation	
		old	new
1-2 GHz	30-15 cm	L	D
2-3 GHz	15-10 cm	S	E
3-4 GHz	10-7.5 cm	S	F
4-6 GHz	7.5-5 cm	C	G
6-8 GHz	5-3.75 cm	C	H
8-10 GHz	3.75-3 cm	X	I
10-12.4 GHz	3-2.42 cm	X	J
12.4-18 GHz	2.42-1.67 cm	Ku	J
18-20 GHz	1.67-1.5 cm	K	J
20-26.5 GHz	1.5-1.13 cm	K	K
26.5-40 GHz	1.13 cm-7.5 mm	Ka	K
40-300 GHz	7.5-1.0 mm	mm	

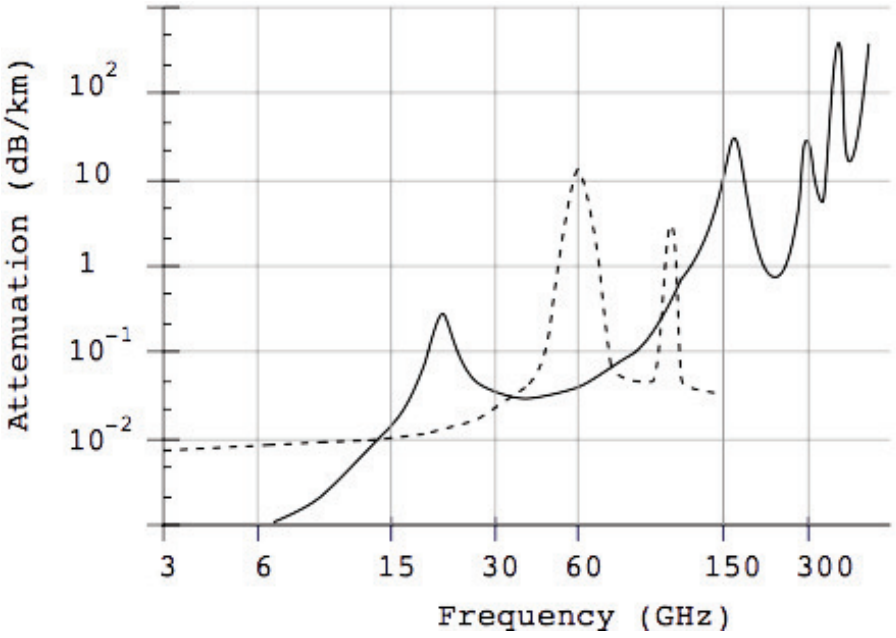


Fig. 1.4. Attenuation by oxygen (dotted line) and water vapour (continuous line) at sea level. The water content is 7.5 g/m^3 and the temperature is T = 20°C.

Unfortunately, as frequency increases so does the average atmospheric attenuation, as shown in Figure 1.4. This fact, together with problems related to electromagnetic interference, prevents the use of free–space as a transmitting media for very large amounts of data. Atmospheric attenuation is also important for many applications; for example, by choosing frequencies for which the atmosphere is opaque, one prevents detection of satellite–to–satellite communications by ground–based receivers. Similarly, terrestrial systems desiring to prevent signal overshoot in range may operate at a frequency of high atmospheric absorption.

Optical fibres are nowadays widely used because of their low-loss, high bandwidth, transmission characteristics which make them ideally suited for carrying voice, data, and video signals between fixed points. Existing optical fibres have now an attenuation of less than 0.2 dB/km for source wavelength near 1.55 μm. Moreover, since optical fibres are made of a dielectric, the link between source and detector is nonmetallic; this avoids ground loop pick–up problems resulting from electromagnetic interference.

Fibers, however, are not applicable to satellite communications, or to connect mobile systems, where wireless communication systems are required. The latter are commonly realized both in the microwave and millimeter–wave frequency ranges, upon consideration of the atmospheric attenuation. In this case, the preferred transmission lines are open waveguides such as microstrip lines, coplanar waveguides, etc.

II.9 Conventional vector analysis

A linear space is one upon which addition and scalar multiplication are defined. Although such a space is often called a "vector space", we will use the term "vector" for the geometric concept of a directed line segment.

A *vector* is a quantity having both direction and magnitude in space. Examples of vector quantities are force, velocity, acceleration, etc. We require linearity, so that for any vectors $\mathbf{a}$ and $\mathbf{b}$ we must be able to define their vector sum $\mathbf{a}+\mathbf{b}$. We define the product of a scalar α and a vector $\mathbf{a}$ as $\alpha\mathbf{a}$. The multiplication of a vector times a scalar leaves the direction unchanged and the resulting vector is simply with a different magnitude. In order to express algebraically the geometric idea of magnitude, it is convenient to define a dot product.

The *dot product* $\mathbf{a}\cdot\mathbf{b}$

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \tag{1.23}$$

is the scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ and is a scalar with magnitude $|\mathbf{a}||\mathbf{b}|\cos\theta$, where $|\mathbf{a}|$ and $|\mathbf{b}|$ are the lengths of the vectors, and θ is the angle between them. This is illustrated in Fig. 1.5.

The *cross product* $\mathbf{a}\times\mathbf{b}$ of two vectors is

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta \tag{1.24}$$

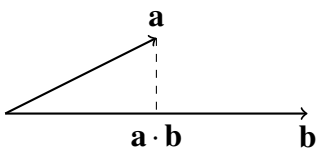


Fig. 1.5.

Fig. 1.6. The dot product of two vectors **a** and **b**. It is like the shadow of **a** when the light is coming from a direction orthogonal to **b** and from the same direction of **a**

.

i.e. a vector of magnitude $|\mathbf{a}||\mathbf{b}|\sin\theta$ in the direction perpendicular to **a** and **b**, such that **a**, **b** and $\mathbf{a}\times\mathbf{b}$ form a right-handed set. However, the vector cross product exists only in our 3-dimensional world; in two dimensions there is simply no direction perpendicular to **a** and **b**, and in four or more dimensions that direction is ambiguous. A more general concept is needed, so that full information about relative directions can still be encoded in all dimensions. Before introducing the external product we would like to summarize the vector identities.

Vector identities

A summary of the vector identities is given in the following

$$a = \sqrt{\mathbf{a}\cdot\mathbf{a}} \tag{1.25}$$

$$\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a} \tag{1.26}$$

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} \tag{1.27}$$

$$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a} \tag{1.28}$$

$$(\mathbf{a}+\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c} \tag{1.29}$$

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \mathbf{b}\cdot\mathbf{c}\times\mathbf{a} = \mathbf{c}\cdot\mathbf{a}\times\mathbf{b} \tag{1.30}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \tag{1.31}$$

Equation (1.27) is called the *commutative law*, while (1.29) is the *distributive law*. It is seen from (1.28) that the cross product is *anticommutative*. In (1.30) it is stated that it is possible to exchange the dot with the cross (naturally the cross product has to be performed first). In the last equation (1.31) it is reported a well known identity (often memorized as bac minus cab).

It is noted that the above expressions are independent from the coordinate system. If one identity is proved in a coordinate system, it is valid in all the coordinate systems. Computation of vector expression may often become quite tedious and it is of considerable advantage to be able to perform such computations, either symbolically and numerically, with a computer algebra system as described next.

Computer algebra system for vector operations

Several Computer Algebra Systems (CAS) are presently available. In the following we will make use of *wxMaxima* for three reasons:

- it is freely available;
- it runs on several operating systems,
- it allows to copy the result as a LaTeX expression.

We will write simple lines of code that can be cut and past in a *wxMaxima* window and are ready to run.

Example 1. The first example will refer to a segment of code for performing the dot and cross product. The file

```
wxm/vectors.wxm
```

provides the details.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 15.04.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
print("In order to perform vector operations use load(vect)")$
load(vect);

/* file_name : vectors_v03.wxm */
print("Given the vectors")$
a:[a1, a2, a3];
b:[b1, b2, b3];
c:[c1, c2, c3];
```

```
print("dot product is obtained by a . b ")$
dab : a . b;

print(" cross product")$
axb : a ~ b;
axb : express(axb);
bxa : express(b ~ a);
print("find  axb + bxa")$
diff : axb + bxa;

print("Perform triple scalar product")$
print("(A X B) . C")$
try1 : express(a ~ b) . c;
print("(C X A) . B")$
try2 : express(c ~ a) . b;
print("(B X C) . a")$
try3 : express(b ~ c) . a;
ratsimp(try1-try2);
ratsimp(try3-try2);

print("Perform triple vector product")$
print("first verify the rule A X ( B X C ) = B(A.C) -C(A.B)")$
print("A X ( B X C ) = ")$
r1 : express(a ~ express(b ~ c));
print(" B(A.C) -C(A.B) = ")$
r2 : b * (a . c) - c*(a . b);
print("Perform their difference")$
ratsimp(r1-r2);

print("numerical example")$
A : [1,-1,2];
B : [0,1,1] ;
C : [-2,0,3] ;

print("Property of cross product")$
print(" A X B")$
apb : express(A~B);
```

```
print(" (A X B) X C")$
apbpc : express(apb~ C);

print(" B X C")$
bpc : express(B~C);
print("A X (B X C)")$
apbpc2 : express(A ~ bpc);
print(" (A X B) X C  -  A X (B X C)")$
diff : (apbpc- apbpc2);

print("End")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

We start by initializing the environment with the kill(all) command. In order to perform vector calculus it is necessary to load the *vect package*. Once this is done we can perform the dot products by simply placing a dot between two elements previously defined (a and b in our case). The cross product is performed by using the expression

```
a ~ b
```

and then using the command express in order to evaluate the previous command.

It is noted that, at the end, we also evaluate $\mathbf{b} \times \mathbf{a}$ and verify (1.28).

By introducing a third vector $\mathbf{c}$ we can perform the operations reported in (1.30) and verify their correctness. The ratsimp command allows to simplify the expressions.

The next part is done in order to verify the rule in (1.31)

It is seen that the final expression, ratsimp(r1-r2), gives zero showing that the identity holds.

So far we have used symbolic calculations, but naturally also numerical computations can be performed. Here following an example is reported. It is shown numerically the important fact that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ is different from $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

Example 2. An example of exercise with vectors is reported next in the file:

```
wxm/example3.1.wxm

/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */
```

```
/* [wxMaxima: input   start ] */
kill(all);
load(vect);
fpprintprec:7$
/* file_name example3.1 */

print("Vector A is the vector from the origin to point P1")$
print("Point P1 has coordinates")$
/* P[1] : [2,3,3]; */
P[1] : [1,2,3];
print("vector B starts from P1 and ends in P2 which is")$
/* P[2] : [1,-2,2]; */
P[2] : [3,-2,2];

print("Find: vector A its modulus and versor a")$
print("the angle of A with the y axes")$
print("vector B")$

print("==========================")$
print("Solution")$
print("vector A is ")$
A : P[1];
print("its modulus is")$
modA : sqrt(A . (A));
print("versor a")$
a : A /modA;

print("define the versors")$
x[0] : [1,0,0];
y[0] : [0,1,0];
z[0] : [0,0,1];

print("The angle between A and y axis: A . y[0] = abs(A) cos(%beta)")$
%beta : float(acos((A. y[0])/modA));
float(%beta *180/%pi);
```

```
print("vector B is ")$
B : P[2]-P[1];
print("with modulus")$
modB : float(sqrt(B . (B)));
print("versor b")$
b : B /modB;

print("theta is the angle between A and B")$
%theta : float(acos((A.B)/(modA * modB))) * float(180/%pi);
print("distance is")$
dist : float(modA * sin(%pi - %theta*%pi/180));

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

III Complex numbers as an introduction to Clifford algebra

III.1 Complex numbers

Often the sinusoidal regime is considered and the phasors representation is used. Note that the time dependent quantity is different from its phasors representation and therefore, when there is chance of ambiguity, a different symbol should be used in order to distinguish them.
It is assumed that the reader is familiar with phasor formalism and the use of complex numbers. We start by recalling the following notation for the square root of -1. Since there are two roots for -1 we associate i with the solution in the upper part of the Argand plane ($\pi/2$ angle) and j with the solution with angle $3\pi/2$ as

$$i = -j \tag{1.32}$$

$$i^2 = j^2 = -1 \tag{1.33}$$

as shown in Fig. 1.7.

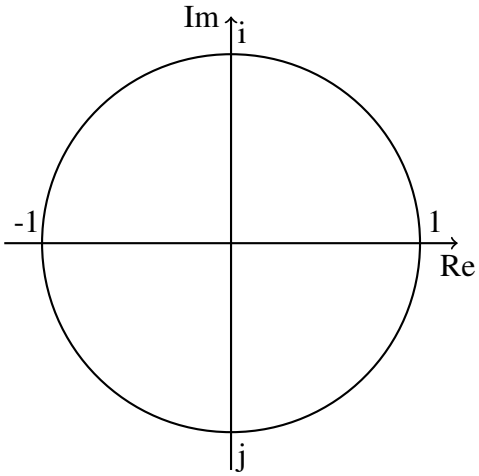


Fig. 1.7. The square root of -1 has two solutions i, j. The solution with i has an angle with the real axis of $\pi/2$, while the solution with j has an angle of $3/2\pi$. Both i and j square to -1.

The fact that real and imaginary numbers map a plane seems to provide way to describe vectors in two dimensions. In fact, given a vector $\mathbf{a} = a_r + i\,a_i$ we can plot it as illustrated in Fig. 1.8.

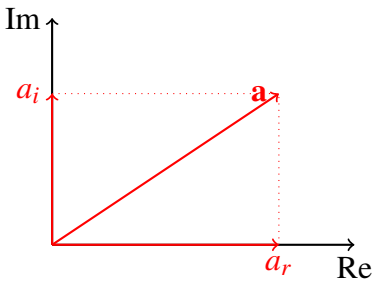


Fig. 1.8. Representation of a vector via complex numbers. The vector $\mathbf{a} = a_r + i\,a_i$ is represented on the Argand plane.

It is also very interesting that if we multiply the vector **a** by i we obtain a rotation as

$$\mathbf{a}' = i\,\mathbf{a} = -a_i + i\,a_r \tag{1.34}$$

as reported in Fig. 1.9. These important facts have lead William Hamilton in 1843 to extend this behavior to three dimensions and to introduce quaternions. In particular a quaternion is defined as

$$q = a + b\mathbf{i} + c\mathbf{j} + z\mathbf{k} \tag{1.35}$$

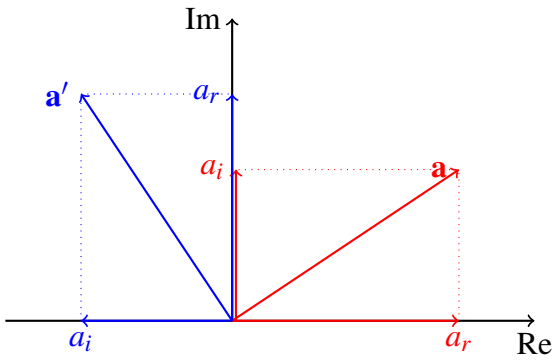


Fig. 1.9. The vector $\mathbf{a} = a_r + i\,a_i$ when multiplied by i is rotated by $\pi/2$ in the counterclockwise direction.

where $\mathbf{i},\mathbf{j},\mathbf{k}$ are imaginary units that obey the following rules: (cite Polar and axial vectors versus quaternions Cibelle Celestino Silvaa and Roberto de Andrade Martins)

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1\,. \tag{1.36}$$

$$\begin{aligned} \mathbf{ij} &= \mathbf{k}, & \mathbf{ji} &= -\mathbf{k}, \\ \mathbf{jk} &= \mathbf{i}, & \mathbf{kj} &= -\mathbf{i}, \\ \mathbf{ki} &= \mathbf{j}, & \mathbf{ik} &= -\mathbf{j}\,. \end{aligned} \tag{1.37}$$

A quaternion $q = a + b\mathbf{i} + c\mathbf{j} + z\mathbf{k}$ contains a scalar part (a) and a vector part ($b\mathbf{i} + c\mathbf{j} + z\mathbf{k}$). A quaternion of the type $q = b\mathbf{i} + c\mathbf{j} + z\mathbf{k}$ is called a pure quaternion and looks like an ordinary vector. What quaternions are in the context of geometric algebra, and more specifically their relation with Pauli matrices, will be illustrated next; for the moment let us continue our discussion on complex numbers and their ability to represent vectors.

It is rather intuitive that we want to have for the measure of $\mathbf{a}$ the following value:

$$a = \sqrt{a_r^2 + a_i^2}\,. \tag{1.38}$$

Since $a = a_r + i\,a_i$ we can try to evaluate its square as

$$a^2 = (a_r + ia_i)\,(a_r + ia_i) \tag{1.39}$$

which produces

$$\begin{aligned} a^2 &= (a_r + ia_i)(a_r + ia_i) \\ &= a_r^2 - a_i^2 + 2\,i\,a_r\,a_i\,. \end{aligned} \tag{1.40}$$

This is not the result we are looking for, i.e. a measure of the length of **a**.

Note that with vectors we are generally accustomed to perform either the dot or cross product. Here, instead, since we are dealing with complex quantities, we can perform their multiplication.

The remedy commonly used is to consider instead the product:

$$\begin{aligned} a\,a^* &= (a_r + ia_i)(a_r - ia_i) \\ &= a_r^2 + a_i^2\,, \end{aligned} \tag{1.41}$$

which correctly gives the square of the length of **a**.

To summarize the concepts seen:

- complex numbers can have two different interpretations:
  - as a two–dimensional vector;
  - as an operator that apply a rotation and a scale factor to a vector.
- It can be observed that the product between two complex numbers

$$zw = |z|e^{i\phi_z}|w|e^{i\phi_w} = |z||w|e^{i(\phi_z+\phi_w)} \tag{1.42}$$

  represents:
  - the operator that results from the composition of z and w if z and w are operators;
  - the vector resulting from a rotation of w by an angle ϕ_z and a scale factor $|z|$, if z is an operator and w is a vector;
- When z and w are two vectors, it is convenient to introduce the following product:

$$\begin{aligned} z^*w = |z|e^{-i\phi_z}|w|e^{i\phi_w} = |z||w|e^{i(\phi_w-\phi_z)} = \\ = |z||w|cos(\phi_w - \phi_z) + i|z||w|sin(\phi_w - \phi_z) \end{aligned} \tag{1.43}$$

  It can be noted that:
  - $Re[z^*w]$ is the scalar product between the vectors z and w;

  - $Im[z^* w]$ is the area of the parallelogram defined by the two vectors with sign:
    * $+$ if w if rotated counter clockwise with respect to z;
    * $-$ if it is rotated clockwise,

    and it will be related to the external product of the two vectors.
  - The result of the product can not be interpreted as a vector because the real part is a scalar product and the imaginary part is an area.
- The fact that complex numbers can be interpreted in two different ways, it is necessary to define different products. This happens, for example, for the study of circuits in AC regime. In DC regime, the Ohm's law and the power are expressed as $v = Ri$ and $p = vi$. In AC regime, with the symbolic method:
  - The symbolic Ohm's law is written as $V = ZI$. In this expression V and I are phasors (i.e. vectors) but Z is an operator that shows haw to modify the amplitude and phase of I to obtain V.
  - The expression of the complex power is written as $N = VI^*$. In this case, V and I are phasors but N is not a phasor.

III.2 Clifford algebra of order two

There is, however, a different way for proceeding. Let us introduce a *base* with two orthogonal elements e_1 and e_2 which satisfy the rule

$$e_1^2 = e_2^2 = 1\,. \tag{1.44}$$

In terms of the base elements e_1, e_2 the vector $\mathbf{a}$ can be written as

$$\mathbf{a} = a_r e_1 + a_i e_2 \tag{1.45}$$

and we can try to evaluate the following product:

$$\begin{aligned} \mathbf{a}\,\mathbf{a} &= (a_r e_1 + a_i e_2)(a_r e_1 + a_i e_2) \\ &= a_r^2 + a_i^2 + e_1 e_2\, a_r a_i + e_2 e_1\, a_r a_i\,. \end{aligned} \tag{1.46}$$

In the multiplication the elements are generally assumed as commutative (i.e. $ab = ba$). But let us now assume that the basis elements are *anticommutative*, i.e. that our base verify the following rule:

$$e_1 e_2 = -e_2 e_1 \,. \tag{1.47}$$

If this is the case, then the product in (1.46) becomes the sought one, i.e.

$$\begin{aligned} \mathbf{a}\,\mathbf{a} &= (a_r e_1 + a_i e_2)(a_r e_1 + a_i e_2) \\ &= a_r^2 + a_i^2 \,. \end{aligned} \tag{1.48}$$

The two rules introduced in (1.44) and (1.47) are just what is sufficient to define the *Clifford algebra* of order 2. Note that in this algebra we have four elements:

- the scalar
- the two vectors basis e_1 and e_2
- and the element $e_1 e_2$.

While we postpone to investigate what is the meaning of the element $e_1 e_2$ we just try to evaluate its square. We have:

$$e_1 e_2\, e_1 e_2 = -e_1 e_2\, e_2 e_1 = -1$$

since both e_2 and e_1 square to 1. We have just found the important relationship that

$$e_1 e_2 = e_{12} = i \tag{1.49}$$

i.e. that the product of the base elements is equal to imaginary number i.

Therefore the real part corresponds to the scalar, the imaginary part corresponds to the product of the base elements $e_1 e_2$ and we have two additional terms which are the base elements e_1 and e_2.

The scalar part is associated with grade zero, the base elements have grade one and the element e_{12} has grade two. It is apparent that by selecting only the even grades we recover the complex numbers from the Clifford algebra of order two.

In addition note that, similarly to the complex numbers, the Clifford algebra can sum together elements of different grades. We can therefore form the multivector $\mathscr{M}$ which contains the

scalar, the two vectors components relative to e_1, e_2 and the component relative to e_{12} as

$$\mathscr{M} = a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_{12} \,. \tag{1.50}$$

In the case of the vector **a** we have therefore two different representations in terms of multivectors:

$$a' = a_r + e_{12} a_i = a_r + i a_i \tag{1.51}$$

$$a'' = a_r e_1 + a_i e_2 \tag{1.52}$$

It is noted that graphically the two representations are very similar (see Fig. 1.10) but obviously refer to different multivectors.

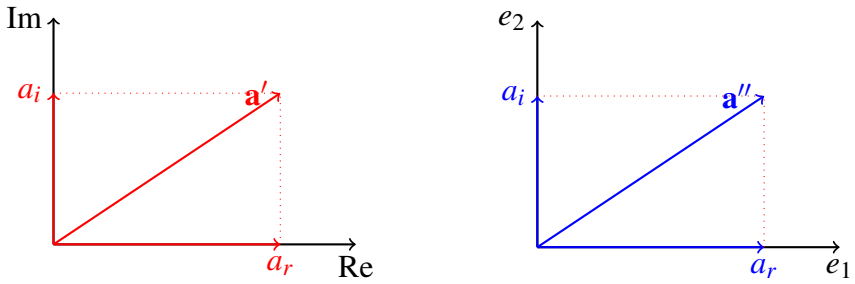


Fig. 1.10. Representation of the multivectors $\mathbf{a}'$ and $\mathbf{a}''$. Although they look similar they represent different multivectors.

Dilation and rotation

A complex number can also be represented in polar form as

$$a = a_r + i a_i = \rho e^{i\alpha} \tag{1.53}$$

with $\rho = \sqrt{a_r^2 + a_i^2}$. In the following, for ease of representation, let us assume that we are dealing with vectors of unit amplitude, i.e. $\rho = 1$. Naturally, multiplication with a real number r only provides a change of scale but the direction remains the same. Let us now consider a complex number b of unit module

$$b = e^{i\beta} \,. \tag{1.54}$$

Multiplication of a by b performs a rotation generating c with

$$c = ab = e^{i(\alpha+\beta)} \tag{1.55}$$

as shown in Fig. 1.11. In particular if $\beta = \pi/2$ then $b = i$ and multiplication by i provides an anticlockwise rotation of $\pi/2$.

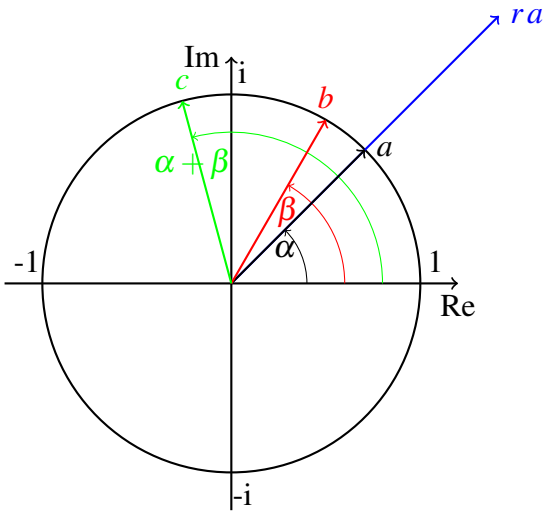


Fig. 1.11. Multiplication of a complex number by a real one produces scaling. Considering complex number of unit modulus a, b their multiplication provides a rotation.

Complex numbers and vectors transformations

With complex numbers we can perform their sum, subtraction, multiplication and division. When performing all of these operations, in general, another complex number is obtained. In addition, given two complex numbers of the form

$$\begin{aligned} a &= \rho_a e^{i\phi_a} \\ b &= \rho_b e^{i\phi_b} \end{aligned} \tag{1.56}$$

it is easy to find the element c that transforms a into b. In fact, from

$$b = ca \tag{1.57}$$

we recover the following expression in polar form:

$$c = \frac{b}{a} = \frac{\rho_b}{\rho_a} e^{i(\phi_b - \phi_a)} . \tag{1.58}$$

It is apparent that the term $\frac{\rho_b}{\rho_a}$ provides the appropriate scaling, while the term $e^{i(\phi_b-\phi_a)}$ gives the phase variation, i.e. the change in direction.

If we use the representation of a vector as given in (1.52) is still feasible to obtain similar results? This question is to be investigated in the next subsection.

III.3 Introducing the geometric product

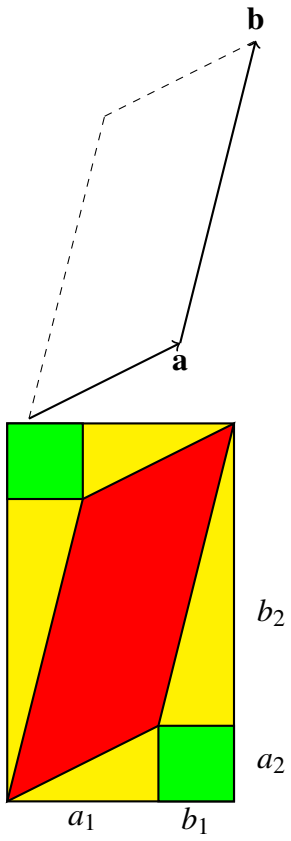


Fig. 1.12. Let us consider two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ with e.g. the following amplitudes $\mathbf{a} = (2,1)$ and $\mathbf{b} = (1,4)$. The product $(a_1 + b_1)(a_2 + b_2)$ is the entire rectangle shown in the lower figure. In order to obtain the part in red we should subtract from the entire rectangle the green parts $(2a_2b_1)$ and the yellow parts $(a_1a_2 + b_1b_2)$.

Let us perform the product of two vectors according to the rules of Clifford algebra. We have a vector $\mathbf{a} = (a_1, a_2)$ and a vector $\mathbf{b} = (b_1, b_2)$ which we may write as:

$$\begin{aligned} \mathbf{a} &= a_1\, e_1 + a_2\, e_2 \\ \mathbf{b} &= b_1\, e_1 + b_2\, e_2\,. \end{aligned} \tag{1.59}$$

By performing the multiplication we get

$$\begin{aligned} \mathbf{a}\,\mathbf{b} &= (a_1\, e_1 + a_2\, e_2)\,(b_1\, e_1 + b_2\, e_2) \\ &= a_1 b_1 + a_2 b_2 + e_{12}\,(a_1 b_2 - a_2 b_1) \\ &= \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} \end{aligned} \tag{1.60}$$

with the dot and wedge products corresponding, respectively, to

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= a_1 b_1 + a_2 b_2 \\ \mathbf{a}\wedge\mathbf{b} &= e_{12}\,(a_1 b_2 - a_2 b_1) \end{aligned} \tag{1.61}$$

for this two dimensional case. While the dot product has the ordinary meaning, let us see the geometric meaning of the wedge product. With reference to Fig. 1.12 it is easy to see that the wedge product of **a** and **b** corresponds to the part in red.
In fact, the product $(a_1 + b_1)(a_2 + b_2)$ is the entire rectangle shown in the lower part of Fig. 1.12. In order to obtain the part in red we should subtract from the entire rectangle the green parts ($2a_2 b_1$) and the yellow parts ($a_1 a_2 + b_1 b_2$). In this way we obtain:

$$\begin{aligned} (a_1 + b_1)(a_2 + b_2) - 2a_2 b_1 - a_1 a_2 - b_1 b_2 = \\ a_1 a_2 + a_1 b_2 + b_1 a_2 + b_1 b_2 - 2a_2 b_1 - a_1 a_2 - b_1 b_2 = \\ a_1 b_2 - a_2 b_1\,. \end{aligned} \tag{1.62}$$

The product introduced in (1.60) is the *geometric product* and is a new entity. Similarly to the case of complex number in the geometric product we sum together a scalar quantity with a surface (like real and imaginary parts). In addition, it is immediately recognized that, contrary

to the case of complex number, the product of two vectors gives something different (i.e. the scalar ($\mathbf{a}\cdot\mathbf{b}$) and a bivector ($\mathbf{a}\wedge\mathbf{b}$)).

It is useful to consider also the following product:

$$\begin{aligned}\mathbf{b}\,\mathbf{a} &= (b_1\,e_1 + b_2\,e_2)\,(a_1\,e_1 + a_2\,e_2)\\ &= a_1 b_1 + a_2 b_2 - e_{12}\,(a_1 b_2 - a_2 b_1)\\ &= \mathbf{b}\cdot\mathbf{a} + \mathbf{b}\wedge\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\wedge\mathbf{b}\,. \end{aligned} \tag{1.63}$$

By comparing (1.60) and (1.63) it is seen that when changing the order of multiplication the part in the wedge product change sign. Equivalently, by summing and subtracting (1.60) and (1.63) we obtain alternative definitions of the dot and wedge product as

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2}\,(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a}) \tag{1.64}$$

$$\mathbf{a}\wedge\mathbf{b} = \frac{1}{2}\,(\mathbf{a}\,\mathbf{b} - \mathbf{b}\,\mathbf{a})\,. \tag{1.65}$$

The above relations are valid not only in two dimensions but apply in general.

Let us also note that the wedge product of a vector with itself is null, i.e.

$$\mathbf{a}\wedge\mathbf{a} = a_1 a_2 - a_2 a_1 = 0\,. \tag{1.66}$$

The geometric product of vectors is *invertible* for all vectors with non-zero square $a^2 \neq 0$

$$\begin{aligned}\mathbf{a}^{-1} &:= \mathbf{a}/a^2, \quad \mathbf{a}\mathbf{a}^{-1} = \mathbf{a}\mathbf{a}/a^2 = 1,\\ \mathbf{a}^{-1}\mathbf{a} &= \frac{\mathbf{a}}{a^2}\mathbf{a} = a^2/a^2 = 1.\end{aligned} \tag{1.67}$$

The inverse vector $\mathbf{a}/a^2$ is a rescaled version (reflected at the unit circle) of the vector $\mathbf{a}$. This invertibility leads to significant simplifications and ease in computations.

Let us now consider again the problem of finding a transformation c (with c denoting an object not yet known) that allows to pass from the vector $\mathbf{a}$ to the vector $\mathbf{b}$. It is apparent that if $\mathbf{a}$ is invertible, from

$$\mathbf{b} = c\mathbf{a} \tag{1.68}$$

we obtain

$$c = \mathbf{b}\,\mathbf{a}^{-1}\,. \tag{1.69}$$

The same result can be obtained by writing:

$$\begin{aligned}
\mathbf{b} &= c\,\mathbf{a} \\
\mathbf{b}\,\mathbf{a} &= c\,\mathbf{a}\,\mathbf{a} \\
\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a} &= c\,(\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\wedge\mathbf{a}) = c\,\mathbf{a}\cdot\mathbf{a} \\
\frac{1}{\mathbf{a}\cdot\mathbf{a}}\,(\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a}) &= c\,.
\end{aligned} \tag{1.70}$$

We have obtained the transformation c which is composed in the following way. There is a term one over $\mathbf{a}\cdot\mathbf{a}$ which is a scalar quantity, therefore representing a scale factor. This quantity is multiplied by a scalar and by a bivector. Considering that the scalar plus the bivector correspond to a complex number (expressed using the identity $e_{12} = i$), we see that in the Clifford algebra it is also feasible to recover the transformation c that transform one vector into another. However, as we will see next there are also other ways to transform vectors. In the following we would like to describe some useful geometric concepts.

III.4 Vector Projection, Rejection and Reflection

For example, the *projection* of one vector $\mathbf{u}$ onto another vector $\mathbf{v}$ is $\mathbf{u}_{\parallel}$, where $\mathbf{u}_{\parallel}$ is the component of $\mathbf{u}$ parallel to $\mathbf{v}$. Naturally we can decompose the vector $\mathbf{u}$ in the parallel and perpendicular components as:

$$\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}\,. \tag{1.71}$$

The component $\mathbf{u}_{\perp}$ is the *rejection* of vector $\mathbf{u}$ onto vector $\mathbf{v}$, as illustrated in Fig. 1.13. Their values can be found by using the geometric product.
By post multiplying both sides of (1.71) by $\mathbf{v}$ we obtain

$$\begin{aligned}
\mathbf{u}\,\mathbf{v} &= \mathbf{u}_{\parallel}\,\mathbf{v} + \mathbf{u}_{\perp}\,\mathbf{v} \\
&= \mathbf{u}_{\parallel}\cdot\mathbf{v} + \mathbf{u}_{\perp}\wedge\mathbf{v} \\
&= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}
\end{aligned} \tag{1.72}$$

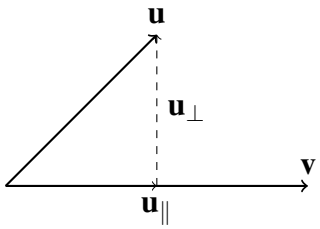


Fig. 1.13. Decomposition of vector **u** in a part parallel to **v** and a part orthogonal to **v**.

since

$$\begin{aligned}\mathbf{u}_{\parallel}\,\mathbf{v} &= \mathbf{u}_{\parallel}\cdot\mathbf{v}+\mathbf{u}_{\parallel}\wedge\mathbf{v} = \mathbf{u}_{\parallel}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{v}\\ \mathbf{u}_{\perp}\,\mathbf{v} &= \mathbf{u}_{\perp}\cdot\mathbf{v}+\mathbf{u}_{\perp}\wedge\mathbf{v} = \mathbf{u}_{\perp}\wedge\mathbf{v} = \mathbf{u}\wedge\mathbf{v}\end{aligned} \tag{1.73}$$

because $\mathbf{u}_{\parallel}\wedge\mathbf{v}=0$ (since they are parallel) and $\mathbf{u}_{\perp}\cdot\mathbf{v}=0$ (since they are orthogonal).
By post multiplying with $\mathbf{v}^{-1}$ the expressions in (1.73) we get

$$\begin{aligned}\mathbf{u}_{\parallel}\,\mathbf{v}\mathbf{v}^{-1} &= \left(\mathbf{u}_{\parallel}\cdot\mathbf{v}\right)\mathbf{v}^{-1} = (\mathbf{u}\cdot\mathbf{v})\,\mathbf{v}^{-1}\\ \mathbf{u}_{\perp}\,\mathbf{v}\mathbf{v}^{-1} &= (\mathbf{u}_{\perp}\wedge\mathbf{v})\,\mathbf{v}^{-1} = (\mathbf{u}\wedge\mathbf{v})\,\mathbf{v}^{-1}\end{aligned} \tag{1.74}$$

or

$$\begin{aligned}\mathbf{u}_{\parallel} &= (\mathbf{u}\cdot\mathbf{v})\,\frac{\mathbf{v}}{v^2}\\ \mathbf{u}_{\perp} &= (\mathbf{u}\wedge\mathbf{v})\,\frac{\mathbf{v}}{v^2}\end{aligned}$$

with $\mathbf{v}\cdot\mathbf{v}=v^2$.

Example 3. As an example let us consider the following two vectors

$$\begin{aligned}\mathbf{u} &= (e_1+e_2)\\ \mathbf{v} &= 2\,e_1\end{aligned}$$

we have

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= 2 \\ \mathbf{u}\wedge\mathbf{v} &= 2\,e_2\,e_1 \\ v^2 &= 4 \end{aligned} \tag{1.75}$$

so that

$$\begin{aligned} \mathbf{u}_{\parallel} &= (\mathbf{u}\cdot\mathbf{v})\,\frac{\mathbf{v}}{v^2} = e_1 \\ \mathbf{u}_{\perp} &= (\mathbf{u}\wedge\mathbf{v})\,\frac{\mathbf{v}}{v^2} = e_2\,. \end{aligned}$$

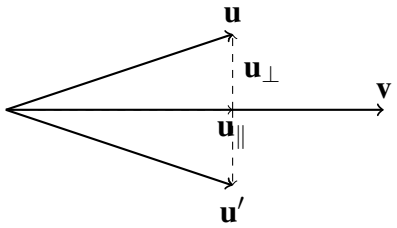


Fig. 1.14. The vector $\mathbf{u}'$ has been reflected with respect to $\mathbf{v}$. It has the same component $\mathbf{u}_{\parallel}$ but has opposite perpendicular part $\mathbf{u}_{\perp}$.

Let us now consider *reflections* as illustrated in Fig. 1.14. We can form the vector $\mathbf{u}'$ which share the same component $\mathbf{u}_{\parallel}$ but has opposite perpendicular part as

$$\mathbf{u}' = \mathbf{u}_{\parallel} - \mathbf{u}_{\perp} \tag{1.76}$$

By substitution of (1.75) we obtain

$$\begin{aligned} \mathbf{u}' &= \frac{1}{v^2}\,(\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\wedge\mathbf{v})\,\mathbf{v} \\ &= \frac{1}{v^2}\,(\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\wedge\mathbf{u})\,\mathbf{v} \\ &= \frac{1}{v^2}\mathbf{v}\,\mathbf{u}\,\mathbf{v} \end{aligned} \tag{1.77}$$

Reflections takes a simplified form when we consider vectors of unit norm denoted e.g. by $\mathbf{n}$. In this case the formula for reflection is very simple and can be derived by (1.77) by setting $v^2 = 1$, thus obtaining

$$\mathbf{u}' = \mathbf{n}\,\mathbf{u}\,\mathbf{n}\,. \tag{1.78}$$

It is noted that $\mathbf{u}$, $\mathbf{u}'$ and $\mathbf{n}$ are all vectors. The form in (1.78) allows to transform a vector into another vector and is different from the form used in (1.70).

A more thorough understanding can be achieved by considering the polar form. We recall the Euler formula for a generic angle ϕ_n

$$e^{i\,\phi_n} = \cos\phi_n + i\,\sin\phi_n \tag{1.79}$$

which may also be written as

$$e^{i\,\phi_n} = \cos\phi_n + e_{12}\,\sin\phi_n = n \tag{1.80}$$

introducing the multivector n as the sum of a scalar and a bivector. For abbreviation it is convenient to define the following quantities

$$\begin{aligned} c_n &= \cos\phi_n \\ s_n &= \sin\phi_n \end{aligned} \tag{1.81}$$

Similarly to what we have seen before we can also form the unit vector $\mathbf{n}$ defined as

$$\mathbf{n} = e_1 n = c_n\,e_1 + s_n\,e_2\,. \tag{1.82}$$

Note that, since the amplitude of $\mathbf{n}$ is specified to be unitary, the vector $\mathbf{n}$ is completely defined by its phase ϕ_n. Let us introduce in a similar way another vector $\mathbf{m}$ with the following definitions:

$$\begin{aligned} c_m &= \cos\phi_m \\ s_m &= \sin\phi_m \\ m &= c_m + e_{12}\,s_m \\ \mathbf{m} &= e_1 m = c_m\,e_1 + s_m\,e_2\,. \end{aligned} \tag{1.83}$$

It is possible to consider the bivector formed by their product as

$$\begin{aligned}\mathbf{nm} &= (c_n e_1 + s_n e_2)(c_m e_1 + s_m e_2) \\ &= c_n c_m + s_n s_m + e_{12}(c_n s_m - s_n c_m) \\ &= \cos(\phi_m - \phi_n) + e_{12} \sin(\phi_m - \phi_n) \end{aligned} \tag{1.84}$$

Note that the angle is $\phi_m - \phi_n$. This is similar to the case when we give two points A and B and the vector going from A to B is obtained by subtracting the coordinates of A from the coordinates of B. It is convenient to define the angle between $\mathbf{m}$ and $\mathbf{n}$ as

$$\frac{\theta}{2} = \phi_m - \phi_n \tag{1.85}$$

and to write

$$\mathbf{nm} = e^{i\theta/2}. \tag{1.86}$$

Since both $\mathbf{n}$ and $\mathbf{m}$ are unit vectors we have

$$\begin{aligned}\mathbf{nn} = \mathbf{mm} &= 1 \\ \mathbf{mnnm} &= 1 \\ \mathbf{mn} &= e^{-i\theta/2}. \end{aligned} \tag{1.87}$$

We can now perform a further reflection of the vector $\mathbf{u}'$ appearing in (1.78) with respect to the vector $\mathbf{m}$, obtaining

$$\mathbf{u}'' = \mathbf{mu}'\mathbf{m} = \mathbf{mnunm} = e^{-i\theta/2}\mathbf{u}e^{i\theta/2}. \tag{1.88}$$

It is useful to evaluate explicitly the above product as

$$
\begin{aligned}
\mathbf{u}'' &= e^{-i\theta/2}\,\mathbf{u}\,e^{i\theta/2} = R_u\left(\frac{\theta}{2}\right) \\
&= \left[\cos\left(\frac{\theta}{2}\right) - e_{12}\sin\left(\frac{\theta}{2}\right)\right](u_1\,e_1 + u_2\,e_2)\left[\cos\left(\frac{\theta}{2}\right) + e_{12}\sin\left(\frac{\theta}{2}\right)\right] \\
&= e_1\cos\left(\frac{\theta}{2}\right)^2 u_1 + 2e_2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)u_1 - e_1\sin\left(\frac{\theta}{2}\right)^2 u_1 \\
&\quad + e_2\cos\left(\frac{\theta}{2}\right)^2 u_2 - 2e_1\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)u_2 - e_2\sin\left(\frac{\theta}{2}\right)^2 u_2 \\
&= \left(e_1\cos(\theta) + e_2\sin(\theta)\right)u_1 + \left(e_2\cos(\theta) - e_1\sin(\theta)\right)u_2 \\
&= (u_1\cos\theta - u_2\sin\theta)\,e_1 + (u_1\sin\theta + u_2\cos\theta)\,e_2
\end{aligned}
\tag{1.89}
$$

We can now examine the result in (1.89). We have performed a first reflection by pre and post multiplying $\mathbf{u}$ with $\mathbf{n}$ and a second reflection by pre and post multiplication with $\mathbf{m}$. These two reflections correspond to a rotation. Interestingly, this holds true also in three dimensions. In fact, the first reflection will change the sign of the component along e_3, but the second reflection will change again the sign of this component thus coming back to the original value. It is noted that while the angle between the vectors $\mathbf{n}$ and $\mathbf{m}$ is $\theta/2$ the rotation produced on the vector $\mathbf{u}$ is θ. We can also define an operator $R_u(\frac{\theta}{2})$ which acts on the vector $\mathbf{u}$ and produces a rotation. Such operator depends on the angle $\theta/2$ and therefore only when θ is increased by 4π produces the same results. In other words its periodicity is 4π.

A different perspective can be obtained by considering the logarithm of $e^{-i\theta/2}\,\mathbf{u}\,e^{i\theta/2}$.

It is noted that the operator R_u when applied to a scalar a or to a bivector e_{12} produces no effect

$$
\begin{aligned}
e^{-i\theta/2}\,a\,e^{i\theta/2} &= a \\
e^{-i\theta/2}\,e_{12}\,e^{i\theta/2} &= e_{12}\,.
\end{aligned}
$$

To conclude it is useful to consider the effect of premultiplication of a bivector times a vector; we have

$$
e_{12}\,(u_1\,e_1 + u_2\,e_2) = u_2\,e_1 - u_1\,e_2 \tag{1.90}
$$

i.e. pre multiplication corresponds to a *clockwise rotation*. While post multiplication gives

$$
(u_1\,e_1 + u_2\,e_2)\,e_{12} = -u_2\,e_1 + u_1\,e_2 \tag{1.91}
$$

i.e. post multiplication corresponds to a *counterclockwise rotation*.

In (1.89) we have post multiplied a vector by $e^{i\theta/2}$, thus performing a counterclockwise rotation of $\theta/2$ and we have pre multiplied by $e^{-i\theta/2}$ which corresponds to a clockwise rotation of $-\theta/2$ which is also a counterclockwise rotation of $\theta/2$.

III.5 Performing Clifford algebra computations with wxMaxima

The two packages *Clifford.mac* and *Cliffordan.mac*, developed by Dimiter Prodanov [29], allows to perform Clifford algebra computations. An example is reported in

```
2D_ab.wxm
```

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 15.04.0 ] */

/* [wxMaxima: input   start ] */
pathmaxima:file_search_maxima$
pathmaxima[2];
kill(all)$

load("clifford.mac");
load("cliffordan.mac");

if get('clifford,'version)=false then load("clifford")$
if get('cliffordan,'version)=false then load("cliffordan")$
("Cliford implements Clifford algebra for Maxima.")$

print("Using Cl(2) with the e basis") $
clifford(e, 2);

print("The space is formed by ")$
EE:cons(1, %elements );

print("Table of direct products")$
mtable1(%elements);

print("Table of external product")$
mtable1o(%elements);
```

```
print("The elements are recovered as")$
EE[1]; EE[2]; EE[3]; EE[4];
/* */

print("--------------------------------------")$
print("Program Starts")$
print("--------------------------------------")$

print("construct vector a")$
ac:cvect(a);
print("construct vector b")$
bc:cvect(b);
print("--------------------------------------")$
"basic products";
print("--------------------------------------")$
adotb  :ac | bc$
awedgeb :ac & bc$
ab :ac . bc$
print("a dot b = ",'adotb," = ",adotb)$
print("a wedge b = ",'awedgeb," = ",awedgeb)$
print("a  b = ",'ab," = ",ab)$
print("expand ab ", expand(ab))$

print("grade")$
gab : grade(ab);
print("grade 0"," = ",gab[1])$
print("grade 1"," = ",gab[2])$
print("grade 2"," = ",gab[3])$
print("The product of two vectors has given a scalar plus a bivector ")$

print("--------------------------------------")$
print ("Start Investigations")$
print("--------------------------------------")$
print("perform e1 . a" )$

e1a : e[1] . ac;
```

```
ela : dotsimpc(expand(ela));
print("Grades")$
gela : grade(ela);

print("------------------------------------")$
print ("Reflection")$
print("------------------------------------")$

print("now perform e1 . a . e1" )$
elael : dotsimpc(expand(ela . e[1]));

print("The e1 component is not changed")$
print("The e2 component has taken the minus sign")$
print("vector a has been reflected w.r.t. e1")$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

III.6 Operators' square root with Clifford algebra

The imaginary number i is the square root of the number -1. With Clifford algebra we can also take the square root of operators. In electromagnetic field analysis we often finds equations of the type

$$\left(\partial_x^2 + \partial_y^2\right) F(x,y) = 0 \tag{1.92}$$

or the equivalent version in three–dimensions. This is a second order equation with an operator $\left(\partial_x^2 + \partial_y^2\right)$. Another possibility to state the same equation is via the use of complex numbers. In fact, we have

$$\left(\partial_x + i\,\partial_y\right)\left(\partial_x - i\,\partial_y\right) F(x,y) = \left(\partial_x^2 + \partial_y^2\right) F(x,y) = 0 \tag{1.93}$$

but the two operators, $\left(\partial_x + i\,\partial_y\right)$ and $\left(\partial_x - i\,\partial_y\right)$ are obviously different.

However, Clifford algebra allows another very useful expression. Let us consider the following expression:

$$\left(\partial_x e_1 + \partial_y e_2\right)\left(\partial_x e_1 + \partial_y e_2\right) F(x,y) = \left(\partial_x^2 + \partial_y^2\right) F(x,y) = 0 \tag{1.94}$$

where use has been made of the anticommutation property. It is apparent that $\left(\partial_x e_1 + \partial_y e_2\right)$ is the square root of $\left(\partial_x^2 + \partial_y^2\right)$ and is an operator of the first order. It is also noted that, in this case, a Clifford algebra, with two basis vectors, e_1, e_2 which are anticommutative and square to unity, has been sufficient to accomplish the task of taking the square root of the operator $\left(\partial_x^2 + \partial_y^2\right)$.

In some other cases the form of the operator may be different. We will later introduce the telegrapher equations, which deals with the amplitude of currents and voltages along a transmission line. Let us consider the voltage V and the current I along a transmission line in the x direction. The telegrapher's equations, for a lossless line, are

$$\begin{aligned} \partial_x V &= -L\,\partial_t I \\ \partial_x I &= -C\,\partial_t V \end{aligned} \tag{1.95}$$

where L and C are the inductances and capacitances per unit length. From these two equations we recover that

$$\left(\partial_x^2 - LC\,\partial_t^2\right) V = 0\,. \tag{1.96}$$

In order to find the square root of the operator $\left(\partial_x^2 - LC\,\partial_t^2\right)$ we can proceed as follows:

- assume that e_1 squares to one, i.e. $e_1^2 = 1$;
- assume that e_2 *squares to minus one, i.e.* $e_2^2 = -1$;
- assume that e_1 and e_2 are anticommutative.

These rules define the Clifford algebra $Cl(1,1)$ with one element that squares to one and one element that squares to minus one. In the future we will encounter cases where one element squares to one and thee basis elements squares to minus one which is denoted by $Cl(1,3)$. It is apparent that we have realized

$$\left(\partial_x e_1 + \sqrt{LC}\,\partial_t e_2\right)\left(\partial_x e_1 + \sqrt{LC}\,\partial_t e_2\right) = \left(\partial_x^2 - LC\partial_t^2\right) \tag{1.97}$$

thus finding the square root of the operator $\left(\partial_x^2 - LC\,\partial_t^2\right)$. In subsection (V.3) of this chapter we will see in more detail how to apply Clifford algebra to telegrapher equations. In addition, we will find application of this technique when writing Maxwell's equation in Dirac form.

III.7 Bivectors and Trivectors in three dimensions

Grassmann extensions

Hermann Grassmann, in 1844, in his fundamental work "Die Lineale Ausdehnungslehre, ein neuer Zweig der Mathematik", has introduced an operation called *extension* or *evolution* which generates higher dimensional objects starting from lower dimensional ones. As an example, let us consider two vectors **a** and **b** (see Fig. 1.15). We can consider **a** as the *generating element* and the vector **b** as the *measure of generation*. The operation of evolution is taken as the process of translation of the generating element **a** through a continuous sequence of intermediate states described by **b** as shown in Fig. 1.15. At each intermediate state the image of the generating element (the dashed vectors) is called an *intermediate element* of the continuous form.
When we consider the collection of the intermediate elements between two vectors a surface is generated. That is the geometrical application of the operation of evolution produces an extension which is a two–dimensional object. The mathematical operation that describes this extension is called external product.

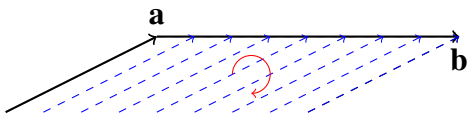


Fig. 1.15. The external product of two vectors **a** and **b**. The vector **a** is moved along the direction of the vector **b** covering a surface. The surface orientation is described by the movement from **a** to **b**. Note that going from **a** to **b** defines the clockwise orientation shown in red.

External product

It is advantageous to consider an *external (or outer) product* which allows, in turn, to define the *bivector* as

$$\hat{\mathbf{B}} = \mathbf{a} \wedge \mathbf{b} \tag{1.98}$$

The latter has the same magnitude of the cross product but it is neither a scalar or a vector

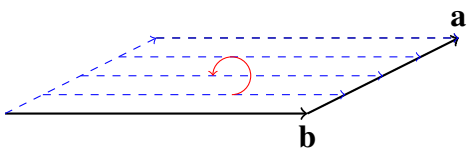


Fig. 1.16. The external product of two vectors **b** and **a**. The vector **b** is moved along the direction of the vector **a** covering a surface. The surface orientation is described by the movement from **b** to **a** and is described by the red circle going in the counterclockwise orientation.

$$\hat{\mathbf{B}} = |\mathbf{a}||\mathbf{b}| \sin\theta \tag{1.99}$$

The outer product is a *bivector* oriented in the plain containing **a** and **b** (see Fig. 1.15 and Fig. 1.16) and corresponds to an oriented surface.

The reader may notice that every bivector can be represented in the form (1.98), but this representation is not unique, since one can construct many parallelograms parallel to themselves with the same area and orientation but not necessarily with parallel sides. The representation (1.98) is called the *factorization of a bivector in the outer product*.

The outer product has the same magnitude as the cross product and shares its anti–commutative property:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}\,. \tag{1.100}$$

A way to visualize the outer product is to imagine $\mathbf{a} \wedge \mathbf{b}$ as the area "swept out" by displacing **a** along **b**, with the orientation given by traversing the parallelogram so formed first along an **a** vector then along a **b** vector. Note that if the vectors **a** and **b** are parallel (we can write this as $\mathbf{a} \parallel \mathbf{b}$) one obtains $\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$.

Bivectors can be added together (insert fig.7-9 of Jancewicz) and they are *distributive* under the addition

$$\mathbf{a} \wedge (\mathbf{b} + \mathbf{c}) = \mathbf{a} \wedge \mathbf{b} + \mathbf{a} \wedge \mathbf{c}\,. \tag{1.101}$$

This property enable us to obtain a bivector as the result of the addition of two bivector (figure 7 jancewicz). Bivectors can also be multiplied by scalars. The outer product of vectors is homogeneous under multiplication by scalar

$$(\lambda \mathbf{a}) \wedge \mathbf{b} = \lambda (\mathbf{a} \wedge \mathbf{b}) = \mathbf{a} \wedge (\lambda \mathbf{b})\,. \tag{1.102}$$

With vectors it is common practice to find a basis and express other vectors in terms of this basis. As an example, in three dimensions, we may assume the vectors $\mathbf{e_1}$, $\mathbf{e_2}$ and $\mathbf{e_3}$ form an orthonormal basis. The vectors $\mathbf{a}$ and $\mathbf{b}$ can be represented as

$$\begin{aligned}\mathbf{a} &= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \\ \mathbf{b} &= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3\,.\end{aligned}$$

We may write their outer product as

$$\begin{aligned}\mathbf{a}\wedge\mathbf{b} = {} & (a_1b_2 - a_2b_1)\mathbf{e}_1\wedge\mathbf{e}_2 + \\ & (a_3b_1 - a_1b_3)\mathbf{e}_3\wedge\mathbf{e}_1 + \\ & (a_2b_3 - a_3b_2)\mathbf{e}_2\wedge\mathbf{e}_3 \end{aligned} \tag{1.103}$$

Thus we have shown that an arbitrary bivector $\hat{\mathbf{B}} = \mathbf{a}\wedge\mathbf{b}$ can be represented as a linear combination of the three bivectors $\mathbf{e}_1\wedge\mathbf{e}_2$, $\mathbf{e}_3\wedge\mathbf{e}_1$ and $\mathbf{e}_2\wedge\mathbf{e}_3$, and they are linearly independent (their planes are orthogonal to each other). Therefore we have that the linear space of bivectors is three–dimensional. Three–dimensional space has the peculiar property that for any plane there is just one perpendicular line. This fact can be used to attach a vector $\mathbf{B}$ to each bivector $\hat{\mathbf{B}}$. However, this operation will be better explained after the introduction of the trivectors.

Trivectors

This notion leads to a generalization to products of objects with higher dimensionality, or grade. Thus, if the bivector $\mathbf{a}\wedge\mathbf{b}$ (grade 2) is swept out along another vector $\mathbf{c}$ (grade 1), we obtain the directed volume element $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$, which is a *trivector* (grade 3). By construction, the outer product is associative:

$$(\mathbf{a}\wedge\mathbf{b})\wedge\mathbf{c} = \mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) = \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} \tag{1.104}$$

We can go no further in 3-dimensional space there is nowhere else to go. Correspondingly, the outer product of any four vectors $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{d}$ is zero.

We have already noted that the inner and outer products have opposite commutation properties

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= \mathbf{b}\cdot\mathbf{a} \\ \mathbf{a}\wedge\mathbf{b} &= -\mathbf{b}\wedge\mathbf{a} \end{aligned} \tag{1.105}$$

and we have seen the *geometric product* $\mathbf{a}\,\mathbf{b}$, defined as

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,. \tag{1.106}$$

It is noted that the geometri product of two vectors results in a scalar part and a bivector. If the two vectors are parallel only the scalar part is present, while if the two vectors are orthogonal only the bivector part is present. The result of adding a scalar to a bivector is an object that has both scalar and bivector parts, in exactly the same way that the addition of real and imaginary numbers yields an object with both real and imaginary parts. We call this latter object a complex number and, in the same way, we shall refer to a (scalar + bivector) as a "multivector", accepting throughout that we are combining objects of different types. It is also noted that

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= \frac{1}{2}\left(\mathbf{ab}+\mathbf{ba}\right) \\ \mathbf{a}\wedge\mathbf{b} &= \frac{1}{2}\left(\mathbf{ab}-\mathbf{ba}\right) \end{aligned} \tag{1.107}$$

it is now possible to apply the above definitions both to the two–dimensional and three–dimensional cases. The three–dimensional case will become particularly interesting when using as a basis the Pauli matrices. As a matter of fact Clifford algebra in three–dimensions is isomorphic to the algebra of Pauli matrices.

IV Pauli matrices and their properties

Three–dimensional vectors are usually represented as three ordered numbers as e.g. $\mathbf{a} = \left(a_x, a_y, a_z\right)^T$.

However, there is also another possible representation in terms of 2x2 matrices. Naturally a 2x2 real matrix is defined by 4 numbers, while a complex one requires 8 numbers. It is therefore fairly natural that we can represent a vector via a matrix. However, there are many possible representations but, one among them, has several particular advantages. The representation we are going to introduce is due to Wolfgang Pauli. We first introduce the Pauli matrices and then discuss some of their properties and show how to operate with this new tool. Note that engineers are quite well trained to operate with matrices and we will see that many standard vectors operation can be simplified and new important elements will be found.

The Pauli matrices are a set of three 2x2 complex matrices which are *Hermitian* and *unitary*.

An Hermitian matrix (or *self-adjoint* matrix) is a complex square matrix that is equal to its own conjugate transpose, that is, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j. Let us consider the matrix a

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

its transpose is given by

$$A^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

and, by denoting with * the complex conjugate, its Hermitian $A^\dagger$ is

$$A^\dagger = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}$$

The *trace* of A is

$$tr(A) = a_{11} + a_{22}\,. \tag{1.108}$$

A complex square matrix U is unitary if its conjugate transpose $U^\dagger$ is also its inverse.
The Pauli matrices have the following form

$$\begin{aligned}
\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,.
\end{aligned} \tag{1.109}$$

Several properties of the Pauli matrices are of interest. It is immediately noted that the trace of these matrices is always zero. The determinant of the Pauli matrices is always -1. The following products hold:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = \sigma_0\,. \tag{1.110}$$

By multiplying e.g. σ_1 with σ_2 the result is $i\sigma_3$ and similarly for the other cases:

$$\begin{aligned}
\sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\
\sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \\
\sigma_3\sigma_1 &= i\sigma_2 = -\sigma_1\sigma_3\,.
\end{aligned} \tag{1.111}$$

The above relations are very important. In fact, they show that, in the three-dimensional case, we can always replace the quantities $\sigma_i\sigma_j$ with the orthogonal vector $i\sigma_k$ (with the appropriate combination given in (1.113)). An equivalent property is also present in the Clifford algebra.
Let us first note that

$$e_{123}\,e_{123} = -1 \tag{1.112}$$

and, therefore $e_{123} = i1$. In fact if we consider the bivectors e_{12}, e_{13}, e_{23} and multiply them by $-i^2 = 1 = -i\,e_{123}$ we have

$$e_{12} = -ie_{123}e_{12} = ie_3$$
$$e_{13} = -ie_{123}e_{13} = -ie_2$$
$$e_{23} = -ie_{123}e_{23} = ie_1\,. \tag{1.113}$$

From the above properties it is seen that, similarly for the Clifford basis, we have

$$(\sigma_1\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = -1\sigma_0 = -\sigma_0 \tag{1.114}$$

i.e.

$$\sigma_1\sigma_2\sigma_3 = i\sigma_0\,. \tag{1.115}$$

The three Pauli matrices, with the addition of the identity matrix σ_0, form a basis in the space of the 2x2 Hermitian matrices and a matrix A can be represented as:

$$A = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3\,. \tag{1.116}$$

It is worthwhile to note that when the coefficients (a_0, a_1, a_2, a_3) are complex, also non Hermitian matrices can be described by the basis of $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$.

IV.1 The Pauli vector

Let us introduce the Pauli vector which is a vector constructed by using the three matrices $(\sigma_1, \sigma_2, \sigma_3)$ as

$$\boldsymbol{\sigma} = (\sigma_1\,\mathbf{x}_0 + \sigma_2\,\mathbf{y}_0 + \sigma_3\,\mathbf{z}_0)\,. \tag{1.117}$$

We can consider the vector $\mathbf{a} = \left(a_x\,\mathbf{x}_0 + a_y\,\mathbf{y}_0 + a_z\,\mathbf{z}_0\right)$ in the three-dimensional space and make the following product:

$$\tilde{a} = \boldsymbol{\sigma}\cdot\mathbf{a} = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix} \tag{1.118}$$

The matrix $\tilde{a}$ is an equivalent description of the vector $\mathbf{a}$ in terms of a 2x2 matrix. Two different symbols have been used to denote the 2x2 matrix representation $\tilde{a}$ and the standard vector representation $\mathbf{a}$. They refer to exactly the same quantity and it is always possible to pass from one to the other.

Inner product

It is advantageous the define the inner product of two matrices $\tilde{a}, \tilde{b}$, not necessarily representing vectors, as

$$\left\langle \tilde{a}, \tilde{b} \right\rangle = \frac{1}{2} trace\left(\tilde{a}\tilde{b}\right). \tag{1.119}$$

We will make use of the inner product in several circumstances, e.g. when transforming the components of one vector from one coordinate system to another.

From the properties of the Pauli matrices it is readily recognized that, for $i \neq j$ we have $\left\langle \sigma_i, \sigma_j \right\rangle = 0$, while $\langle \sigma_i, \sigma_i \rangle = 1$. This provides a simple way to retrieve the coefficients of the Pauli matrices for a given matrix A. In fact, we have

$$\langle A, \sigma_0 \rangle = a_0 \langle \sigma_0, \sigma_0 \rangle + a_1 \langle \sigma_1, \sigma_0 \rangle + a_2 \langle \sigma_2, \sigma_0 \rangle + a_3 \langle \sigma_3, \sigma_0 \rangle = a_0 \tag{1.120}$$

and similarly for other components.

Pauli matrices properties determination with a computer algebra system

It is instructive to try to verify the properties previously illustrated by using a computer algebra system (CAS). Here following we use wxMaxima to define the 2x2 Pauli matrices and to perform some computations. Please refer to the file:

`wxm/Pauli_properties.wxm`

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 15.04.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
/* Pauli_properties_v01 */

print("------------------------------------------------------------")$
print("Pauli Matrices Definition")$
print("------------------------------------------------------------")$
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
```

```
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print('%sigma[0]," = ",%sigma[0])$
print('%sigma[1]," = ",%sigma[1])$
print('%sigma[2]," = ",%sigma[2])$
print('%sigma[3]," = ",%sigma[3])$

print("------------------------------------------------------")$
print("Pauli Matrices Properties")$
print("------------------------------------------------------")$
print("")$
print("------------------------------------------------------")$
print("Squared give the identity matrix %sigma[i].%sigma[i]=sigma[0]")$
print("------------------------------------------------------")$

print("involutory (it is also its inverse)")$
ss1 : %sigma[1] . %sigma[1];
ss2 : %sigma[2] . %sigma[2];
ss3 : %sigma[3] . %sigma[3];

print("------------------------------------------------------")$
print(" %sigma[1] . %sigma[2] = i %sigma[3]")$
print("------------------------------------------------------")$
print("Let us try multiply %sigma[1].%sigma[2]")$
%sigma[1] . %sigma[2];
print("It is apparent that the result is i %sigma[3]")$
print("------------------------------------------------------")$
print("Let us try multiply %sigma[1].%sigma[3]")$
%sigma[1] . %sigma[3];
print("It is apparent that the result is - i %sigma[2]")$
print("------------------------------------------------------")$
print("Let us try multiply %sigma[2].%sigma[3]")$
%sigma[2] . %sigma[3];
print("It is apparent that the result is i %sigma[1]")$

print("------------------------------------------------------")$
```

```
print("Anticommutative")$
print("------------------------------------------------------")$
print("Commutation : %sigma[1] . %sigma[2] - %sigma[2] . %sigma[1]")$
sc12 : %sigma[1] . %sigma[2] - %sigma[2] . %sigma[1];
print("The result is 2i %sigma[3]")$
print("------------------------------------------------------")$
print("If we add instead of subtracting")$
print(" %sigma[1] . %sigma[2] + %sigma[2] . %sigma[1]")$
sc12 : %sigma[1] . %sigma[2] + %sigma[2] . %sigma[1];
print("the result is %sigma[1] . %sigma[2] = - %sigma[2] . %sigma[1]")$

print("------------------------------------------------------")$
print("(%sigma[i] . %sigma[j]) squared gives -1")$
print("------------------------------------------------------")$
print("Squared %sigma[1] . %sigma[2]")$
sc12 : (%sigma[1] . %sigma[2]) . (%sigma[1] . %sigma[2]);
print("The result is -1 !!")$

print("------------------------------------------------------")$
print("%sigma[1] . %sigma[2] . %sigma[3] = i")$
print("------------------------------------------------------")$
print("Squared %sigma[1] . %sigma[2] . %sigma[3]")$
sc12 : (%sigma[1].%sigma[2].%sigma[3]).(%sigma[1].%sigma[2].%sigma[3]);
print("The result is -1 !!")$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Pauli Library with definitions

Instead of repeating in each code the definitions of the Pauli matrices it is convenient to create a library (a file named .mac) containing the various definitions and procedures. This file is

`wxm/Pauli_v02.wxm.`

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
/*
print("Definitions introduced in Pauli_v02")$
print("innerproduct(A,B)")$

print("----------------------------------------")$
print("RECTANGULAR COORDINATES ")$
print("----------------------------------------")$
print("%sigma[0], %sigma[1], %sigma[2], %sigma[3]")$
print("Gradesep(M)")$
print("Nablarect(Ap)")$

print("----------------------------------------")$
print("CYLINDRICAL COORDINATES ")$
print("----------------------------------------")$
print("%sigma[%rho], %sigma[phi], %sigma[z]")$
print("Gradecyl(M)")$
print("Nablacyl(Ap)")$

print("----------------------------------------")$
print("SPHERICAL COORDINATES ")$
print("----------------------------------------")$
print("%sigma[r], %sigma[%theta], %sigma[phi]")$
print("Gradesph(M)")$
print("Nablasph(Ap)")$
*/

/*
print("------------------------------------------------------")$
print("Pauli Matrices Cartesian")$
print("------------------------------------------------------")$
*/
%sigma[0] : matrix([1,0],[0,1])$
```

```
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

/*
print("------------------------------------------------------------")$
print("Pauli Matrices Cylindrical")$
print("------------------------------------------------------------")$
*/
%sigma[%rho] : matrix([0,%e^(-%i * phi)],[%e^(%i * phi),0])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$
%sigma[z] : matrix([1,0],[0,-1])$

/*
print("------------------------------------------------------------")$
print("Pauli Matrices Definition Spherical")$
print("------------------------------------------------------------")$
*/

ct : cos(%theta)$
st : sin(%theta)$

%sigma[r] : matrix([ct,st * %e^(-%i * phi)],[st * %e^(%i * phi),-ct])$
%sigma[%theta] : matrix([-st,ct  * %e^(-%i * phi)],[ct * %e^(%i * phi),st])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$


/***************Block Start*****************************/
innerproduct(amat,bmat,tr) :=
block([],
cctr : amat .bmat,
tr : ratsimp((cctr[1,1]+cctr[2,2])/2)
 )$
/***************Block End*******************************/


depends([a11,a12,a21,a22],[x,y,z])$
```

```
Ap : matrix([a11,a12],[a21,a22])$

/*
The block Nablarect perform the nabla operation
in rectangular coordinates on a Pauli matrix Ap
*/

/***************Block Start*****************************/
Nablarect(Ap) :=
block([],
NAp : %sigma[1].(diff(Ap,x)) + %sigma[2].(diff(Ap,y)) + %sigma[3].(diff(Ap,z)),
NAp : factor(NAp)
)$
/***************Block End*******************************/

/* Gradesep  */

/***************Block Start*****************************/
Gradesep(M):=
block([],
/*
print("----------------------------------------"),
print("separate the components"),
print("----------------------------------------"),
*/
tr0 : innerproduct(M,%sigma[0],tr0),
tr1 : innerproduct(M,%sigma[1],tr1),
tr2 : innerproduct(M,%sigma[2],tr2),
tr3 : innerproduct(M,%sigma[3],tr3),
a0r : factor(realpart(tr0)),
a1r : factor(realpart(tr1)),
a2r : factor(realpart(tr2)),
a3r : factor(realpart(tr3)),
B1r : factor(imagpart(tr1)),
B2r : factor(imagpart(tr2)),
B3r : factor(imagpart(tr3)),
t0r : factor(imagpart(tr0)),
```

```
print("------------------------------------------------------"),
print("Grade 0"),
print('a0r," = ",a0r),
print("------------------------------------------------------"),
print("Grade 1"),
print('a1r," = ",a1r),
print('a2r," = ",a2r),
print('a3r," = ",a3r),
print("------------------------------------------------------"),
print("Grade 2"),
print('B1r," = ",B1r),
print('B2r," = ",B2r),
print('B3r," = ",B3r),
print("------------------------------------------------------"),
print("Grade 3"),
print('t0r," = ",t0r)
 )$
/****************Block End*******************************/


/* Nabla_cyl */
/* print("Introduce functional dependence")$ */
depends([a11,a12,a21,a22],[%rho,phi,z])$
Ap : matrix([a11,a12],[a21,a22])$

/**********************Block start**********************/
/* ------------------------------------------------- */
/* This part is the function that compute the nabla
of a Pauli matrix in cylindrical coordinates */

Nablacyl(Ap) :=
block([],
NAp : %sigma[%rho].(diff(Ap,%rho)) + 1/%rho * %sigma[phi].(diff(Ap,phi)) + %sigma[3].(diff(Ap,z
NAp : factor(NAp)
)$
/**********************Block end***********************/
```

```
/**********************Block start**********************/
Gradecyl(M) :=
block([],

/* print("finding the scalar part")$ */
NdivA : expand(trigsimp(1/2*(M[1,1]+M[2,2]))),
a0 :realpart(NdivA ),
t0 :imagpart(NdivA ),

/* print("finding the rho component")$ */
NArho : M . %sigma[%rho],
NAr : expand(trigsimp(1/2*(NArho[1,1]+NArho[2,2]))),
arho : realpart(NAr),
Brho : imagpart(NAr),

/* print("finding the phi component")$ */
NAphi : M . %sigma[phi],
NAp : expand(trigsimp(1/2*(NAphi[1,1]+NAphi[2,2]))),
ap : realpart(NAp),
Bp : imagpart(NAp),

/* print("finding the z component")$ */
NAz : M . %sigma[3],
NAzz : expand(trigsimp(1/2*(NAz[1,1]+NAz[2,2]))),
az : realpart(NAzz),
Bz : imagpart(NAzz),

print("------------------------------------------------------"),
print("Grade structure"),
print("Grade 0"),
print('a0," = ",a0),
print("------------------------------------------------------"),
print("Grade 1"),
print('arho," = ",arho),
```

```
print('ap," = ",ap),
print('az," = ",az),
print("------------------------------------------------------------"),
print("Grade 2"),
print('Brho," = ",Brho),
print('Bp," = ",Bp),
print('Bz," = ",Bz),
print("------------------------------------------------------------"),
print("Grade 3"),
print('t0," = ",t0)
)$
/**********************Block end ********************/


depends([a11,a12,a21,a22],[r,%theta,phi])$
Ap : matrix([a11,a12],[a21,a22])$

/* ------------------------------------------------------------------*/
/* This block computes the nabla operator in spherical coordinates */
/* ------------------------------------------------------------------*/
Nablasph(Ap) :=
block([],
NAp: %sigma[r].(diff(Ap,r)) + 1/r * %sigma[%theta].(diff(Ap,%theta)) +
1/(r*sin(%theta)) * %sigma[phi].(diff(Ap,phi)),
NAp : expand(NAp),
NAp : factor(trigsimp(NAp))
)$
/* ------------------------------------------------------------------*/
/* Block end */
/* ------------------------------------------------------------------*/


/* ------------------------------------------------------------------*/
/* This block computes the grades in spherical coordinates */
/* ------------------------------------------------------------------*/
Gradesph(M) :=
block([],
```

```
/* print("Divergence"),*/
DivA : expand(1/2 * (M[1,1]+M[2,2])),

/* print("Dot Multiplication with %sigma[r] and taking the trace")$ */
Asph_sr : (M . %sigma[r]),
Asr : expand(1/2 * (Asph_sr[1,1]+Asph_sr[2,2])),
Asr : trigsimp(Asr),
Asr : expand(Asr),
/* print('Asr," = ",Asr), */

/* print("Dot Multiplication with %sigma[%theta] and taking the trace")$ */
Asph_st : (M . %sigma[%theta]),
Ast : 1/2 * (Asph_st[1,1]+Asph_st[2,2]),
Ast : trigsimp(expand(Ast)),
Ast : expand(Ast),
/* print('Ast," = ",Ast), */

/* print("Dot Multiplication with %sigma[phi] and taking the trace")$ */
Asph_sp : (M . %sigma[phi]),
Asp : 1/2 * (Asph_sp[1,1]+Asph_sp[2,2]),
Asp : trigsimp(expand(Asp)),
Asp : expand(Asp),
/* print('Asp," = ",Asp), */

print("------------------------------------------------------"),
print("Grade structure"),
print("Grade 0"),
a0 : realpart(DivA),
print('a0," = ",a0),
print("Grade 1"),
ar : realpart(Asr),
at : realpart(Ast),
ap : realpart(Asp),
ar : factor(trigsimp(ar)),
at : factor(trigsimp(at)),
ap : factor(trigsimp(ap)),
```

```
print('ar," = ",ar),
print('at," = ",at),
print('ap," = ",ap),
print("Grade 2"),
Bar : imagpart(Asr),
Bat : imagpart(Ast),
Bap : imagpart(Asp),
Bar : factor(trigsimp(Bar)),
Bat : factor(trigsimp(Bat)),
Bap : factor(trigsimp(Bap)),
print('Bar," = ",Bar),
print('Bat," = ",Bat),
print('Bap," = ",Bap),
print("Grade 3"),
t0 : imagpart(DivA),
print('t0," = ",t0)
)$
/*----------------------------------------------------------------------*/
/* Block end */
/*----------------------------------------------------------------------*/


/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Product of two vectors with CAS

In the previous section we have seen that a vector has been represented in terms of a 2x2 matrix. Naturally, several operations are possible with matrices: we can multiply them, we can compute the determinant, the inverse etc. With vectors, so far, we are only able to make the dot and cross products. We will therefore try to investigate what happens when we perform the product of two matrices $\tilde{a}$, $\tilde{b}$, representing two vectors (say **a** and **b**). In order to do that we take advantage of the CAS in the following way.

We introduce the Pauli matrices and then the representation of two vectors **a** and **b** via their Pauli matrices $\tilde{a}, \tilde{b}$. Then we perform their product obtaining $\tilde{c} = \tilde{a}\tilde{b}$ as illustrated in the code :

wxm/Pauli_ab.wxm

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load("Pauli_v02.mac");

print("------------------------------------------------------")$
print("Product of two vectors via Pauli Matrices")$
print("------------------------------------------------------")$

declare([a1,a2,a3],scalar)$
declare([b1,b2,b3],scalar)$

a : a1 . %sigma[1] + a2 . %sigma[2] + a3 . %sigma[3]$
print("Express vector ",'a," as a Pauli matrix ",a)$

b : b1 . %sigma[1] + b2 . %sigma[2] + b3 . %sigma[3]$
print("Express vector ",'b," as a Pauli matrix ",b)$

print("Find their product")$
c : ratsimp(a . b);

print("The compoments of ab are found as follows")$
print("The  scalar part (inner product) is the trace/2")$
sc : ratsimp((c[1,1]+c[2,2])/2);

print("------------------------------------------------------")$
print("The above is the dot product a . b  ")$
print("and it is the trace/2 of the matrix ")$
print("------------------------------------------------------")$
```

```
print("The  x component is given by ",('c[21]+'c[12])/2)$
cx : factor(ratsimp((c[2,1]+c[1,2])/2));

print("The  y component is given by ", ('c[21]-'c[12])/(2*%i))$
cy : factor(ratsimp((c[2,1]-c[1,2])/(2*%i)));

print("The  z component is given by ", ('c[11]-'c[22])/2)$
cz : factor(ratsimp((c[1,1]-c[2,2])/2));

print("one can recognizes that the above components,")$
print("apart for an i factor, are the same of a x b ")$

print("This new part is called external product")$
print("The external product ",'a ,"^", 'b)$
awb : factor(ratsimp(c - sc.%sigma[0]));

print("------------------------------------------------------------")$
print("The external product a ^ b = i a x b")$
print("------------------------------------------------------------")$

print("It is also possible to compute the determinant of a vector")$
deta : ratsimp(determinant(a));
print("its modulus is the square root of the abs value of the determinant")$
sqrt(abs(deta));
print("and its inverse is")$
inva : ratsimp(invert(a));
print("note that the direction is the same of the original vector")$
print("but it is scaled by the absolute value of the determinant")$
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

In the above code we have defined two matrices $\tilde{a}, \tilde{b}$ and evaluated their product $\tilde{c} = \tilde{a}\tilde{b}$,

$$\tilde{a} = \begin{pmatrix} a_3 & a_1 - i a_2 \\ i a_2 + a_1 & -a_3 \end{pmatrix}$$
$$\tilde{b} = \begin{pmatrix} b_3 & b_1 - i b_2 \\ i b_2 + b_1 & -b_3 \end{pmatrix}$$
$$\tilde{c} = \begin{pmatrix} a_1 b_1 + a_2 b_2 + i\,(-b_1 a_2 + a_1 b_2) + a_3 b_3 & b_1 a_3 - a_1 b_3 + i\,(-b_2 a_3 + a_2 b_3) \\ -b_1 a_3 + a_1 b_3 + i\,(-b_2 a_3 + a_2 b_3) & a_1 b_1 + a_2 b_2 + i\,(b_1 a_2 - a_1 b_2) + a_3 b_3 \end{pmatrix}$$

The trace of the matrix $\tilde{a}\tilde{b}$ divided by 2 is the dot product

As it is possible to note, if we perform $(\tilde{c}_{11} + \tilde{c}_{22})/2$ we obtain $a_1 b_1 + a_2 b_2 + a_3 b_3$ i.e. the dot product $\mathbf{a} \cdot \mathbf{b}$. Let us recall that the sum of the elements on the diagonal of a matrix is its *trace*. Therefore we have the following important result:

- *the trace of $\tilde{c} = \tilde{a}\tilde{b}$ divided by 2 gives us the dot product.*

$$(\tilde{c}_{11} + \tilde{c}_{22})/2 = a_1 b_1 + a_2 b_2 + a_3 b_3 \tag{1.121}$$

IV.2 Retrieving the other vector components

It is straightforward to recognize that the component along x, y, z can be easily retrieved by the following operations

$$\begin{aligned} c_x &= \langle \tilde{c}, \sigma_1 \rangle = \frac{c_{21} + c_{12}}{2} \\ c_y &= \langle \tilde{c}, \sigma_2 \rangle = -\frac{i\,(c_{21} - c_{12})}{2} \\ c_z &= \langle \tilde{c}, \sigma_3 \rangle = \frac{c_{11} - c_{22}}{2} \end{aligned} \tag{1.122}$$

By writing them explicitly we find:

$$\begin{aligned} c_x &= i\,(a_2 b_3 - a_3 b_2) \\ c_y &= -i\,(a_1 b_3 - a_3 b_1) \\ c_z &= i\,(a_1 b_2 - a_2 b_1) \end{aligned} \tag{1.123}$$

It is now easy to recognize that, apart for the i factor, this is equal to the well known cross product. This part is called *external product* and is denoted by $\mathbf{a} \wedge \mathbf{b}$. We have just obtained the important identity:

$$\mathbf{a} \wedge \mathbf{b} = i\,\mathbf{a} \times \mathbf{b}\,. \tag{1.124}$$

Naturally, to represent the wedge product in terms of the Pauli matrices it is sufficient to subtract the dot product from $\tilde{a}\tilde{b}$, thus obtaining

$$\mathbf{a} \wedge \mathbf{b} = \begin{pmatrix} i\,(a_1 b_2 - a_2 b_1) & i a_2 b_3 - a_1 b_3 - i a_3 b_2 + a_3 b_1 \\ i a_2 b_3 + a_1 b_3 - i a_3 b_2 - a_3 b_1 & -i\,(a_1 b_2 - a_2 b_1) \end{pmatrix}. \tag{1.125}$$

The quantity $\mathbf{a} \wedge \mathbf{b}$ is often called *external product* and is a *bivector* . The properties, meaning and characteristics of bivectors will be discussed in a separate section. For now it is important to recognize that the product $\tilde{a}\tilde{b}$ has provided us with both the dot product and the cross product.

But there is more...

In fact once we have represented the vector $\mathbf{a}$ as a matrix $\tilde{a}$ it is possible to compute its determinant and its inverse.

The determinant of $\tilde{a}$

It is immediately recognized that we have for the determinant

$$det(\tilde{a}) = -(a_1^2 + a_2^2 + a_3^2) \tag{1.126}$$

from which it is evident that, by taking the square root of the absolute value, we can recover the modulus of the vector.

The inverse of a vector

In standard vector algebra the inverse of a vector is not defined. However, we have seen that in the Clifford algebra of 2D we have the inverse. We can perform the inverse of $\tilde{a}$ obtaining

$$\tilde{a}^{-1} = \frac{1}{{a_1}^2+{a_2}^2+{a_3}^2}\begin{pmatrix} a_3 & -ia_2+a_1 \\ ia_2+a_1 & -a_3 \end{pmatrix}$$
$$= \frac{1}{\mathbf{a}\cdot\mathbf{a}}\tilde{a} \tag{1.127}$$

As in Cl_2 this is just the same vector divided by the square of its modulus!

Indeed there is much more...

Not only it is possible to take the inverse of a vector but, as we will see later, also of a multivector! It is also possible to take the inverse of an electromagnetic field! Now we can start to take ratios of electromagnetic fields. We can use these ratios to scale one electromagnetic field to match another at one chosen point.
To investigate further it is appropriate, at this point, to make an observation. Since it is well known that in matrix product the order is important we can ask to ourselves what comes out if we perform the product $\tilde{b}\tilde{a}$. This is shown in the next section.

Finding the operator which transforms one vector into another

Let us assume that the vectors $\mathbf{a},\mathbf{b}$ are given and we want to find the operator that transforms vector $\mathbf{a}$ into vector $\mathbf{b}$. In conventional vector analysis, apart for particular cases, this is not a simple operation. Using Pauli matrices it is quite trivial. In fact, by calling with $\tilde{c}$ the transformation we have that

$$\tilde{b} = \tilde{c}\tilde{a} \tag{1.128}$$

and the transformation is simply

$$\tilde{c} = \tilde{b}\tilde{a}^{-1} = \frac{1}{\mathbf{a}\cdot\mathbf{a}}\tilde{b}\tilde{a}. \tag{1.129}$$

The product $\tilde{b}\tilde{a}$

It is immediate, by applying the rules of matrix multiplication, to compute the product $\tilde{b}\tilde{a}$. An example of code doing this is reported in:

```
Pauli_ba.wxm
```

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 15.04.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$

load("Pauli_v02.mac");

declare([a1,a2,a3],scalar)$
declare([b1,b2,b3],scalar)$

a : a1 . %sigma[1] + a2 . %sigma[2] + a3 . %sigma[3]$
print("Express vector ",'a," as a Pauli matrix ",a)$
b : b1 . %sigma[1] + b2 . %sigma[2] + b3 . %sigma[3]$
print("Express vector ",'b," as a Pauli matrix ",b)$

print("Find the product ba")$
ba : ratsimp(b . a);

print("The compoments of ba are found as follows")$
print("The  scalar part (inner product) is the trace/2")$
sba : ratsimp((ba[1,1]+ba[2,2])/2);

print("------------------------------------------------------------")$
print("The above is the dot product b . a = a . b")$
print("      and it is the trace/2 of the matrix ")$
print("------------------------------------------------------------")$

print("The  x component is given by ",('ba[2,1]+'ba[1,2])/2)$
bax : factor(ratsimp((ba[2,1]+ba[1,2])/2));

print("The  y component is given by ", ('ba[2,1]-'ba[1,2])/(2*%i))$
bay : factor(ratsimp((ba[2,1]-ba[1,2])/(2*%i)));

print("The  z component is given by ", ('ab[1,1]-'ab[2,2])/2)$
baz : factor(ratsimp((ba[1,1]-ba[2,2])/2));
```

```
print("the above components are the same of - i(a x b)")$

print("This new part is called external product")$
print("The external product ",'b ,"^", 'a)$
bwa : factor(ratsimp(ba - sba.%sigma[0]));

print("------------------------------------------------------")$
print("Note that the external product b ^ a = - a ^ b")$
print("The external product is anticommutative")$
print("------------------------------------------------------")$

print("------------------------------------------------------")$
print("Therefore the dot product a . b can be obtained as (ab + ba)/2")$
print("While the external product a ^ b can be obtained as (ab - ba)/2")$
print("------------------------------------------------------")$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Naturally, the Pauli matrices and the matrices $\tilde{a}, \tilde{b}$ are the same as before. However, now we have:

$$\tilde{d} = \tilde{b}\tilde{a} = \begin{pmatrix} a_1 b_1 + a_2 b_2 + i\left(b_1 a_2 - a_1 b_2\right) + a_3 b_3 & -b_1 a_3 + a_1 b_3 + i\left(b_2 a_3 - a_2 b_3\right) \\ b_1 a_3 - a_1 b_3 + i\left(b_2 a_3 - a_2 b_3\right) & a_1 b_1 + a_2 b_2 + i\left(-b_1 a_2 + a_1 b_2\right) + a_3 b_3 \end{pmatrix}$$

It is apparent that the trace is the same as the one from $\tilde{a}\tilde{b}$. By writing explicitly the various components we find:

$$\begin{aligned} d_x &= -i\left(a_2 b_3 - a_3 b_2\right) \\ d_y &= i\left(a_1 b_3 - a_3 b_1\right) \\ d_z &= -i\left(a_1 b_2 - a_2 b_1\right) \end{aligned} \tag{1.130}$$

It is now easy to recognize that these are the coefficient of $-i\,\mathbf{a}\times\mathbf{b}$. The *external product* $\mathbf{b}\wedge\mathbf{a}$ thus satisfy the rule:

$$\mathbf{b}\wedge\mathbf{a} = -\mathbf{a}\wedge\mathbf{b} = -i\,\mathbf{a}\times\mathbf{b}\,. \tag{1.131}$$

We have another relevant property: *the external product between two vectors is anti–commutative*.

We have therefore found that the dot product can also be obtained as:

$$\mathbf{a}\cdot\mathbf{b} = \frac{\tilde{a}b + \tilde{b}a}{2}\,, \tag{1.132}$$

while for the external product we have:

$$\mathbf{a}\wedge\mathbf{b} = \frac{\tilde{a}b - \tilde{b}a}{2}\,. \tag{1.133}$$

The relations (1.132), (1.133) can be also taken as the definitions of dot and external products, respectively.

They also allow to introduce the *fundamental identity* i.e. the *geometric* or *Clifford* product:

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b}\,. \tag{1.134}$$

What has been called external product is a new quantity, which is not anymore a vector, but is an oriented surface. In 3D this oriented surface can be uniquely associated to a vector (the perpendicular at the surface).

As in the case of standard vector analysis it is interesting to see what happens when we consider triple products, e.g. $\tilde{a}b\tilde{c}$. Please note that in our 3D space we should have:

- one *scalar* (σ_0);
- 3 basis *vectors* ($\sigma_1, \sigma_2, \sigma_3$) corresponding to three directions;
- 3 basis *bivectors* ($\sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_3$) corresponding to three surfaces;
- one *pseudo–scalar* ($i\,\sigma_0$)

Vector and Bivector product in GA

Let us now come back to Clifford algebra in 3D, and consider a generic vector **B** expressed in terms of the Clifford representation as

$$\mathbf{B} = e_1 B_x + e_2 B_y + e_3 B_z \tag{1.135}$$

where the elements e_i constitutes the basis elements. The bivector $\hat{B} = i\mathbf{B}$ is expressed, using the identity $i = e_{123}$, as

$$\begin{aligned} \hat{B} &= e_{123}\left(e_1 B_x + e_2 B_y + e_3 B_z\right) \\ &= e_{23} B_x + e_{31} B_y + e_{12} B_z \end{aligned} \tag{1.136}$$

A vector **a**, in cartesian coordinates, is given by:

$$\mathbf{a} = e_1 a_x + e_2 a_y + e_3 a_z\,. \tag{1.137}$$

Although we are referring to cartesian coordinates, the results derived next are valid in general. The dot product between **a** and **B** is readily expressed as usual as:

$$\mathbf{a}\cdot\mathbf{B} = a_x B_x + a_y B_y \; + a_z B_z\,. \tag{1.138}$$

The external product of **a** with the bivector $\hat{B}$ is

$$\begin{aligned} \mathbf{a}\wedge\hat{B} &= \left(e_1 a_x + e_2 a_y + e_3 a_z\right)\wedge\left(e_{23} B_x + e_{31} B_y + e_{12} B_z\right) \\ &= e_{123}\left(a_x B_x + a_y B_y \; + a_z B_z\right) \\ &= i\,\mathbf{a}\cdot\mathbf{B}\,. \end{aligned} \tag{1.139}$$

Note that $\mathbf{a}\cdot\mathbf{B}$ is a scalar; when multiplied by i it becomes a pseudoscalar. Therefore, when performing the external product of **a** with the bivector $\hat{B}$ a pseudoscalar is obtained. This pseudoscalar represent the volume element.
The dot product of **a** with the bivector $\hat{B}$ is

$$\begin{aligned}\mathbf{a}\cdot\hat{B} &= \left(e_1 a_x + e_2 a_y + e_3 a_z\right)\cdot\left(e_{23} B_x + e_{31} B_y + e_{12} B_z\right)\\ &= -e_3 a_x B_y + e_2 a_x B_z + e_3 a_y B_x - e_1 a_y B_z - e_2 a_z B_x + e_1 a_z B_y\\ &= -\mathbf{a}\times\mathbf{B} = i\mathbf{a}\wedge\mathbf{B}. \end{aligned} \tag{1.140}$$

In summary we have derived the following important identities:

$$\mathbf{a}\wedge\hat{B} = i\,\mathbf{a}\cdot\mathbf{B} \tag{1.141}$$

$$\mathbf{a}\cdot\hat{B} = -\mathbf{a}\times\mathbf{B} = i\,\mathbf{a}\wedge\mathbf{B} \tag{1.142}$$

$$\mathbf{a}\hat{B} = i\,\mathbf{a}\mathbf{B}. \tag{1.143}$$

IV.3 Multivectors

Triple products with Pauli matrices

Let us now investigate the triple product. We start by considering the product $\mathbf{a}(\mathbf{b}\wedge\mathbf{c})$ and by expressing $\mathbf{b}\wedge\mathbf{c}$. In (1.125) the geometric product has been written for $\mathbf{a}\wedge\mathbf{b}$ and it is now repeated giving:

$$\mathbf{b}\wedge\mathbf{c} = \begin{pmatrix} i\left(b_1 c_2 - b_2 c_1\right) & i\left(b_2 c_3 - b_3 c_2\right) - b_1 c_3 + b_3 c_1 \\ i\left(b_2 c_3 - b_3 c_2\right) + b_1 c_3 - b_3 c_1 & i\left(b_2 c_1 - b_1 c_2\right) \end{pmatrix} \tag{1.144}$$

The next step is to obtain $d = \mathbf{a}(\mathbf{b}\wedge\mathbf{c})$ which simply reduces to the matrix multiplication of $\mathbf{a}$ and $\mathbf{b}\wedge\mathbf{c}$. This operation, together with a few others is reported in the following code. Note that we have used the symbol d to denote the result but we still need to identify what type of result we are going to get. It will be a 2x2 matrix but the meaning of its components is still to be found. Please refer to the following code as an example:

```
wxm/Pauli_abc.wxm
```

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */


/* [wxMaxima: input   start ] */
```

```
kill(all)$
load(vect);
load(Pauli_v02);

declare([a1,a2,a3],scalar)$
declare([b1,b2,b3],scalar)$
declare([c1,c2,c3],scalar)$

a : a1 . %sigma[1] + a2 . %sigma[2] + a3 . %sigma[3]$
print("Express vector ",'a," as a Pauli matrix ",a)$

b : b1 . %sigma[1] + b2 . %sigma[2] + b3 . %sigma[3]$
print("Express vector ",'b," as a Pauli matrix ",b)$

c : c1 . %sigma[1] + c2 . %sigma[2] + c3 . %sigma[3]$
print("Express vector ",'c," as a Pauli matrix ",c)$

print("------------------------------------------------------")$
print("geometric (Clifford) product bc (between two vectors)   ")$
print("------------------------------------------------------")$

print("Find their product")$
bc : ratsimp(b . c);

print("The  scalar part (inner product) is the trace/2")$
sbc : ratsimp((bc[1,1]+bc[2,2])/2);

print("The external product ",'b ,"^", 'c)$
bwc : ratsimp(bc - sbc . %sigma[0]);
print("one recognizes that the above is %i b x c")$
print("------------------------------------------------------")$
print(" The following identity holds b ^ c = %i b x c  ")$
print("------------------------------------------------------")$
print("Now perform a (b ^ c)")$
print("------------------------------------------------------")$
abwc : a . bwc;
```

```
print("The  scalar part (inner product) is the trace/2")$
sbc : factor(ratsimp((abwc[1,1]+abwc[2,2])/2));
print("Note that the above is a trivector (a pseudoscalar)")$
print("It corresponds to the volume element a ^ b ^ c")$

print("If we consider the matrix ")$
abc : matrix([a1,a2,a3],[b1,b2,b3],[c1,c2,c3])$
print('abc," = ",abc)$
print("its determinat is ")$
dabc : determinant(abc);
print("The determinant multiplied by i is equal to the scalar part,")$
print("their difference being")$
ratsimp(%i * dabc - sbc);

print("------------------------------------------------------------")$
print("Therefore we have a ^ b ^ c = i det(abc)")$
print("------------------------------------------------------------")$

print("The remaining part corresponds to a dot (b ^ c)")$
awbwc : ratsimp(abwc - sbc . %sigma[0]);

print("i.e. a vector with components")$
ws : ratsimp((awbwc[1,1]+awbwc[2,2])/2)$
wx : ratsimp((awbwc[2,1]+awbwc[1,2])/2)$
wy : ratsimp((awbwc[2,1]-awbwc[1,2])/(2*%i))$
wz : ratsimp((awbwc[1,1]-awbwc[2,2])/2)$

print('ws, " = ",ws )$
print('wx, " = ",wx )$
print('wy, " = ",wy )$
print('wz, " = ",wz )$

print("------------------------------------------------------------")$
print(" (a.b)c - (a.c)b  (identity bac- cab = a x b c )  ")$
print("------------------------------------------------------------")$

print("trying a different approach")$
```

```
print("evaluating dot products e.g. a dot b as (ab + ba)/2")$
adb : ratsimp((a . b + b . a)/2);
adc : ratsimp((a . c + c . a)/2);
w : adb . c - adc . b;

as : ratsimp((w[1,1]+w[2,2])/2)$
ax : ratsimp((w[2,1]+w[1,2])/2)$
ay : ratsimp((w[2,1]-w[1,2])/(2*%i))$
az : ratsimp((w[1,1]-w[2,2])/2)$

print("i.e. a vector with components")$
print('as, " = ",as )$
print('ax, " = ",ax )$
print('ay, " = ",ay )$
print('az, " = ",az )$

print("Proof that are equal")$
ratsimp(wx-ax);
ratsimp(wy-ay);
ratsimp(wz-az);

print("------------------------------------------------------------")$
print("Therefore w = a . (b ^ c) = (a.b)c - (a.c)b")$
print("------------------------------------------------------------")$

print("======================GIBBS=======================")$
a:[a1, a2, a3];
b:[b1, b2, b3];
c : [c1, c2, c3];
/*
print(" dot product")$
c : a . b;

print(" cross product")$
 axb : a ~ b;
 axb : express(axb);
```

```
bxa : express(b ~ a);
print("find the difference axb - bxa")$
diff : axb + bxa;
*/

print("A X ( B X C ) = ")$
r1 : express(a ~ express(b ~ c))$
r1 :ratsimp(r1);

print("The sum of these coefficients with")$
print("those previously found gives zero ")$
ratsimp(r1[1] + ax);
ratsimp(r1[2] + ay);
ratsimp(r1[3] + az);

print("------------------------------------------------------------")$
print("Therefore a . (b ^ c) = (a.b)c - (a.c)b = - a x b x c  ")$
print("------------------------------------------------------------")$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

After performing the matrix multiplication one get

$$d = \mathbf{a}(\mathbf{b}\wedge\mathbf{c}) \tag{1.145}$$

$$d_{11} = (a_1 - ia_2)\left((ib_2 + b_1)c_3 - ib_3c_2 - b_3c_1\right) + a_3\left(ib_1c_2 - ib_2c_1\right)$$

$$d_{12} = a_3\left((ib_2 - b_1)c_3 - ib_3c_2 + b_3c_1\right) + (a_1 - ia_2)\left(ib_2c_1 - ib_1c_2\right)$$

$$d_{21} = (ia_2 + a_1)\left(ib_1c_2 - ib_2c_1\right) - a_3\left((ib_2 + b_1)c_3 - ib_3c_2 - b_3c_1\right)$$

$$d_{22} = (ia_2 + a_1)\left((ib_2 - b_1)c_3 - ib_3c_2 + b_3c_1\right) - a_3\left(ib_2c_1 - ib_1c_2\right)$$

Scalar part

As before the scalar part of d is obtained by taking half of the trace and is denoted here (for reasons that will become clear in the following) as $\langle d \rangle_3$

$$\langle d \rangle_3 = i\,(a_1 b_2 c_3 - a_2 b_1 c_3 - a_1 b_3 c_2 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_3 b_2 c_1) \tag{1.146}$$

It is noted that there is the factor i in front of the expression. This part corresponds to a *trivector*, i.e. to the volume element obtained by performing $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$. It is also observed that if we consider the matrix

$$abc = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \tag{1.147}$$

its determinant is

$$det(abc) = a_1\,(b_2 c_3 - b_3 c_2) - a_2\,(b_1 c_3 - b_3 c_1) + a_3\,(b_1 c_2 - b_2 c_1) \tag{1.148}$$

and, when multiplied by i coincides with (1.146). We have therefore derived the important property:

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} = i\,det\,(abc) = \langle d \rangle_3 \tag{1.149}$$

It is noted that $\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c}$ is often called *pseudoscalar* i.e. is a scalar quantity multiplied by i. In the 3D space this is the element of the third grade (this is the reason why we have used $\langle d \rangle_3$). This is also the highest grade in 3D. A few observations are in order:

$$(\mathbf{a} \wedge \mathbf{b}) \wedge \mathbf{c} = \mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) \tag{1.150}$$

$$\mathbf{a} \wedge \mathbf{b} \wedge \mathbf{c} \wedge \mathbf{d} = 0 \tag{1.151}$$

In (1.150) it is noted that, *unlikely to the cross product*, the wedge product can be computed without a particular order.

The other property in (1.151) tell us the following. If we have assumed that the three vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ span the 3D space (i.e. they do not lay on a plane, than a vector $\mathbf{d}$ cannot be external to them, i.e. its external part is zero.
Let us introduce the bivector $\hat{\mathbf{B}} = \mathbf{b} \wedge \mathbf{c}$. It is apparent (see (1.149)) that we have that *the external product between a vector and a bivector is commutative*:

$$\mathbf{a} \wedge \hat{\mathbf{B}} = \hat{\mathbf{B}} \wedge \mathbf{a}\,. \tag{1.152}$$

Vector part $\mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$

Once from the matrix d in (1.145) the scalar part is removed, the remaining part corresponds to $\mathbf{w} = \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c})$. If we express the components relative to this matrix we have:

$$\begin{aligned} w_x &= -a_3\, b_1\, c_3 - a_2\, b_1\, c_2 + (a_3\, b_3 + a_2\, b_2)\, c_1 \\ w_y &= -a_3\, b_2\, c_3 + (a_3\, b_3 + a_1\, b_1)\, c_2 - a_1\, b_2\, c_1 \\ w_z &= (a_2\, b_2 + a_1\, b_1)\, c_3 - a_2\, b_3\, c_2 - a_1\, b_3\, c_1 \end{aligned} \tag{1.153}$$

it is noted that they are the same coefficients, apart for a sign, that we would have obtained by performing the vector product $\mathbf{v} = \mathbf{a} \times \mathbf{b} \times \mathbf{c}$. In fact, in the last part of the code, we perform the triple vector product obtaining:

$$\begin{aligned} v_x &= a_3\, b_1\, c_3 + a_2\, b_1\, c_2 + (-a_3\, b_3 - a_2\, b_2)\, c_1 \\ v_y &= a_3\, b_2\, c_3 + (-a_3\, b_3 - a_1\, b_1)\, c_2 + a_1\, b_2\, c_1 \\ v_z &= (-a_2\, b_2 - a_1\, b_1)\, c_3 + a_2\, b_3\, c_2 + a_1\, b_3\, c_1\,. \end{aligned} \tag{1.154}$$

By comparing (1.154) with (1.153) it is readily recognized that we have found another important relationship i.e.

$$\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = -\mathbf{a} \times \mathbf{b} \times \mathbf{c} \tag{1.155}$$

In passing, it is reminded that we have already introduced the relationship $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. It is therefore apparent that we can also compute $\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c}$ as

$$\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c} = -\mathbf{a}\times\mathbf{b}\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} \tag{1.156}$$

This computation is also performed in the code, where the dot products are obtained as e.g. $\mathbf{a}\cdot\mathbf{b} = \left(\tilde{a}b + \tilde{b}a\right)/2$ and then proceeding with standard matrix multiplication.
So far we have considered the external product of a bivector with a vector (or viceversa) and we have obtained a trivector. Then we have considered the dot product of a vector with a bivector, obtaining a vector. It is thus natural to pose the question: what is the result of a product of the type $\mathbf{b}\wedge\mathbf{c}\cdot\mathbf{a}$? In other words, the dot product of a *bivector* with a vector is commutative or anticommutative? In order to answer to this question we can consider the following code.

wxm/Pauli_bca.wxm

which is listed in the following.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect);
load(Pauli_v02);
/* code name: Pauli_bca.wxm */

declare([a1,a2,a3],scalar)$
declare([b1,b2,b3],scalar)$
declare([c1,c2,c3],scalar)$

a : a1 . %sigma[1] + a2 . %sigma[2] + a3 . %sigma[3]$
print("Express vector ",'a," as a Pauli matrix ",a)$
b : b1 . %sigma[1] + b2 . %sigma[2] + b3 . %sigma[3]$
print("Express vector ",'b," as a Pauli matrix ",b)$
c : c1 . %sigma[1] + c2 . %sigma[2] + c3 . %sigma[3]$
print("Express vector ",'c," as a Pauli matrix ",c)$

print("-------------------------------------------------------")$
print("geometric (Clifford) product bc (between two vectors)   ")$
print("-------------------------------------------------------")$
```

```
print("Find their product")$
bc : ratsimp(b . c);

print("The  scalar part (inner product) is the trace/2")$
sbc : ratsimp((bc[1,1]+bc[2,2])/2);

print("The external product ",'b ,"^", 'c)$
bwc : ratsimp(bc - sbc . %sigma[0]);
print("one recognizes that the above is %i b x c")$
print("-------------------------------------------------------")$
print(" The following identity holds b ^ c = %i b x c  ")$
print("-------------------------------------------------------")$
print("Now perform  (b ^ c) a")$
print("-------------------------------------------------------")$
abwc :  bwc . a ;

print("The  scalar part (inner product) is the trace/2")$
sbc : factor(ratsimp((abwc[1,1]+abwc[2,2])/2));
print("Note that the above is a trivector (a pseudoscalar)")$
print("It corresponds to the volume element  b ^ c ^ a")$

print("If we consider the matrix ")$
abc : matrix([a1,a2,a3],[b1,b2,b3],[c1,c2,c3])$
print('abc," = ",abc)$
print("its determinat is ")$
dabc : determinant(abc);
print("and the determinant multiplied by i is")$
print("equal to  the scalar part, their difference being")$
ratsimp(%i * dabc - sbc);

print("-------------------------------------------------------")$
print("Therefore we have  b ^ c ^ a = i det(abc)")$
print("-------------------------------------------------------")$

print("The remaining part corresponds to  (b ^ c) dot a")$
awbwc : ratsimp(abwc - sbc . %sigma[0]);
```

```
print("i.e. a vector with components")$
ws : ratsimp((awbwc[1,1]+awbwc[2,2])/2)$
wx : ratsimp((awbwc[2,1]+awbwc[1,2])/2)$
wy : ratsimp((awbwc[2,1]-awbwc[1,2])/(2*%i))$
wz : ratsimp((awbwc[1,1]-awbwc[2,2])/2)$

print('ws, " = ",ws )$
print('wx, " = ",wx )$
print('wy, " = ",wy )$
print('wz, " = ",wz )$



print("------------------------------------------------------")$
print(" (a.b)c - (a.c)b  (identity bac- cab = a x b c )  ")$
print("------------------------------------------------------")$

print("trying a different approach")$
print("evaluating dot products e.g. a dot b as (ab + ba)/2")$
adb : ratsimp((a . b + b . a)/2);
adc : ratsimp((a . c + c . a)/2);
w : adb . c - adc . b;

as : ratsimp((w[1,1]+w[2,2])/2)$
ax : ratsimp((w[2,1]+w[1,2])/2)$
ay : ratsimp((w[2,1]-w[1,2])/(2*%i))$
az : ratsimp((w[1,1]-w[2,2])/2)$

print("i.e. a vector with components")$
print('as, " = ",as )$
print('ax, " = ",ax )$
print('ay, " = ",ay )$
print('az, " = ",az )$

print("Proof that their sum (e.g. wx + ax) vanishes")$
ratsimp(wx+ax);
ratsimp(wy+ay);
```

```
ratsimp(wz+az);



print("-------------------------------------------------------")$
print("Therefore w = a . (b ^ c) = (a.b)c - (a.c)b = -(b ^ c) . a")$
print("The dot product of a vector with a bivector is anticommutative")$
print("-------------------------------------------------------")$

/*
print("======================GIBBS=======================")$
a:[a1, a2, a3];
b:[b1, b2, b3];
c:[c1, c2, c3];

print("A X ( B X C ) = ")$
r1 : express(a ~ express(b ~ c));

print("The sum of these coefficients with")$
print("those previously found gives zero ")$
ratsimp(r1[1] + ax);
ratsimp(r1[2] + ay);
ratsimp(r1[3] + az);

print("-------------------------------------------------------")$
print("Therefore a . (b ^ c) = (a.b)c - (a.c)b = - a x b x c  ")$
print("-------------------------------------------------------")$
*/
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

The initial part is identical as before, the difference only coming out when we compute $(\mathbf{b}\wedge\mathbf{c})\mathbf{a}$. The result is a 2x2 matrix with half of the trace that now corresponds to $\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{a}$. We thus have $\mathbf{b}\wedge\mathbf{c}\wedge\mathbf{a} = \mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}$. By subtracting this part (times σ_0) from the matrix $(\mathbf{b}\wedge\mathbf{c})\mathbf{a}$ we obtain the matrix representing $(\mathbf{b}\wedge\mathbf{c})\cdot\mathbf{a}$ with the components which are the same as in (1.153) but with a

minus sign. Therefore, *the dot product for bivectors is anticommutative* or

$$\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c} = -\mathbf{b}\wedge\mathbf{c}\cdot\mathbf{a}\,. \tag{1.157}$$

Therefore for a bivector $\hat{\mathbf{B}}$ we can write

$$\mathbf{a}\cdot\hat{\mathbf{B}} = \frac{1}{2}\left(\mathbf{a}\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{a}\right) = -\mathbf{a}\times\mathbf{b}\times\mathbf{c} \tag{1.158}$$

$$\mathbf{a}\wedge\hat{\mathbf{B}} = \frac{1}{2}\left(\mathbf{a}\hat{\mathbf{B}} + \hat{\mathbf{B}}\mathbf{a}\right) = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\,i \tag{1.159}$$

The last equality has not been shown so far, but the reader can easily prove it.
By introducing the vector $\mathbf{B} = \mathbf{b}\times\mathbf{c}$ we have

$$\hat{\mathbf{B}} = \mathbf{b}\wedge\mathbf{c} = i\,\mathbf{b}\times\mathbf{c} = i\,\mathbf{B} \tag{1.160}$$

which, by using (1.158, 1.159), allows to write

$$\mathbf{a}\cdot\hat{\mathbf{B}} = -\mathbf{a}\times\mathbf{B} \tag{1.161}$$

$$\mathbf{a}\wedge\hat{\mathbf{B}} = \mathbf{a}\cdot\mathbf{B}\,i \tag{1.162}$$

Equation (1.160) shows that a bivector $\hat{\mathbf{B}}$ can be obtained from a vector $\mathbf{B}$ simply by multiplication times i (and viceversa). It is also noted that a scalar t when multiplied by i becomes a trivector (and viceversa). The relationships (1.161, 1.162) can be used for relating classical vector analysis with the Pauli algebra.

Summary of the results of vector analysis with Pauli matrices

Although several aspects have still to be further elucidated, we can make a summary of what we have obtained so far.

- Three–dimensional vectors can be represented using 2x2 Pauli matrices.

$$\tilde{a} = \begin{pmatrix} a_z & a_x - ia_y \\ a_x + ia_y & -a_z \end{pmatrix} \tag{1.163}$$

- Multiplication of two matrices corresponding to two vectors give us both the dot product and another new part, the external product. This multiplication corresponds, in 3D, to the geometric or Clifford product.

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} = \tilde{a}\tilde{b}. \tag{1.164}$$

- dot product between two vectors is commutative:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} = \frac{\tilde{a}\tilde{b}+\tilde{b}\tilde{a}}{2}, \tag{1.165}$$

- the external product between two vectors is anticommutative

$$\mathbf{a}\wedge\mathbf{b} = -\mathbf{b}\wedge\mathbf{a} = \frac{\tilde{a}\tilde{b}-\tilde{b}\tilde{a}}{2}. \tag{1.166}$$

- the external product is related to the cross product as:

$$\mathbf{a}\wedge\mathbf{b} = i\,\mathbf{a}\times\mathbf{b}. \tag{1.167}$$

- the external product between two vectors introduces a new subject: *the bivector*.
- the bivector can be expressed either showing the two vector components or a single vector component but multiplied by i

$$\begin{aligned}
\sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\
\sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \\
\sigma_3\sigma_1 &= i\sigma_2 = -\sigma_1\sigma_3 .
\end{aligned} \tag{1.168}$$

- a bivector can be multiplied by a vector giving rise to: a vector and a *trivector*:

$$\mathbf{a}\hat{\mathbf{B}} = \mathbf{a}\cdot\hat{\mathbf{B}} + \mathbf{a}\wedge\hat{\mathbf{B}} \tag{1.169}$$

- the internal product of a vector with a bivector is given by

$$\mathbf{a}\cdot\hat{\mathbf{B}}=\frac{1}{2}\left(\mathbf{a}\hat{\mathbf{B}}-\hat{\mathbf{B}}\mathbf{a}\right)=-\mathbf{a}\times\mathbf{b}\times\mathbf{c}=-\mathbf{a}\times\mathbf{B} \tag{1.170}$$

- the external product of a vector and a bivector is

$$\mathbf{a}\wedge\hat{\mathbf{B}}=\frac{1}{2}\left(\mathbf{a}\hat{\mathbf{B}}+\hat{\mathbf{B}}\mathbf{a}\right)=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\,i=\mathbf{a}\cdot\mathbf{B}\,i \tag{1.171}$$

Space description

It is noted that elements three-dimensional space are described by eight numbers (i.e. a complex 2x2 matrix). In particular they are:

- one *scalar* (σ_0). Grade 0
- 3 basis *vectors* ($\sigma_1,\sigma_2,\sigma_3$) corresponding to three directions. Grade 1
- 3 basis *bivectors* ($\sigma_1\sigma_2,\sigma_1\sigma_3,\sigma_2\sigma_3$). Grade 2
- one *pseudoscalar* ($i\,\sigma_0$). Grade 3

All these elements are contained in a matrix and, similarly to what we do for complex numbers, they can be written together in a *multivector M* as

$$M=a_0+\mathbf{a}+\hat{\mathbf{B}}+\hat{t} \tag{1.172}$$

where a_0 is a scalar, $\mathbf{a}$ is a vector, $\hat{\mathbf{B}}$ is a bivector and $\hat{t}$ is a pseudoscalar.
We have started this section showing that a vector can be represented by a Pauli matrix. It is now possible to conclude that, in the three–dimensional space, a Pauli matrix not only can represent a vector, but it can encode all the information of the eight-dimensional base of a multivector! In other words a multivector can be represented as a Pauli matrix. Since matrix algebra is well–known, we can also multiply, take the inverse, etc of multivectors with ease.

Pauli matrix representation of a multivvector

Let us see with more details the Pauli matrix representation of a multivector. The corresponding matrices of the multivector in (1.172) are given next:

$$\tilde{a_0} = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}$$
$$\tilde{a} = \begin{pmatrix} a_3 & a_1 - ia_2 \\ ia_2 + a_1 & -a_3 \end{pmatrix}$$
$$\tilde{B} = \begin{pmatrix} iB_3 & B_2 + iB_1 \\ iB_1 - B_2 & -iB_3 \end{pmatrix}$$
$$\tilde{t} = \begin{pmatrix} it & 0 \\ 0 & it \end{pmatrix}. \tag{1.173}$$

The matrices in (1.173) can be summed together giving for the the multivector M

$$\tilde{M} = \begin{pmatrix} iB_3 + it + a_3 + a_0 & B_2 + iB_1 - ia_2 + a_1 \\ -B_2 + iB_1 + ia_2 + a_1 & -iB_3 + it - a_3 + a_0 \end{pmatrix}. \tag{1.174}$$

Naturally, for a given Pauli matrix, it is possible to retrieve the elements of the different grades as described next.

Retrieving the elements of a multivector

Let us assume that the matrix in (1.174) is given and we want to retrieve the various elements. It is convenient to extract the real and imaginary part of $\tilde{M}$ as

$$\tilde{M_r} = \mathrm{Re}\{\tilde{M}\} = \begin{pmatrix} a_3 + a_0 & B_2 + a_1 \\ a_1 - B_2 & a_0 - a_3 \end{pmatrix}$$
$$\tilde{M_i} = \mathrm{Im}\{\tilde{M}\} = \begin{pmatrix} B_3 + t & B_1 - a_2 \\ B_1 + a_2 & t - B_3 \end{pmatrix}. \tag{1.175}$$

By inspection, it is seen that we have the following identities:

$$
\begin{aligned}
a_0 &= \frac{1}{2}\left(\tilde{M}_{r11} + \tilde{M}_{r22}\right) \\
a_1 &= \frac{1}{2}\left(\tilde{M}_{r12} + \tilde{M}_{r21}\right) \\
a_2 &= \frac{1}{2}\left(\tilde{M}_{i21} - \tilde{M}_{i12}\right) \\
a_3 &= \frac{1}{2}\left(\tilde{M}_{r11} - \tilde{M}_{r22}\right) \\
B_1 &= \frac{1}{2}\left(\tilde{M}_{i21} + \tilde{M}_{i12}\right) \\
B_2 &= \frac{1}{2}\left(\tilde{M}_{r12} - \tilde{M}_{r21}\right) \\
B_3 &= \frac{1}{2}\left(\tilde{M}_{i11} - \tilde{M}_{i22}\right) \\
t &= \frac{1}{2}\left(\tilde{M}_{i11} + \tilde{M}_{i22}\right) .
\end{aligned} \tag{1.176}
$$

The code for converting multivectors into their Pauli matrix equivalent and viceversa is given in the following lines.

wxm/Pauli_multivectors.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02);
/* code name: Pauli_multivectors.wxm */

declare([a1,a2,a3],scalar)$
declare([B1,B2,B3],scalar)$
declare([a0,t],scalar)$

/* Construct multivector */
A :  a0 . %sigma[0] $
print("Scalar ",'a0," Pauli matrix ",A)$
a : a1 . %sigma[1] + a2 . %sigma[2] + a3 . %sigma[3]$
print("Vector ",'a," Pauli matrix ",a)$
B : %i * (B1 . %sigma[1] + B2 . %sigma[2] + B3 . %sigma[3])$
B : ratsimp(B)$
```

```
print("Bivector ",'B," Pauli matrix ",B)$
T : %i * t . %sigma[0] $
print("Pseudoscalar ",'t," Pauli matrix ",T)$

print("Multivector M")$
M : A + a + B + T ;

print("------------------------------------------------------")$
print("Retrieving the elements of the multivector")$
print("for a given Pauli matrix")$
print("------------------------------------------------------")$

Gradesep(M)$
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

V Clifford Algebra

General definition of Clifford's geometric algebra (GA)

Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, e_{p+q+1}, \ldots, e_n\}$, with $n = p+q+r$, $e_k^2 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1, \ldots, p$, $\varepsilon_k = -1$ for $k = p+1, \ldots, p+q$, $\varepsilon_k = 0$ for $k = p+q+1, \ldots, n$, be an *orthonormal base* of the inner product vector space $\mathbf{R}^{p,q,r}$ with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k,l = 1, \ldots n, \tag{1.177}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$. This non-commutative product and the additional axiom of *associativity* generate the 2^n-dimensional Clifford geometric algebra $Cl(p,q,r)$. The definition of Clifford's GA is fundamentally *coordinate system independent*, i.e. *coordinate free*.

V.1 Clifford's GA in two dimensions

In order to demonstrate how to compute with Clifford numbers, we begin with a low dimensional example.

Example of $Cl(2,0)$

A Euclidean plane is spanned by $e_1, e_2 \in \mathbf{R}^2$ with

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 1, \quad e_1 \cdot e_2 = 0. \tag{1.178}$$

$\{e_1, e_2\}$ is an *orthonormal* vector basis of $\mathbf{R}^2$.
Under Clifford's *associative* geometric product we set

$$\begin{aligned} e_1^2 = e_1 e_1 &:= e_1 \cdot e_1 = 1, \\ e_2^2 = e_2 e_2 &:= e_2 \cdot e_2 = 1, \end{aligned} \tag{1.179}$$

and

$$e_1 e_2 + e_2 e_1 = 0 \;\Leftrightarrow\; e_1 e_2 = -e_2 e_1, \tag{1.180}$$

i.e. the geometric product of orthogonal vectors forms a new entity, called unit *bi-vector* $e_{12} = e_1 e_2$ by Grassmann, and is *anti-symmetric*. General bivectors in $Cl(2,0)$ are e.g. βe_{12}. For orthogonal vectors the geometric product equals Grassmann's anti-symmetric outer product (exterior product, symbol $\wedge$)

$$\begin{aligned} e_{12} &= e_1 e_2 = e_1 \wedge e_2 \\ &= -e_2 \wedge e_1 = -e_2 e_1 = -e_{21}. \end{aligned} \tag{1.181}$$

Using associativity, we can compute the products

$$e_1 e_{12} = e_1 e_1 e_2 = e_1^2 e_2 = e_2, \quad e_2 e_{12} = -e_2 e_{21} = -e_1, \tag{1.182}$$

which represent a mathematically *positive* (anti-clockwise) 90° *rotation*. The opposite order gives

$$e_{12} e_1 = -e_{21} e_1 = -e_2, \quad e_{12} e_2 = e_1, \tag{1.183}$$

which represents a mathematically *negative* (clockwise) 90° *rotation*. The bivector e_{12} acts like a *rotation operator*, and we observe the general anti-commutation property

$$a e_{12} = -e_{12} a, \quad \forall a = a_1 e_1 + a_2 e_2 \in \mathbf{R}^2,\ a_1, a_2 \in \mathbf{R}. \tag{1.184}$$

The square of the unit bivector is -1,

$$e_{12}^2 = e_1 e_2 e_{12} = e_1(-e_1) = -1, \tag{1.185}$$

just like the imaginary unit j of complex numbers $\mathbf{C}$.

Table 1.6 is the complete multiplication table of the Clifford algebra $Cl(\mathbf{R}^2) = Cl(2,0,0) = Cl(2,0)$ with algebra basis elements $\{1, e_1, e_2, e_{12}\}$ (which includes the vector basis of $\mathbf{R}^2$). The even subalgebra spanned by $\{1, e_{12}\}$ (closed under geometric multiplication), consisting of even grade scalars (0-vectors) and bivectors (2-vectors), is isomorphic to $\mathbb{C}$.

Algebraic unification and vector inverse

The general geometric product of two vectors $a, b \in \mathbf{R}^2$

$$\begin{aligned} ab &= (a_1 e_1 + a_2 e_2)(b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 + a_2 b_2 + (a_1 b_2 - a_2 b_1) e_{12} \\ &= \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) = a \cdot b + a \wedge b, \end{aligned} \tag{1.186}$$

has therefore a scalar *symmetric* inner product part

$$\begin{aligned} \frac{1}{2}(ab + ba) &= a \cdot b = a_1 b_1 + a_2 b_2 \\ &= |a||b| \cos \theta_{a,b}, \end{aligned} \tag{1.187}$$

and a bi-vector *skew-symmetric* outer product part

$$\frac{1}{2}(ab - ba) = a \wedge b = (a_1 b_2 - a_2 b_1) e_{12} = |a||b| e_{12} \sin \theta_{a,b}. \tag{1.188}$$

We observe that parallel vectors ($\theta_{a,b} = 0$) commute, $ab = a \cdot b = ba$, and orthogonal vectors ($\theta_{a,b} = 90^\circ$) anti-commute, $ab = a \wedge b = -ba$. The outer product part $a \wedge b$ represents the *oriented area* of the parallelogram spanned by the vectors a, b in the plane of $\mathbf{R}^2$, with oriented magnitude

$$\det(a, b) = |a||b| \sin \theta_{a,b} = (a \wedge b) e_{12}^{-1}, \tag{1.189}$$

where $e_{12}^{-1} = -e_{12}$, because $e_{12}^2 = -1$.

Table 1.6. Multiplication table of plane Clifford algebra $Cl(2,0)$.

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

With the *Euler* formula we can rewrite the geometric product as

$$\begin{aligned} ab &= |a||b|(\cos\theta_{a,b} + e_{12}\sin\theta_{a,b}) \\ &= |a||b|e^{\theta_{a,b}e_{12}}, \end{aligned} \tag{1.190}$$

again because $e_{12}^2 = -1$.

The geometric product of vectors is *invertible* for all vectors with non-zero square $a^2 \neq 0$

$$\begin{aligned} a^{-1} &:= a/a^2, \quad aa^{-1} = aa/a^2 = 1, \\ a^{-1}a &= \frac{a}{a^2}a = a^2/a^2 = 1. \end{aligned} \tag{1.191}$$

The inverse vector a/a^2 is a rescaled version (reflected at the unit circle) of the vector a. This invertibility leads to significant simplifications and ease in computations.

Geometric operations and transformations

For example, the *projection* of one vector $x \in \mathbf{R}^2$ onto another $a \in \mathbf{R}^2$ is

$$x_{\|} = |x|\cos\theta_{a,x}\frac{a}{|a|} = (x\cdot\frac{a}{|a|})\frac{a}{|a|} = (x\cdot a)\frac{a}{|a|^2} = (x\cdot a)a^{-1}. \tag{1.192}$$

The *rejection* (perpendicular part) is

$$\begin{aligned} x_{\perp} &= x - x_{\|} = xaa^{-1} - (x\cdot a)a^{-1} \\ &= (xa - x\cdot a)a^{-1} = (x\wedge a)a^{-1}. \end{aligned} \tag{1.193}$$

We can now use $x_{\|}, x_{\perp}$ to compute the reflection of $x = x_{\|} + x_{\perp}$ at the line with normal vector a, which means to reverse $x_{\|} \to -x_{\|}$

$$\begin{aligned} x' &= -x_{\|} + x_{\perp} = -a^{-1}ax_{\|} + a^{-1}ax_{\perp} \\ &= -a^{-1}x_{\|}a - a^{-1}x_{\perp}a = -a^{-1}(x_{\|} + x_{\perp})a = -a^{-1}xa. \end{aligned} \tag{1.194}$$

The combination of two reflections at two lines (hyperplanes) with normals a, b

$$x'' = -b^{-1}x'b = b^{-1}a^{-1}xab = (ab)^{-1}xab = R^{-1}xR, \tag{1.195}$$

gives a rotation. The rotation angle is $\alpha = 2\theta_{a,b}$ and the *rotor*

$$R = e^{\theta_{a,b} e_{12}} = e^{\frac{1}{2}\alpha e_{12}}, \tag{1.196}$$

where the lengths $|a||b|$ of ab cancel against $|a|^{-1}|b|^{-1}$ in $(ab)^{-1}$. The rotor R gives the *spinor* form of rotations, fully replacing rotation matrices, and introducing the same elegance to *real* rotations in $\mathbf{R}^2$, like in the complex plane.

In 2D, the product of three reflections, i.e. of a rotation and a reflection, leads to another reflection. In 2D the product of an *odd* number of reflections always results in a *reflection*. That the product of an *even* number of reflections leads to a *rotation* is true in general dimensions. These transformations are in Clifford algebra simply described by the products of the vectors normal to the lines (hyperplanes) of reflection and called versors.

An example of computations involving $Cl(2,0)$ are in the following listing

`wxm/2D_demo.wxm`

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
pathmaxima:file_search_maxima$
kill(all)$

load("clifford.mac");
load("cliffordan.mac");

if get('clifford,'version)=false then load("clifford")$
if get('cliffordan,'version)=false then load("cliffordan")$
print("Cliford implements Clifford algebra for Maxima.")$

print("---------------------------------------")$
print("Program Starts")$
print("---------------------------------------")$

print("Construct the Clifford basis providing the symbol and ")$
print("space dimensionality")$
clifford(e, 2);

print("Note that it is possible to use different symbols")$
/*
It is possible to use different symbols
clifford(%sigma, 2);
*/

print("and it is also possible to have different signatures")$
/*
It is possible to define the value of the squares
if 1,-1
clifford(e,0, 2);
clifford(e,1, 1);
*/

print("---------------------------------------")$
```

```
print("Construction of Tables of products")$
print("------------------------------------")$
print("Geometric product")$
mtable1(%elements);
print("inner (dot) product ")$
mtable1i(%elements);
print("outer (wedge) product ")$
mtable1o(%elements);
print("outer")$
mtable2o();

inprotype : lc ;
mtable2i () ;

print("The function elements provides the basis elements")$
el : %elements;
print("In order to retrieve one element of the basis do")$
el[1];

print("The function cons add the scalar to the base")$
EE:cons(1, %elements );

print("construct vector a")$
ac:cvect(a);
print("construct vector b")$
bc:cvect(b);

print("------------------------------------")$
print("basic products")$
print("------------------------------------")$
adotb  :ac | bc$
awedgeb :ac & bc$
ab :ac . bc$
print("a dot b = ",'adotb," = ",adotb)$
print("a wedge b = ",'awedgeb," = ",awedgeb)$
print("a  b = ",'ab," = ",ab)$
print("expand ab ", expand(ab))$
```

```
print("--------------------------------------")$
print("GRADE FUNCTIONS")$
print("--------------------------------------")$
print("grade")$
gab : grade(ab);
print("grade 0"," = ",gab[1])$
print("grade 1"," = ",gab[2])$
print("grade 2"," = ",gab[3])$
print("The product of two vectors has given a scalar plus a bivector ")$

/*
print("If we want to retrieve the coefficient")$
print(" e.g. of the bivector it is sufficient to divide and simplify")$
cliffsimpall(gab[3]/el[3]);
*/

print("--------------------------------------")$
print("scalar part")$
scalarpart(ab);

print("--------------------------------------")$
print("vector part")$
vectorpart(ab);

print("--------------------------------------")$
print("grpart (2)")$
grpart(ab,2);

print("--------------------------------------")$
print("mvectorpart")$
mvectorpart(ab);

print("--------------------------------------")$
print("blade decomposition bdecompose")$
bdecompose(ab);
```

```
ac & bc;

print("-------------------------------------")$
print("Several operations are feasible")$
print("-------------------------------------")$

am : a[0]+ac+a[12]*el[3];
print("blade decomposition of the multivector")$
bdecompose(am);

print("ctranspose")$
ctranspose(ab);
print("creverse")$
creverse(ab);
print("cinvolve")$
cinvolve(ab);
print("cconjugate")$
cconjugate(ab);
print("cnorm")$
cnorm(ab);
print("cinv")$
cinv(ab);

print("-------------------------------------")$
print("INVOLUTIONS")$
print("-------------------------------------")$

print("creverse")$
creverse(am);

print("dotreverse")$
dotreverse(am);

print("cinvolve")$
cinvolve(am);
print("cconjugate")$
cconjugate(am);
```

```
print("dotconjugate")$
dotconjugate(am);

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

V.2 Geometric algebra of 3D Euclidean space

The Clifford algebra $Cl(\mathbf{R}^3) = Cl(3,0)$ is probably the most thoroughly studied and applied GA. In physics it is also known as *Pauli algebra*, since Pauli's spin matrices provide a 2×2 matrix representation. This shows how GA unifies *classical* with *quantum* mechanics.
Given an orthonormal vector basis $\{e_1, e_2, e_3\}$ of $\mathbf{R}^3$, the eight-dimensional ($2^3 = 8$) Clifford algebra $Cl(\mathbf{R}^3) = Cl(3,0)$ has a basis of one scalar, three vectors, three bivectors and one trivector

$$\{1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123}\}, \tag{1.197}$$

where as before $e_{23} = e_2 e_3, e_{123} = e_1 e_2 e_3$, etc. All basis bivectors square to -1, and the product of two basis bivectors gives the third

$$e_{23} e_{31} = e_{21} = -e_{12}, \text{ etc.} \tag{1.198}$$

Therefore the even subalgebra $Cl^+(3,0)$ with basis[1] $\{1, -e_{23}, -e_{31}, -e_{12}\}$ is indeed found to be isomorphic to quaternions $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. This isomorphism is not incidental. As we have learned already for $Cl(2,0)$, also in $Cl(3,0)$, the even subalgebra is the algebra of rotors (rotation operators) or spinors, and describes rotations in the same efficient way as do quaternions [30]. We therefore gain a *real geometric* interpretation of quaternions, as the oriented bi-vector side faces of a unit cube, with edge vectors $\{e_1, e_2, e_3\}$.
In $Cl(3,0)$ a reflection at a plane (=hyperplane) is specified by the plane's normal vector $a \in \mathbf{R}^3$

$$x' = -a^{-1} x a, \tag{1.199}$$

the proof is identical to the one in (1.194) for $Cl(2,0)$. The combination of two such reflections leads to a rotation by $\alpha = 2\theta_{a,b}$

$$\begin{aligned} x'' &= R^{-1} x R, \\ R &= ab = |a||b| e^{\theta_{a,b} \mathbf{i}_{a,b}} = |a||b| e^{\frac{1}{2} \alpha \mathbf{i}_{a,b}}, \end{aligned} \tag{1.200}$$

[1] The minus signs are only chosen, to make the product of two bivectors identical to the third, and not minus the third.

where $\mathbf{i}_{a,b} = a \wedge b/(|a \wedge b|)$ specifies the oriented unit bivector of the plane spanned by $a, b \in \mathbf{R}^3$. The unit trivector $i = e_{123}$ also squares to -1

$$\begin{aligned} i^2 &= e_1 e_2 e_3 e_1 e_2 e_3 = -e_1 e_2 e_1 e_3 e_2 e_3 \\ &= e_1 e_2 e_1 e_2 e_3 e_3 = (e_1 e_2)^2 (e_3)^2 = -1, \end{aligned} \tag{1.201}$$

where we only used that the permutation of two orthogonal vectors in the geometric product produces a minus sign. Hence $i^{-1} = -i$. We further find, that i commutes with every vector, e.g.

$$\begin{aligned} e_1 i &= e_1 e_1 e_2 e_3 = e_{23}, \\ i e_1 &= e_1 e_2 e_3 e_1 = -e_1 e_2 e_1 e_3 = e_1 e_1 e_2 e_3 = e_{23}, \end{aligned} \tag{1.202}$$

and the like for $e_2 i = i e_2$, $e_3 i = i e_3$. If i commutes with every vector, it also commutes with every bivector $a \wedge b = \frac{1}{2}(ab - ba)$, hence i commutes with every element of $Cl(3,0)$. i changes bivectors into orthogonal vectors

$$e_{23} i = e_2 e_3 e_1 e_2 e_3 = e_1 e_{23}^2 = -e_1\,, \text{ etc.} \tag{1.203}$$

Yet writing the basis in the simple product form (1.197), fully preserves the *geometric interpretation* in terms of scalars, vectors, bivectors and trivectors, and allows to *reduce* all products to elementary geometric products of basis vectors.

Table 1.7. Multiplication table of Clifford algebra $Cl(3,0)$ of Euclidean 3D space $\mathbf{R}^3$.

	1	e_1	e_2	e_3	e_{23}	e_{31}	e_{12}	e_{123}
1	1	e_1	e_2	e_3	e_{23}	e_{31}	e_{12}	e_{123}
e_1	e_1	1	e_{12}	$-e_{31}$	e_{123}	$-e_3$	e_2	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	e_3	e_{123}	$-e_1$	e_{31}
e_3	e_3	e_{31}	$-e_{23}$	1	$-e_2$	e_1	e_{123}	e_{12}
e_{23}	e_{23}	e_{123}	$-e_3$	e_2	-1	$-e_{12}$	e_{31}	$-e_1$
e_{31}	e_{31}	e_3	e_{123}	$-e_1$	e_{12}	-1	$-e_{23}$	$-e_2$
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	$-e_{31}$	e_{23}	-1	$-e_3$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	$-e_1$	$-e_2$	$-e_3$	-1

Multiplication table and subalgebras of $Cl(3,0)$

For the full multiplication table of $Cl(3,0)$ we still need the geometric products of vectors and bivectors. By changing labels in Table 1.6 ($1 \leftrightarrow 3$ or $2 \leftrightarrow 3$), we get that

$$e_2 e_{23} = -e_{23} e_2 = e_3,$$
$$e_3 e_{23} = -e_{23} e_3 = -e_2 \tag{1.204}$$
$$e_1 e_{31} = -e_{31} e_1 = -e_3,$$
$$e_3 e_{31} = -e_{31} e_3 = e_1, \tag{1.205}$$

which shows that in general a vector and a bivector, which includes the vector, anti-commute. The products of a vector with its orthogonal bivector always gives the trivector i

$$e_1 e_{23} = e_{23} e_1 = i, \quad e_2 e_{31} = e_{31} e_2 = i,$$
$$e_3 e_{12} = e_{12} e_3 = i, \tag{1.206}$$

which also shows that in general vectors and orthogonal bivectors necessarily commute. Commutation relationships therefore clearly depend on both *orthogonality* properties and on the *grades* of the factors, which can frequently be exploited for computations even without the explicit use of coordinates.

Table 1.7 gives the *multiplication table* of $Cl(3,0)$. The elements on the left most column are to be multiplied from the left in the geometric product with the elements in the top row. Every subtable of Table 1.7, that is closed under the geometric product represents a *subalgebra* of $Cl(3,0)$.

The grade structure of $Cl(3,0)$ and duality

A general multivector in $Cl(3,0)$, can be represented as

$$M = m_0 + m_1 e_1 + m_2 e_2 + m_3 e_3 + m_{23} e_{23} + m_{31} e_{31} + m_{12} e_{12}$$
$$+ m_{123} e_{123}, \quad m_0, \ldots, m_{123} \in \mathbf{R}. \tag{1.207}$$

We have a scalar part $\langle M\rangle_0$ of grade 0, a vector part $\langle M\rangle_1$ of grade 1, a bivector part $\langle M\rangle_2$ of grade 2, and a trivector part $\langle M\rangle_3$ of grade 3

$$\begin{aligned} M &= \langle M\rangle_0 + \langle M\rangle_1 + \langle M\rangle_2 + \langle M\rangle_3, \\ \langle M\rangle_0 &= m_0, \quad \langle M\rangle_1 = m_1 e_1 + m_2 e_2 + m_3 e_3, \\ \langle M\rangle_2 &= m_{23} e_{23} + m_{31} e_{31} + m_{12} e_{12}, \quad \langle M\rangle_3 = m_{123} e_{123}. \end{aligned} \tag{1.208}$$

The set of all grade k elements, $0 \le k \le 3$, is denoted $Cl^k(3,0)$.

The multiplication table of $Cl(3,0)$, Table 1.7, reveals that multiplication with i (or $i^{-1} = -i$) consistently changes an element of grade k, $0 \le k \le 3$, into an element of grade $3-k$, i.e. scalars to trivectors (also called pseudoscalars) and vectors to bivectors, and vice versa.

An example of computations involving $Cl(3,0)$ are in the following listing

`wxm/3D_multivector.wxm`

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 15.04.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$

pathmaxima:file_search_maxima$

file_search_maxima;
maxima_userdir;

kill(all)$

load("clifford.mac");
load("cliffordan.mac");

if get('clifford,'version)=false then load("clifford")$
if get('cliffordan,'version)=false then load("cliffordan")$
print("Cliford implements Clifford algebra for Maxima.")$

print("--------------------------------------")$
print("Program Starts")$
print("--------------------------------------")$

print("Construct the Clifford basis providing the symbol and ")$
print("space dimensionality")$
print("we select a dimension 3 with elements which squares to")$
clifford(e, 3);

print("The space is formed by ")$
EE:cons(1, %elements );

print("Table of direct products")$
mtable1(%elements);

print("Table of external product")$
mtable1o(%elements);
```

```
print("The elements are recovered as")$
EE[1]; EE[2]; EE[3]; EE[4];
EE[5]; EE[6]; EE[7]; EE[8];

declare([%alpha,%beta,v,t],scalar);

print("The function elements provides the basis elements")$
el : %elements;

print("Construct the following vectors")$
ac:cvect(a,[x,y,z]);
Ac:cvect(A,[x,y,z]);
bc:cvect(b,[x,y,z]);
Bc:cvect(B,[x,y,z]);
cc:cvect(c,[x,y,z]);

print("multivectors have the form:")$
multi1 : %alpha + ac + %iv . Ac + t *%iv, expand, dotsimpc;
multi2 : %beta + bc + %iv . Bc + v *%iv, expand, dotsimpc;

/* bivector multiplication only */
print("Let us now consider two Bivectors")$
multi1 : %iv . Ac , expand, dotsimpc;
multi2 : %iv . Bc , expand, dotsimpc;

/*
multi1 . multi2 + multi2 . multi1, expand, dotsimpc;
*/
print("Form the product of two bivectors")$
mm : multi1 . multi2, expand, dotsimpc;

print("Grades")$
mc0 : grpart(mm,0);
mc0 : grpart(mm,1);
mc0 : grpart(mm,2);
mc0 : grpart(mm,3);
```

```
print("Now consider two vectors")$
ac; bc;
print("Their product is")$
mc : ac.bc, expand, dotsimpc;
print("Grades")$
gela : grade(mc);
mc0 : grpart(mc,0)$
mc1 : grpart(mc,1)$
mc2 : grpart(mc,2)$
mc3 : grpart(mc,3)$
print("grade 0 ",mc0)$
print("grade 1 ",mc1)$
print("grade 2 ",mc2)$
print("grade 3 ",mc3)$


cliffsimpall(mc2);

print("components in ",el[4])$
mc2 & el[3]/%iv, expand, dotsimpc  ; /* looking at plane */

print("components in ",el[5])$
- mc2 & el[2]/%iv, expand, dotsimpc  ; /* looking at plane */

print("components in ",el[6])$
 mc2 & el[1]/%iv, expand, dotsimpc  ; /* looking at plane */
/* */
print("--------------------------------")$
print("Proving relevant identities")$
print("--------------------------------")$

print("Performing ac & (%iv . Bc) wedge product")$
ac & (%iv . Bc), expand, dotsimpc;
print("The external product of a vector a with a bivector %iv . B is %i a dot B")$
print("Commutative")$
print("ac & (%iv . Bc) = (%iv . Bc)& ac")$
```

```
ac & (%iv . Bc)- (%iv . Bc) & ac, expand, dotsimpc;



print("--------------------------------")$
print("performing ac | (%iv . Bc) dot product")$
ac | (%iv . Bc), expand, dotsimpc;
print("The dot product of a vector a with a bivector %iv . B is -
a cross B = %i a ^ B")$
print("It should be expected to be anticommutative, as the cross product")$
ac | (%iv . Bc) + (%iv . Bc) | ac, expand, dotsimpc;

print("--------------------------------")$
print("Performing ac . (%iv.Bc)")$
acBc : ac . (%iv.Bc), expand, dotsimpc;
grade(acBc);
print("ac . (%iv.Bc) = ac | (%iv . Bc) + ac & (%iv . Bc)")$
print("--------------------------------")$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

V.3 The Telegrapher's equations: an example of $Cl(1,1)$

Let us consider the voltage V and the current I along a transmission line in the x direction. The telegrapher's equations, for a lossless line, are

$$\begin{aligned} \partial_x V &= -L\,\partial_t I \\ \partial_x I &= -C\,\partial_t V \end{aligned} \tag{1.209}$$

where L and C are the inductances and capacitances per unit length. It is convenient to introduce the velocity v and the impedance η defined as

$$\begin{aligned} v &= \frac{1}{\sqrt{LC}} \\ \eta &= \sqrt{\frac{L}{C}} \end{aligned} \tag{1.210}$$

or, equivalently

$$\begin{aligned} L &= \frac{\eta}{v} \\ C &= \frac{1}{\eta v}\,. \end{aligned} \tag{1.211}$$

By substituting (1.211) into (1.212) we get

$$\begin{aligned} \partial_x V &= -\frac{1}{v}\,\partial_t\,(\eta\, I) \\ \partial_x\,(\eta\, I) &= -\frac{1}{v}\,\partial_t V \end{aligned} \tag{1.212}$$

It is also convenient to introduce the two variables x_0, x_1 as

$$\begin{aligned} x_0 &= v t \\ x_1 &= x \end{aligned} \tag{1.213}$$

so that

$$\begin{aligned}\frac{1}{v}\partial_t &= \frac{\partial}{\partial x_0} = \partial_0 \\ \partial_x &= \frac{\partial}{\partial x_1} = \partial_1 \,.\end{aligned} \tag{1.214}$$

The equations in (1.214) can be written in matrix form as

$$\begin{pmatrix} \partial_0 & \partial_1 \\ \partial_1 & \partial_0 \end{pmatrix} \begin{pmatrix} V \\ \eta I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{1.215}$$

or, equivalently, by changing the sign in the second row, as

$$\begin{pmatrix} \partial_0 & \partial_1 \\ -\partial_1 & -\partial_0 \end{pmatrix} \begin{pmatrix} V \\ \eta I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{1.216}$$

An example of $Cl(1,1)$

By making use of Pauli matrices we can therefore write the telegrapher equations in (1.216) as

$$(\sigma_3\partial_0 + i\,\sigma_2\partial_1)\,\psi = 0 \tag{1.217}$$

where we have introduced the quantity ψ defined as

$$\psi = \begin{pmatrix} V \\ \eta I \end{pmatrix} . \tag{1.218}$$

It is convenient, when the time variable is considered as in this case, to denote the elements of the Clifford basis starting from zero instead of one. We can now identify the elements of the basis $\{e_0, e_1\}$ as

$$\begin{aligned} e_0 &= \sigma_3 \\ e_1 &= i\,\sigma_2 \end{aligned} \tag{1.219}$$

and note that

$$
\begin{aligned}
e_0^2 &= \sigma_3\sigma_3 = 1\,\sigma_0 \\
e_1^2 &= -\sigma_2\sigma_2 = -1\,\sigma_0 \\
e_0 e_1 &= \sigma_3\, i\,\sigma_2 = -i\,\sigma_2\,\sigma_3 = -e_1 e_0\,.
\end{aligned}
\tag{1.220}
$$

we have therefore realized an example of $Cl(1,1)$. Therefore, the Clifford algebra $Cl(1,1)$ with the identification of the basis as in (1.219), is well suited to describe the telegrapher's equation. The telegrapher's equation, in a geometric algebra form, is:

$$
(e_0\partial_0 + e_1\partial_1)\,\psi = 0 \tag{1.221}
$$

Naturally, since e_0 squares to $1\sigma_0$ and e_1 squares to $-1\sigma_0$ and they anticommute we also have

$$
(e_0\partial_0 + e_1\partial_1)\,(e_0\partial_0 + e_1\partial_1) = \left(\partial_0^2 - \partial_1^2\right)\sigma_0 \tag{1.222}
$$

which provides the operator of the wave equation

$$
\left(\partial_0^2 - \partial_1^2\right)\sigma_0\psi = 0\,. \tag{1.223}
$$

Conventional procedure compared to GA

The conventional procedure to find the wave equation corresponding to (1.223) is the following. One starts from

$$
\partial_0 V + \partial_1(\eta I) = 0 \tag{1.224}
$$

$$
-\partial_1 V - \partial_0(\eta I) = 0 \tag{1.225}
$$

perform a derivative of the first equation (1.224) w.r.t. ∂_0. Then perform a derivative of (1.225) w.r.t. ∂_1 and then substitute $\partial_0\partial_1(\eta I)$ so as to obtain the equation in V. Then repeat again the procedure for obtaining the other second order equation in I. With (1.222) in just one passage we have obtained (1.223)!

Equivalent equations

As an alternative we could have considered (1.215) and write it in terms of Pauli matrices as:

$$(\sigma_0\partial_0 + \sigma_1\partial_1)\,\psi = 0\,. \tag{1.226}$$

Note, however, that the σ_0 matrix is not anticommutative and therefore cannot be used to create the geometric algebra basis. Nonetheless, it is still feasible to obtain the operator of the wave equation in the following way:

$$(\sigma_0\partial_0 + \sigma_1\partial_1)(\sigma_0\partial_0 - \sigma_1\partial_1) = \partial_0^2 - \partial_1^2\,. \tag{1.227}$$

But there is one more systematic way to generate Dirac–like equation when σ_0 is present. The expression appearing in (1.226) can be transformed without needing to change the sign at one equation, as we did before. In fact, we can multiply (1.226) by one of the sigma not appearing in the equation (therefore either σ_2 or σ_3) and obtain a different equation composed exclusively by anti–commuting matrices.
As an example, by pre mutltiplying (1.226) with σ_3 one obtains

$$\sigma_3\,(\sigma_0\partial_0 + \sigma_1\partial_1)\,\psi = (\sigma_3\partial_0 + i\sigma_2\partial_1)\,\psi\,. \tag{1.228}$$

i.e. (1.217).
Pre and post multiplication of (1.226) with σ_2 leads to other possible equations as

$$\begin{aligned}\sigma_2\,(\sigma_0\partial_0 + \sigma_1\partial_1) &= \sigma_2\partial_0 - i\sigma_3\partial_1\\ (\sigma_0\partial_0 + \sigma_1\partial_1)\,\sigma_2 &= \sigma_2\partial_0 + i\sigma_3\partial_1\,.\end{aligned} \tag{1.229}$$

It is left as an exercise to perform the following computation:

$$\frac{1}{2}\,(\sigma_0 + i\,\sigma_2)\,(\sigma_2\partial_0 + i\,\sigma_3\partial_1)\,(\sigma_1 + \sigma_3) = i\,\sigma_1\partial_0 - \sigma_2\partial_1 \tag{1.230}$$

and to recognize that the result is an anti–diagonal matrix, thus leading to two separated problems. Similarly, if an expression is formed only by employing σ_0 and σ_3 will lead again to two separated problems. The technique to obtain such diagonalization is called Weyl decomposition.

V.4 The Weyl decomposition

Let us introduce the quantities a, b defined as

$$\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \psi = \begin{pmatrix} V \\ \eta I \end{pmatrix} . \tag{1.231}$$

By using (1.231) in (1.216) we obtain the following two equations

$$\partial_0 (a+b) + \partial_1 (a-b) = 0 \tag{1.232}$$

$$-\partial_1 (a+b) - \partial_0 (a-b) = 0 . \tag{1.233}$$

By summing and subtracting (1.232) and (1.233) we obtain two independent expressions as

$$\begin{aligned} \partial_0 a + \partial_1 a &= 0 \\ \partial_0 b - \partial_1 b &= 0 \end{aligned} \tag{1.234}$$

or, in matrix form,

$$\begin{pmatrix} \partial_0 + \partial_1 & 0 \\ 0 & \partial_0 - \partial_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 . \tag{1.235}$$

The a and b correspond to progressive and regressive waves, respectively. What we have obtained here is of *noticeable importance*. If we consider our original problem (1.216) we have a *system of two coupled equations of the first order*. By contrast, when considering (1.235) we have two *independent* first order equations. Therefore, the traveling waves (both progressive and regressive) are the natural basis for having uncoupled equations!

Alternatively, we can also proceed in a different manner for obtaining the same result. We can introduce the matrix A defined as

$$A = \frac{1}{\sqrt{2}}(\sigma_1 + \sigma_3) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{1.236}$$

which has the following properties

$$A = A^T = A^{-1}. \tag{1.237}$$

Since the matrix A is equal to its inverse it squares to the identity matrix.

Similarly, we can introduce the matrix B defined as

$$B = A\sigma_1 = \frac{1}{\sqrt{2}}(\sigma_0 + i\,\sigma_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \tag{1.238}$$

Since the matrix B contains an imaginary part we can also introduce

$$B^\dagger = \sigma_1 A = \frac{1}{\sqrt{2}}(\sigma_0 - i\,\sigma_2) \tag{1.239}$$

which is the inverse of B, giving

$$B^\dagger B = I_2. \tag{1.240}$$

We note that the following equality holds:

$$\begin{pmatrix} \partial_0 & \partial_1 \\ -\partial_1 & -\partial_0 \end{pmatrix} = B \begin{pmatrix} \partial_0 + \partial_1 & 0 \\ 0 & \partial_0 - \partial_1 \end{pmatrix} A. \tag{1.241}$$

By premultiplying both sides of the above equation with $B^\dagger$ and post multiplying times A we get

$$B^\dagger \begin{pmatrix} \partial_0 & \partial_1 \\ -\partial_1 & -\partial_0 \end{pmatrix} A = \begin{pmatrix} \partial_0 + \partial_1 & 0 \\ 0 & \partial_0 - \partial_1 \end{pmatrix}. \tag{1.242}$$

It is also easy to show that we have:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\sqrt{2}} A \begin{pmatrix} a+b \\ a-b \end{pmatrix}. \tag{1.243}$$

or, equivalently

$$\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \sqrt{2}A \begin{pmatrix} a \\ b \end{pmatrix} \tag{1.244}$$

In the following, the Weyl decomposition will be applied also to Maxwell's equations expressed in Dirac form. A code fraction reporting the above theory is given next.

wxm/TL_Weyl.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02);

print("original matrix")$
delta :matrix ([%delta[0],%delta[1]],[-%delta[0],-%delta[1]]);

print("diagonal matrix dd")$
dd : matrix ([%delta[0]+%delta[1],0],[0,%delta[0]-%delta[1]]);

print("matrix A = 1/sqrt(2) * (%sigma[1]+ %sigma[3])")$
A : 1/sqrt(2) *(%sigma[1]+ %sigma[3]);
print("A is equal to its inverse")$
A.A;

print("matrix B = 1/sqrt(2) * (%sigma[0] + %i * %sigma[2])")$
B : 1/sqrt(2) *(%sigma[0] + %i * %sigma[2]);
print("B dagger")$
Bd : 1/sqrt(2) *(%sigma[0] - %i * %sigma[2]);
print("Bdagger . B gives identity")$
Bd.B ;

print("matrix a+b, a-b")$
apmb : matrix([a+b],[a-b]);

print("1/sqrt(2)*ratsimp(A . apmb)")$
ab : 1/sqrt(2)*ratsimp(A . apmb);

print("sqrt(2)*A . ab")$
ratsimp(sqrt(2)*A . ab);

print("B . dd . A")$
ratsimp(B . dd . A);
/* [wxMaxima: input   end   ] */
```

```
/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

V.5 Time–harmonic solution

We have seen that propagation along the transmission line can be described by the equations (1.235) here repeated for convenience:

$$\begin{pmatrix} \partial_0 + \partial_1 & 0 \\ 0 & \partial_0 - \partial_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0\,. \tag{1.245}$$

The a and b correspond to progressive and regressive waves, respectively. In order to find a solution it is advantageous to apply separation of variables and to consider a time–harmonic solution.

Separation of variables

The solution for the propagating wave a can be written in terms of two different functions α_0, α_1 as

$$a(x_0, x_1) = \alpha_0(x_0)\,\alpha_1(x_1) \tag{1.246}$$

In addition we can assume a time–harmonic behavior for the α_0 part. In particular, it is typically chosen the following expansion

$$\alpha_0(x_0) = A_0 e^{j\omega t} = A_0 e^{jkx_0} \tag{1.247}$$

with $k = \omega/v$ and x_0 defined in (1.213). It is noted that the derivative w.r.t. x_0 of a gives

$$\partial_0 a(x_0, x_1) = jk\,\alpha_0(x_0)\alpha_1(x_1)\,. \tag{1.248}$$

and therefore the equation becomes

$$jk\,\alpha_0(x_0)\alpha_1(x_1) + \partial_1\,\alpha_0(x_0)\alpha_1(x_1) = 0\,. \tag{1.249}$$

Since $\alpha_0(x_0)$ is present in all members can be eliminated, obtaining the following equation

$$\partial_1\,\alpha_1(x_1) = -jk\,\alpha_1(x_1) \tag{1.250}$$

with solution

$$\alpha_1(x_1) = A_1 e^{-jkx_1}\,. \tag{1.251}$$

Therefore the wave a can be written as

$$a(x_0,x_1) = A_0 e^{jkx_0} A_1 e^{-jkx_1} = A e^{jk(x_0-x_1)} \tag{1.252}$$

with $A = A_0 A_1$. This solution represent a progressive wave. By a similar procedure the solution for b can be obtained as

$$b(x_0,x_1) = B e^{jk(x_0+x_1)}\,. \tag{1.253}$$

V.6 Interface between two media

Let us consider the case when we have two different transmission lines, the one on the left with an impinging wave denoted by a_0 and reference impedance η_0. The medium on the right has an impedance η_1. In terms of waves the discontinuity gives rise to a progressive wave in medium 1 (on the right) a_1 and to a reflected wave b_0 in medium 0 on the left. By placing the discontinuity at $x_1 = 0$ we have

$$\begin{aligned} V_0 &= a_0 + b_0 \\ \eta_0 I_0 &= a_0 - b_0 \end{aligned} \tag{1.254}$$

and

$$\begin{aligned} V_1 &= a_1 \\ \eta_1 I_1 &= a_1\,. \end{aligned} \tag{1.255}$$

The continuity conditions tell us that

$$\begin{aligned} V_1 &= V_0 \\ I_1 &= I_0\,. \end{aligned} \tag{1.256}$$

After solving for a_1, b_0 we get

$$\begin{aligned} b_0 &= \Gamma a_0 \\ a_1 &= \tau a_0 \end{aligned} \tag{1.257}$$

with

$$\begin{aligned} \Gamma &= \frac{\eta_1 - \eta_0}{\eta_1 + \eta_0} \\ \tau &= \frac{2\eta_1}{\eta_1 + \eta_0} \end{aligned} \tag{1.258}$$

as shown in the code reported next.

wxm/bc1.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$

/* bc1 */
/* we  find the boundary conditions for a wave */

print("Let us consider a transmission line with an incident wave", a[0], " in medium 0")$
print("when a discontinuity is present a reflected wave ", b[0]," is generated")$
print("and also a transmitted wave", a[1], " in medium 1 is generated. ")$

print("")$
print("In medium 0")$
print("The incident and reflected waves in medium 0 are related to valtages as")$

print('V[0]," = ", a[0] + b[0])$
print("and the current is" )$
print(%eta[0]*I[0]," = ",a[0] - b[0])$

print("")$
print("In medium 1")$
print("The transmitted waves in medium 1 are related to valtages as")$

print('V[1]," = ", a[1] )$
print("and the current is" )$
print(%eta[1]*I[1]," = ",a[1] )$

print("")$
print("By equating the voltages and currents at the interface we obtain")$
print('V[0], " = ",'V[1])$
print('I[0], " = ",I[1])$

print("which gives the following equations")$

eq1 : a[1] - b[0] -a[0];
```

```
eq2 : %eta[0]* a[1] + %eta[1] * b[0] -a[0]* %eta[1];

print("with solutions")$
sol : solve([eq1,eq2],[a[1],b[0]]);
print("Transmission coefficient")$
T : rhs(sol[1][1])/a[0];
print("Reflection coefficient")$
%Gamma : rhs(sol[1][2])/a[0];
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

VI Pauli matrices for cylindrical and spherical coordinates

VI.1 Coordinate systems

Coordinates

In Electromagnetic field engineering, whenever possible, we orient rectangular (x,y,z), circular cylindrical (ρ,ϕ,z), and spherical coordinates as shown in Fig. 1.17 (to be completed). Rules for transformation among coordinates are then for the cylindrical to rectangular

$$\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned} \tag{1.259}$$

and for the spherical to rectangular

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\,. \end{aligned} \tag{1.260}$$

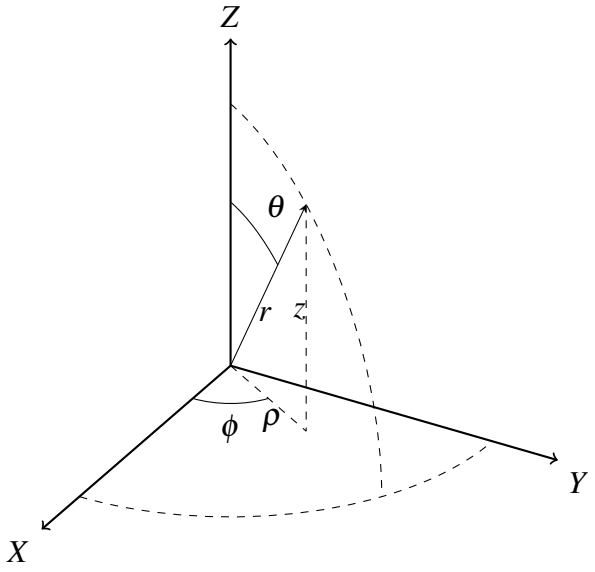


Fig. 1.17. Convention for coordinate orientation.

When passing from rectangular to cylindrical/spherical coordinates the following transformations apply:

$$\begin{aligned} \rho &= \sqrt{x^2+y^2} = r\sin\theta \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= r\cos\theta \end{aligned} \tag{1.261}$$

and

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} = \sqrt{\rho^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) = \tan^{-1}\left(\frac{\rho}{z}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right). \end{aligned} \tag{1.262}$$

The following files perform the transformations both numerically and symbolically.

```
wxm/coordinate_transf/r2c.wxm
wxm/coordinate_transf/c2r.wxm
wxm/coordinate_transf/r2s.wxm
wxm/coordinate_transf/s2r.wxm
wxm/coordinate_transf/c2s.wxm
wxm/coordinate_transf/s2c.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$

/* numerical
fpprintprec:5$
ratprint : false$
x :5.0 ;
y :2.0 ;
```

```
z :4.6 ;
*/

/* rectangular to cylindrical */
%rho : sqrt(x^2+y^2)$
phi : atan2(y,x)$
z :z$
print("rectangular to cylindrical")$
print('%rho, " = ",%rho)$
print('phi, " = ",phi)$
print('z, " = ",z)$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$
/* c2r.wxm */

/* numerical
fpprintprec:5$
ratprint : false$
%rho :5.0 ;
phi : 3.14/4.0 ;
z :4.6 ;
 */

/* cylindrical to rectangular */
x : %rho * cos(phi)$
y : %rho * sin(phi)$
z : z$

print("cylindrical to rectangular")$
print('x, " = ",x)$
print('y, " = ",y)$
print('z, " = ",z)$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$
/* r2s.wxm */

/* numerical
fpprintprec:5$
ratprint : false$
x :5.0 ;
y :2.0 ;
z :4.6 ;
*/

r : sqrt(x^2+y^2+z^2)$
theta : atan2(sqrt(x^2 +y^2),z)$
phi : atan2(y,x)$

print("rectangular to spherical")$
print('r, " = ",r)$
print('theta, " = ",theta)$
print('phi, " = ",phi)$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$
/* r2s.wxm */

/* numerical
fpprintprec:5$
ratprint : false$
r :5.0 ;
theta :float(%pi)/4 ;
phi :4.6 ;
*/

x : r * sin(theta) * cos(phi)$
y : r * sin(theta) * sin(phi)$
z : r *cos(theta)$

print("spherical to rectangular")$
print('x, " = ",x)$
print('y, " = ",y)$
print('z, " = ",z)$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$
/* c2s.wxm */

/* numerical
fpprintprec:5$
ratprint : false$
%rho :5.0 ;
phi :4.6 ;
z :2.0;
*/

/* cylindrical to spherical */
r : sqrt(%rho^2 + z^2)$
theta  :atan2(%rho,z) $
phi : phi$

print("cylindrical to spherical")$
print('r, " = ",r)$
print('theta, " = ",theta)$
print('phi, " = ",phi)$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$
/* s2c.wxm */

/* numerical */
fpprintprec:5$
ratprint : false$
r :5.0 ;
theta :float(%pi)/4 ;
phi :0.6 ;



%rho : r * sin(theta) $
phi : phi$
z : r *cos(theta)$

print("spherical to cylindrical")$
print('%rho, " = ",%rho)$
print('phi, " = ",phi)$
print('z, " = ",z)$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Unit vectors

The coordinate unit vectors are defined as vectors of unit length pointing along coordinate lines in the direction of increasing coordinate variables. It is left as an exercise to draw them in the three coordinate systems. The following rules are applicable to transformations among unit coordinate vectors.

In order to introduce the Pauli matrices for cylindrical and spherical coordinate systems we only need the expression of the unit vectors in terms of the cartesian ones.

Transformations from rectangular to cylindrical (insert figures):

$$\begin{aligned}\mathbf{u}_\rho &= \mathbf{u}_x \cos\phi + \mathbf{u}_y \sin\phi \\ \mathbf{u}_\phi &= -\mathbf{u}_x \sin\phi + \mathbf{u}_y \cos\phi\end{aligned}$$

Transformations from rectangular/cylindrical to spherical (insert figures):

$$\begin{aligned}\mathbf{u}_r &= \mathbf{u}_x \sin\theta\cos\phi + \mathbf{u}_y \sin\theta\sin\phi + \mathbf{u}_z \cos\theta \\ \mathbf{u}_\theta &= \mathbf{u}_x \cos\theta\cos\phi + \mathbf{u}_y \cos\theta\sin\phi - \mathbf{u}_z \sin\theta \\ \mathbf{u}_\phi &= -\mathbf{u}_x \sin\phi + \mathbf{u}_y \cos\phi\end{aligned} \tag{1.263}$$

Once the Pauli matrices for the cylindrical and spherical components have been found, the coefficients for all the possible transformations can be recovered in a direct algebraic way.

VI.2 Pauli matrices in Cylindrical coordinates

We have already introduced the Pauli matrices for the rectangular coordinate system σ_i, with $i = 0, \ldots, 3$. When expressing a vector can identify σ_1 with $\mathbf{u}_x$, σ_2 with $\mathbf{u}_y$ and σ_3 with $\mathbf{u}_z$. By using (1.284) we can write:

$$\begin{aligned}\mathbf{u}_\rho &= \mathbf{u}_x \cos\phi + \mathbf{u}_y \sin\phi \\ \sigma_\rho &= \cos\phi\,\sigma_1 + \sin\phi\,\sigma_2 \\ &= \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}.\end{aligned} \tag{1.264}$$

Similarly, for the unit vector along ϕ we can write

$$\begin{aligned}\mathbf{u}_\phi &= -\mathbf{u}_x \sin\phi + \mathbf{u}_y \cos\phi \\ \sigma_\phi &= -\sin\phi\,\sigma_1 + \cos\phi\,\sigma_2 \\ &= \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}. \end{aligned} \tag{1.265}$$

Naturally, for the z component we have that $\sigma_z = \sigma_3$. Summarizing, we have the following three matrices in cylindrical coordinates

$$\begin{aligned}\sigma_\rho &= \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix} \\ \sigma_\phi &= \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix} \\ \sigma_z &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{1.266}$$

While in rectangular coordinates the Pauli matrices are not dependent on the coordinate, we note that $\sigma_\rho = \sigma_\rho(\phi)$ and $\sigma_\phi = \sigma_\phi(\phi)$. Therefore we have

$$\begin{aligned}\partial_\phi \sigma_\rho &= \sigma_\phi \\ \partial_\phi \sigma_\phi &= -\sigma_\rho \end{aligned} \tag{1.267}$$

It is readily proved that the matrices $\sigma_\rho, \sigma_\phi, \sigma_z$ when multiplied by themselves give the identity matrix σ_0, their trace is null, their determinant is always -1 and their dot product is zero if they are not the same. In addition we have

$$\begin{aligned} i\sigma_z &= \sigma_\rho \sigma_\phi \\ i\sigma_\rho &= \sigma_\phi \sigma_z \\ i\sigma_\phi &= \sigma_z \sigma_\rho . \end{aligned} \tag{1.268}$$

Therefore the generic vector **A** can be expressed in cylindrical coordinates in terms of Pauli matrices as

$$\begin{aligned}\tilde{A} &= A_\rho \sigma_\rho + A_\phi \sigma_\phi + A_z \sigma_z \\ &= \begin{pmatrix} A_z & e^{-i\phi}\left(A_\rho - iA_\phi\right) \\ e^{i\phi}\left(A_\rho + iA_\phi\right) & -A_z \end{pmatrix}. \end{aligned} \tag{1.269}$$

It is noted that since the vector is the same when expressed in rectangular or cylindrical components we have the identity

$$A_\rho \sigma_\rho + A_\phi \sigma_\phi + A_z \sigma_z = A_x \sigma_1 + A_y \sigma_2 + A_z \sigma_3 . \tag{1.270}$$

By performing the inner product of the above expression with a selected sigma matrix we can recover the desired field component in terms of the other basis. As an example let us consider to retrieve the expression of A_ρ in terms of the rectangular components. It is sufficient to perform the inner productof both sides of (1.278) times σ_ρ to retrieve

$$\begin{aligned} A_\rho &= A_x \sigma_1 \cdot \sigma_\rho + A_y \sigma_2 \cdot \sigma_\rho + A_z \sigma_3 \cdot \sigma_\rho \\ &= A_x \cos\phi + A_y \sin\phi \end{aligned} \tag{1.271}$$

where we remind that performing the inner product corresponds to making the matrix product and taking half of the trace. It is left as an exercise for the reader to recover the results in the previous section by using Pauli matrices multiplication.

Example 4. Given the point P_1 with coordinates $(3,-4,3)$ and a vector $\mathbf{A} = \mathbf{x}_0 2 - \mathbf{y}_0 3 + \mathbf{z}_0 4$, defined in Cartesian coordinates, express P_1 and **A** in cylindrical coordinates and evaluate **A** in the point P_1. Solution: the point P_1 is described by the coordinates $\rho = \sqrt{x^2+y^2} = 5$ and the angle $\phi = 306.87$ (in degrees). The z component is the same. Let us now express the vector **A** in terms of Pauli matrices. We have

$$\tilde{A} = A_x \sigma_1 + A_y \sigma_2 + A_z \sigma_3 = \begin{pmatrix} 4 & 3i+2 \\ 2-3i & -4 \end{pmatrix} \tag{1.272}$$

Noticeably, we obtain the same representation also using cylindrical coordinates as

$$\tilde{A} = A_\rho\,\sigma_\rho + A_\phi\,\sigma_\phi + A_z\,\sigma_3 = \begin{pmatrix} 4 & 3i+2 \\ 2-3i & -4 \end{pmatrix}. \tag{1.273}$$

In order to retrieve the components in cylindrical coordinates it is sufficient to multiply for the considered sigma and take half of the trace. By doing so we get

$$\begin{aligned} A_\rho &= 3.60 \\ A_\phi &= -0.20 \\ A_z &= 4\,. \end{aligned}$$

The following code perform computations in cylindrical coordinates.

wxm/Pauli_cyl.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
/* Pauli_cyl_v2 */
print("------------------------------------------------------")$
print("Pauli Matrices Definition")$
print("------------------------------------------------------")$

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print('%sigma[0]," = ",%sigma[0])$
print('%sigma[1]," = ",%sigma[1])$
print('%sigma[2]," = ",%sigma[2])$
print('%sigma[3]," = ",%sigma[3])$

print("------------------------------------------------------")$
print("Obtaining Pauli matrices in cylindrical coordinates")$
print("------------------------------------------------------")$
print("Representation of the unit vector rho as")$
print(cos(phi)* '%sigma[1] + sin(phi)* '%sigma[2])$
rho : cos(phi)* %sigma[1] + sin(phi)* %sigma[2]$
rho : ratsimp(exponentialize(rho));

print("Representation of the unit vector \phi as")$
print(- sin(phi)* '%sigma[1] + cos(phi)* '%sigma[2])$
uphi : - sin(phi)* %sigma[1] + cos(phi)* %sigma[2]$
uphi : ratsimp(exponentialize(uphi));

print("------------------------------------------------------")$
print("Pauli Matrices Definition Cylindrical")$
print("------------------------------------------------------")$
```

```
%sigma[%rho] : matrix([0,%e^(-%i * phi)],[%e^(%i * phi),0])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$
%sigma[z] : matrix([1,0],[0,-1])$

print('%sigma[%rho],"  = ",%sigma[%rho])$
print('%sigma[phi],"  = ",%sigma[phi])$
print('%sigma[z],"  = ",%sigma[z])$

print("Squared matrices")$
%sigma[%rho] . %sigma[%rho];
%sigma[phi] . %sigma[phi];
%sigma[z] . %sigma[z];

print("Alternating rules")$
%sigma[%rho] . %sigma[phi] - %i * %sigma[z];
%sigma[phi] . %sigma[z] - %i * %sigma[%rho];
%sigma[z] . %sigma[%rho] - %i * %sigma[phi];

print("Determinants")$
determinant(%sigma[%rho]);
determinant(%sigma[phi]);
determinant(%sigma[z]);

print("vector in cylindrical coordinate")$
Acyl : factor(A[%rho]*%sigma[%rho]+A[phi]*%sigma[phi]+A[z]*%sigma[3]);

print("vector decompositions")$
print("obtaining A[rho] components")$

print("Dot multiplication with %sigma[%rho] allows to retrieve the rho component")$
sr1 :A[x] * trigsimp(%sigma[%rho] . %sigma[1])
   + A[y] * trigsimp(%sigma[%rho] . %sigma[2])
   + A[z] * trigsimp(%sigma[%rho] . %sigma[3])  ;
Arho : factor(realpart(1/2 * (sr1[1,1]+sr1[2,2])));

print("Dot multiplication with %sigma[%phi] allows to retrieve the phi component")$
sr1 : A[x] * trigsimp(%sigma[phi] . %sigma[1])
```

```
    + A[y] * trigsimp(%sigma[phi] . %sigma[2])
    + A[z] * trigsimp(%sigma[phi] . %sigma[3])  ;
Aphi : ratsimp(factor(realpart(1/2 * (sr1[1,1]+sr1[2,2]))));

sr1 : A[x] * trigsimp(%sigma[3] . %sigma[1])
    + A[y] * trigsimp(%sigma[3] . %sigma[2])
    + A[z] * trigsimp(%sigma[3] . %sigma[3])  ;
Az : ratsimp(factor(realpart(1/2 * (sr1[1,1]+sr1[2,2]))));

print("------------------------------------------------------------")$
print("Numerical example ")$
print("------------------------------------------------------------")$

fpprintprec :5$
ratprint : false$
print("For a given point P1")$
P1[x] : 3.0$
P1[y] : -4.0$
P1[z] : 3.0$

print("and a given vector A defined in cartesian coordinates")$
A[x] : 2.0$
A[y] : -3.0$
A[z] : 4.0$

print("------------------------------------------------------------")$
print("Transformation of coordinates !!!")$
print("------------------------------------------------------------")$

print("phi value for P1")$
phi : float(atan2(P1[y],P1[x]));

print("point P1 as column vector")$
P1xyz : matrix([P1[x]],[P1[y]],[P1[z]]);

print("vector A as column vector")$
Axyz : matrix([A[x]],[A[y]],[A[z]]);
```

```
print("From rectangular to cylindrical ...")$
print("Point P1 expressed in rectangular is transformed in point P1 in cylindrical")$

P1rho : sqrt(P1[x]^2+P1[y]^2);
P1phi : float(atan2(P1[y],P1[x]));
P1z : P1[z];

print("the phi component in degrees is",phi*180.0/float(%pi))$
print(" or equivalently ", 360.0 + phi*180.0/float(%pi))$

print("------------------------------------------------------------")$
print("Transformation of vector components !!!")$
print("------------------------------------------------------------")$

print("A is considered as a vector")$
print("transformation matrix from rect to cyl")$
r2c : matrix([cos(phi),sin(phi),0],[-sin(phi),cos(phi),0],[0,0,1]);

print("Vector in cylindrical coordinates")$
Arpz : r2c . Axyz;

print("------------------------------------------------------------")$
print("Using Pauli matrices")$
print("------------------------------------------------------------")$

print("Pauli vector in rectangular coordinates")$
/* Avect : matrix([A[x], A[y], A[z]]); */
Aprect : ratsimp(A[x] * %sigma[1] + A[y] * %sigma[2]+ A[z] * %sigma[3]);


print("Pauli matrices for the cylindrical coordinate system")$
print("They are dependent on the phi value!")$
%sigma[%rho] : matrix([0,%e^(-%i * phi)],[%e^(%i * phi),0]);
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0]);
%sigma[z] : matrix([1,0],[0,-1]);
```

```
print("write the A[%rho] , A[phi] , A[z] components from conventional analysis")$
Aarho : Arpz[1][1]$
Aaphi : Arpz[2][1]$
Aaz :   Arpz[3][1]$

print("Pauli vector in cylindrical coordinates")$
Apcyl : rectform(Aarho * %sigma[%rho] + Aaphi * %sigma[phi] + Aaz * %sigma[3]);
print("and the difference between Pauli matrix in cylindrical and in rectangular coordinates")$

(Aprect-Apcyl);

print("THEY ARE THE SAME !!!")$

print("How to derive the cylindrical components from the Pauli matrix?")$
print("In order to obtain A[rho] just multiply the Pauli matrix ")$
print("with %sigma[%rho] and take half of the trace")$
Apsr : rectform(Aprect . %sigma[%rho]);
Aprho : 1.0/2.0 *(Apsr[1,1]+Apsr[2,2]);

print("proceed similarly for the components along phi")$
Apsp : rectform(Aprect . %sigma[phi]);
App : 1.0/2.0 *(Apsp[1,1]+Apsp[2,2]);

print("and finally")$
Apsz : rectform(Aprect . %sigma[3]);
Apz : 1.0/2.0 *(Apsz[1,1]+Apsz[2,2]);

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

VI.3 Pauli matrices in Spherical coordinates

So far we have introduced the Pauli matrices for the rectangular and cylindrical coordinate systems. We can proceed in the same way that we have followed for the cylindrical coordinate system. By using (1.285) we can write:

$$\begin{aligned}\mathbf{u}_r &= \mathbf{u}_x \sin\theta\cos\phi + \mathbf{u}_y \sin\theta\sin\phi + \mathbf{u}_z \cos\theta \\ \sigma_r &= \sin\theta\cos\phi\,\sigma_1 + \sin\theta\sin\phi\,\sigma_2 + \cos\theta\,\sigma_3 \\ &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}.\end{aligned} \tag{1.274}$$

Similarly, for the unit vector along θ we can write

$$\begin{aligned}\mathbf{u}_\theta &= \mathbf{u}_x \cos\theta\cos\phi + \mathbf{u}_y \cos\theta\sin\phi - \mathbf{u}_z \sin\theta \\ \sigma_\theta &= \cos\theta\cos\phi\,\sigma_1 + \cos\theta\sin\phi\,\sigma_2 - \sin\theta\,\sigma_3 \\ &= \begin{pmatrix} -\sin\theta & e^{-i\phi}\cos\theta \\ e^{i\phi}\cos\theta & \sin\theta \end{pmatrix}.\end{aligned} \tag{1.275}$$

Not surprisingly for the ϕ coordinate the result is the same as in cylindrical coordinates. Summarizing, we have the following three matrices in spherical coordinates

$$\begin{aligned}\sigma_r &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} \\ \sigma_\theta &= \begin{pmatrix} -\sin\theta & e^{-i\phi}\cos\theta \\ e^{i\phi}\cos\theta & \sin\theta \end{pmatrix} \\ \sigma_\phi &= \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}.\end{aligned} \tag{1.276}$$

It is readily proved that the matrices $\sigma_r, \sigma_\theta, \sigma_\phi$ when multiplied by themselves give the identity matrix σ_0, their trace is null, their determinant is always -1 and their dot product is zero if they are not the same. In addition we have

$$
\begin{aligned}
i\,\sigma_\phi &= \sigma_r\sigma_\theta \\
i\,\sigma_r &= \sigma_\theta\sigma_\phi \\
i\,\sigma_\theta &= \sigma_\phi\sigma_r\,.
\end{aligned}
\tag{1.277}
$$

Therefore, the generic vector $\mathbf{A}$ can be expressed in spherical coordinates in terms of Pauli matrices as

$$
\begin{aligned}
\tilde{A} &= A_r\sigma_r + A_\theta\sigma_\theta + A_\phi\sigma_\phi \\
&= \begin{pmatrix} A_r\cos\theta - A_\theta\sin\theta & e^{-i\phi}\left(A_r\sin\theta - iA_\phi + A_\theta\cos\theta\right) \\ e^{i\phi}\left(A_r\sin\theta + iA_\phi + A_\theta\cos\theta\right) & -A_r\cos\theta + A_\theta\sin\theta \end{pmatrix}
\end{aligned}
\tag{1.278}
$$

It is noted that since the vector is the same when expressed in rectangular, cylindrical or spherical components we have the identity

$$
\begin{aligned}
\tilde{A} &= A_x\sigma_1 + A_y\sigma_2 + A_z\sigma_3 \\
&= A_\rho\sigma_\rho + A_\phi\sigma_\phi + A_z\sigma_z \\
&= A_r\sigma_r + A_\theta\sigma_\theta + A_\phi\sigma_\phi\,.
\end{aligned}
\tag{1.279}
$$

By performing the inner product of the above expression with a selected sigma matrix we can recover the desired field component in terms of the other basis. Thus all the transformation between vectors in different coordinate systems can be simply obtained by matrix multiplication and trace operation.

```
wxm/Pauli_sph_1.wxm
wxm/example_Pauli_sph.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$

print("------------------------------------------------------------")$
print("Pauli Matrices Definition")$
print("------------------------------------------------------------")$

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print('%sigma[0]," = ",%sigma[0])$
print('%sigma[1]," = ",%sigma[1])$
print('%sigma[2]," = ",%sigma[2])$
print('%sigma[3]," = ",%sigma[3])$

Axyz : A[x] * %sigma[1] + A[y] * %sigma[2]+ A[z] * %sigma[3]$
print("vector A in cartesian coordinates"," = ",Axyz)$

print("------------------------------------------------------------")$
print("Obtaining Pauli matrices in spherical coordinates")$
print("------------------------------------------------------------")$

print("Representation of the unit vector ur as")$
print(sin(%theta) * cos(phi)* '%sigma[1]
+ sin(%theta) * sin(phi)* '%sigma[2]
+ cos(%theta) * '%sigma[3])$

ur : sin(%theta) * cos(phi)* %sigma[1]
+ sin(%theta) * sin(phi)* %sigma[2]
+ cos(%theta) * %sigma[3]$

ur : factor(ur);
```

```
udst: ratsimp(exponentialize((ur[2,1])/sin(%theta)))$
ur[2,1] : udst*sin(%theta)$
udst: ratsimp(exponentialize((ur[1,2])/sin(%theta)))$
ur[1,2] : udst*sin(%theta)$
print("or")$
ur;

print("Representation of the unit vector utheta as")$
print(sin(%theta) * cos(phi)* '%sigma[1]
+ sin(%theta) * sin(phi)* '%sigma[2]
+ cos(%theta) * '%sigma[3])$

utheta : cos(%theta) * cos(phi)* %sigma[1]
+ cos(%theta) * sin(phi)* %sigma[2]
- sin(%theta) * %sigma[3]$

utheta : factor(utheta);
udst: ratsimp(exponentialize((utheta[2,1])/cos(%theta)))$
utheta[2,1] : udst*cos(%theta)$
udst: ratsimp(exponentialize((utheta[1,2])/cos(%theta)))$
utheta[1,2] : udst*cos(%theta)$
print("or")$
utheta;

print("Representation of the unit vector \phi as")$
print(- sin(phi)* '%sigma[1] + cos(phi)* '%sigma[2])$
uphi : - sin(phi)* %sigma[1] + cos(phi)* %sigma[2]$
uphi : ratsimp(exponentialize(uphi));

print("------------------------------------------------------------")$
print("Pauli Matrices Definition Spherical")$
print("------------------------------------------------------------")$

ct : cos(%theta)$
st : sin(%theta)$

%sigma[r] : matrix([ct,st * %e^(-%i * phi)],[st * %e^(%i * phi),-ct])$
```

```
%sigma[%theta] : matrix([-st,ct  * %e^(-%i * phi)],[ct * %e^(%i * phi),st])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$

print('%sigma[r],"  = ",%sigma[r])$
print('%sigma[%theta],"  = ",%sigma[%theta])$
print('%sigma[phi],"  = ",%sigma[phi])$

print("------------------------------------------------------------")$
print("vector in spherical coordinates")$
print("------------------------------------------------------------")$
Asph : ratsimp(factor(A[r]*%sigma[r]+A[%theta]*%sigma[%theta]+A[phi]*%sigma[phi]));

print("------------------------------------------------------------")$
print("properties")$
print("------------------------------------------------------------")$

print("Squared matrices")$
trigsimp(%sigma[r] . %sigma[r]);
trigsimp(%sigma[%theta] . %sigma[%theta]);
%sigma[phi] . %sigma[phi];

print("------------------------------------------------------------")$
print("Alternating rules")$
trigsimp(%sigma[r] . %sigma[%theta] - %i * %sigma[phi]);
%sigma[%theta] . %sigma[phi] - %i * %sigma[r];
%sigma[phi] . %sigma[r] - %i * %sigma[%theta];
print("%sigma[r] . %sigma[%theta] = %i * %sigma[phi]")$
print("%sigma[%theta] . %sigma[phi] = %i * %sigma[r]")$
print("%sigma[phi] . %sigma[r] = %i * %sigma[%theta]")$

print("------------------------------------------------------------")$
print("Determinants")$
trigsimp(determinant(%sigma[r]));
trigsimp(determinant(%sigma[%theta]));
determinant(%sigma[phi]);

/*
```

```
print("------------------------------------------------------------")$
print("Transformation matrices")$
print("------------------------------------------------------------")$
%sigma[r] . %sigma[1]$
%sigma[%theta] . %sigma[2]$
%sigma[phi] . %sigma[3]$
*/

print("------------------------------------------------------------")$
print("vector decompositions")$
print("------------------------------------------------------------")$
print("obtaining A vector components")$
print("------------------------------------------------------------")$
print("From rectangular to spherical")$
print("------------------------------------------------------------")$
print("original vector ",Axyz)$
print("Dot multiplication with %sigma[r] provides the r component")$
sr1 :A[x] * trigsimp(%sigma[r] . %sigma[1]) +
     A[y] * trigsimp(%sigma[r] . %sigma[2]) +
     A[z] * trigsimp(%sigma[r] . %sigma[3]) $
Ar : factor(realpart(1/2 * (sr1[1,1]+sr1[2,2])))$
print('Ar, " = ",Ar)$

print("Dot multiplication with %sigma[%theta] provides the %theta component")$
sr1 :A[x] * trigsimp(%sigma[%theta] . %sigma[1]) +
     A[y] * trigsimp(%sigma[%theta] . %sigma[2]) +
     A[z] * trigsimp(%sigma[%theta] . %sigma[3])$
Atheta : ratsimp(factor(realpart(1/2 * (sr1[1,1]+sr1[2,2]))))$
print('Atheta, " = ",Atheta)$

print("Dot multiplication with %sigma[phi] provides the phi component")$
sr1 :A[x] * trigsimp(%sigma[phi] . %sigma[1]) +
     A[y] * trigsimp(%sigma[phi] . %sigma[2]) +
     A[z] * trigsimp(%sigma[phi] . %sigma[3])$
Aphi : ratsimp(factor(realpart(1/2 * (sr1[1,1]+sr1[2,2]))))$
print('Aphi, " = ",Aphi)$
```

```
print("------------------------------------------------------------")$
print("From spherical to rectangular")$
print("------------------------------------------------------------")$

print("Dot multiplication with %sigma[1] provides the x component")$
sr1 :A[r] * trigsimp(%sigma[r] . %sigma[1]) +
     A[%theta] * trigsimp(%sigma[%theta] . %sigma[1]) +
     A[phi] * trigsimp(%sigma[phi] . %sigma[1])  $
Ax : factor(realpart(1/2 * (sr1[1,1]+sr1[2,2])))$
print('Ax, " = ",Ax)$

print("Dot multiplication with %sigma[2] provides the y component")$
sr1 :A[r] * trigsimp(%sigma[r] . %sigma[2]) +
     A[%theta] * trigsimp(%sigma[%theta] . %sigma[2]) +
     A[phi] * trigsimp(%sigma[phi] . %sigma[2])  $
Ay : factor(realpart(1/2 * (sr1[1,1]+sr1[2,2])))$
print('Ay, " = ",Ay)$

print("Dot multiplication with %sigma[3] provides the z component")$
sr1 :A[r] * trigsimp(%sigma[r] . %sigma[3]) +
     A[%theta] * trigsimp(%sigma[%theta] . %sigma[3]) +
     A[phi] * trigsimp(%sigma[phi] . %sigma[3])  $
Az : factor(realpart(1/2 * (sr1[1,1]+sr1[2,2])))$
print('Az, " = ",Az)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

An example of application is provided with the following listing.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(vect)$
/* example3.8.wxm */

/* */
print("Example of of transformation from rectangular to spherical")$
print("given the vector")$
print(Ax, " = ",x+y)$
print(Ay, " = ",-x+y)$
print(Az, " = ",z)$
print("Find the vector in spherical coordinates")$

print("---------------------------------")$
print(" Solution")$
print("---------------------------------")$
print("Express x,y,z in terms of r, theta ,phi")$
x : r * sin(theta) * cos(phi)$
y : r * sin(theta) * sin(phi)$
z : r *cos(theta)$

print("spherical to rectangular")$
print('x, " = ",x)$
print('y, " = ",y)$
print('z, " = ",z)$

print("write the vector A in rectangular coordinates")$
Ax : trigsimp(x+y) ;
Ay : trigsimp(y-x);
Az : trigsimp(z);

print("Transformation matrix s2rmat")$
r2smat :matrix([sin(theta)* cos(phi),sin(theta)*sin(phi),cos(theta)],
```

```
        [cos(theta)*cos(phi),cos(theta)*sin(phi),-sin(theta)],
        [-sin(phi),cos(phi),0]);

print("vector A in rectangular coordinates")$
Arect : [Ax,Ay,Az];

print("Transform vector A in spherical coordinates")$
print("rectangular to spherical")$
Aspher : r2smat . Arect ;
print("after semplification we recover")$
Aspher : trigsimp(Aspher);

print("----------------------------------------")$
print(" same excercise but with Pauli matrices")$
print("----------------------------------------")$

print("------------------------------------------------------------")$
print("Pauli Matrices Definition")$
print("------------------------------------------------------------")$

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print('%sigma[0]," = ",%sigma[0])$
print('%sigma[1]," = ",%sigma[1])$
print('%sigma[2]," = ",%sigma[2])$
print('%sigma[3]," = ",%sigma[3])$

print("Vector A in rectangular coordinates")$
Axyz : Ax*%sigma[1] + Ay*%sigma[2] + Az*%sigma[3];


print("------------------------------------------------------------")$
print("Pauli Matrices Definition Spherical")$
print("------------------------------------------------------------")$
```

```
ct : cos(theta)$
st : sin(theta)$

%sigma[r] : matrix([ct,st * %e^(-%i * phi)],[st * %e^(%i * phi),-ct])$
%sigma[theta] : matrix([-st,ct  * %e^(-%i * phi)],[ct * %e^(%i * phi),st])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$

print('%sigma[r]," = ",%sigma[r])$
print('%sigma[theta]," = ",%sigma[theta])$
print('%sigma[phi]," = ",%sigma[phi])$

A[x] : Ax$
A[y] : Ay$
A[z] : Az$

print("Dot multiplication with %sigma[r] allows to retrieve the r component")$
sr1 :A[x] * trigsimp(%sigma[r] . %sigma[1])
+ A[y] * trigsimp(%sigma[r] . %sigma[2])
+ A[z] * trigsimp(%sigma[r] . %sigma[3])  $
sr1 :trigsimp(sr1)$
Ar : factor(realpart(1/2 * (sr1[1,1]+sr1[2,2])))$
Ar : trigsimp(Ar)$
print('Ar," = ",Ar)$


print("Dot multiplication with %sigma[theta] allows to retrieve the theta component")$
sr1 :A[x] * trigsimp(%sigma[theta] . %sigma[1])
+ A[y] * trigsimp(%sigma[theta] . %sigma[2])
+ A[z] * trigsimp(%sigma[theta] . %sigma[3])  $
Atheta : ratsimp(factor(realpart(1/2 * (sr1[1,1]+sr1[2,2]))))$
Atheta : trigsimp(Atheta)$
print('Atheta," = ",Atheta)$


print("Dot multiplication with %sigma[phi] allows to retrieve the phi component")$
sr1 :A[x] * trigsimp(%sigma[phi] . %sigma[1])
```

```
+ A[y] * trigsimp(%sigma[phi] . %sigma[2])
+ A[z] * trigsimp(%sigma[phi] . %sigma[3])  $
Aphi : ratsimp(factor(realpart(1/2 * (sr1[1,1]+sr1[2,2]))))$
Aphi : trigsimp(Aphi)$
print('Aphi," = ",Aphi)$

print("------------------------------------------------------------")$
print("vector in spherical coordinates")$
print("------------------------------------------------------------")$
Asph : ratsimp(factor(Ar*%sigma[r]+Atheta*%sigma[theta]+Aphi*%sigma[phi]));

print("Try to find the difference with the cartesian representation")$
ratsimp(demoivre(Asph - Axyz));
print("There is no difference!")$
print("The vector representation with Pauli is independent from the coordinate system")$

print("bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

VI.4 Transformation of a vector from one coordinate system to another

It is noted that since the vector is the same when expressed in rectangular, cylindrical or spherical components we have the identity

$$\begin{aligned}\tilde{A} &= A_x\sigma_1 + A_y\sigma_2 + A_z\sigma_3 = \tilde{A}_{xyz} \\ &= A_\rho\sigma_\rho + A_\phi\sigma_\phi + A_z\sigma_z = \tilde{A}_{rpz} \\ &= A_r\sigma_r + A_\theta\sigma_\theta + A_\phi\sigma_\phi = \tilde{A}_{rtp}\,. \end{aligned} \tag{1.280}$$

By performing the inner product of the above expression with a selected sigma matrix we can recover the desired field component in terms of the other basis. Thus all the transformation between vectors in different coordinate systems can be simply obtained by matrix multiplication and trace operation.

As an example let us say that we want to find A_r when the vector is expressed in rectangular coordinates and $\tilde{A}_{xyz}$ is given. We simply perform the internal product of $\tilde{A}_{xyz}$ with σ_r obtaining

$$A_r = \left\langle \tilde{A}_{xyz}, \sigma_r \right\rangle = \cos(\theta)\, A_z + \sin(\theta)\sin(\phi)\, A_y + \sin(\theta)\cos(\phi)\, A_x \tag{1.281}$$

This is performed by a simple block as follows:

```
coordtransf(Ap,ss) :=
block([],
Apss : Ap . ss,
res : trigsimp(1/2*(Apss[1,1]+Apss[2,2]))
/* print('res," = ",res)*/
)$
```

where the matrix Ap represents the vector in the original coordinate system, while the *ss* is the Pauli matrix of the component we want to retrieve. Therefore all the transformations between different coordinate systems can be achieved with the single equation

$$A_i = \left\langle \tilde{A}, \sigma_i \right\rangle . \tag{1.282}$$

An example of application is provided in the code

```
wxm/vector_transf_Pauli.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
/* vector_transf_Pauli */
/*
This code provides the Pauli matrices in the
rectangular, cylindrical and spherical coordinate systems
Then all the possible transformation are considered
*/

print("------------------------------------------------------------")$
print("Pauli Matrices Definition")$
print("------------------------------------------------------------")$

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print('%sigma[0]," = ",%sigma[0])$
print('%sigma[1]," = ",%sigma[1])$
print('%sigma[2]," = ",%sigma[2])$
print('%sigma[3]," = ",%sigma[3])$

print("------------------------------------------------------------")$
print("Pauli Matrices Definition Cylindrical")$
print("------------------------------------------------------------")$

%sigma[%rho] : matrix([0,%e^(-%i * phi)],[%e^(%i * phi),0])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$
%sigma[z] : matrix([1,0],[0,-1])$

print('%sigma[%rho]," = ",%sigma[%rho])$
print('%sigma[phi]," = ",%sigma[phi])$
print('%sigma[z]," = ",%sigma[z])$
```

```
print("------------------------------------------------------")$
print("Pauli Matrices Definition Spherical")$
print("------------------------------------------------------")$

ct : cos(%theta)$
st : sin(%theta)$

%sigma[r] : matrix([ct,st * %e^(-%i * phi)],[st * %e^(%i * phi),-ct])$
%sigma[%theta] : matrix([-st,ct  * %e^(-%i * phi)],[ct * %e^(%i * phi),st])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$

print('%sigma[r],"  = ",%sigma[r])$
print('%sigma[%theta],"  = ",%sigma[%theta])$
print('%sigma[phi],"  = ",%sigma[phi])$

print("------------------------------------------------------")$
print("vector in rectangular coordinates")$
Axyz : ratsimp(factor(A[x]*%sigma[1]+A[y]*%sigma[2]+A[z]*%sigma[3]));
print("------------------------------------------------------")$
print("vector in cylindrical coordinates")$
Arpz : ratsimp(factor(A[%rho]*%sigma[%rho]+A[z]*%sigma[3]+A[phi]*%sigma[phi]));
print("------------------------------------------------------")$
print("vector in spherical coordinates")$
Artp : ratsimp(factor(A[r]*%sigma[r]+A[%theta]*%sigma[%theta]+A[phi]*%sigma[phi]));
print("------------------------------------------------------")$
coordtransf(Ap,ss) :=
block([],
Apss : Ap . ss,
res : trigsimp(1/2*(Apss[1,1]+Apss[2,2]))
/* print('res,"  = ",res)*/
)$

print("------------------------------------------------------")$
print("Rectangular to cylindrical")$
print("------------------------------------------------------")$
Ap :Axyz;
```

```
ss : rectform(%sigma[%rho])$
coordtransf(Ap,ss)$
Arhoxyz : res$
print('Arhoxyz," = ",Arhoxyz)$

ss : rectform(%sigma[phi])$
coordtransf(Ap,ss)$
Apxyz : res$
print('Apxyz," = ",Apxyz)$

ss : rectform(%sigma[3])$
coordtransf(Ap,ss)$
Azxyz : res$
print('Azxyz," = ",Azxyz)$

print("------------------------------------------------------------")$
print("Rectangular to spherical")$
print("------------------------------------------------------------")$
Ap :Axyz;
ss : rectform(%sigma[r])$
coordtransf(Ap,ss)$
Arxyz : res$
print('Arxyz," = ",Arxyz)$

ss : rectform(%sigma[%theta])$
coordtransf(Ap,ss)$
Atxyz : res$
print('Atxyz," = ",Atxyz)$

ss : rectform(%sigma[phi])$
coordtransf(Ap,ss)$
Apxyz : res$
print('Apxyz," = ",Apxyz)$

print("------------------------------------------------------------")$
print("Cylindrical to rectangular")$
print("------------------------------------------------------------")$
```

```
Ap :demoivre(Arpz);
ss : rectform(%sigma[1])$
coordtransf(Ap,ss)$
Axrpz : res$
print('Axrpz," = ",Axrpz)$

ss : rectform(%sigma[2])$
coordtransf(Ap,ss)$
Ayrpz : res$
print('Ayrpz," = ",Ayrpz)$

ss : rectform(%sigma[3])$
coordtransf(Ap,ss)$
Azrpz : res$
print('Azrpz," = ",Azrpz)$

print("------------------------------------------------------------")$
print("Cylindrical to spherical")$
print("------------------------------------------------------------")$
Ap :demoivre(Arpz);
ss : rectform(%sigma[r])$
coordtransf(Ap,ss)$
Arrpz : res$
print('Arrpz," = ",Arrpz)$

ss : rectform(%sigma[%theta])$
coordtransf(Ap,ss)$
Atrpz : res$
print('Atrpz," = ",Atrpz)$

ss : rectform(%sigma[phi])$
coordtransf(Ap,ss)$
Aprpz : res$
print('Aprpz," = ",Aprpz)$

print("------------------------------------------------------------")$
print("Spherical to rectangular")$
```

```
print("------------------------------------------------------------")$
Ap :demoivre(Artp);
ss : rectform(%sigma[1])$
coordtransf(Ap,ss)$
Axrtp : res$
print('Axrtp," = ",Axrtp)$

ss : rectform(%sigma[2])$
coordtransf(Ap,ss)$
Ayrtp : res$
print('Ayrtp," = ",Ayrtp)$

ss : rectform(%sigma[3])$
coordtransf(Ap,ss)$
Azrtp : res$
print('Azrtp," = ",Azrtp)$

print("------------------------------------------------------------")$
print("Spherical to cylindrical")$
print("------------------------------------------------------------")$
Ap :demoivre(Artp);
ss : rectform(%sigma[%rho])$
coordtransf(Ap,ss)$
Arrtp : res$
print('Arrtp," = ",Arrtp)$

ss : rectform(%sigma[phi])$
coordtransf(Ap,ss)$
Aprtp : res$
print('Aprtp," = ",Aprtp)$

ss : rectform(%sigma[3])$
coordtransf(Ap,ss)$
Azrtp : res$
print('Azrtp," = ",Azrtp)$
```

```
print("end")$
/* [wxMaxima: input   end   ] */


/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

VII Summary of conventional coordinate transformations for unit vectors and vector components

When replacing the vector component with the unit versors we have the transformation between unit vectors.

VII.1 Unit vectors

The coordinate unit vectors are defined as vectors of unit length pointing along coordinate lines in the direction of increasing coordinate variables. It is left as an exercise to draw them in the three coordinate systems. The following rules are applicable to transformations among unit coordinate vectors.

Transformations from cylindrical/spherical to rectangular:

$$\begin{aligned}
\mathbf{u}_x &= \mathbf{u}_\rho \cos\phi - \mathbf{u}_\phi \sin\phi \\
&= \mathbf{u}_r \sin\theta \cos\phi + \mathbf{u}_\theta \cos\theta \cos\phi - \mathbf{u}_\phi \sin\phi \\
\mathbf{u}_y &= \mathbf{u}_\rho \sin\phi + \mathbf{u}_\phi \cos\phi \\
&= \mathbf{u}_r \sin\theta \sin\phi + \mathbf{u}_\theta \cos\theta \sin\phi + \mathbf{u}_\phi \cos\phi \\
\mathbf{u}_z &= \mathbf{u}_r \cos\theta - \mathbf{u}_\theta \sin\theta
\end{aligned} \tag{1.283}$$

Transformations from rectangular/spherical to cylindrical:

$$\begin{aligned}
\mathbf{u}_\rho &= \mathbf{u}_x \cos\phi + \mathbf{u}_y \sin\phi = \mathbf{u}_r \sin\theta + \mathbf{u}_\theta \cos\theta \\
\mathbf{u}_\phi &= -\mathbf{u}_x \sin\phi + \mathbf{u}_y \cos\phi \\
\mathbf{u}_z &= \mathbf{u}_r \cos\theta - \mathbf{u}_\theta \sin\theta
\end{aligned} \tag{1.284}$$

Transformations from rectangular/cylindrical to spherical:

$$\begin{aligned}
\mathbf{u}_r &= \mathbf{u}_x \sin\theta\cos\phi + \mathbf{u}_y \sin\theta\sin\phi + \mathbf{u}_z \cos\theta \\
&= \mathbf{u}_\rho \sin\theta + \mathbf{u}_z \cos\theta \\
\mathbf{u}_\theta &= \mathbf{u}_x \cos\theta\cos\phi + \mathbf{u}_y \cos\theta\sin\phi - \mathbf{u}_z \sin\theta \\
&= \mathbf{u}_\rho \cos\theta - \mathbf{u}_z \sin\theta \\
\mathbf{u}_\phi &= -\mathbf{u}_x \sin\phi + \mathbf{u}_y \cos\phi
\end{aligned} \tag{1.285}$$

VII.2 Coordinate transformations for vector components

A list of the coordinate transformation for vector components is provided in the following.

a) Rectangular to cylindrical components

$$\begin{aligned}
F_\rho &= F_x \cos\phi + F_y \sin\phi \\
F_\phi &= -F_x \sin\phi + F_y \cos\phi \\
F_z &= F_z
\end{aligned} \tag{1.286}$$

b) Rectangular to spherical components

$$\begin{aligned}
F_r &= F_x \sin\theta\cos\phi + F_y \sin\theta\sin\phi + F_z \cos\theta \\
F_\theta &= F_x \cos\theta\cos\phi + F_y \cos\theta\sin\phi - F_z \sin\theta \\
F_\phi &= -F_x \sin\phi + F_y \cos\phi
\end{aligned} \tag{1.287}$$

c) Cylindrical to rectangular components

$$\begin{aligned}
F_x &= F_\rho \cos\phi - F_\phi \sin\phi \\
F_y &= F_\rho \sin\phi + F_\phi \cos\phi \\
F_z &= F_z
\end{aligned} \tag{1.288}$$

d) Cylindrical to spherical components

$$\begin{aligned} F_r &= F_\rho \sin\theta + F_z \cos\theta \\ F_\theta &= F_\rho \cos\theta - F_z \sin\theta \\ F_\phi &= F_\phi \end{aligned} \tag{1.289}$$

e) Spherical to rectangular components

$$\begin{aligned} F_x &= F_r \sin\theta \cos\phi + F_\theta \cos\theta \cos\phi - F_\phi \sin\phi \\ F_y &= F_r \sin\theta \sin\phi + F_\theta \cos\theta \sin\phi + F_\phi \cos\phi \\ F_z &= F_r \cos\theta - F_\theta \sin\phi \end{aligned} \tag{1.290}$$

f) Spherical to cylindrical components

$$\begin{aligned} F_\rho &= F_r \sin\theta + F_\theta \cos\theta \\ F_\phi &= F_\phi \\ F_z &= F_r \cos\theta - F_\theta \sin\theta \end{aligned} \tag{1.291}$$

VIII Unified representation of vectors with Pauli algebra

Vector are generally represented in the various coordinate system as

$$\begin{aligned} \mathbf{A} &= A_x \mathbf{x}_0 + A_y \mathbf{y}_0 + A_z \mathbf{z}_0 \\ &= A_\rho \boldsymbol{\rho}_0 + A_\phi \boldsymbol{\phi}_0 + A_z \mathbf{z}_0 \\ &= A_r r_0 + A_\theta \boldsymbol{\theta}_0 + A_\phi \boldsymbol{\phi}_0 \end{aligned} \tag{1.292}$$

Let us now represent them with the following form

$$\mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3 \tag{1.293}$$

and let us impose that for $i = 1,2,3$

$$(\mathbf{e}_i)^2 = 1 \tag{1.294}$$

and that for $i \neq j$

$$\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i \tag{1.295}$$

These two conditions are sufficient to establish a Clifford algebra $Cl(3,0)$.

The use of the representation in (1.292) has several drawbacks as:

- Vectors cannot be multiplied
- it is not possible to find the inverse
- given two vectors **a** and **b** it is not possible to find the transformation which leads from **a** to **b**
- the cross product is misleading; instead a bivector, which is an oriented surface, should be used.
- no volume elements are present.
- As we have seen the representation of a vector with Pauli matrices is the same in all coordinate systems, while with versors we only have the components.

By using the correspondence in Table 2.7 the conditions (1.294) and (1.295) are realized.

Table 1.8. Basis vector in rectangular, cylindrical and spherical coordinate systems. The $\mathbf{e}_i$ are a Clifford basis and correspond to the appropriate Pauli matrices.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
rectangular	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
cylindrical	$\sigma_\rho = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$	$\sigma_\phi = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$	$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
spherical	$\sigma_r = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$	$\sigma_\theta = \begin{pmatrix} -\sin\theta & e^{-i\phi}\cos\theta \\ e^{i\phi}\cos\theta & \sin\theta \end{pmatrix}$	$\sigma_\phi = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$

In addition, by using the correspondence shown in Table 1.9, eq. (1.293) can be used to refer to all the three coordinate systems.

Table 1.9. Vector **A** equivalence for rectangular, cylindrical and spherical coordinate systems.

	A_1	A_2	A_3
rectangular	A_x	A_y	A_z
cylindrical	A_ρ	A_ϕ	A_z
spherical	A_r	A_θ	A_ϕ

Example 5. Let us consider the vector **A** with components $A_x = 1, A_y = 2, A_z = 3$ evaluated at the position $P_x = 4, P_y = 5, P_z = 6$. In rectangular coordinates vector **A** represented as a Pauli matrices is

$$A_{xyz} = A_x\sigma_1 + A_y\sigma_2 + A_z\sigma_3 = \begin{pmatrix} 3 & 1-2i \\ 2i+1 & -3 \end{pmatrix}$$

The position P corresponds the following angles in degrees:

$$\theta = 43.138^\circ$$
$$\phi = 51.34^\circ .$$

At this point the Pauli matrices are:

$$\sigma_\rho = \begin{pmatrix} 0 & 0.6247 - 0.78087\,i \\ 0.78087\,i + 0.6247 & 0 \end{pmatrix}$$
$$\sigma_\phi = \begin{pmatrix} 0 & -0.6247\,i - 0.78087 \\ 0.6247\,i - 0.78087 & 0 \end{pmatrix}$$
$$\sigma_r = \begin{pmatrix} 0.7297 & 0.42714 - 0.53393\,i \\ 0.53393\,i + 0.42714 & -0.7297 \end{pmatrix}$$
$$\sigma_\theta = \begin{pmatrix} -0.68376 & 0.45584 - 0.5698\,i \\ 0.5698\,i + 0.45584 & 0.68376 \end{pmatrix}$$

The components in cylindrical coordinates are:

$$A_\rho = 2.1864$$

$$A_\phi = 0.46852$$

$$A_z = 3.0$$

When we perform

$$A_{rpz} = A_\rho \sigma_\rho + A_\phi \sigma_\phi + A_z \sigma_3 = \begin{pmatrix} 3.0 & 1.0-2.0i \\ 2.0i+1.0 & -3.0 \end{pmatrix}$$

we get the same matrix as before, i.e. $A_{xyz} = A_{rpz}$, showing that the vector is independent from the coordinate system. Similarly, the components in spherical coordinates are:

$$A_r = 3.6841$$

$$A_\theta = -0.45584$$

$$A_\phi = 0.46852$$

and again the vector is given by

$$A_{rtp} = A_r \sigma_r + A_\theta \sigma_\theta + A_\phi \sigma_\phi = \begin{pmatrix} 3.0 & 1.0-2.0i \\ 2.0i+1.0 & -3.0 \end{pmatrix}.$$

It is noted that such representation of a vector, independent from a coordinate system, is not existent in the conventional approach.
An example of code is provided with the following listing.

wxm/vector_example_Pauli.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
/* vector_transf_Pauli */
/*
This code provides the Pauli matrices in the
```

```
rectangular, cylindrical and spherical coordinate systems
Then all the possible transformation are considered
*/

print("------------------------------------------------------------")$
print("Pauli Matrices Definition")$
print("------------------------------------------------------------")$

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print('%sigma[0],"=",%sigma[0])$
print('%sigma[1],"=",%sigma[1])$
print('%sigma[2],"=",%sigma[2])$
print('%sigma[3],"=",%sigma[3])$

print("------------------------------------------------------------")$
print("Pauli Matrices Definition Cylindrical")$
print("------------------------------------------------------------")$

%sigma[%rho] : matrix([0,%e^(-%i * phi)],[%e^(%i * phi),0])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$
%sigma[z] : matrix([1,0],[0,-1])$

print('%sigma[%rho],"=",%sigma[%rho])$
print('%sigma[phi],"=",%sigma[phi])$
print('%sigma[z],"=",%sigma[z])$

print("------------------------------------------------------------")$
print("Pauli Matrices Definition Spherical")$
print("------------------------------------------------------------")$

ct : cos(%theta)$
st : sin(%theta)$
```

```
%sigma[r] : matrix([ct,st * %e^(-%i * phi)],[st * %e^(%i * phi),-ct])$
%sigma[%theta] : matrix([-st,ct  * %e^(-%i * phi)],[ct * %e^(%i * phi),st])$
%sigma[phi] : matrix([0,-%i * %e^(-%i * phi)],[%i * %e^(%i * phi),0])$

print('%sigma[r],"  = ",%sigma[r])$
print('%sigma[%theta],"  = ",%sigma[%theta])$
print('%sigma[phi],"  = ",%sigma[phi])$

coordtransf(Ap,ss) :=
block([],
Apss : Ap . ss,
res : rectform(1/2*(Apss[1,1]+Apss[2,2]))
/* print('res,"  = ",res)*/
)$

print("------------------------------------------------------------")$
print("vector in rectangular coordinates")$
print("Given the vector of coordinates")$
fpprintprec:5$
ratprint : false$
A[x] : 1.0$
A[y] : 2.0$
A[z] : 3.0$
print('A[x],"  = ",A[x])$
print('A[y],"  = ",A[y])$
print('A[z],"  = ",A[z])$
Axyz : ratsimp(factor(A[x]*%sigma[1]+A[y]*%sigma[2]+A[z]*%sigma[3]));

print("Evaluate in P")$
Px : 4.0$
Py : 5.0$
Pz : 6.0$
print('P[x],"  = ",Px)$
print('P[y],"  = ",Py)$
print('P[z],"  = ",Pz)$

print("At point P we have")$
```

```
phi_angle : float(atan2(Py,Px))$
%sigma[phi] : ev(%sigma[phi],phi = phi_angle)$
%sigma[phi] : rectform(%sigma[phi])$
%sigma[%rho] : ev(%sigma[%rho],phi = phi_angle)$
phi_deg : float(phi_angle * 180 /%pi)$
print('phi_deg," = ",phi_deg)$
print('%sigma[%rho]," = ",%sigma[%rho])$
print('%sigma[phi]," = ",%sigma[phi])$
print('%sigma[z]," = ",%sigma[z])$

theta_angle : atan2(Pz,sqrt(Px^2+Py^2))$
%sigma[r] : ev(%sigma[r],phi = phi_angle)$
%sigma[%theta] : ev(%sigma[%theta],phi = phi_angle)$
%sigma[r] : ev(%sigma[r],%theta = theta_angle)$
%sigma[%theta] : ev(%sigma[%theta],%theta = theta_angle)$
%sigma[r] : rectform(%sigma[r])$
%sigma[%theta] : rectform(%sigma[%theta])$
theta_deg : float(theta_angle * 180 /%pi)$
print('theta_deg," = ",theta_deg)$

print('%sigma[r]," = ",%sigma[r])$
print('%sigma[%theta]," = ",%sigma[%theta])$
print('%sigma[phi]," = ",%sigma[phi])$

print("------------------------------------------------------------")$
print("Rectangular to cylindrical")$
print("------------------------------------------------------------")$
Ap :Axyz;
ss : rectform(%sigma[%rho])$
coordtransf(Ap,ss)$
A[%rho] : rectform(res)$
print('A[%rho]," = ",A[%rho])$

ss : rectform(%sigma[phi])$
coordtransf(Ap,ss)$
A[phi] : rectform(res)$
print('A[phi] ," = ",A[phi] )$
```

```
ss : rectform(%sigma[3])$
coordtransf(Ap,ss)$
A[z] : rectform(res)$
print('A[z],"  = ",A[z])$

print("------------------------------------------------------------")$
print("vector in cylindrical coordinates")$
Arpz : rectform((A[%rho]*%sigma[%rho]+A[z]*%sigma[3]+A[phi]*%sigma[phi]));

print("------------------------------------------------------------")$
print("Rectangular to spherical")$
print("------------------------------------------------------------")$
Ap :Axyz;
ss : rectform(%sigma[r])$
coordtransf(Ap,ss)$
A[r] : res$
print('A[r],"  = ",A[r])$

ss : rectform(%sigma[%theta])$
coordtransf(Ap,ss)$
A[%theta] : res$
print('A[%theta],"  = ",A[%theta])$

ss : rectform(%sigma[phi])$
coordtransf(Ap,ss)$
A[phi] : res$
print('A[phi],"  = ",A[phi])$

print("------------------------------------------------------------")$
print("vector in spherical coordinates")$
Artp : rectform((A[r]*%sigma[r]+A[%theta]*%sigma[%theta]+A[phi]*%sigma[phi]));
print("------------------------------------------------------------")$

print("end")$
/* [wxMaxima: input   end   ] */
```

```
/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

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2

Nabla operator with Pauli matrices

In order to introduce the nabla operator with a geometric algebra approach, it is first convenient to recall the gradient definition.

I Gradient

Let us consider a scalar function $\psi(\mathbf{r})$, which is single valued and continuous in a volume V. Physically, the function ψ may represent an electric potential, a temperature, etc. At a point P the function will take the value ψ_P. Now suppose to draw a sphere of radius Δs centered in P. We can check the values of $\psi(\mathbf{r})$ on this sphere and, in general, there will be a point Q on which the variation $\Delta\psi = \psi_P - \psi_Q$ is maximum. This defines also the direction from P to Q. This direction is taken as the direction of a new vector which is called the *gradient*. The magnitude of the gradient is defined as the value $\Delta\psi/\Delta s$ in this preferred direction. Thus the gradient of a scalar may be defined as

$$grad\,\psi = \nabla\psi = \mathbf{a}_{max} \lim_{\Delta s\to 0} \left(\frac{\Delta\psi}{\Delta s}\right)_{max} \tag{2.1}$$

where $\mathbf{a}_{max}$ is a unit vector pointing in the direction of maximum $\frac{\Delta\psi}{\Delta s}$. It is noted that the definition (2.1) is not dependent on a a particular coordinate system.

In rectangular coordinates, (2.1) reduces to

$$\nabla\psi = \mathbf{a}_x\frac{\partial\psi}{\partial x} + \mathbf{a}_y\frac{\partial\psi}{\partial y} + \mathbf{a}_z\frac{\partial\psi}{\partial z}\,. \tag{2.2}$$

In circular–cylinder coordinates , Δs in the angular direction ϕ is not equal to $\Delta\phi$ but $\delta s = \rho\Delta\phi$. Thus we have

$$\nabla\psi = \mathbf{a}_\rho \frac{\partial\psi}{\partial\rho} + \frac{\mathbf{a}_\psi}{\rho}\frac{\partial\psi}{\partial\phi} + \mathbf{a}_z \frac{\partial\psi}{\partial z}\,. \tag{2.3}$$

Similarly, in spherical coordinates,

$$\nabla\psi = \mathbf{a}_r \frac{\partial\psi}{\partial r} + \frac{\mathbf{a}_\phi}{r\sin\theta}\frac{\partial\psi}{\partial\phi} + \frac{\mathbf{a}_\theta}{r}\frac{\partial\psi}{\partial\theta}\,. \tag{2.4}$$

II Nabla definition

The nabla operator may be written in general as

$$\nabla = \sum_{i=1}^{3} c_i\,\mathbf{e}_i\,\partial_i \tag{2.5}$$

where we have introduced appropriate scaling coefficients c_i, a vector base composed by $\mathbf{e}_i$, and the partial derivatives ∂_i. Note that the $\mathbf{e}_i$ are a *Clifford basis* and they satisfy the properties of a Clifford basis, i.e. the $\mathbf{e}_i\mathbf{e}_i = 1$ and $\mathbf{e}_i\mathbf{e}_j = -\mathbf{e}_j\mathbf{e}_i$ for $i \neq j$. Depending on the coordinate system we will identify the basis vector as reported in Table 2.1.

Table 2.1. Basis vector in rectangular, cylindrical and spherical coordinate systems. The $\mathbf{e}_i$ are a Clifford basis and correspond to the appropriate Pauli matrices.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
rectangular	σ_1	σ_2	σ_3
cylindrical	σ_ρ	σ_ϕ	σ_3
spherical	σ_r	σ_θ	σ_ϕ

The scaling coefficients are reported in Table 2.2. The partial derivatives symbols ∂_i assume the meaning reported in Table 2.3

Table 2.2. Scaling coefficients in rectangular, cylindrical and spherical coordinate systems.

	c_1	c_2	c_3
rectangular	1	1	1
cylindrical	1	$1/\rho$	1
spherical	1	$1/r$	$1/(r\sin(\theta))$

Table 2.3. Partial derivatives for rectangular, cylindrical and spherical coordinate systems.

	∂_1	∂_2	∂_3
rectangular	∂_x	∂_y	∂_z
cylindrical	∂_ρ	∂_ϕ	∂_z
spherical	∂_r	∂_θ	∂_ϕ

From the fundamental identity (the geometric product) we have:

$$\nabla \mathbf{F} = \nabla \cdot \mathbf{F} + \nabla \wedge \mathbf{F}. \tag{2.6}$$

The term $\mathbf{F}$ can be a multivector composed, in general, by a scalar part (grade 0), a vector part (grade 1), a bivector part (grade 2) and a trivector or pseudo scalar (grade 3). In Table 2.4 it is reported the application of the nabla operator to a scalar function, while in Table 2.5 the nabla operator has been applied to a vector function.

Table 2.4. For a scalar function ϕ application of the nabla operator gives a vector (i.e. the gradient), here assumed equal to the external product of ∇ and ϕ. Note that in the geometric product expansion of ϕ also a term of the type $\nabla \cdot \phi$ is present, but this term is equal to zero. Further application of the nabla operator provides the Laplacian (which is of 0 grade) and the term $\nabla \wedge \nabla \wedge \phi = 0$.

Grade	0	1	2	3
quantity	ϕ			
$\nabla\phi$	0	$\nabla \wedge \phi = \nabla\phi$		
$\nabla\nabla\phi = \nabla^2\phi$	$\nabla \cdot \nabla\phi$		$\nabla \wedge \nabla \wedge \phi = 0$	

Table 2.5. For a vector function $\mathbf{A}$ application of the nabla operator gives a scalar (i.e. the divergence), and a bivector. From $\nabla \cdot \mathbf{A}$ a further application of the nabla operator gives the vector term $\nabla\nabla \cdot \mathbf{A}$, while the nabla operator applied to $\nabla \wedge \mathbf{A}$ gives the term $\nabla \cdot \nabla \wedge \mathbf{A}$ and the term $\nabla \wedge \nabla \wedge \mathbf{A}$ which is equal to zero.

Grade	0	1	2	3
quantity		$\mathbf{A}$		
$\nabla\mathbf{A}$	$\nabla \cdot \mathbf{A}$		$\nabla \wedge \mathbf{A}$	
$\nabla\nabla\mathbf{A} = \nabla^2\mathbf{A}$		$\nabla\nabla \cdot \mathbf{A} + \nabla \cdot \nabla \wedge \mathbf{A}$		$\nabla \wedge \nabla \wedge \mathbf{A} = 0$

It is noted that by substituting the vector $\mathbf{A}$ with $i\mathbf{B}$ we have the corresponding Table for nabla operating on a bivector. Similarly, by substituting ϕ with $i\psi$ we have nabla operating on a pseudoscalar.

III Rectangular Coordinates

By using Pauli matrices a field vector $\mathbf{F}$ may be written as

$$\begin{aligned}\tilde{F} &= \sigma_1 F_x + \sigma_2 F_y + \sigma_3 F_z \\ &= \begin{pmatrix} F_z & F_x - i F_y \\ i F_y + F_x & -F_z \end{pmatrix}\end{aligned} \tag{2.7}$$

Similarly, the Pauli matrix representation of the ∇ operator takes the form

$$\begin{aligned}\tilde{\nabla} &= \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z \\ &= \begin{pmatrix} \partial_z & \partial_x - i\partial_y \\ i\partial_y + \partial_x & -\partial_z \end{pmatrix}.\end{aligned} \tag{2.8}$$

When evaluating $\nabla \mathbf{F}$ via Pauli matrices we simply need to perform the following matrix product:

$$\begin{aligned}\tilde{\nabla}\tilde{F} &= \begin{pmatrix} \partial_z & \partial_x - i\partial_y \\ i\partial_y + \partial_x & -\partial_z \end{pmatrix} \begin{pmatrix} F_z & F_x - iF_y \\ iF_y + F_x & -F_z \end{pmatrix} \\ &= \begin{pmatrix} \partial_z F_z + \partial_y F_y + \partial_x F_x & 0 \\ 0 & \partial_z F_z + \partial_y F_y + \partial_x F_x \end{pmatrix} + \\ &+ \begin{pmatrix} i\left(\partial_x F_y - \partial_y F_x\right) & i\left(\partial_y F_z\right) - \partial_x F_z - i\left(\partial_z F_y\right) + \partial_z F_x \\ i\left(\partial_y F_z\right) + \partial_x F_z - i\left(\partial_z F_y\right) - \partial_z F_x & -i\left(\partial_x F_y - \partial_y F_x\right) \end{pmatrix}.\end{aligned} \tag{2.9}$$

In (2.9) we have separated the scalar part corresponding to $\nabla \cdot \mathbf{F}$ (diagonal matrix) from the external product $\nabla \wedge \mathbf{F}$.

In several instances it is necessary to form the second order expressions, e.g. $\nabla\nabla\mathbf{F}$. Computation of this quantity via Pauli matrices is embarrassing simple, since only matrix multiplication is required

$$\begin{aligned}\tilde{\nabla}\tilde{\nabla}\tilde{F} &= \begin{pmatrix} \partial_z & \partial_x - i\partial_y \\ i\partial_y + \partial_x & -\partial_z \end{pmatrix} \begin{pmatrix} \partial_z & \partial_x - i\partial_y \\ i\partial_y + \partial_x & -\partial_z \end{pmatrix} \begin{pmatrix} F_z & F_x - iF_y \\ iF_y + F_x & -F_z \end{pmatrix} \\ &= \begin{pmatrix} \partial_z^2 + (\partial_x - i\partial_y)\,(i\partial_y + \partial_x) & 0 \\ 0 & \partial_z^2 + (\partial_x - i\partial_y)\,(i\partial_y + \partial_x) \end{pmatrix} \begin{pmatrix} F_z & F_x - iF_y \\ iF_y + F_x & -F_z \end{pmatrix} \\ &= \begin{pmatrix} \Delta F_z & \Delta F_x - i\Delta F_y \\ i\Delta F_y + \Delta F_x & \Delta F_z \end{pmatrix} \end{aligned} \tag{2.10}$$

where we have introduced the laplacian Δ defined as

$$\Delta = \partial_z^2 + \partial_y^2 + \partial_x^2 \,. \tag{2.11}$$

In the following code it is shown how to compute, for a vector $\mathbf{E}$, the two geometric products $\nabla\mathbf{E}$ and $\nabla\nabla\mathbf{E}$.

wxm/1Pauli_nabla_v7.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
load(Pauli_v02)$
functions;
print("----------------------------------------")$
print("Consider a vector E with components Ex, Ey, Ez")$
depends([Ex,Ey,Ez],[x,y,z]);

/*
print("Consider an Electric Field of the form")$
print("It is important to use j instead of i in order to preserve the grades")$
Ex : %e^(-j * k * z)$
Ey : 0$
Ez : 0$
*/

[Ex, Ey, Ez];
```

```
print("Nabla as Pauli matrix in rectangular coordinates")$
Np : dx * %sigma[1] + dy * %sigma[2] + dz * %sigma[3];

print("vector E as Pauli matrix")$
Ep : Ex * %sigma[1] + Ey * %sigma[2] + Ez * %sigma[3];

print("nabla E is")$
NEp  : Nablarect(Ep);

print("nabla nabla E is")$
NNEp : Nablarect(NEp);

print("Finding the grade elements of the following Pauli matrix")$
M : NEp;
print("By calling Gradsep(M) we have")$
Gradesep(M)$

print("Finding the grade elements of the following Pauli matrix")$
M : NNEp;
print("By calling Gradsep(M) we have")$
Gradesep(M)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Examples in rectangular coordinates

Example 6. We consider the scalar function

$$V = V_0 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{c}\right)$$

it is required to evaluate the gradient.

First we write the scalar function as a Pauli matrix:

$$\tilde{V} = V\sigma_0 = \begin{pmatrix} V_0 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{c}\right) & 0 \\ 0 & V_0 \cos\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{c}\right) \end{pmatrix}$$

Then we perform the nabla operation obtaining

$$\tilde{\nabla}\tilde{V} = \begin{pmatrix} \frac{2\pi V_0 \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi z}{c}\right)}{c} & -\frac{\pi V_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi z}{c}\right)}{a} \\ -\frac{\pi V_0 \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi z}{c}\right)}{a} & -\frac{2\pi V_0 \cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi z}{c}\right)}{c} \end{pmatrix}$$

Then, by calling the block for the grade separation(Gradesep in the listing) we see that only grade 1 elements are present. In particular, by calling with **a** this vector, we have

$$\begin{aligned} a_x &= -\frac{\pi V_0 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{2\pi z}{c}\right)}{a} \\ a_y &= 0 \\ a_z &= \frac{2\pi V_0 \cos\left(\frac{\pi x}{a}\right) \cos\left(\frac{2\pi z}{c}\right)}{c} \end{aligned}$$

The relative listing is

```
wxm/1Example3.10mod1.wxm
```

Example 7. The next example requires the calculation of the nabla operator on the following vector **A**

$$\begin{aligned} A_x &= 3y^2 \\ A_y &= 2z \\ A_z &= y^2 z \end{aligned}$$

We start by expressing the vector **A** in terms of Pauli matrices as

$$\tilde{A} = \begin{pmatrix} y^2 z & 3y^2 - 2iz \\ 2iz + 3y^2 & -y^2 z \end{pmatrix}$$

from which $\tilde{\nabla}\tilde{A}$ is readily evaluated as

$$\tilde{\nabla}\tilde{A} = \begin{pmatrix} y\,(y-6\,i) & 2\,i\,(yz-1) \\ 2\,i\,(yz-1) & y\,(y+6\,i) \end{pmatrix}.$$

Next the grade evaluation is performed giving for grade 0

$$a_0 = \nabla \cdot \mathbf{A} = y^2 \tag{2.12}$$

and the only other components are the bivector $\hat{\mathbf{B}} = i\mathbf{B} = i\left(B_x + B_y + B_z\right)$ of grade 2 with elements

$$B_x = 2yz - 2$$
$$B_y = 0$$
$$B_z = -6y$$

The relative listing is

```
wxm/1Example3.11mod1.wxm
```

Note that in the code it also reported how to compute the divergence and the curl in the conventional way. However, it is noted that the geometric product keeps together the divergence and the curl.

Example 8. Let us consider a vector $\mathbf{A}$ with only the z component

$$A_z = \frac{1}{\sqrt{z^2+y^2+x^2}}$$

Apply the nabla operator in rectangular coordinates.
The Pauli matrix of the vector $\mathbf{A}$ is

$$\tilde{A} = \begin{pmatrix} \frac{1}{\sqrt{z^2+y^2+x^2}} & 0 \\ 0 & -\frac{1}{\sqrt{z^2+y^2+x^2}} \end{pmatrix},$$

and when performing the nabla operator we obtain

$$\tilde{\nabla}\tilde{A} = \begin{pmatrix} -\frac{z}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}} & -\frac{iy-x}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}} \\ -\frac{iy+x}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}} & -\frac{z}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}} \end{pmatrix}.$$

By looking at the various grades we retrieve for grade 0 (divergence term)

$$a_0 = -\frac{z}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$

for grade 1 (vector)

$$a_x = 0$$
$$a_y = 0$$
$$a_z = 0$$

for grade 2 (bivector)

$$B_x = -\frac{y}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$
$$B_y = \frac{x}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$
$$B_z = -0$$

and for grade 3 (trivector)

$$t_0 = 0\,.$$

The relative listing is

```
wxm/1Nabla_rectangular1.wxm
```

So far we have seen application of the nabla operators on scalars and on vectors. However the nabla operator can be applied directly on multivectors as shown in the next example.

Example 9. Let us consider a scalar function

$$f = \frac{1}{\sqrt{z^2+y^2+x^2}}$$

and a vector with only the z component

$$A_z = \frac{1}{\sqrt{z^2+y^2+x^2}}$$

It is required to apply the nabla operator on the multivector $M = f + \mathbf{A}$.
The Pauli matrix of the scalar function is

$$\tilde{f} = \begin{pmatrix} \frac{1}{\sqrt{z^2+y^2+x^2}} & 0 \\ 0 & \frac{1}{\sqrt{z^2+y^2+x^2}} \end{pmatrix}$$

and the Pauli matrix of the vector $\mathbf{A}$ is

$$\tilde{A} = \begin{pmatrix} \frac{1}{\sqrt{z^2+y^2+x^2}} & 0 \\ 0 & -\frac{1}{\sqrt{z^2+y^2+x^2}} \end{pmatrix},$$

their sum giving the multivector

$$\tilde{M} = \begin{pmatrix} \frac{2}{\sqrt{z^2+y^2+x^2}} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.13}$$

By applying the nabla operator on such multivector $\tilde{M}$ one obtains

$$\tilde{\nabla}\tilde{M} = \begin{pmatrix} -\frac{2z}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}} & 0 \\ -\frac{2(iy+x)}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}} & 0 \end{pmatrix}$$

By looking at the various grades we retrieve for grade 0 (divergence term)

$$a_0 = -\frac{z}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$

for grade 1 (vector)

$$a_x = -\frac{x}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$
$$a_y = -\frac{y}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$
$$a_z = -\frac{z}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$

for grade 2 (bivector)

$$B_x = -\frac{y}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$
$$B_y = \frac{x}{\left(z^2+y^2+x^2\right)^{\frac{3}{2}}}$$
$$B_z = -0$$

and for grade 3 (trivector)

$$t_0 = 0\,.$$

The relative listing is

```
wxm/1Nabla_multivector1.wxm
```

IV Relating vector algebra to conventional vector analysis

We have previously shown that the following identities between vectors hold:

$$\mathbf{a}\wedge\mathbf{b} = i\,\mathbf{a}\times\mathbf{b} \tag{2.14}$$

$$\mathbf{a}\cdot\mathbf{b}\wedge\mathbf{c} = -\,\mathbf{a}\times\mathbf{b}\times\mathbf{c} \tag{2.15}$$

$$\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = i\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\,. \tag{2.16}$$

These equations allows to translate into conventional vector algebra (VA) the identities obtained in geometric algebra (GA) and viceversa. As an example, consider the triple geometric product $\mathbf{a}\,\mathbf{b}\,\mathbf{c}$ expressed as:

$$\begin{aligned}\mathbf{a}\,\mathbf{b}\,\mathbf{c} &= \mathbf{a}\,(\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\wedge\mathbf{c})\\ &= \mathbf{a}\cdot(\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\wedge\mathbf{c})+\mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\wedge\mathbf{c})\\ &= \mathbf{a}\cdot\mathbf{b}\cdot\mathbf{c}+\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c})+\mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{c})+\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c}\\ &= -\mathbf{a}\times\mathbf{b}\times\mathbf{c}+\mathbf{a}(\mathbf{b}\cdot\mathbf{c})+i\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) \qquad (2.17)\end{aligned}$$

where the term $\mathbf{a}\cdot\mathbf{b}\cdot\mathbf{c}$ evidently is equal to zero.

Computationally, by using the Pauli matrices $(\tilde{a},\tilde{b},\tilde{c})$ the triple product is trivial and is simply a matrix product. However, from direct interpretation of the result in (2.17) one may also recognize:

- the term $\mathbf{a}\cdot(\mathbf{b}\wedge\mathbf{c}) = -\mathbf{a}\times\mathbf{b}\times\mathbf{c}$ is the dot product of a vector and a bivector giving as a result a vector;
- the term $\mathbf{a}\wedge(\mathbf{b}\cdot\mathbf{c}) = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ corresponds also to a vector;
- the last term $\mathbf{a}\wedge\mathbf{b}\wedge\mathbf{c} = i\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is a trivector (or pseudoscalar).

The above identities are also useful in order to find some relationships concerning the nabla operator.

The nabla operator in geometric algebra

It is instructive to start from (2.14)–(2.16) and to formally substitute **a** with the nabla operator ∇, thus obtaining:

$$\nabla\wedge\mathbf{b} = i\nabla\times\mathbf{b} \qquad (2.18)$$

$$\nabla\cdot(\mathbf{b}\wedge\mathbf{c}) = -\nabla\times(\mathbf{b}\times\mathbf{c}) \qquad (2.19)$$

$$\nabla\wedge(\mathbf{b}\wedge\mathbf{c}) = i\nabla\cdot(\mathbf{b}\times\mathbf{c})\,. \qquad (2.20)$$

Equation (2.18) simply relates the external product of ∇ with a vector **b** with the curl operation. The other two equations (2.19), (2.20) are interesting, since they refer to the divergence and the external product of a *bivector*.

Equation (2.19) states that the divergence of a bivector $\mathbf{b}\wedge\mathbf{c}$ is equal to minus the curl of $\mathbf{b}\times\mathbf{c}$., i.e. to a vector. An alternative way to prove (2.19) is the following. Let us assume

$$\nabla = \mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3 \tag{2.21}$$

and consider a bivector $\hat{\mathbf{B}}$ given by

$$\hat{\mathbf{B}} = i\mathbf{B} = i\left(\mathbf{e}_1 B_1 + \mathbf{e}_2 B_2 + \mathbf{e}_3 B_3\right) = \mathbf{e}_{23}B_1 + \mathbf{e}_{31}B_2 + \mathbf{e}_{12}B_3\,. \tag{2.22}$$

Dot multiplication provides:

$$\begin{aligned}
\nabla\cdot\hat{\mathbf{B}} &= (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3)\cdot(\mathbf{e}_{23}B_1 + \mathbf{e}_{31}B_2 + \mathbf{e}_{12}B_3) \\
&= -\mathbf{e}_3\partial_1 B_2 + \mathbf{e}_2\partial_1 B_3 + \mathbf{e}_3\partial_2 B_1 - \mathbf{e}_1\partial_2 B_3 - \mathbf{e}_2\partial_3 B_1 + \mathbf{e}_1\partial_3 B_2 \\
&= -\left[\mathbf{e}_1\left(\partial_2 B_3 - \partial_3 B_2\right) + \mathbf{e}_2\left(\partial_3 B_1 - \partial_1 B_3\right) + \mathbf{e}_3\left(\partial_1 B_2 - \partial_2 B_1\right)\right] \\
&= -\nabla\times\mathbf{B}
\end{aligned} \tag{2.23}$$

which proves the relation (2.18).

Equation (2.20) tells us that the external product of ∇ with a bivector $\mathbf{b}\wedge\mathbf{c}$ is equal to the divergence of $\mathbf{b}\times\mathbf{c}$ multiplied by i. The latter can be verified by considering

$$\begin{aligned}
\nabla\wedge\hat{\mathbf{B}} &= (\mathbf{e}_1\partial_1 + \mathbf{e}_2\partial_2 + \mathbf{e}_3\partial_3)\wedge(\mathbf{e}_{23}B_1 + \mathbf{e}_{31}B_2 + \mathbf{e}_{12}B_3) \\
&= \mathbf{e}_{123}\partial_1 B_1 + \mathbf{e}_{231}\partial_2 B_2 + \mathbf{e}_{312}\partial_3 B_3 \\
&= i\nabla\cdot\mathbf{B}
\end{aligned} \tag{2.24}$$

since

$$i = \mathbf{e}_{123} = \mathbf{e}_{231} = \mathbf{e}_{312}\,. \tag{2.25}$$

Hence, Table 2.5 can be extended so as to include the conventional $\nabla\times\nabla\times\mathbf{A}$ term as

In Table 2.6 we have used the equivalences:

$$\nabla\cdot\nabla\wedge\mathbf{A} = -\nabla\times\nabla\times\mathbf{A} \tag{2.26}$$

$$\nabla\wedge\nabla\wedge\mathbf{A} = \nabla\cdot\nabla\times\mathbf{A} = 0\,. \tag{2.27}$$

Table 2.6. For a vector function $\mathbf{A}$ application of the nabla operator gives a scalar (i.e. the divergence), and a bivector. From $\nabla\cdot\mathbf{A}$ a further application of the nabla operator gives the vector term $\nabla\nabla\cdot\mathbf{A}$, while the nabla operator applied to $\nabla\wedge\mathbf{A}$ gives the term $\nabla\cdot\nabla\wedge\mathbf{A}$ and the term $\nabla\wedge\nabla\wedge\mathbf{A}$ which is equal to zero.

Grade	0	1	2	3
quantity		$\mathbf{A}$		
$\nabla\mathbf{A}$	$\nabla\cdot\mathbf{A}$		$\nabla\wedge\mathbf{A}$	
$\nabla\nabla\mathbf{A}=\nabla^2\mathbf{A}$		$\nabla\nabla\cdot\mathbf{A}+\nabla\cdot\nabla\wedge\mathbf{A}$		$\nabla\wedge\nabla\wedge\mathbf{A}=0$
$\nabla^2\mathbf{A}$		$\nabla\nabla\cdot\mathbf{A}-\nabla\times\nabla\times\mathbf{A}$		$i\nabla\cdot\nabla\times\mathbf{A}=0$

Geometric product of $\nabla(\mathbf{b}\,\mathbf{c})$

By formally substituting $\mathbf{a}$ with ∇ and considering the latter acting on $\mathbf{b}\,\mathbf{c}$ one derives from (2.17):

$$\begin{aligned}\nabla(\mathbf{b}\,\mathbf{c}) &= \nabla\cdot(\mathbf{b}\wedge\mathbf{c})+\nabla\wedge(\mathbf{b}\cdot\mathbf{c})+\nabla\wedge(\mathbf{b}\wedge\mathbf{c})\\ &= -\nabla\times(\mathbf{b}\times\mathbf{c})+\nabla(\mathbf{b}\cdot\mathbf{c})+i\nabla\cdot(\mathbf{b}\times\mathbf{c})\,.\end{aligned}\tag{2.28}$$

Naturally $\mathbf{b}\,\mathbf{c}$ is a product of two vectors and, as such, is a scalar plus a bivector. The operation $\nabla\wedge(\mathbf{b}\wedge\mathbf{c})$ provides a trivector, while the other two operations, $\nabla\cdot(\mathbf{b}\wedge\mathbf{c})$ and $\nabla\wedge(\mathbf{b}\cdot\mathbf{c})$, return a vector. Note that we have made use of the fact (cite Jancewicz) that, for a scalar function ϕ, we may interpret $\nabla\phi=\nabla\wedge\phi$ i.e. the gradient is obtained as an external product of a vector with a scalar.

Geometric product of $\nabla\nabla\,\mathbf{c}$

By formally substituting also $\mathbf{b}$ with ∇ and considering the latter acting on $\mathbf{c}$ one derives from (2.17):

$$\begin{aligned}\nabla(\nabla\,\mathbf{c}) = \nabla^2\mathbf{c} &= \nabla\cdot(\nabla\wedge\mathbf{c})+\nabla\wedge(\nabla\cdot\mathbf{c})+\nabla\wedge(\nabla\wedge\mathbf{c})\\ &= -\nabla\times(\nabla\times\mathbf{c})+\nabla(\nabla\cdot\mathbf{c})\,.\end{aligned}\tag{2.29}$$

Naturally $\nabla\,\mathbf{c}$ is a product of two vectors and, as such, is a scalar plus a bivector. The operation $\nabla\wedge(\nabla\wedge\mathbf{c})$ provides a trivector of zero value, while the other two operations, $\nabla\cdot(\nabla\wedge\mathbf{c})$ and $\nabla\wedge(\nabla\cdot\mathbf{c})$, return a vector.

Second order derivatives

Always referring to (2.17) it is also possible to make the additional substitution of **b** with ∇. It is noted that in GA we have (the external part of a vector to itself is null)

$$\nabla \wedge \nabla \wedge \mathbf{c} = 0\,, \tag{2.30}$$

and therefore we get the identity:

$$\nabla \nabla \mathbf{c} = \nabla \cdot (\nabla \wedge \mathbf{c}) + \nabla \wedge (\nabla \cdot \mathbf{c}) \tag{2.31}$$

$$= \nabla^2 \mathbf{c} = \Delta \mathbf{c} \tag{2.32}$$

where Δ is the *Laplacian*. It is straightforward to prove that the above identity also holds for multivectors.

Noticeable cases

From (2.30) applied to a scalar function function ϕ and taking into account (2.14) one gets

$$\nabla \wedge \nabla \wedge \phi = 0 = i \left[\nabla \times (\nabla \phi)\right]\,, \tag{2.33}$$

i.e. the classical curl(grad $\phi = 0$). Moreover, from (2.30) and (2.16) we get:

$$\nabla \wedge \nabla \wedge \mathbf{c} = 0 = i \left[\nabla \cdot (\nabla \times \mathbf{c})\right]\,, \tag{2.34}$$

i.e. div (curl $\mathbf{c} = 0$).

The above identities suggest that if we have e.g. a vector field **E**, which has to satisfy $\nabla \wedge \mathbf{E} = 0$, such a field can be expressed as $\mathbf{E} = \nabla \phi$ and (2.33) will be automatically satisfied. Similarly, if we have a bivector field $\hat{\mathbf{H}}$, which has to satisfy $\nabla \wedge \hat{\mathbf{H}}$, the bivector field $\hat{\mathbf{H}}$ can be expressed as $\hat{\mathbf{H}} = \nabla \wedge \mathbf{A}$, with (2.34) satisfied. Note that in (2.34) we can also give another interpretation by looking at the r.h.s.. In fact, we can say that if a vector **H** has to have zero divergence than it can be expressed as the curl of a vector **A**.

V Nabla operator in circular cylindrical coordinate using Pauli matrices

The gradient operator of a scalar function w in circular cylindrical coordinates has the following form:

$$\nabla w = \mathbf{u}_\rho \frac{\partial w}{\partial \rho} + \mathbf{u}_\phi \frac{1}{\rho}\frac{\partial w}{\partial \phi} + \mathbf{u}_z \frac{\partial w}{\partial z}, \tag{2.35}$$

from which we can infer the Pauli matrix representation of the nabla operator $\tilde{\nabla}$ in cylindrical coordinates. By substituting the unit versors with the matrices in (1.266), and using the abbreviated notation e.g. $\partial_\rho = \partial/\partial\rho$, we obtain

$$\begin{aligned}
\tilde{\nabla} &= \sigma_\rho\, \partial_\rho + \frac{1}{\rho}\sigma_\phi\, \partial_\phi + \sigma_z\, \partial_z \\
&= \begin{pmatrix} \partial_z & \frac{e^{-i\phi}}{\rho}\left(\rho\, \partial_\rho - i\, \partial_\phi\right) \\ \frac{e^{i\phi}}{\rho}\left(\rho\, \partial_\rho + i\, \partial_\phi\right) & -\partial_z \end{pmatrix} \\
&= \begin{pmatrix} \partial_z & \nabla_t^* \\ \nabla_t & -\partial_z \end{pmatrix}
\end{aligned} \tag{2.36}$$

where we have introduced the operator ∇_t and its complex conjugate ∇_t^* defined as

$$\begin{aligned}
\nabla_t &= \frac{e^{i\phi}}{\rho}\left(\rho\, \partial_\rho + i\, \partial_\phi\right) \\
\nabla_t^* &= \frac{e^{-i\phi}}{\rho}\left(\rho\, \partial_\rho - i\, \partial_\phi\right).
\end{aligned} \tag{2.37}$$

Let us now consider a vector $\mathbf{A}$ expressed in terms of cylindrical Pauli matrices as:

$$\tilde{A} = \begin{pmatrix} A_z & e^{-i\phi}\left(A_\rho - iA_\phi\right) \\ e^{i\phi}\left(A_\rho + iA_\phi\right) & -A_z \end{pmatrix} \tag{2.38}$$

and let us recall that, due to the fundamental identity of geometric algebra, we also have:

$$\nabla\mathbf{A} = \nabla\cdot\mathbf{A} + \nabla\wedge\mathbf{A}. \tag{2.39}$$

By performing the matrix multiplication of (2.36) with (2.38), and by denoting with $\left(\tilde{\nabla}\tilde{A}\right)_{ij}$ the ij element of the matrix, we obtain:

$$\begin{aligned}
\left(\tilde{\nabla}\tilde{A}\right)_{11} &= \nabla\cdot\mathbf{A} + i\left[\frac{A_\phi}{\rho} + \frac{\partial A_\phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right] \\
\left(\tilde{\nabla}\tilde{A}\right)_{12} &= \left[-\left(\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\rho}{\partial z}\right) + i\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\right] e^{-i\phi} \\
\left(\tilde{\nabla}\tilde{A}\right)_{21} &= \left[\left(\frac{\partial A_z}{\partial\rho} - \frac{\partial A_\rho}{\partial z}\right) + i\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\right] e^{i\phi} \\
\left(\tilde{\nabla}\tilde{A}\right)_{22} &= \nabla\cdot\mathbf{A} - i\left[\frac{A_\phi}{\rho} + \frac{\partial A_\phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi}\right].
\end{aligned} \tag{2.40}$$

In (2.40), the divergence term

$$\nabla\cdot\mathbf{A} = \frac{\partial A_\rho}{\partial\rho} + \frac{A_\rho}{\rho} + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{2.41}$$

has been singled out.

The matrix $\tilde{\nabla}\tilde{A}$ contains therefore the divergence term, which can be obtained by dividing by two the matrix trace, and contains also all the terms relative to the external product o$\tilde{\nabla}$ and $\tilde{A}$. The specific component of the part corresponding to the external product, i.e. to the bivector, is obtained in the following manner. First the matrix $\tilde{\nabla}\wedge\tilde{A}$ is obtained as

$$\tilde{\nabla}\wedge\tilde{A} = \tilde{\nabla}\tilde{A} - (\nabla\cdot\mathbf{A})\,\sigma_0\,. \tag{2.42}$$

Then the component are retrieved by dot multiplication for the appropriate Pauli matrix. The dot multiplication is obtained from the multiplication of the matrices and then by taking half of the trace. As an example for the ρ component we have

$$\begin{aligned}
(\nabla\times\mathbf{A})_\rho &= \frac{1}{2i}trace\left\{\left(\tilde{\nabla}\wedge\tilde{A}\right)\sigma_\rho\right\} \\
&= \frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}
\end{aligned} \tag{2.43}$$

while for the ϕ component one has

$$(\nabla \times \mathbf{A})_\phi = \frac{1}{2i} trace \left\{ \left(\tilde{\nabla} \wedge \tilde{A} \right) \sigma_\phi \right\} = \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \tag{2.44}$$

and, finally, for the z component we obtain

$$(\nabla \times \mathbf{A})_z = \frac{1}{2i} trace \left\{ \left(\tilde{\nabla} \wedge \tilde{A} \right) \sigma_z \right\} = \frac{A_\phi}{\rho} + \frac{\partial A_\phi}{\partial \rho} + \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} . \tag{2.45}$$

Second order operators

Further application of the nabla operator allows to obtain

$$\tilde{\nabla} \tilde{\nabla} \tilde{A} = \tilde{\nabla}^2 \tilde{A} \tag{2.46}$$

which, in terms of Pauli matrices, can be be computed by matrix multiplication. In vector terms the second order operator gives the vector Laplacian $\nabla^2 \mathbf{A}$:

$$\nabla (\nabla \mathbf{A}) = \nabla^2 \mathbf{A} = \nabla \cdot \nabla \wedge \mathbf{A} + \nabla \wedge \nabla \cdot \mathbf{A} \tag{2.47}$$

which shows that the final result is a vector.
It is convenient to introduce the scalar Laplacian operator ∇_s^2 defined as:

$$\nabla_s^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \tag{2.48}$$

As a result of the matrix multiplication, and by identifying the various components, the following results are obtained:

$$\begin{aligned} \left(\tilde{\nabla}^2 \tilde{A} \right)_\rho &= \nabla_s^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2} \frac{\partial A_\phi}{\partial \phi} \\ \left(\tilde{\nabla}^2 \tilde{A} \right)_\phi &= \nabla_s^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2} \frac{\partial A_\rho}{\partial \phi} \\ \left(\tilde{\nabla}^2 \tilde{A} \right)_\rho &= \nabla_s^2 A_z . \end{aligned} \tag{2.49}$$

It is important to note the differences between the scalar Laplacian (2.48) and the vector Laplacian (2.47). In conventional vector algebra the vector Laplacian is expressed as

$$\nabla^2 \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla \times \nabla \times \mathbf{A} \,. \tag{2.50}$$

A code illustrating also the grade behavior is shown next in

```
wxm/2Nabla_cyl_5.wxm
```

In this code the block Nablacyl performs the nabla operation on a Pauli matrix, while the block Gradecyl perform the grade extraction on a Pauli matrix in cylindrical coordinates.
It is assumed that the multivector M has the following form

$$M = a_0 + a_\rho \sigma_\rho + a_\phi \sigma_\phi + a_z \sigma_3 + i \left(B_\rho \sigma_\rho + B_\phi \sigma_\phi + B_z \sigma_3 \right) + i t_0 \tag{2.51}$$

Optionally, these operations are used to find, in the following code, the gradient of a line charge distribution and the magnetic field on a current line.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
load(Pauli_v02)$
functions;

depends([Arho,Aphi,Az],[%rho,phi,z]);
Arpz : Arho*%sigma[%rho] + Aphi * %sigma[phi] + Az * %sigma[z];

NArpz :Nablacyl(Arpz);

M:NArpz$
Gradecyl(M) $

print("------------------------------------------------------------")$
print("Second order")$
print("------------------------------------------------------------")$
NNArpz : Nablacyl(NArpz);
M:NNArpz$
Gradecyl(M) $

/*
print("------------------------------------------------------------")$
print("An example")$
print("------------------------------------------------------------")$
print("Electric charge distribution along a line (z) Harrington pag. 107")$
print("Potential given as ")$
V : log(%rho);
AV : V*%sigma[0];
NAV :Nablacyl(AV);
M : NAV$
Gradecyl(M) $

print("------------------------------------------------------------")$
print("Another example")$
```

```
print("------------------------------------------------------")$
/* Harrington pag. 136 eq. 6-44 */
Az : log(%rho);
PotA :  Az * %sigma[z];
NAV :Nablacyl(PotA);
M : NAV$
Gradecyl(M) $
*/
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

In the following code the relevant quantities are computed first in terms of conventional operators. Successively, they are computed in terms of Pauli matrices. When feasible a test has been introduced showing that the results are coincident.

wxm/2Pnabla_cyl.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
/* Code name: Pnabla_cyl.wxm */

kill(all)$
assume(%rho>0)$
load(Pauli_v02)$
functions;

print("------------------------------------------------------------")$
print("Conventional Vector Analysis")$
print("------------------------------------------------------------")$

print("the package vect allows to perform vector operations")$
load(vect);

print("select the coordinate system by using the appropriate scalefactors")$
print("Better to use phi since %phi is reserved")$

scalefactors([[%rho*cos(phi),%rho*sin(phi),z],%rho,phi,z]);
depends(w,[%rho,phi,z]);
depends([Arho,Aphi,Az],[%rho,phi,z]);

print("Cylindrical COORDINATES")$
print("Gradient of w")$
g : grad (w)$
express (g)$
gradw : ev (%, diff);

print("Divergence")$
d : div ([Arho,Aphi,Az])$
express (d)$
divA1 : ev (%, diff);

print("Curl")$
```

```
curl ([Arho,Aphi,Az])$
express (%)$
curl_A1 : ev (%, diff)$
curl_A1_rho : curl_A1[1]$
curl_A1_phi : curl_A1[2]$
curl_A1_z   : curl_A1[3]$
print('curl_A1_rho," = ",curl_A1_rho)$
print('curl_A1_phi," = ",curl_A1_phi)$
print('curl_A1_z," = ",curl_A1_z)$

print("grad(divA1)")$
g : grad (divA1);
express (g)$
graddivA1 : factor(ev (%, diff))$
graddivA1_rho : graddivA1[1]$
graddivA1_phi : graddivA1[2]$
graddivA1_z   : graddivA1[3]$
print('graddivA1_rho," = ",graddivA1_rho)$
print('graddivA1_phi," = ",graddivA1_phi)$
print('graddivA1_z,"   = ",graddivA1_z)$

print("curlcurlA1")$
ccA1 : curl(curl_A1);
express (%)$
curlcurlA1 : ev (%, diff)$
curlcurlA1_rho : curlcurlA1[1]$
curlcurlA1_phi : curlcurlA1[2]$
curlcurlA1_z   : curlcurlA1[3]$
print('curlcurlA1_rho," = ",curlcurlA1_rho)$
print('curlcurlA1_phi," = ",curlcurlA1_phi)$
print('curlcurlA1_z,"   = ",curlcurlA1_z)$

print("nabla^2 A = grad (div A) -nabla x nabla xA")$
N2A_rho : ratsimp(graddivA1_rho - curlcurlA1_rho)$
N2A_phi : ratsimp(graddivA1_phi - curlcurlA1_phi)$
N2A_z : ratsimp(graddivA1_z - curlcurlA1_z)$
print('N2A_rho," = ",N2A_rho)$
```

```
print('N2A_phi," = ",N2A_phi)$
print('N2A_z,"   = ",N2A_z)$

print("------------------------------------------------------------")$
print("End conventional Vector Analysis")$
print("------------------------------------------------------------")$



[Arho,Aphi,Az];

print("Nabla as Pauli matrix")$
Np : dd[%rho] * %sigma[%rho] + dd[phi]/%rho * %sigma[phi] + dd[z] * %sigma[3]$
Np : factor(Np);

print("vector A as Pauli matrix")$
Ap : factor(Arho * %sigma[%rho] + Aphi * %sigma[phi] + Az * %sigma[3]);

print("Nabla A is")$

NEp: Nablacyl(Ap)$

NEp : expand(NEp);

print("The divergence is ")$
divA : expand(1/2*(NEp[1,1]+NEp[2,2]));
print("Test divergence")$
ratsimp(divA-divA1);

/* The external product is */
NeEp : ratsimp(NEp -divA * %sigma[0])$

print("The rho component is ")$
rr : NeEp.%sigma[%rho]$
CurlA_rho : expand(1/2*(rr[1,1]+rr[2,2]));
print("Test curl_rho")$
ratsimp(CurlA_rho - curl_A1_rho*%i);
```

```
print("The phi component is ")$
rr : NeEp.%sigma[phi]$
CurlA_phi : expand(1/2*(rr[1,1]+rr[2,2]));
print("Test curl_phi")$
ratsimp(CurlA_phi - curl_A1_phi*%i);

print("The z component is ")$
CurlA_z : expand(1/2*(NEp[1,1]-NEp[2,2]));
print("Test curl_z")$
ratsimp(CurlA_z - curl_A1_z*%i);

print("------------------------------------------------------------")$
print("Second order operator")$
print("------------------------------------------------------------")$
print("Nabla Nabla A is")$

NNEp : Nablacyl(NEp)$

NNEp : ratsimp(NNEp);

print("The divergence is ")$
NdivA : expand(1/2*(NNEp[1,1]+NNEp[2,2]));

/* The external product is */
NNeEp : ratsimp(NNEp -NdivA * %sigma[0])$

print("The rho component is ")$
rr : NNeEp.%sigma[%rho]$
NCurlA_rho : expand(1/2*(rr[1,1]+rr[2,2]));
print("Test ")$
ratsimp(NCurlA_rho - N2A_rho) ;

print("The phi component is ")$
rr : NNeEp.%sigma[phi]$
NCurlA_phi : expand(1/2*(rr[1,1]+rr[2,2]));
print("Test ")$
ratsimp(NCurlA_phi - N2A_phi) ;
```

```
print("The z component is ")$
NCurlA_z : expand(1/2*(NNEp[1,1]-NNEp[2,2]));
print("Test ")$
ratsimp(NCurlA_z - N2A_z) ;

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

It is noted that so far we have considered all the components with all the spatial dependencies. In several cases noticeable simplifications can occur. Consider e.g. the case of infinite filament of constant current, placed in the z direction. The potential A will be of the form $A_z(\rho)$ and therefore the problem is considerably simpler.

Example 10. We start by looking at the gradient of the scalar function

$$V = V_0\, e^{-2\rho} \sin(3\phi) \tag{2.52}$$

which may be written in terms of Pauli matrices as

$$\tilde{V} = \begin{pmatrix} V_0\, e^{-2\rho} \sin(3\phi) & 0 \\ 0 & V_0\, e^{-2\rho} \sin(3\phi) \end{pmatrix} \tag{2.53}$$

After application of the nabla operator and grade extraction one get that all components are zero apart for the following ones

$$\begin{aligned} a_\rho &= -2V_0\, e^{-2\rho} \sin(3\phi) \\ a_\phi &= \frac{3V_0\, e^{-2\rho} \cos(3\phi)}{\rho} \end{aligned} \tag{2.54}$$

Example 11. It is required to evaluate the divergence for the following vector:

$$A_\rho = \rho\cos(\phi)$$
$$A_\phi = \rho\sin(\phi)$$
$$A_\rho = 3z$$

Solution: the vector in terms of Pauli matrix is

$$\tilde{A} = \begin{pmatrix} 3z & \rho\, e^{-i\phi}\cos(\phi) - i\rho\, e^{-i\phi}\sin(\phi) \\ i\rho\, e^{i\phi}\sin(\phi) + \rho\, e^{i\phi}\cos(\phi) & -3z \end{pmatrix} \tag{2.55}$$

Application of the nabla operator gives

$$\tilde{\nabla}\tilde{A} = \begin{pmatrix} 3\left(i\sin(\phi) + \cos(\phi) + 1\right) & 0 \\ 0 & -3\left(i\sin(\phi) - \cos(\phi) - 1\right) \end{pmatrix}$$

Analysis of the grade structure provides

$$a_0 = \nabla\cdot\mathbf{A} = 3\cos(\phi) + 3$$
$$B_z = 3\sin(\phi) \tag{2.56}$$

with all other elements equal to zero.

Example 12. It is required to evaluate the curl for the following vector:

$$A_\rho = e^{-2\rho}\cos(\phi)$$
$$A_\phi = 0$$
$$A_\rho = \sin(\phi)$$

Solution: the vector in terms of Pauli matrix is

$$\tilde{A} = \begin{pmatrix} \sin(\phi) & e^{-i\phi-2\rho}\cos(\phi) \\ e^{i\phi-2\rho}\cos(\phi) & -\sin(\phi) \end{pmatrix} \tag{2.57}$$

Application of the nabla operator gives

$$\tilde{\nabla}\tilde{A} = \begin{pmatrix} \frac{e^{-2\rho}\left(i\sin(\phi)-2\rho\cos(\phi)+\cos(\phi)\right)}{\rho} & \frac{ie^{-i\phi}\cos(\phi)}{\rho} \\ \frac{ie^{i\phi}\cos(\phi)}{\rho} & -\frac{e^{-2\rho}\left(i\sin(\phi)+2\rho\cos(\phi)-\cos(\phi)\right)}{\rho} \end{pmatrix}$$

Analysis of the grade structure provides

$$\begin{aligned} a_0 &= \nabla\cdot\mathbf{A} = \frac{e^{-2\rho}\cos(\phi)}{\rho} - 2e^{-2\rho}\cos(\phi) \\ B_\rho &= \frac{\cos(\phi)}{\rho} \\ B_z &= \frac{e^{-2\rho}\sin(\phi)}{\rho} \end{aligned}$$

with all other elements equal to zero.

The code with all the last three examples is given

```
wxm/2Example_cyl1.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
load(Pauli_v02)$
functions;

/* Nabla_cyl */
print("Introduce functional dependence")$
depends([a11,a12,a21,a22],[%rho,phi,z]);

Ap : matrix([a11,a12],[a21,a22]);

depends(w,[%rho,phi,z]);

print("--------------------------------------------")$
print("Example of gradient in cylindrical coordinates")$
print("--------------------------------------------")$
w : V[0]*%e^(-2*%rho)*sin(3*phi);

a11 : w$
a22: a11$
a12:0 $
a21 : 0$
Ap : matrix([a11,a12],[a21,a22]);

NCAp : Nablacyl(Ap);

print("retrieving the components")$

M:NCAp$
Gradecyl(M) $

print("--------------------------------------------")$
print("Example of Divergence")$
print("--------------------------------------------")$
```

```
print("given the vector A")$
Arho : %rho * cos(phi) ;
Aphi : %rho * sin(phi) ;
Az : 3*z;

print("compute the divergence in (2,0,3)")$
print("Write the vector in Pauli matrix form")$
Arpz : Arho * %sigma[%rho] + Aphi * %sigma[phi] + Az * %sigma[3];

print("apply the nabla operator")$
NCAp : Nablacyl(Arpz);

M:NCAp$
Gradecyl(M) $

print("-----------------------------------------------")$
print("Example of curl")$
print("-----------------------------------------------")$
print("compute curl A with A given by")$
Arho : %e^(-2*%rho)*cos(phi);
Aphi : 0;
Az : sin(phi);

print("Solution: ")$
print("Express vector A in Pauli form")$
Arpz : Arho * %sigma[%rho] + Aphi * %sigma[phi] + Az * %sigma[3];

print("apply the nabla operator")$
NCAp : Nablacyl(Arpz);

M:NCAp$
Gradecyl(M) $

print("remember that nabla wedge A = i nabla x A")$
print("end")$
/* [wxMaxima: input   end   ] */
```

```
/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

VI Nabla operator in spherical coordinates using Pauli matrices

The gradient operator of a scalar function w in spherical coordinates has the following form:

$$\nabla w = \mathbf{u}_r \frac{\partial w}{\partial r} + \mathbf{u}_\theta \frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{1}{r\sin\theta}\mathbf{u}_\phi \frac{\partial w}{\partial \phi}, \tag{2.58}$$

from which we can infer the Pauli matrix representation of the nabla operator $\tilde{\nabla}$ in spherical coordinates. By substituting the unit versors with the matrices in (1.276), here repeated for convenience,

$$\begin{aligned}
\sigma_r &= \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix} \\
\sigma_\theta &= \begin{pmatrix} -\sin\theta & e^{-i\phi}\cos\theta \\ e^{i\phi}\cos\theta & \sin\theta \end{pmatrix} \\
\sigma_\phi &= \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}.
\end{aligned} \tag{2.59}$$

and by using the abbreviated notation e.g. $\partial_r = \partial/\partial r$, we obtain

$$\begin{aligned}
\tilde{\nabla} &= \sigma_r \partial_r + \frac{1}{r}\sigma_\theta \partial_\theta + \frac{1}{r\sin\theta}\sigma_\phi \partial_\phi \\
&= \begin{pmatrix} \cos\theta\,\partial_r - \sin\theta\,\partial_\theta & e^{-i\phi}\left(\sin\theta\,\partial_r + \cos\theta\,\partial_\theta - i\,\partial_\phi\right) \\ e^{i\phi}\left(\sin\theta\,\partial_r + \cos\theta\,\partial_\theta + i\,\partial_\phi\right) & -\left(\cos\theta\,\partial_r - \sin\theta\,\partial_\theta\right) \end{pmatrix}
\end{aligned} \tag{2.60}$$

Let us now consider a vector $\mathbf{A}$ expressed in terms of spherical Pauli matrices as:

$$\tilde{A} = \begin{pmatrix} A_r\cos\theta - A_\theta\sin\theta & e^{-i\phi}\left(A_r\sin\theta - iA_\phi + A_\theta\cos\theta\right) \\ e^{i\phi}\left(A_r\sin\theta + iA_\phi + A_\theta\cos\theta\right) & -A_r\cos\theta + A_\theta\sin\theta \end{pmatrix} \tag{2.61}$$

and let us recall that, due to the fundamental identity of geometric algebra, we also have:

$$\nabla \mathbf{A} = \nabla \cdot \mathbf{A} + \nabla \wedge \mathbf{A} \,. \tag{2.62}$$

By performing the matrix multiplication of (2.60) with (2.61), we obtain

$$\tilde{\nabla} \tilde{A} = \nabla \cdot \mathbf{A}\, \sigma_0 + i \left(n_r\, \sigma_r + n_\theta\, \sigma_\theta + n_\phi\, \sigma_\phi \right) , \tag{2.63}$$

where the following symbols have been used:

$$\begin{aligned} \nabla \cdot \mathbf{A} &= \partial_r A_r + \frac{2A_r}{r} + \frac{\partial_\theta A_\theta}{r} + \frac{\cos\theta\, A_\theta}{r \sin\theta} + \frac{\partial_\phi A_\phi}{r \sin\theta} \\ n_r &= \frac{1}{r \sin\theta} \left(-\partial_\phi A_\theta + \sin\theta\, \partial_\theta A_\phi + \cos\theta\, A_\phi \right) \\ n_\theta &= \frac{\partial_\phi\, A_r}{r \sin\theta} - \frac{A_\phi}{r} - \partial_r A_\phi \\ n_\phi &= -\frac{\partial_\theta A_r}{r} + \frac{A_\theta}{r} + \partial_r A_\theta \,. \end{aligned} \tag{2.64}$$

In (2.64) the term $\nabla \cdot \mathbf{A}$ is the conventional divergence. Since $\nabla \wedge \mathbf{A} = i \nabla \times \mathbf{A}$ the other terms n_r, n_θ, n_ϕ are simply the components along r, θ and ϕ of the curl operator multiplied by i.
It is noted that while in conventional algebra there is no single operator providing both the divergence and the curl, by using Pauli matrices a single operator (2.60) exists for the nabla representation.

Second order operators

Further application of the nabla operator allows to obtain

$$\tilde{\nabla} \tilde{\nabla} \tilde{A} = \tilde{\nabla}^2 \tilde{A} \tag{2.65}$$

which, in terms of Pauli matrices, can be be computed by matrix multiplication. In vector terms the second order operator gives the vector Laplacian $\nabla^2 \mathbf{A}$:

$$\nabla (\nabla \mathbf{A}) = \nabla^2 \mathbf{A} = \nabla \cdot \nabla \wedge \mathbf{A} + \nabla \wedge \nabla \cdot \mathbf{A} \tag{2.66}$$

which shows that the final result is a vector.

It is convenient to introduce the scalar Laplacian operator for spherical coordinates ∇_s^2 defined as:

$$\nabla_s^2 w = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial\phi^2} \tag{2.67}$$

and the coefficients

$$\begin{aligned}
N_r &= \nabla_s^2 A_r - \frac{2}{r^2\sin\theta}\left(A_r\sin\theta + A_\theta\cos\theta + \sin\theta\,\partial_\theta A_\theta + \partial_\phi A_\phi\right)\\
N_\theta &= \nabla_s^2 A_\theta + \frac{1}{r^2\sin^2\theta}\left(2\sin^2\theta\,\partial_\theta A_r - A_\theta - 2\cos\theta\partial_\phi A_\phi\right)\\
N_\phi &= \nabla_s^2 A_\theta + \frac{1}{r^2\sin^2\theta}\left(2\,\partial_\phi A_r + 2\cos\theta\partial_\phi A_\theta - A_\phi\right).
\end{aligned} \tag{2.68}$$

The second order nabla operator can finally be expressed as:

$$\tilde{\nabla}^2\tilde{A} = N_r\,\sigma_r + N_\theta\,\sigma_\theta + N_\phi\,\sigma_\phi\,, \tag{2.69}$$

It is important to note the differences between the scalar Laplacian (2.67) and the vector Laplacian (2.69). In conventional vector algebra the vector Laplacian is expressed as

$$\nabla^2\mathbf{A} = \nabla\nabla\cdot\mathbf{A} - \nabla\times\nabla\times\mathbf{A}\,. \tag{2.70}$$

An efficient way to perform the evaluation of the nabla operator in the spherical coordinate system and to ascertain the various grades is provided in the following code.

```
wxm/3Nabla_sph_3.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02)$
functions;

print("introducing the dependencies")$
depends([a11,a12,a21,a22],[r,%theta,phi]);
Ap : matrix([a11,a12],[a21,a22]);

depends([Ar,Atheta,Aphi],[r,%theta,phi])$
print("------------------------------------------------------------")$
print("vector A in spherical coordinates")$
print("------------------------------------------------------------")$
Asph : trigsimp(factor(Ar*%sigma[r]+Atheta*%sigma[%theta]+Aphi*%sigma[phi]));

Ap : Asph$

print("------------------------------------------------------------")$
print("Performing Nabla A")$
NA  : Nablasph(Ap);

print("------------------------------------------------------------")$
M : NA$
print("Extracting the Grades")$
Gradesph(M)$

print("------------------------------------------------------------")$
print(" ATTENTION : SECOND ORDER OPERATORS")$
print("------------------------------------------------------------")$
print("Performing Nabla Nabla A")$
NNA  : Nablasph(NA);

print("------------------------------------------------------------")$
print("Extracting the Grades")$
```

```
M : NNA$
Gradesph(M)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

In the following code the relevant quantities are computed first in terms of conventional operators. Then they are computed in terms of Pauli matrices. When feasible a test has been introduced showing that the results are coincident.

wxm/3Pnabla_sph2.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
/* Code name: Pnabla_sph.wxm */

kill(all)$
assume(r>0);
assume(sin(%theta) > 0);

print("------------------------------------------------------")$
print("Conventional Vector Analysis")$
print("------------------------------------------------------")$

print("the package vect allows to perform vector operations")$
load(vect);

print("select the coordinate system with the appropriate scalefactors")$
print("Better to use phi since %phi is reserved")$

scalefactors([[r*sin(%theta)*cos(phi),r*sin(%theta)*sin(phi),r*cos(%theta)]
                ,r,%theta,phi]);

depends(w,[r,%theta,phi]);
depends([Ar,Atheta,Aphi],[r,%theta,phi]);

print("SPHERICAL COORDINATES")$
print("Gradient of w")$
g : grad (w)$
express (g)$
gradw : ev (%, diff);

print("Divergence")$
d : div ([Ar,Atheta,Aphi])$
express (d)$
divA1 : expand(ev (%, diff));
```

```
/* divergence according to Harrington
divA1r : 1/r^2 * diff(r^2*Ar,r)$
divA1t : 1/(r * sin(%theta)) * diff(Atheta * sin(%theta),%theta)$
divA1f : 1/(r * sin(%theta))  * diff(Aphi,phi)$
divA2 : divA1r + divA1t + divA1f;
print("test")$
ratsimp(divA1 - divA2);
*/

print("Curl")$
curl ([Ar,Atheta,Aphi])$
express (%)$
curl_A1 : ev (%, diff)$
curl_A1_r : expand(curl_A1[1])$
curl_A1_theta : expand(curl_A1[2])$
curl_A1_phi   : expand(curl_A1[3])$
print('curl_A1_r," = ",curl_A1_r)$
print('curl_A1_theta," = ",curl_A1_theta)$
print('curl_A1_phi," = ",curl_A1_phi)$

/* curl according to Harrington pag 299
curl_A2_r : 1/(r * sin(%theta))  * (diff(Aphi*sin(%theta),%theta)
          - diff(Atheta,phi));
ratsimp(curl_A1_r-curl_A2_r);
curl_A2_theta : 1/(r )  * (1/sin(%theta) * diff(Ar,phi)
              - diff(Aphi*r,r));
ratsimp(curl_A1_theta-curl_A2_theta);
curl_A2_phi : 1/(r )  * ( diff(Atheta*r,r) - diff(Ar,%theta));
ratsimp(curl_A1_phi-curl_A2_phi);
*/

/*  computation of grad div A - curl curl A */
print("grad(divA1)")$
g : grad (divA1)$
express (g)$
graddivA1 : factor(ev (%, diff))$
graddivA1_r : graddivA1[1]$
```

```
graddivA1_theta : graddivA1[2]$
graddivA1_phi   : graddivA1[3]$
print('graddivA1_r," = ",graddivA1_r)$
print('graddivA1_theta," = ",graddivA1_theta)$
print('graddivA1_phi,"   = ",graddivA1_phi)$

print("curlcurlA1")$
ccA1 : curl(curl_A1);
express (%)$
curlcurlA1 : ev (%, diff)$
curlcurlA1_r : curlcurlA1[1]$
curlcurlA1_theta : curlcurlA1[2]$
curlcurlA1_phi   : curlcurlA1[3]$
print('curlcurlA1_r," = ",curlcurlA1_r)$
print('curlcurlA1_theta," = ",curlcurlA1_theta)$
print('curlcurlA1_phi,"   = ",curlcurlA1_phi)$


print("nabla^2 A = grad (div A) -nabla x nabla xA")$
N2A_r : ratsimp(graddivA1_r - curlcurlA1_r)$
N2A_theta : ratsimp(graddivA1_theta - curlcurlA1_theta)$
N2A_phi : ratsimp(graddivA1_phi - curlcurlA1_phi)$
N2A_r : expand(N2A_r)$
N2A_theta : expand(N2A_theta)$
N2A_phi : expand(N2A_phi)$
print('N2A_r," = ",N2A_r)$
print('N2A_theta," = ",N2A_theta)$
print('N2A_phi,"   = ",N2A_phi)$



print("------------------------------------------------------------")$
print("End conventional Vector Analysis")$
print("------------------------------------------------------------")$

load(Pauli_v02)$
functions;
```

```
[Ar,Atheta,Aphi];

print("Nabla as Pauli matrix")$
Np : dd[r] * %sigma[r] + dd[%theta]/r * %sigma[%theta]
    + dd[phi]/(r *sin(%theta)) * %sigma[phi];

print("Nabla A from div + i*curl")$
NablaA : %i * curl_A1_r * %sigma[r]
    + %i*curl_A1_theta * %sigma[%theta]
    + %i * curl_A1_phi * %sigma[phi] + divA1 * %sigma[0];

print("vector A as Pauli matrix")$
Ap : factor(Ar * %sigma[r] + Atheta * %sigma[%theta] + Aphi * %sigma[phi]);



/* computation of nabla A */
NEp : Nablasph(Ap)$
print("Nabla A is")$

NEp : expand(NEp);

print("Verification of nabla A expression")$
trigsimp(ratsimp(NEp - NablaA));



print("The divergence is ")$
divA : expand(1/2*(NEp[1,1]+NEp[2,2]));
print("Test divergence")$
ratsimp(divA-divA1);

/* The external product is */
NeEp : ratsimp(NEp -divA * %sigma[0])$

print("The r component is ")$
rr : NeEp.%sigma[r]$
CurlA_r : trigsimp(expand(1/2*(rr[1,1]+rr[2,2])))$
```

```
CurlA_r : expand(CurlA_r );
print("Test curl_r")$
trigsimp(ratsimp(CurlA_r - curl_A1_r*%i));

print("The theta component is ")$
rr : NeEp.%sigma[%theta]$
CurlA_theta : trigsimp(expand(1/2*(rr[1,1]+rr[2,2])))$
CurlA_theta : expand(CurlA_theta);
print("Test curl_theta")$
trigsimp(ratsimp(CurlA_theta - curl_A1_theta*%i));

print("The phi component is ")$
rr : NeEp.%sigma[phi]$
CurlA_phi : trigsimp(expand(1/2*(rr[1,1]+rr[2,2])))$
CurlA_phi : expand(CurlA_phi);
print("Test curl_phi")$
trigsimp(ratsimp(CurlA_phi - curl_A1_phi*%i));
 /* */

print("------------------------------------------------------------")$
print("Second order operator")$
print("------------------------------------------------------------")$

print("vector N2A as Pauli matrix")$
N2Ap : factor(N2A_r * %sigma[r] + N2A_theta * %sigma[%theta]
        + N2A_phi * %sigma[phi]);

print("Nabla Nabla A is")$

NNEp : Nablasph(NEp)$

NNEp : ratsimp(NNEp);
/* NNEp : expand(NNEp)$ */

trigsimp(N2Ap - NNEp);

print("The divergence is ")$
```

```
NdivA : expand(1/2*(NNEp[1,1]+NNEp[2,2]));

/* The external product is */
NNeEp : ratsimp(NNEp -NdivA * %sigma[0])$

print("The r component is ")$
rr : NNeEp . %sigma[r]$
NCurlA_r : trigsimp(expand(1/2*(rr[1,1]+rr[2,2])))$
NCurlA_r : expand(NCurlA_r) ;
print("Test ")$
trigsimp(NCurlA_r - N2A_r) ;

print("The theta component is ")$
rr : NNeEp . %sigma[%theta]$
NCurlA_theta : trigsimp(1/2*(rr[1,1]+rr[2,2]))$
NCurlA_theta : expand(NCurlA_theta) ;
print("Test ")$
trigsimp(NCurlA_theta - N2A_theta) ;

print("The phi component is ")$
rr : NNeEp.%sigma[phi]$
NCurlA_phi : trigsimp(1/2*(rr[1,1]+rr[2,2]))$
NCurlA_phi : expand(NCurlA_phi) ;
print("Test ")$
trigsimp(NCurlA_phi - N2A_phi) ;

print("Scalar Laplacian in spherical coordinates for Ar")$
l_spher : express(laplacian(Ar))$
l_spher : ev(l_spher,diff)$
l_spher :ratexpand(l_spher)$

print("Difference between NCurlA_r and laplacian")$
expand(ratsimp(NCurlA_r - l_spher)) ;

print("Scalar Laplacian in spherical coordinates for Atheta")$
l_spher : express(laplacian(Atheta))$
l_spher : ev(l_spher,diff)$
```

```
l_spher :ratexpand(l_spher)$

print("Difference between NCurlA_theta and laplacian")$
expand(ratsimp(NCurlA_theta - l_spher)) ;

print("Scalar Laplacian in spherical coordinates for Aphi")$
l_spher : express(laplacian(Aphi))$
l_spher : ev(l_spher,diff)$
l_spher :ratexpand(l_spher)$

print("Difference between NCurlA_phi and laplacian")$
expand(ratsimp(NCurlA_phi - l_spher)) ;

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Example 13. In this example we consider a vector which has only the A_z components and this component is

$$A_z = \frac{1}{r}$$

or, in terms of Pauli matrices,

$$\tilde{A} = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$$

By using Pauli matrices it is not necessary to translate this vector into the spherical coordinate system, since the vectors have the same representation in all the coordinate systems. The first operation is to perform the nabla operation in spherical coordinate system. By doing this operation we obtain

$$\tilde{\nabla}\tilde{A} = \begin{pmatrix} -\frac{\cos(\theta)}{r^2} & \frac{\sin(\theta)e^{-i\phi}}{r^2} \\ -\frac{\sin(\theta)e^{i\phi}}{r^2} & -\frac{\cos(\theta)}{r^2} \end{pmatrix} \tag{2.71}$$

After grade extraction we have

$$a_0 = -\frac{\cos(\theta)}{r^2}$$
$$B_\phi = \frac{\sin(\theta)}{r^2}$$

with all other components equal to zero. It is then required to obtain $\nabla \wedge \mathbf{A}$ which is readily obtained as

$$\nabla \wedge \mathbf{A} = \begin{pmatrix} 0 & \frac{\sin(\theta)e^{-i\phi}}{r^2} \\ -\frac{\sin(\theta)e^{i\phi}}{r^2} & 0 \end{pmatrix}.$$

Finally, it is required to compute $\nabla\nabla \wedge \mathbf{A}$ which gives:

$$\nabla\nabla \wedge \mathbf{A} = \begin{pmatrix} -\frac{3\cos(\theta)^2-1}{r^3} & -\frac{3\cos(\theta)\sin(\theta)e^{-i\phi}}{r^3} \\ -\frac{3\cos(\theta)\sin(\theta)e^{i\phi}}{r^3} & \frac{3\cos(\theta)^2-1}{r^3} \end{pmatrix}. \tag{2.72}$$

By analyzing the grade structure of this Pauli matrix we notice that the only components different from zero are

$$a_r = -\frac{2\cos(\theta)}{r^3}$$
$$a_\theta = -\frac{\sin(\theta)}{r^3}$$

The code for this example is reported in

```
wxm/3Dipole_sph_static2.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02)$
functions;

print("introducing the dependencies")$
depends([a11,a12,a21,a22],[r,%theta,phi]);
Ap : matrix([a11,a12],[a21,a22]);

depends([Ar,Atheta,Aphi],[r,%theta,phi])$
print("------------------------------------------------------------")$
print("vector A in spherical coordinates")$
print("------------------------------------------------------------")$

Az : 1/r;
print("Vector A for a dipole along z")$
Ar : Az * ct;
Atheta : - Az *st;
Aphi : 0;

Asph : trigsimp(factor(Ar*%sigma[r]+Atheta*%sigma[%theta]+Aphi*%sigma[phi]));
Ap : Asph$

print("------------------------------------------------------------")$
print("Performing Nabla A")$
NA  : Nablasph(Ap);

print("------------------------------------------------------------")$
M : NA$
print("Extracting the Grades")$
Gradesph(M)$
divA : a0$
print("Computing nabla wedge A")$
NAmdivA : NA - divA * %sigma[0];
```

```
/* */
print("------------------------------------------------------------")$
print(" ATTENTION : SECOND ORDER OPERATORS")$
print("------------------------------------------------------------")$

print("Performing Nabla Nabla wedge A")$
NNA  : Nablasph(NAmdivA);

print("------------------------------------------------------------")$
print("Extracting the Grades")$
M : NNA$
Gradesph(M)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Example 14. In the following it is understood that j,k,v are just symbols. Let us now assume that the considered vector $\mathbf{A}$ is given by

$$A_z = \frac{e^{-jkr}}{r}$$

and therefore the corresponding Pauli matrix is

$$\tilde{A} = \begin{pmatrix} \frac{e^{-jkr}}{r} & 0 \\ 0 & -\frac{e^{-jkr}}{r} \end{pmatrix} \tag{2.73}$$

It is first required to compute $\tilde{\nabla}\tilde{A}$. By using the standard approach we have that

$$\tilde{\nabla}\tilde{A} = \begin{pmatrix} -\frac{\cos(\theta)\,(jkr+1)\,e^{-jkr}}{r^2} & \frac{\sin(\theta)\,(jkr+1)\,e^{-jkr-i\phi}}{r^2} \\ -\frac{\sin(\theta)\,(jkr+1)\,e^{i\phi-jkr}}{r^2} & -\frac{\cos(\theta)\,(jkr+1)\,e^{-jkr}}{r^2} \end{pmatrix} \tag{2.74}$$

The grades different from zero relative to $\tilde{\nabla}\tilde{A}$ are:

$$a_0 = -\frac{\cos(\theta)\; jke^{-jkr}}{r} - \frac{\cos(\theta)\, e^{-jkr}}{r^2}$$
$$B_\phi = \frac{\sin(\theta)\; jke^{-jkr}}{r} + \frac{\sin(\theta)\, e^{-jkr}}{r^2}$$

We will see that this is a field proportional to the magnetic field of a dipole.
It is then required to compute the field evaluated from the potentials as

$$\Phi = -\frac{v}{jk}\nabla\cdot\mathbf{A}$$
$$\mathbf{E} = -jkv\mathbf{A} - \nabla\Phi$$

it is readily seen that the potential Φ is

$$\Phi = \frac{\left(\cos(\theta)\; jkr + \cos(\theta)\right) e^{-jkr}\, v}{jkr^2} \tag{2.75}$$

The field $\mathbf{E}$ is evaluated by performing the nabla operator on Φ By extracting the grades of $\mathbf{E}$ we have that the only grades different from zero are

$$a_r = \frac{2\cos(\theta)\, e^{-jkr} v}{r^2} + \frac{2\cos(\theta)\, e^{-jkr} v}{jkr^3}$$
$$a_\theta = \frac{\sin(\theta)\, jke^{-jkr} v}{r} + \frac{\sin(\theta)\, e^{-jkr} v}{r^2} + \frac{\sin(\theta)\, e^{-jkr} v}{jkr^3}$$

The code for this example is reported in

```
wxm/3Dipole_sph_3.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02)$
functions;
/* 3Dipole_sph_2.wxm */

print("introducing the dependencies")$
depends([a11,a12,a21,a22],[r,%theta,phi]);
Ap : matrix([a11,a12],[a21,a22]);

print("------------------------------------------------------------")$
print("vector A in spherical coordinates")$
print("------------------------------------------------------------")$

print("It is assumed that the dipole is oriented along z")$
print("also the potential will be oriented along z")$
Az : %e^(-j * k *r)/r;

print("Remeber that the Pauli matrix of a vector is independent of the coordinates")$
print("Accordingly, we write it in rectangular coordinates,")$
print("but it can be used for computation also in spherical coordinates")$

print("The constant c1 is also defined")$
c1 : %mu * I*dl/(4*%pi);
Ap : matrix([Az,0],[0,-Az]);

print("------------------------------------------------------------")$
print("Solution procedure")$
print("------------------------------------------------------------")$
print("The solution procedure is organized in 2 steps")$
print("------------------------------------------------------------")$
print("------------------------------------------------------------")$
print("1)  The magnetic field is obtained from i %eta H = v nabla wedge A ")$
print("------------------------------------------------------------")$
```

```
NA  : Nablasph(Ap);
print("The divergence is")$
divA : 1/2 *(NA[1,1]+NA[2,2]);
print("Nabla wedge A is")$
NWA : NA - divA*%sigma[0];

M : NWA$
print("Extracting the Grades")$
Gradesph(M)$
print("It is seen that i %eta H is only in the phi direction and is ")$
vBap : v * Bap$
Hr : factor(1/%mu * Bar)*c1$
Ht : factor(1/%mu * Bat)*c1$
Hphi : factor(1/%mu * Bap)*c1$
print('Hr," = ",Hr)$
print('Ht," = ",Ht)$
print('Hphi," = ",Hphi)$

print("------------------------------------------------------------")$
print("2)  The electric field is recoverd from j/k * nabla(i %eta H) ")$
print("------------------------------------------------------------")$
print("The Pauli matrix representing H is ")$
HP : %i * vBap * %sigma[phi];
EFA : j/k * Nablasph(HP)$
EFA :subst(-1,j^2,EFA );
M : EFA$
print("Extracting the Grades")$
Gradesph(M)$
print("The electric field is equal to")$
Er : ar$
Et : at$
Ep : ap$
Er : subst(-1,j^2,Er)$
Et : subst(-1,j^2,Et)$
Er : factor(Er)*c1$
Er : subst(%eta/v, %mu,Er)$
Et : factor(Et)*c1$
```

```
Et : subst(%eta/v, %mu,Et)$
print('Er," = ",Er)$
print('Et," = ",Et)$
print('Ep," = ",Ep)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

VII Summary of Nabla operator with Pauli matrices

Table 2.7. Basis vector in rectangular, cylindrical and spherical coordinate systems. The $\mathbf{e}_i$ are a Clifford basis and correspond to the appropriate Pauli matrices.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
rectangular	$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
cylindrical	$\sigma_\rho = \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}$	$\sigma_\phi = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$	$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
spherical	$\sigma_r = \begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$	$\sigma_\theta = \begin{pmatrix} -\sin\theta & e^{-i\phi}\cos\theta \\ e^{i\phi}\cos\theta & \sin\theta \end{pmatrix}$	$\sigma_\phi = \begin{pmatrix} 0 & -ie^{-i\phi} \\ ie^{i\phi} & 0 \end{pmatrix}$

$$\begin{aligned}\tilde{A} &= \sigma_1 A_x + \sigma_2 A_y + \sigma_3 A_z \\ &= \begin{pmatrix} A_z & A_x - iA_y \\ iA_y + A_x & -A_z \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{A} &= \sigma_\rho A_\rho + \sigma_\phi A_\phi + \sigma_3 A_z \\ &= \begin{pmatrix} A_z & e^{-i\phi}\left(A_\rho - iA_\phi\right) \\ e^{i\phi}\left(A_\rho + iA_\phi\right) & -A_z \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\tilde{A} &= \sigma_r A_r + \sigma_\theta A_\theta + \sigma_\phi A_\phi \\ &= \begin{pmatrix} A_r\cos\theta - A_\theta\sin\theta & e^{-i\phi}\left(A_r\sin\theta - iA_\phi + A_\theta\cos\theta\right) \\ e^{i\phi}\left(A_r\sin\theta + iA_\phi + A_\theta\cos\theta\right) & -A_r\cos\theta + A_\theta\sin\theta \end{pmatrix}\end{aligned}$$

Rectangular:

$$\begin{aligned}\tilde{\nabla} &= \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z \\ &= \begin{pmatrix} \partial_z & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & -\partial_z \end{pmatrix}.\end{aligned}$$

Cylindrical:

$$\begin{aligned}\tilde{\nabla} &= \sigma_\rho\,\partial_\rho + \frac{1}{\rho}\sigma_\phi\,\partial_\phi + \sigma_z\,\partial_z \\ &= \begin{pmatrix} \partial_z & \frac{e^{-i\phi}}{\rho}\left(\rho\,\partial_\rho - i\,\partial_\phi\right) \\ \frac{e^{i\phi}}{\rho}\left(\rho\,\partial_\rho + i\,\partial_\phi\right) & -\partial_z \end{pmatrix}\end{aligned}$$

Spherical:

$$\begin{aligned}\tilde{\nabla} &= \sigma_r\,\partial_r + \frac{1}{r}\sigma_\theta\,\partial_\theta + \frac{1}{r\sin\theta}\sigma_\phi\,\partial_\phi \\ &= \begin{pmatrix} \cos\theta\,\partial_r - \sin\theta\,\partial_\theta & e^{-i\phi}\left(\sin\theta\,\partial_r + \cos\theta\,\partial_\theta - i\,\partial_\phi\right) \\ e^{i\phi}\left(\sin\theta\,\partial_r + \cos\theta\,\partial_\theta + i\,\partial_\phi\right) & -\left(\cos\theta\,\partial_r - \sin\theta\,\partial_\theta\right) \end{pmatrix}\end{aligned}$$

VIII Nabla operator: conventional approach

Gradient

The gradient has been already introduced in section I.

Divergence

Divergence is a scalar function. Let us consider a field, at each point of which a vector **F** is specified. We can associate with each point P a scalar quantity, the divergence of **F**, defined as

$$div\,\mathbf{F} = \nabla \cdot \mathbf{F} = \lim_{\Delta\tau \to 0} \frac{1}{\Delta\tau} \oiint \mathbf{F} \cdot d\mathbf{s}. \tag{2.76}$$

The point P can be considered enclosed in a surface **s** of any shape, the volume within the surface being $\Delta\tau$. Take the total outward flux of the vector **F** through the bounding surface. The limit of the total outward flux, per unit volume, as the surface shrinks about P, is defined as div **F** at the point P. In general, div **F** is different for each point in the field.
In rectangular coordinates the divergence take the following form

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}. \tag{2.77}$$

Equation (2.91) is often used as the definition of divergence. However, since this equation is only valid in rectangular coordinates, it is better to use (2.76) as the definition for the divergence.
In circular cylindrical coordinates the divergence has the following expression

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}. \tag{2.78}$$

For spherical coordinates,

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}. \tag{2.79}$$

In many circumstances it is possible to evaluate the divergence without calculation. Whenever physical intuition indicates that the flux entering $\Delta\tau$ is the same as the flux leaving it, the integral in (2.76) must be zero and consequently also the divergence is zero. In electrostatics,

for example, one visualizes electric flux lines between charges. At any point P in a region without charges, the integral in (2.76) must be zero, and we can say immediately that $\nabla \cdot \mathbf{D} = 0$ at such a point. Only if there is a charge distribution of density ρ (coulomb m^{-3}) at P we will have $\nabla \cdot \mathbf{D} = \rho$, i.e. a *measure of the strength of the source* at P.
Similarly, one may say immediately that for the magnetic field $\nabla \cdot \mathbf{B} = 0$ always since there are no magnetic charges and the flux lines invariably form closed loops.

Divergence theorem or Gauss theorem

Let $\mathbf{A}(\mathbf{r})$ be any vector function of position, continuous together with its first derivative throughout a volume V bounded by a surface S. The divergence theorem states that

$$\oint_S \mathbf{A}(\mathbf{r}) \cdot \mathbf{n}\, dS = \int_V \nabla \cdot \mathbf{A}(\mathbf{r})\, dV \tag{2.80}$$

As a matter of fact the Gauss theorem is therefore used to define the divergence.

Curl

Consider an incremental element of area Δs and denote the unit normal to it by $\mathbf{u}_n$. It is possible to define the vector curl $\mathbf{F}$, represented symbolically as $\nabla \times \mathbf{F}$, as

$$curl\, \mathbf{F} = \nabla \times \mathbf{F} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\mathbf{u}_n \oint \mathbf{F} \cdot d\mathbf{l} \right]_{max} . \tag{2.81}$$

Note that in general we should repeat this procedure for three orthogonal surfaces and sum their contributions. However, assuming assuming to know the direction of the curl (i.e. considering the surface such that the result is max), we can use the above expression.
In rectangular coordinates the curl takes the following form:

$$\nabla \times \mathbf{F} = \mathbf{u}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{u}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{u}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) . \tag{2.82}$$

The above expressions can be easily memorized by forming a determinant with the versors on the first line, the partial derivatives on the second line and the vector itself on the third line.
For cylindrical coordinates, the following expression holds:

$$\begin{aligned}\nabla \times \mathbf{F} = \mathbf{u}_\rho &\left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) \\ &+\mathbf{u}_\phi \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right) \\ &+\mathbf{u}_z \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\phi\right) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right]\end{aligned} \tag{2.83}$$

For spherical coordinates:

$$\begin{aligned}\nabla \times \mathbf{F} = \mathbf{u}_r &\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial \phi}\right] \\ &+\mathbf{u}_\theta \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}\left(rF_\phi\right)\right] \\ &+\mathbf{u}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial \theta}\right]\end{aligned} \tag{2.84}$$

Curl theorem or Stokes theorem

Let $\mathbf{A}(\mathbf{r})$ be any vector function of position, continuous together with its first derivative throughout an arbitrary surface S bounded by a contour C, assumed to be resolvable into a finite number of regular arcs. The Stokes theorem states that

$$\oint_C \mathbf{A}(\mathbf{r}) \cdot d\ell = \int_S \left[\nabla \times \mathbf{A}(\mathbf{r})\right] \cdot \mathbf{n}\, dS \tag{2.85}$$

where $d\ell$ is an element of length along C and $\mathbf{n}$ is a unit vector normal to the positive side of the element area dS. This relationship is an equation defining the curl.

Scalar Laplacian

The scalar Laplacian is a second order operator, corresponding to the notation $\nabla \cdot \nabla w$ or div grad w, that is we have to take the divergence of the gradient of w

$$div\,(grad\, w) = \nabla^2 w = \nabla \cdot \nabla w \tag{2.86}$$

In the following are reported the Laplacian expressions in cartesian, circular-cylindrical and spherical coordinate systems, respectively.

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \tag{2.87}$$

$$\nabla^2 w = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial^2 w}{\partial z^2} \tag{2.88}$$

$$\nabla^2 w = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial w}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial w}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 w}{\partial \phi^2} \tag{2.89}$$

IX Summary of gradient, divergence and curl in different coordinate systems

Rectangular coordinates

$$\nabla w = \mathbf{u}_x \frac{\partial w}{\partial x} + \mathbf{u}_y \frac{\partial w}{\partial y} + \mathbf{u}_z \frac{\partial w}{\partial z} \tag{2.90}$$

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{2.91}$$

$$\nabla \times \mathbf{F} = \mathbf{u}_x \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{u}_y \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{u}_z \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \tag{2.92}$$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \tag{2.93}$$

Cylindrical coordinates

$$\nabla w = \mathbf{u}_\rho \frac{\partial w}{\partial \rho} + \mathbf{u}_\phi \frac{1}{\rho}\frac{\partial w}{\partial \phi} + \mathbf{u}_z \frac{\partial w}{\partial z} \tag{2.94}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\rho\right) + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z} \tag{2.95}$$

$$\begin{aligned}\nabla \times \mathbf{F} = \mathbf{u}_\rho &\left(\frac{1}{\rho}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right) \\ &+\mathbf{u}_\phi \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho}\right) \\ &+\mathbf{u}_z \left[\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho F_\phi\right) - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right]\end{aligned} \tag{2.96}$$

$$\nabla^2 w = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial w}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \phi^2} + \frac{\partial^2 w}{\partial z^2} \tag{2.97}$$

Spherical coordinates

$$\nabla w = \mathbf{u}_r \frac{\partial w}{\partial r} + \mathbf{u}_\theta \frac{1}{r}\frac{\partial w}{\partial \theta} + \mathbf{u}_\phi \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi} \tag{2.98}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(F_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi} \tag{2.99}$$

$$\begin{aligned}\nabla \times \mathbf{F} = \mathbf{u}_r &\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(F_\phi \sin\theta\right) - \frac{\partial F_\theta}{\partial \phi}\right] \\ &+\mathbf{u}_\theta \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r}\left(rF_\phi\right)\right] \\ &+\mathbf{u}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial \theta}\right]\end{aligned} \tag{2.100}$$

$$\nabla^2 w = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial w}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial w}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 w}{\partial \phi^2} \tag{2.101}$$

The codes for the different coordinate systems are reported in

```
wxm/vect_op_rectangular.wxm
wxm/vect_op_cyl.wxm
wxm/vect_op_spherical.wxm
```

IX.1 Vector operation with CAS: Cartesian coordinates

A code for performing vector algebra with wxMaxima in cartesian coordinates is reported next. With small modifications the code can be used to compute divergence gradient curl etc. of specific functions.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

/* vect_op_rectangular */

print("the package vect allows to perform vector operations")$
load(vect);

print("select the coordinate system by using the appropriate scalefactors")$

scalefactors(cartesian3d);
/*
scalefactors(polar);
scalefactors(spherical);
*/

print("Version for symbolic general expressions")$

/* give either the w expression or its dependence */

/* w  : 1/sqrt(x^2 +z^2 +y^2);
example of function in rectangular coordinates */

depends(w,[x,y,z]); /*  */

/*  specific function
Fx : x/(x^2+y^2+z^2)^(3/2);
Fy : x*y*z + y/(x^2+y^2+z^2)^(3/2);
Fz : z/(x^2+y^2+z^2)^(3/2);
```

```
*/
 /* */
depends (Fx,[x,y,z]);
depends (Fy,[x,y,z]);
depends (Fz,[x,y,z]);

Fvect: matrix([Fx, Fy, Fz]);

print("-------------------------------------------------")$
print(" RECTANGULAR COORDINATES")$
print("-------------------------------------------------")$
print("Gradient of w")$
g : grad (w);
express (g);
ev (%, diff)$



print("Divergence")$
d : div ([Fx, Fy, Fz]);
express (d);
ev (%, diff)$

print("Curl")$
curl ([Fx, Fy, Fz]);
express (%);
ev (%, diff)$

print("-------------------------------------------------")$
print("second order operators")$
print("-------------------------------------------------")$
print("Expression of the Laplacian in rectangular coordinates")$
aa:laplacian w;
/* scalefactors(cartesian3d); */
express(aa);
ev (%, diff)$

print("laplacian is div(grad(w))")$
```

```
g : grad (w);
express (g);
gradw : ev (%, diff)$

d : div (gradw);
express (d);
ev (%, diff)$

print("Now evaluate curl curl of a vector")$
print("Curl")$
curl ([Fx, Fy, Fz]);
express (%);
curlF : ev (%, diff)$

curl(curlF);
express (%);
curlcurlF : ev (%, diff)$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

IX.2 Cylindrical coordinates

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
assume(r>0);     /* Arghh */
assume(%rho>0); /* Arghh */

print("the package vect allows to perform vector operations")$
load(vect);

print("select the coordinate system by using the appropriate scalefactors")$

print("Better to use phi since %phi is reserved")$

scalefactors([[%rho*cos(phi),%rho*sin(phi),z],%rho,phi,z]);
depends(w,[%rho,phi,z]);

/*  this allow to select the desired scalefactors
scalefactors(cartesian3d);  rectangular
scalefactors(polar);     cylindrical 2D
scalefactors([[%rho*cos(%phi),%rho*sin(%phi),z],%rho,%phi,z]);    cylindrical 3D
scalefactors(spherical);
*/

/* give either the w expression or its dependence */

w : log(%rho);  /* example of scalar function */
 /*  depends(t,[%rho,phi,z]);  */

/*  specific function
A[%rho] : 0 $
A[%phi] : 0$
A[z] : log(%rho)$
```

```
*/

/* set the first secon and third component */
F[x] : A[%rho];
F[y] : A[%phi];
F[z] : A[z];



/* depends(f,[r,theta,phi]) */

print(" Cylindrical COORDINATES")$
print("Gradient of w")$
g : grad (w);
express (g);
gradw : ev (%, diff);



print("Divergence")$
d : div ([F[x], F[y], F[z]]);
express (d);
ev (%, diff);

print("Curl")$
curl ([F[x], F[y], F[z]]);
express (%);
curl_F : ev (%, diff);

print("second order operators")$
print("Expression of the Laplacian in cylindrical coordinates")$
aa:laplacian w;
/* scalefactors(cartesian3d); */
express(aa);
ev (%, diff);

print("laplacian is div(grad(w))")$
```

```
g : grad (w);
express (g);
gradw : ev (%, diff);

d : div (gradw);
express (d);
ev (%, diff);

print("Now evaluate curl curl of a vector")$
print("Curl")$
curl ([F[x], F[y], F[z]]);
express (%);
curlF : ev (%, diff);

curl(curlF);
express (%);
curlcurlF : ev (%, diff);
print("-----------------------------------------")$

print("curl of gradient is always zero")$
curl (gradw);
express (%);
curlgradw : ev (%, diff);

print("-----------------------------------------")$
print("divergence of a curl is always zero");
d_curl_F : div (curl_F);
express (d);
ev (%, diff);

print("Bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

IX.3 Spherical coordinates

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
assume(r>0);     /* Arghh */
assume(%rho>0); /* Arghh */

print("the package vect allows to perform vector operations")$
load(vect);

print("select the coordinate system by using the appropriate scalefactors")$

scalefactors(spherical);

/*  this allow to select the desired scalefactors
scalefactors(cartesian3d);
scalefactors(polar);
scalefactors(spherical);
*/

/* give either the w expression or its dependence */

  w : 1/r; /*   example of scalar function */
 /*  depends(w,[x,y,z]);  */

/*  specific function * %e^{-%i * k * r} */
A[r] : 1/r * cos(%theta)  $
A[%theta] : - 1/r * sin(%theta)  $
A[%phi] : 0$

A[%theta] : - 1/r * sin(%theta) *sin(phi)  $
A[%phi] : cos(phi) + sin(phi)$
```

```
/* set the first secon and third component */
F[x] : A[r];
F[y] : A[%theta];
F[z] : A[%phi];

Avect: matrix([A[r], F[y], F[z]]);

/* depends(f,[r,theta,phi]) */

print(" SPHERICAL COORDINATES")$
print("Gradient of w")$
g : grad (w);
express (g);
gradw : ev (%, diff);



print("Divergence")$
d : div ([F[x], F[y], F[z]]);
express (d);
ev (%, diff);

print("Curl")$
curl ([F[x], F[y], F[z]]);
express (%);
curl_F : ev (%, diff);

print("second order operators")$
print("Expression of the Laplacian in rectangular coordinates")$
aa:laplacian w;
/* scalefactors(cartesian3d); */
express(aa);
ev (%, diff);

print("laplacian is div(grad(w))")$

g : grad (w);
```

```
express (g);
gradw : ev (%, diff);

d : div (gradw);
express (d);
ev (%, diff);

print("Now evaluate curl curl of a vector")$
print("Curl")$
curl ([F[x], F[y], F[z]]);
express (%);
curlF : ev (%, diff);

curl(curlF);
express (%);
curlcurlF : ev (%, diff);
print("----------------------------------------")$

print("curl of gradient is always zero")$
curl (gradw);
express (%);
curlgradw : ev (%, diff);

print("----------------------------------------")$
print("divergence of a curl is always zero");
d_curl_F : div (curl_F);
express (d);
ev (%, diff);
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

X Dirac and gamma matrices

X.1 Direct or Kronecher product

Before introducing the Dirac matrices it is convenient to recall a second procedure to multiply matrices, often referred to as direct or Kronecker product. Let A be a $m \times m$ matrix and B be a $n \times n$ matrix, then the direct product is

$$A \otimes B = C \tag{2.102}$$

with matrix C and $m \times n$ by $m \times n$ matrix. For instance, if A and B are 2x2 matrices, we have

$$A \otimes B = \begin{pmatrix} a_{11}B\ a_{12}B \\ a_{21}B\ a_{22}B \end{pmatrix} \tag{2.103}$$

i.e. a 4x4 matrix. The direct product is associative but not commutative.
Let us now apply the notion of direct product to Pauli matrices. We can define a g_{ij} matrix obtained by the various combinations of the Pauli matrices as:

$$g_{ij} = \sigma_i \otimes \sigma_j \tag{2.104}$$

with the indices i, j running from 0 to 3. The various possibilities arising are reported, along with the standard nomenclatures, in Tables 2.9 and in Table 2.8.

Product of two Dirac matrices, g_{ij} and g_{kl}

While for Pauli matrices it is easy to remember their product (as e.g. $\sigma_1\sigma_2 = i\,\sigma_3$, etc.), for Dirac matrices the task may be more difficult. However there is a rule that can guide us in finding the product of two gamma matrices. Imagine that we would like to obtain the product of g_{ij} and g_{kl}. This may be written as

$$\begin{aligned} \left(g_{ij}\right)\left(g_{kl}\right) &= \left(\sigma_i \otimes \sigma_j\right)\left(\sigma_k \otimes \sigma_l\right) \\ &= \left(\sigma_i\sigma_k\right) \otimes \left(\sigma_j\sigma_l\right) . \end{aligned} \tag{2.105}$$

In words we need to make the ordinary matrix multiplication between the first two terms in each matrix (in this case σ_i, σ_k) and perform the Kronecher product with the product of last two terms (in this case $\sigma_j \sigma_l$). A code fraction reporting part of the above theory is given next.

wxm/Dirac_v05.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

print("Define the Pauli matrices")$

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print("Define the Dirac matrices according to Arfken pag. 213")$

/* print("First row of Arfken table")$ */
print("These are the sigmas here called s")$
s[0] : kronecker_product(%sigma[0],%sigma[0])$
s[1] : kronecker_product(%sigma[0],%sigma[1])$
s[2] : kronecker_product(%sigma[0],%sigma[2])$
s[3] : kronecker_product(%sigma[0],%sigma[3])$

g[00]: s[0]$
g[01]: s[1]$
g[02]: s[2]$
g[03]: s[3]$

/* print("second row of Arfken table")$ */
%alpha[0] : kronecker_product(%sigma[1],%sigma[0])$
%alpha[1] : kronecker_product(%sigma[1],%sigma[1])$
%alpha[2] : kronecker_product(%sigma[1],%sigma[2])$
%alpha[3] : kronecker_product(%sigma[1],%sigma[3])$
g[10]: %alpha[0]$
g[11]: %alpha[1]$
g[12]: %alpha[2]$
g[13]: %alpha[3]$
```

```
/* print("third row of Arfken table")$ */
%gamma[0] : kronecker_product (%sigma[2],%sigma[0])$
%gamma[1] : kronecker_product (%sigma[2],%sigma[1])$
%gamma[2] : kronecker_product (%sigma[2],%sigma[2])$
%gamma[3] : kronecker_product (%sigma[2],%sigma[3])$
g[20]: %gamma[0]$
g[21]: %gamma[1]$
g[22]: %gamma[2]$
g[23]: %gamma[3]$

/* print("fourth row of Arfken table")$ */
%delta[0] : kronecker_product (%sigma[3],%sigma[0])$
%delta[1] : kronecker_product (%sigma[3],%sigma[1])$
%delta[2] : kronecker_product (%sigma[3],%sigma[2])$
%delta[3] : kronecker_product (%sigma[3],%sigma[3])$
g[30]: %delta[0]$
g[31]: %delta[1]$
g[32]: %delta[2]$
g[33]: %delta[3]$

/* Example */

i : 2$
j : 3$

k : 0$
l : 1$

print("Rule for the product of gamma matrices")$
print(" The gamma[ij] is obtained as the kronecher product of %sigma[i] and %sigma[j]")$
print(" The gamma[lk] is obtained as the kronecher product of %sigma[k] and %sigma[k]")$


print("gijgkl product by direct multiplication")$
gijgkl : g[23] . g[01];

print("result obtained with the rule")$
```

```
first : %sigma[2].%sigma[0];
second : %sigma[3].%sigma[1];
res : kronecker_product (first,second);
 print("test")$
res - gijgkl;
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

X.2 A set of four anti–commuting matrices

As noted in Arfken, in 1927 P. A. M. Dirac was looking for a set of four anticommuting matrices. The three Pauli matrices plus the unit matrix form a complete set, but this set presents only three anticommuting matrices. By extending the Pauli matrices to 4 by 4 matrices it is possible to find this set. In 1928, building on 2 by 2 spin matrices which Dirac discovered independently of Wolfgang Pauli's work on non-relativistic spin systems, (Abraham Pais quoted Dirac as saying "I believe I got these (matrices) independently of Pauli and possibly Pauli got these independently of me") Dirac obtained the 4 by 4 matrices.
These matrices can be written in terms of the Pauli matrices as reported in Table 2.8.

Table 2.8. Dirac matrices in terms of Pauli matrices.

σ_0	σ_1	σ_2	σ_3
$\rho_0 \begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}$	$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$	$\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$	$\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$
$g_{00}, 1, \alpha_0$	g_{01}, σ_1	g_{02}, σ_2	g_{03}, σ_3
$\rho_1 \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$
$g_{10}, \rho_1, -\gamma_5$	g_{11}, α_1	g_{12}, α_2	g_{13}, α_3
$\rho_2 \, i \begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$	$i \begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$	$i \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$	$i \begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$
$g_{20}, \gamma_0, \rho_2, \alpha_5$	g_{21}, γ_1	g_{22}, γ_2	g_{23}, γ_3
$\rho_3 \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$	$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}$	$\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$	$\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$
$g_{30}, \delta_0, \rho_3, \alpha_4, \gamma_4, \beta$	g_{31}, δ_1	g_{32}, δ_2	g_{33}, δ_3

From these 16 Hermitian matrices we can form six anti-commuting sets of matrices each. We have the following sets:

1. $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$
2. $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$
3. $\delta_1, \delta_2, \delta_3, \rho_1, \rho_2$
4. $\alpha_1, \gamma_1, \delta_1, \sigma_2, \sigma_3$
5. $\alpha_2, \gamma_2, \delta_2, \sigma_1, \sigma_3$
6. $\alpha_3, \gamma_3, \delta_3, \sigma_1, \sigma_2$

The sixteen forms, which form a complete basis for representing four by four matrices, are reported explicitly in Table 2.9.

Table 2.9. Dirac Matrices as reported in Arfken, mathematical methods for physicists, pag. 213, third edition, academic press. In addition the notation with the g_{ij} has been introduced for compactness.

	σ_0	σ_1	σ_2	σ_3
ρ_0	$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&-i&0&0\\ i&0&0&0\\ 0&0&0&-i\\ 0&0&i&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&1&0\\ 0&0&0&-1 \end{pmatrix}$
	$g_{00}, 1, \alpha_0$	g_{01}, σ_1	g_{02}, σ_2	g_{03}, σ_3
ρ_1	$\begin{pmatrix} 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&-i\\ 0&0&i&0\\ 0&-i&0&0\\ i&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&-1&0&0 \end{pmatrix}$
	$g_{10}, \rho_1, -\gamma_5$	g_{11}, α_1	g_{12}, α_2	g_{13}, α_3
ρ_2	$\begin{pmatrix} 0&0&-i&0\\ 0&0&0&-i\\ i&0&0&0\\ 0&i&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&-i\\ 0&0&-i&0\\ 0&i&0&0\\ i&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&-1\\ 0&0&1&0\\ 0&1&0&0\\ -1&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&-i&0\\ 0&0&0&i\\ i&0&0&0\\ 0&-i&0&0 \end{pmatrix}$
	$g_{20}, \gamma_0, \rho_2, \alpha_5$	g_{21}, γ_1	g_{22}, γ_2	g_{23}, γ_3
ρ_3	$\begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0&0\\ 1&0&0&0\\ 0&0&0&-1\\ 0&0&-1&0 \end{pmatrix}$	$\begin{pmatrix} 0&-i&0&0\\ i&0&0&0\\ 0&0&0&i\\ 0&0&-i&0 \end{pmatrix}$	$\begin{pmatrix} 1&0&0&0\\ 0&-1&0&0\\ 0&0&-1&0\\ 0&0&0&1 \end{pmatrix}$
	$g_{30}, \delta_0, \rho_3, \alpha_4, \gamma_4, \beta$	g_{31}, δ_1	g_{32}, δ_2	g_{33}, δ_3

X.3 Dirac Gamma matrices

The Gamma matrices, also known as Dirac matrices, are often used. They are defined, for $j = 1,2,3$ as

$$\gamma^0 = \sigma_3 \otimes \sigma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$
$$\gamma^j = i\,\sigma_2 \otimes \sigma_j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} \tag{2.106}$$

Note that we have used the superscript in order to differentiate from the gamma matrices used in the Arfken book. The relationship between the gamma is clearly the following

$$\gamma^j = i\gamma_j\,. \tag{2.107}$$

Since we have that

$$\gamma_j\gamma_j = I_4 \tag{2.108}$$

i.e. squares to plus one, the γ^j square to minus one.

For the gamma matrices the following relations hold:

$$
\begin{aligned}
\gamma^0\gamma^0 &= \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} = I_4 \\
\gamma^i\gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = -I_4 \\
\gamma^0\gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
\gamma^i\gamma^0 &= -\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
\gamma^i\gamma^j &= -\begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} \\
\gamma^j\gamma^i &= \begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} .
\end{aligned} \tag{2.109}
$$

From the first relation in (4.79) we note that γ^0 squares to plus one, while from the second relation we see that all the γ^i with $i = 1,2,3$ square to minus one. The remaining four relationships shows that they anticommute. As a consequence the set of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ form a Clifford basis $Cl(1,-3)$.

X.4 Weyl Gamma matrices

In the literature oftentimes the name of gamma matrices refers to the Weyl gamma matrices. They are different from the Dirac gamma matrices and they are defined as:

$$
\begin{aligned}
\gamma^0_{Weyl} &= \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \\
\gamma^j_{Weyl} &= \begin{pmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{pmatrix}
\end{aligned} \tag{2.110}
$$

The Dirac and Weyl gamma matrices are related from the following similarity transformation:

$$\gamma^j_{Dirac} = A\left(\gamma^j_{Weyl}\right)A^{-1} \tag{2.111}$$

with the matrix A similar to the matrix introduced in (1.236), and sharing the same properties, but this time having dimension four:

$$A = A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix}\sigma_0 & \sigma_0\\ \sigma_0 & -\sigma_0\end{pmatrix} = \frac{1}{\sqrt{2}}(\sigma_1+\sigma_3)\otimes\sigma_0\,. \tag{2.112}$$

A code fraction reporting part of the above theory is given next.

wxm/Weyl_Dirac_v02.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
%sigma[0] : matrix ([1,0],[0,1])$
%sigma[1] : matrix ([0,1],[1,0])$
%sigma[2] : matrix ([0,-%i],[%i,0])$
%sigma[3] : matrix ([1,0],[0,-1])$

print("matrix A = 1/sqrt(2) * (%sigma[1]+ %sigma[3])")$
A : 1/sqrt(2) *(%sigma[1]+ %sigma[3]);
print("A is equal to its inverse")$
A.A;
print("Dirac gamma0 matrix (s0 stands for sigma0)")$
%gamma[0] : matrix([s[0],0],[0,-s[0]]);
print("pass to Weyl gamma0 matrix")$
Wg[0]: A. %gamma[0] .A ;

print("Dirac gammai matrix (si stands for sigmai)")$
%gamma[i] : matrix([0,s[i]],[-s[i],0]);
print("pass to Weyl gammai matrices")$
Wg[i]: A. %gamma[i] .A ;

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

3

Transmission Lines

I Transmission lines theory

I.1 Voltage and Current variation along a transmission line

Before using the transmission lines in any circuit, we will derive the basic equations that describe the voltage across and the current flowing in the transmission line as a function of the time t and the position z. A trasmission line can be modeled as the cascade of many short lines, each of length Δz, as shown in Fig. 3.1(a) and Fig. 3.1(b). The behaviour of a single element Δz long can be derived by using the equivalent circuit in Fig. 3.1(c)

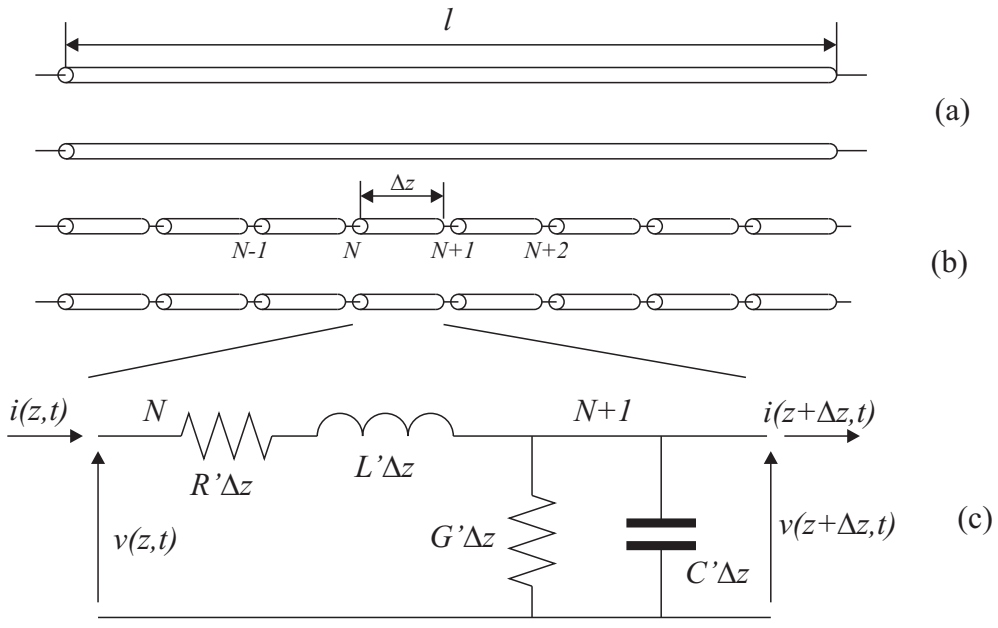

Fig. 3.1. Equivalent circuit of a transmission line of length Δz

The term $R'\Delta z$ is the resistance of the wires, due to conductor losses. The term $L'\Delta z$ is the inductance of the wires. The term $G'\Delta z$ is the conductance between wires, due to dielectric losses. The therm $C'\Delta z$ is the capaciutance between wires. The quantities with a prime superscript (R', L', G', C') are differential ones whose units are per unit length. The units are $[\Omega/m]$, $[H/m]$, $[S/m]$, $[F/m]$ respectively.

In Fig. 3.1(c), $v(z,t)$ and $i(z,t)$ are the voltage and current at the left end of the circuit, whereas $v(z+\Delta z,t)$ and $i(z+\Delta z,t)$ are the voltage and current at the right side. It is worth noting that if $v(z,t)$ and $i(z,t)$ are the quantitnes at the input of the N_{th} section of line, then $v(z+\Delta z,t)$ and $i(z+\Delta z,t)$ are the input at the $(N+1)_{th}$ section. If we apply the Kirchhoff's voltage law we obtain:

$$v(z,t) - R'\Delta z i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \tag{3.1}$$

Dividing by Δz we obtain:

$$\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t} \tag{3.2}$$

now, in the limit as $\Delta z \to 0$:

$$\frac{\partial v(z,t)}{\partial z} = -R' i(z,t) - L' \frac{\partial i(z,t)}{\partial t} \tag{3.3}$$

On the other hand, if we apply the Kirchhoff's current law at the node $(N+1)$, we obtain:

$$i(z,t) - G'\Delta z v(z+\Delta z,t) - C'\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0 \tag{3.4}$$

dividing by Δz

$$\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = -G' v(z+\Delta z,t) - C' \frac{\partial v(z+\Delta z,t)}{\partial t} \tag{3.5}$$

in the limit as $\Delta z \to 0$:

$$\frac{\partial i(z,t)}{\partial z} = -G' v(z,t) - C' \frac{\partial v(z,t)}{\partial t} \tag{3.6}$$

The first-order differential equations (3.3) and (3.6) are known as the *telegrapher's equations*. They are the time-domain form of the trasmission line equations.

In most cases we are interested in the steady state condition, where the time dependance of these quantities if of the type $\cos\omega t$, so that voltages and currents can be expressed by using phasors. Voltage and current on the line can written as:

$$\begin{aligned} v(z,t) &= \mathrm{Re}\{\bar{V}(z)e^{j\omega t}\} \\ i(z,t) &= \mathrm{Re}\{\bar{I}(z)e^{j\omega t}\} \end{aligned} \tag{3.7}$$

where $\bar{V}(z)$ and $\bar{I}(z)$ are the phasors of voltage and current respectively. Using phasor quantities, all the derivative with respect to time can be replaced by $j\omega$ and the term $e^{j\omega t}$ can be neglected as it multiplies all terms. The telegrapher's equation become:

$$\begin{aligned} \tfrac{d\bar{V}(z)}{dz} &= -(R' + j\omega L')\bar{I}(z) \\ \tfrac{d\bar{I}(z)}{dz} &= -(G' + j\omega C')\bar{V}(z) \end{aligned} \tag{3.8}$$

Wave equations

Starting from equation 3.8 and deriving the first equation and substituting the second, we have:

$$\frac{d^2\bar{V}(z)}{dz^2} = -(R' + j\omega L')\frac{d\bar{I}(z)}{dz} = (R' + j\omega L')(G' + j\omega C')\bar{V}(z) \tag{3.9}$$

then

$$\frac{d^2\bar{V}(z)}{dz^2} - (R' + j\omega L')(G' + j\omega C')\bar{V}(z) = 0 \tag{3.10}$$

which is a wave equation. It can be written in a compact form defining the *prapagation constant* γ as:

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta \tag{3.11}$$

the (3.10) becomes:

$$\frac{d^2\bar{V}(z)}{dz^2} - \gamma^2\bar{V}(z) = 0 \tag{3.12}$$

In (3.11) α $[N_p/m]$ and β $[rad/m]$ are definded respectively *attenuation constant* and *phase constant*. As their names suggest, they take into account the attenuation and the phase variation of the signal along the line. Deriving the second equation in (3.8) and substituting the first we obtain a similar equation for the current:

$$\frac{d^2\bar{I}(z)}{dz^2} = -(G' + j\omega C')\frac{d\bar{V}(z)}{dz} = (G' + j\omega C')(R' + j\omega L')\bar{I}(z) \tag{3.13}$$

then

$$\frac{d^2\bar{I}(z)}{dz^2} - (G' + j\omega C')(R' + j\omega L')\bar{I}(z) = \frac{d^2\bar{I}(z)}{dz^2} - \gamma^2\bar{I}(z) = 0 \tag{3.14}$$

Equations (3.12) and (3.14) are Helmolz' equations whose solutions are of the type:

$$\begin{aligned} \bar{V}(z) &= \bar{V}_0^+ e^{-\gamma z} + \bar{V}_0^- e^{\gamma z} \\ \bar{I}(z) &= \bar{I}_0^+ e^{-\gamma z} + \bar{I}_0^- e^{\gamma z} \end{aligned} \tag{3.15}$$

Equations in (3.15) show that along the line there are two waves of voltage and current. The waves associated with the term $e^{-\gamma z}$ are called *progressive waves*, since they are propagating along the positive z direction, and it is usually identified by a $+$ superscript. The waves associ-

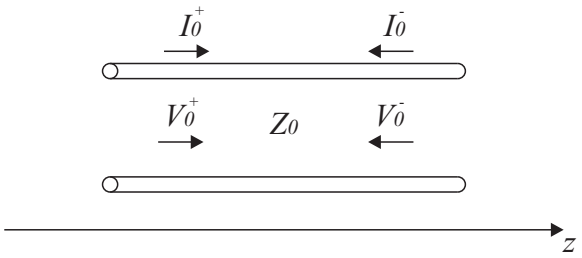

Fig. 3.2. Voltage and current waves propagating along the line

ated with the term $e^{\gamma z}$ are called *regressive waves*, identified by a $-$ superscript (Fig. 3.2) that are propagating along the negative z direction.

The amplitudes of voltage and current waves, however, are not independent, but are tied toghether. In fact, by substituting (3.15) into (3.8) we obtain:

$$\bar{V}_0^+(-\gamma)e^{-\gamma z} + \bar{V}_0^- \gamma e^{\gamma z} = -(R' + j\omega L')\bar{I}(z) \tag{3.16}$$

from this we can derive the current:

$$\bar{I}(z) = \frac{\gamma}{R' + j\omega L'}\left[\bar{V}_0^+ e^{-\gamma z} - \bar{V}_0^- e^{\gamma z}\right] = \bar{I}_0^+ e^{-\gamma z} + \bar{I}_0^- e^{\gamma z} \tag{3.17}$$

then the amplitude of the progressive and regressive current waves are:

$$\begin{aligned} \bar{I}_0^+ &= \tfrac{\gamma}{R'+j\omega L'}\bar{V}_0^+ \\ \bar{I}_0^- &= -\tfrac{\gamma}{R'+j\omega L'}\bar{V}_0^- \end{aligned} \tag{3.18}$$

Note that the reverse current amplitude is negative since it is flowing from right to left. Dimensionally, the term $\gamma/(R' + j\omega L')$ is an admittance, so we can define:

$$Z_0 = \frac{R' + j\omega L'}{\gamma} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \tag{3.19}$$

as the *characteristic impedance* of the transmission line. It is called characteristic because it depends only on the intrinsic parameters of the line and, of course, frequency. The amplitude of voltage and current waves are then related by each other through such impedance:

$$Z_0 = \frac{\bar{V}_0^+}{\bar{I}_0^+} = -\frac{\bar{V}_0^-}{\bar{I}_0^-} \tag{3.20}$$

With (3.20), the equations (3.15) can be rewritten as:

$$\begin{aligned} \bar{V}(z) &= \bar{V}_0^+ e^{-\gamma z} + \bar{V}_0^- e^{\gamma z} \\ \bar{I}(z) &= \frac{\bar{V}_0^+}{Z_0} e^{-\gamma z} - \frac{\bar{V}_0^-}{Z_0} e^{\gamma z} \end{aligned} \tag{3.21}$$

In (3.21) we have now only two unknowns ($\bar{V}_0^+$ and $\bar{V}_0^-$) that can be found using the boundary conditions. If we replace γ with $\alpha + j\beta$

$$\begin{aligned} \bar{V}(z) &= \bar{V}_0^+ e^{-(\alpha+j\beta)z} + \bar{V}_0^- e^{(\alpha+j\beta)z} \\ \bar{I}(z) &= \frac{\bar{V}_0^+}{Z_0} e^{-(\alpha+j\beta)z} - \frac{\bar{V}_0^-}{Z_0} e^{(\alpha+j\beta)z} \end{aligned} \tag{3.22}$$

thus explicitating the attenuation term ($-\alpha z$) and the propagation term ($-j\beta z$). It is worth noting that the amplitudes are phasors, so they are complex number that can be written in the form:

$$\begin{aligned} \bar{V}_0^+ &= |\bar{V}_0^+| e^{j\phi^+} \\ \bar{V}_0^- &= |\bar{V}_0^-| e^{j\phi^-} \end{aligned} \tag{3.23}$$

The time domain expression of the voltage can be derived as

$$\begin{aligned} v(z,t) &= \mathrm{Re}\{\bar{V}(z) e^{j\omega t}\} = \mathrm{Re}\{(\bar{V}_0^+ e^{-\gamma z} + \bar{V}_0^- e^{\gamma z}) e^{j\omega t}\} \\ &= \mathrm{Re}\{|\bar{V}_0^+| e^{j\phi^+} e^{j\omega t} e^{-\alpha z - j\beta z} + |\bar{V}_0^-| e^{j\phi^-} e^{j\omega t} e^{\alpha z + j\beta z}\} \\ &= |\bar{V}_0^+| e^{-\alpha z} \cos(\omega t - \beta z + \phi^+) + |\bar{V}_0^-| e^{\alpha z} \cos(\omega t + \beta z + \phi^-) \end{aligned} \tag{3.24}$$

I.2 Lossless case

Let now consider a perfect conductor (so that $R' = 0$) and a perfect dielectric between conductors ($G' = 0$). The telegrapher's equations can be simplified as:

$$\begin{aligned} \frac{\partial v(z,t)}{\partial z} &= -L' \frac{\partial i(z,t)}{\partial t} \\ \frac{\partial i(z,t)}{\partial z} &= -C' \frac{\partial v(z,t)}{\partial t} \end{aligned} \tag{3.25}$$

and in the phasor form:

$$\begin{aligned} \frac{d\bar{V}(z)}{dz} &= -j\omega L' \bar{I}(z) \\ \frac{d\bar{I}(z)}{dz} &= -j\omega C' \bar{V}(z) \end{aligned} \tag{3.26}$$

Equation (3.11) can be simplified as:

$$\gamma = \sqrt{j\omega L' j\omega C'} = j\omega\sqrt{L'C'} = j\beta \tag{3.27}$$

The γ constant in this case is purely imaginary, so that only the propagation constant is different from zero and the attenuation constant α is zero.

$$\begin{aligned} \alpha &= 0 \\ \beta &= \omega\sqrt{L'C'} \end{aligned} \tag{3.28}$$

The wave equations are:

$$\begin{aligned} \frac{d^2\bar{V}(z)}{dz^2} + \beta^2 \bar{V}(z) &= 0 \\ \frac{d^2\bar{I}(z)}{dz^2} + \beta^2 \bar{I}(z) &= 0 \end{aligned} \tag{3.29}$$

In such a case, the waves (progressive and regressive) along the line are propagating without attenuation. We can rewrite (3.21) as

$$\begin{aligned}\bar{V}(z) &= \bar{V}_0^+ e^{-j\beta z} + \bar{V}_0^- e^{j\beta z} \\ \bar{I}(z) &= \frac{\bar{V}_0^+}{Z_0} e^{-j\beta z} - \frac{\bar{V}_0^-}{Z_0} e^{j\beta z}\end{aligned} \tag{3.30}$$

Moreover, the characteristic impedance can be rewritten as

$$Z_0 = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}} \tag{3.31}$$

It can be seen that, for a lossless transmission line, Z_0 is purely real and is independent from the frequency. In time domain voltage and current are equal to:

$$v(z,t) = |\bar{V}_0^+|\cos(\omega t - \beta z + \phi^+) + |\bar{V}_0^-|\cos(\omega t + \beta z + \phi^-) \tag{3.32}$$

and

$$i(z,t) = \frac{|\bar{V}_0^+|}{Z_0}\cos(\omega t - \beta z + \phi^+) - \frac{|\bar{V}_0^-|}{Z_0}\cos(\omega t + \beta z + \phi^-) \tag{3.33}$$

The voltage (or current) is observed to be the same at any two points along the line that are separated in z such that βz is equal to 2π. The distance between such points is defined *wavelength* and is equal to

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{L'C'}} \tag{3.34}$$

The argument of a sinusoid is called its *phase* and the velocity for which phase is constant is the *phase velocity*:

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}} \tag{3.35}$$

For TEM transmission lines, this is also equal to

$$v_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\varepsilon\mu}} \tag{3.36}$$

If v_p is independent on the frequency, then the line is *non dispersive*. In this case sinusoidal waves of different frequencies travel on the line at the same speed. On the contrary if v_p depends on the frequency, the line is called *dispersive*. Moreover, considering the relative permettivity and permeability of the medium, the phase velocity can be written as:

$$v_p = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0\mu_r}} = \frac{c}{\sqrt{\varepsilon_r\mu_r}} \tag{3.37}$$

where c is the speed of light in vacuum (free space). In this view v_p is the speed of light in the medium. Considering the wavelength, we can also find:

$$\lambda = \frac{v_p}{f} = \frac{c}{f}\frac{1}{\sqrt{\varepsilon_r\mu_r}} = \frac{\lambda_0}{\sqrt{\varepsilon_r\mu_r}} \tag{3.38}$$

It is worth noting that for real transmission lines, in many cases the conditions $R' \ll \omega L'$ and $G' \ll \omega C'$ are satisfied. In such a case, the analysis of the propagation along the line can be effectively approximated using (3.27) and (3.31), provided that the line is short enough. The losses, and consequently the propagation constant γ can be approximated in the following way.

$$\begin{aligned}\gamma &= \sqrt{(R'+j\omega L')(G'+j\omega C')} = \sqrt{j\omega L'(1+\tfrac{R'}{j\omega L'})j\omega C'(1+\tfrac{G'}{j\omega C'})} \\ &= j\omega\sqrt{L'C'}\sqrt{1+\tfrac{R'}{j\omega L'}}\sqrt{1+\tfrac{G'}{j\omega C'}}\end{aligned} \tag{3.39}$$

The quantities under the square roots can be approximated with a first order Taylor expansion:

$$\gamma \simeq j\omega\sqrt{L'C'}(1+\frac{R'}{2j\omega L'})(1+\frac{G'}{2j\omega C'}) \tag{3.40}$$

then

$$\gamma \simeq j\omega\sqrt{L'C'}\left(1+\frac{R'}{2j\omega L'}+\frac{G'}{2j\omega C'}-\frac{R'G'}{4\omega^2 L'C'}\right) \tag{3.41}$$

The last term is small compared to the others and is neglected.

$$\gamma \simeq j\omega\sqrt{L'C'}+\frac{R'}{2}\sqrt{\frac{C'}{L'}}+\frac{G'}{2}\sqrt{\frac{L'}{C'}}=\frac{R'}{2Z_0}+\frac{G'Z_0}{2}+j\omega\sqrt{L'C'}=\alpha_c+\alpha_d+j\beta \tag{3.42}$$

In the latter equation we have three terms. Two of them take into account the losses due to the conductor and the dielectric (α_c and α_d respectively). The last term is the phase constant. This approximation is good enough for low losses and relatively short transmission lines.

I.3 Voltage reflection coefficient

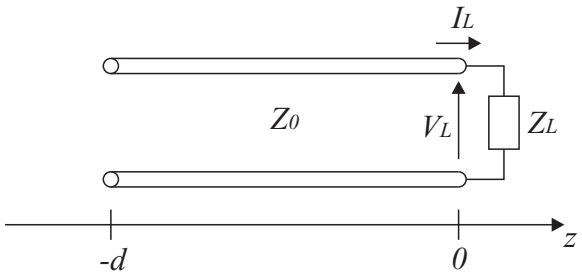

Fig. 3.3. Transmission line terminated to a load impedance Z_L

Let us consider a transmission line closed to a load impedance Z_L (Fig. 3.3). The line is lossless so its characteristic impedance is purely real, whereas the load can be complex. In the circuit the origin of the z-axis is conveniently placed at the load section. Starting from equation (3.30), we can write the voltage and the current at the load we obtain:

$$\begin{aligned}\bar{V}(0)&=\bar{V}_0^+ + \bar{V}_0^- = V_L\\ \bar{I}(0)&=\frac{\bar{V}_0^+}{Z_0}-\frac{\bar{V}_0^-}{Z_0}=\frac{V_L}{Z_L}\end{aligned} \tag{3.43}$$

The load impedance Z_L is then a boundary condition for the telegrapher's equations as it fixes the ratio between voltage and current in $z = 0$. Solving (3.43), we can derive the ratio between the amplitudes of the regressive and prograssive waves:

$$\Gamma_L \triangleq \frac{\bar{V}_0^-}{\bar{V}_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{3.44}$$

where Γ_L is defined as *reflection coefficient*. It relates the amplitude of the regressive voltage wave as function of theamplitude of the progressive wave. It can also be seen that the current reflection coefficient is:

$$\frac{\bar{I}_0^-}{\bar{I}_0^+} = -\Gamma_L \tag{3.45}$$

Inverting (3.44) we have:

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{3.46}$$

We can also define the *transmission coefficient* as the ratio between the voltage at the load and the amplitude of the progressive wave:

$$T \triangleq \frac{V_L}{\bar{V}_0^+} = \frac{2Z_L}{Z_L + Z_0} = 1 + \Gamma_L \tag{3.47}$$

It is worth noting that when the transmission line is terminated to a load, often progressive and regressive waves are also called *incident* and *reflected* waves respectively. Note also that the current reflection coefficient is exactly the same $\Gamma_L = \bar{I}_0^- / \bar{I}_0^+$.
Using the reflection coefficient equation (3.26) can be rewritten as:

$$\begin{aligned}\bar{V}(z) &= \bar{V}_0^+ e^{-j\beta z} + \Gamma_L \bar{V}_0^+ e^{j\beta z} \\ \bar{I}(z) &= \frac{\bar{V}_0^+}{Z_0} e^{-j\beta z} - \Gamma_L \frac{\bar{V}_0^+}{Z_0} e^{j\beta z}\end{aligned} \tag{3.48}$$

showing that the only remaining unknown is the amplitude of the progressive (or incident) wave $\bar{V}_0^+$. Its value depends on the generator that is feeding the transmission line.

Looking at equation (3.44) if $Z_L = Z_0$ then the reflection coefficient is equal to zero as well as the amplitude of the reflected wave. In this case there is only the incident wave propagating along the line and we say that the line is closed to its *matched load*.

Since the load impedance is in general a complex value $Z_L = R_L + jX_L$, the reflection coefficient is also complex:

$$\Gamma_L = |\Gamma_L| e^{j\phi_r} \tag{3.49}$$

It can be demonstrated that if the real part of the load is greater or equal to zero ($R_L \geq 0$) then the modulus of the reflection coefficient is $0 \leq |\Gamma_L| \leq 1$.

The reflection coefficient can be also computed starting from the characteristic admittance of the line and the load admittance, since they are $Y_0 = 1/Z_0$ and $Y_L = 1/Y_0$ respectively. It is easy to demonstrate that

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} \tag{3.50}$$

Exercices

3.1. It is given a transmission line of characteristic impedance equal to $Z_0 = 50\Omega$. It is closed to a load impedance $Z_L = 75 + j100\Omega$. Compute the reflection coefficient in amplitude and phase.

Solution

The reflection coefficient can be computed by using 3.44:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{75 + j100 - 50}{75 + j100 + 50} = 0.512 + j0.39$$

The amplitude of the reflection coefficient is:

$$|\Gamma_L| = \sqrt{0.512^2 + 0.39^2} = 0.644$$

and the phase is

$$\phi_r = \text{atan}\frac{0.39}{0.512} = 37.3^\circ$$

I.4 Power flow

Considering the voltage and current phasors along the line $\bar{V}(z)$ and $\bar{I}(z)$, the time-average power can be computed by

$$P_{av} = \frac{1}{2}\text{Re}[\bar{V}(z)\bar{I}^*(z)] \qquad [W] \tag{3.51}$$

where $\bar{I}^*(z)$ is the complex conjugate of the current. Substituting the expression of the voltage and current along the line:

$$\begin{aligned} P_{av} &= \tfrac{1}{2}\text{Re}\left[\left(\bar{V}_0^+ e^{-j\beta z} + \Gamma_L \bar{V}_0^+ e^{j\beta z}\right)\left(\frac{\bar{V}_0^{+*}}{Z_0} e^{j\beta z} - \Gamma_L^* \frac{\bar{V}_0^{+*}}{Z_0} e^{-j\beta z}\right)\right] \\ &= \tfrac{1}{2}\text{Re}\left[\frac{|\bar{V}_0^+|^2}{Z_0} - \Gamma_L^* \frac{|\bar{V}_0^+|^2}{Z_0} e^{-2j\beta z} + \Gamma_L \frac{|\bar{V}_0^+|^2}{Z_0} e^{2j\beta z} - |\Gamma_L|^2 \frac{|\bar{V}_0^+|^2}{Z_0}\right] \end{aligned} \tag{3.52}$$

In the previous equation the sum of the second and the third terms gives an imaginary number, then it can be written off when extracting the real part. Finally, the averaging power is:

$$P_{av} = \frac{|\bar{V}_0^+|^2}{2Z_0}\left[1 - |\Gamma_L|^2\right] \qquad [W] \tag{3.53}$$

where the term

$$P_{av}^i = \frac{|\bar{V}_0^+|^2}{2Z_0} \qquad [W] \tag{3.54}$$

is the incident average power and the term

$$P_{av}^{r} = |\Gamma_L|^2 \frac{|\bar{V}_0^+|^2}{2Z_0} \qquad [W] \tag{3.55}$$

is the reflected average power. As can be seen, the reflected power is proportional to the incident power by a factor $|\Gamma_L|^2$. This can be considered as a *power reflection coefficient* for the circuit. It is interesting also to note that if the transmission line is closed to its matched load (i.e. $Z_L = Z_0$) then the reflected power is zero and all the incident power is absorbed by the load. This means that the matched load is the optimal condition in order to transfer all the power flowing along the line to the load. In a following section we will see some methods to match the line to an unmatched load.

It is worth noting that the last three expressions do not depend on the position but only on the incident voltage amplitude and the reflection coefficient. Also, the average power is not intended as a power dissipated by the line itself but instead is a flow of power travelling from the source to the load.

The same results can be achieved considering the time domain signals. Using the equation (3.48) and considering the section $z = 0$, the istantaneus incident power carried by the progressive wave is:

$$\begin{aligned} P^i(t) &= v^i(t) i^i(t) \\ &= \mathrm{Re}\left(\bar{V}_0^+ e^{j\omega t}\right) \mathrm{Re}\left(\bar{I}_0^+ e^{j\omega t}\right) \\ &= \mathrm{Re}\left(|\bar{V}_0^+| e^{j\phi^+} e^{j\omega t}\right) \mathrm{Re}\left(\tfrac{|\bar{V}_0^+|}{Z_0} e^{j\phi^+} e^{j\omega t}\right) \\ &= |\bar{V}_0^+| \cos(\omega t + \phi^+) \tfrac{|\bar{V}_0^+|}{Z_0} \cos(\omega t + \phi^+) \\ &= \tfrac{|\bar{V}_0^+|^2}{Z_0} \cos^2(\omega t + \phi^+) \qquad [W] \end{aligned} \tag{3.56}$$

With a similar procedure the instantaneus reflected power can also be computed

$$\begin{aligned} P^r(t) &= v^r(t) i^r(t) \\ &= |\Gamma_L|^2 \tfrac{|\bar{V}_0^+|^2}{Z_0} \cos^2(\omega t + \phi^+ + \phi_r) \qquad [W] \end{aligned} \tag{3.57}$$

Now, the average power can be computed by integrating the former quantities over a time interval $T = 1/f = 2\pi/\omega$:

$$P_{av}^{(i)} = \frac{1}{T}\int_0^T P^i(t)dt = \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} P^i(t)dt \tag{3.58}$$

integrating this quantity we have:

$$P_{av}^{(i)} = \frac{|\bar{V}_0^+|^2}{2Z_0} \qquad [W] \tag{3.59}$$

Integrating the reverse power we have:

$$P_{av}^{(r)} = \frac{1}{T}\int_0^T P^r(t)dt = \Gamma_L^2\frac{|\bar{V}_0^+|^2}{2Z_0} \qquad [W] \tag{3.60}$$

I.5 Input impedance

The equations in (3.48) describes the voltage and current along the line. In the section $z = -d$ (Fig. 3.3) voltage and current are:

$$\begin{aligned}\bar{V}(-d) &= \bar{V}_0^+ e^{j\beta d} + \Gamma_L\bar{V}_0^+ e^{-j\beta d}\\ \bar{I}(-d) &= \frac{\bar{V}_0^+}{Z_0}e^{j\beta d} - \Gamma_L\frac{\bar{V}_0^+}{Z_0}e^{-j\beta d}\end{aligned} \tag{3.61}$$

The ratio between these two gives us the input impedance of a section of transmission line closed to a load Z_L:

$$Z_{in}(-d) = \frac{\bar{V}(-d)}{\bar{I}(-d)} = Z_0\frac{e^{j\beta d} + \Gamma_L e^{-j\beta d}}{e^{j\beta d} - \Gamma_L e^{-j\beta d}} \qquad [\Omega] \tag{3.62}$$

using the expression of the reflection coefficient (3.44) and substituting the exponential with

$$\begin{aligned}e^{j\beta d} &= \cos(\beta d) + j\sin(\beta d)\\ e^{-j\beta d} &= \cos(\beta d) - j\sin(\beta d)\end{aligned} \tag{3.63}$$

the input impedance becomes:

$$\begin{aligned} Z_{in}(-d) &= Z_0 \frac{\cos(\beta d)+j\sin(\beta d)+\frac{Z_L-Z_0}{Z_L+Z_0}\big(\cos(\beta d)-j\sin(\beta d)\big)}{\cos(\beta d)+j\sin(\beta d)-\frac{Z_L-Z_0}{Z_L+Z_0}\big(\cos(\beta d)-j\sin(\beta d)\big)} \\ &= Z_0 \frac{(Z_L+Z_0)\big(\cos(\beta d)+j\sin(\beta d)\big)+(Z_L-Z_0)\big(\cos(\beta d)-j\sin(\beta d)\big)}{(Z_L+Z_0)\big(\cos(\beta d)+j\sin(\beta d)\big)-(Z_L-Z_0)\big(\cos(\beta d)-j\sin(\beta d)\big)} \\ &= Z_0 \frac{Z_L\cos(\beta d)+jZ_0\sin(\beta d)}{Z_0\cos(\beta d)+jZ_L\sin(\beta d)} = Z_0 \frac{Z_L+jZ_0\tan(\beta d)}{Z_0+jZ_L\tan(\beta d)} \end{aligned} \tag{3.64}$$

Equation (3.64) says that the input impedance at a distance d from the load depends on the load itself, the characteristic impedance Z_0 and the position. In general it is different for every position along the line, with a notable exception when the line is closed to its matched load. In fact, when $Z_L = Z_0$ the input impedance is always $Z_{in} = Z_0$ at every position along the line.
The product βd is often defined as *electrical length* of the transmission line section and it is expressed by the quantity θ:

$$\theta = \beta d \tag{3.65}$$

This quantity is expressed in radians.
With the definition of input impedance we can substitute a line section closed to a load impedance with its input impedance Z_{in}. We can also define a reflection coefficient at $z = -d$ of the transmissione line as a function of the load reflection coefficient. In fact

$$\begin{aligned} \Gamma(-d) &\triangleq \frac{\tilde{V}^-(-d)}{\tilde{V}^+(-d)} = \frac{\tilde{V}_0^- e^{-j\beta d}}{\tilde{V}_0^+ e^{j\beta d}} \\ &= \Gamma_L e^{-2j\beta d} \end{aligned} \tag{3.66}$$

Equation (3.66) gives us the relation between the Γ_L and the reflection coefficient at any section of the line. It can be seen that, for a lossless line, only the phase of the coefficient is changed by the exponential term. Explicitating Γ_L we also obtain:

$$\Gamma(-d) = |\Gamma_L| e^{j\phi_r} e^{-2j\beta d} = |\Gamma_L| e^{j(\phi_r - 2\beta d)} \tag{3.67}$$

As can be easily understood, this coefficient can be computed by using the input impedance at the section $z = -d$ with a formula that is a generalization of (3.44):

$$\Gamma(-d) = \frac{Z_{in}(-d) - Z_0}{Z_{in}(-d) + Z_0} \tag{3.68}$$

and inverting this relation we have

$$Z_{in}(-d) = Z_0 \frac{1 + \Gamma(-d)}{1 - \Gamma(-d)} \tag{3.69}$$

that is another way to compute the input impedance.

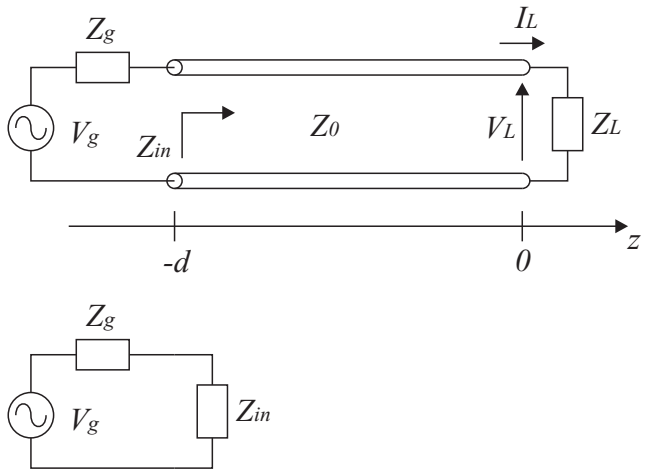

Fig. 3.4. Transmission line fed by a generator and closed to a load

With the definition of the input impedance we can then easily apply the boundary condition at the input of the line in order to compute the amplitude of the progressive wave. Considering Fig. 3.4 and using the input impedance at section $z = -d$ we can compute the voltage at the input of the line:

$$\bar{V}(-d) = \bar{V}_g \frac{Z_{in}(-d)}{Z_g + Z_{in}(-d)} \tag{3.70}$$

but we also know that

$$\bar{V}(-d) = \bar{V}_0^+ \left(e^{j\beta d} + \Gamma_L e^{-j\beta d} \right) \tag{3.71}$$

then

$$\bar{V}_0^+ = \bar{V}_g \frac{Z_{in}(-d)}{Z_g + Z_{in}(-d)} \cdot \frac{1}{e^{j\beta d} + \Gamma_L e^{-j\beta d}} \tag{3.72}$$

Exercices

3.2. It is given a transmission line of characteristic impedance equal to $Z_0 = 50\Omega$ closed to a load impedance $Z_L = 75 + j100\Omega$. Compute the input impedance at $d = \lambda/3$.

Solution

Because the distance is expressed in term of the wavelength λ the electrical length βd can be easily computed:

$$\theta = \beta d = \frac{2\pi}{\lambda}\frac{\lambda}{3} = \frac{2\pi}{3}$$

Using the equation 3.64 we have:

$$\begin{aligned} Z_{in} &= Z_0 \frac{Z_L \cos(\beta d) + jZ_0 \sin(\beta d)}{Z_0 \cos(\beta d) + jZ_L \sin(\beta d)} \\ &= Z_0 \frac{Z_L \cos(\frac{2\pi}{3}) + jZ_0 \sin(\frac{2\pi}{3})}{Z_0 \cos(\frac{2\pi}{3}) + jZ_L \sin(\frac{2\pi}{3})} \\ &= Z_0 \frac{Z_L(-0.5) + jZ_0(0.866)}{Z_0(-0.5) + jZ_L(0.866)} \\ &= 11.245 + j9.456 \qquad [\Omega] \end{aligned}$$

I.6 Standing wave ratio

Let us consider the voltage along a transmission line in phasor form:

$$\bar{V} = \bar{V}_0^+ e^{-j\beta z} + \bar{V}_0^- e^{j\beta z} = \bar{V}_0^+ e^{-j\beta z} + \Gamma_L \bar{V}_0^+ e^{j\beta z} \tag{3.73}$$

This equation shows that along the line the total voltage is the sum of the progressive and the regressive wave. The interference between such waves create a standing wave pattern. In fact, manipulating the equation of the total voltage we can derive an important result:

$$\begin{aligned}\bar{V}(z) &= \bar{V}_0^+ \left(e^{-j\beta z} + \Gamma_L e^{j\beta z}\right) \\ &= \bar{V}_0^+ \left(e^{-j\beta z} + \Gamma_L e^{-j\beta z} - \Gamma_L e^{-j\beta z} + \Gamma_L e^{j\beta z}\right) \\ &= \bar{V}_0^+ \left[(1+\Gamma_L)\, e^{-j\beta z} + 2j\Gamma_L \sin\beta z\right]\end{aligned} \tag{3.74}$$

Therefore, if the reflection coefficient is different from zero, the total voltage is always a combination of a propagating wave and a stationary wave. A similar result can be obtained for the current.

This is always true and only in particular cases the amplitude of the propagating or the standing term goes to zero (more details will be illustrated in a subsequent section). For example, if the line is closed to its matched load ($Z_L = Z_0$), $\Gamma_L = 0$ so that there is only the propagating wave. Moreover, it is straightforward to see that if $\Gamma_L = -1$ the propagating term is zero and the only term left is the stationary wave $\sin\beta z$.

At this point it is interesting to see the variation of the amplitude of the voltage along the line. At a given position $z = -d$, the voltage is

$$\bar{V}(-d) = \bar{V}_0^+ \left(e^{j\beta d} + \Gamma_L e^{-j\beta d}\right) \tag{3.75}$$

the amplitude $|\bar{V}(-d)|$ can be computed using

$$|\bar{V}(-d)| = |\bar{V}_0^+||e^{j\beta d}||1 + \Gamma_L e^{-2j\beta d}| = |\bar{V}_0^+||1 + \Gamma_L e^{-2j\beta d}| \tag{3.76}$$

since the modulus of the exponential is clearly one. The second term is the combination of the propagating wave and the reflected wave at the position $z = -d$. Graphically it can be shown as in Fig. 3.5. The vector of length 1 lies in the real axis, whereas the vector $\Gamma_L e^{-2j\beta d}$ rotates clockwise while moving toward the generator along the transmission line.

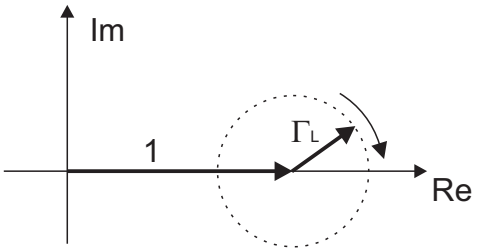

Fig. 3.5. Vector representation of the progressive and regressive waves

During the rotation, it can be seen that there are two locations where the vectors are in phase and in opposition of phase. In the former the voltage is maximum whereas in the latter the voltage is minimum. In particular:

$$V_{max} = |\bar{V}_0^+|(1+|\Gamma_L|) \tag{3.77}$$

and

$$V_{min} = |\bar{V}_0^+|(1-|\Gamma_L|) \tag{3.78}$$

The standing wave pattern is shown in Fig. 3.6. It is plotted considering a a reflection coefficient $\Gamma_L = 0.3$.

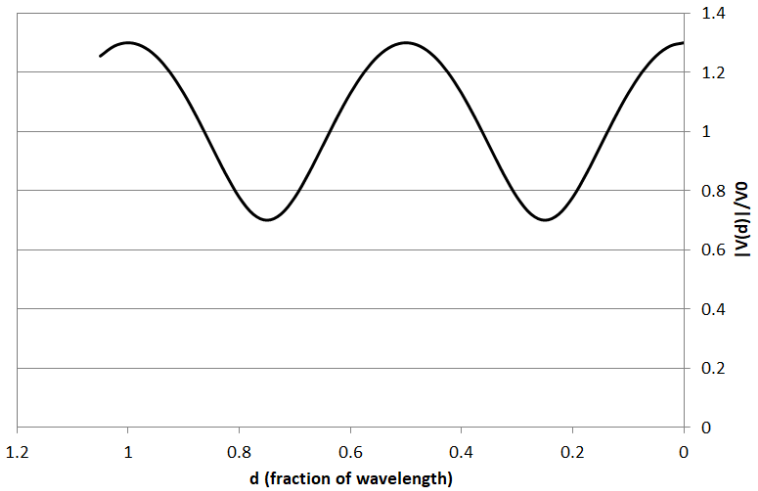

Fig. 3.6. Amplitude of the voltage along the line

The ratio between maximum and minimum voltages is defined as the *Voltage Standing Wave Ratio* or $VSWR$:

$$VSWR \triangleq \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \tag{3.79}$$

The standing wave ration is a quantity related to the amplitude of the reflection coefficient. It is always ≥ 1. In particular, in case of a line closed to its matched load ($\Gamma_L = 0$) the $VSWR = 1$. If the line is completely mismatched ($\Gamma_L = 1$ or $\Gamma_L = -1$) the $VSWR$ goes to ∞.
Writing the reflection coefficient in modulus and phase:

$$\Gamma_L = |\Gamma_L| e^{j\phi_r} \tag{3.80}$$

The position of the maxima can be found imposing that the phase of the reflected wave is zero or multiple of 2π.

$$\phi_r - 2\beta d = -2n\pi \tag{3.81}$$

whereas the minima can be found by imposing that the phase of the reflected wave is $-\pi$:

$$\phi_r - 2\beta d = -(2n+1)\pi \tag{3.82}$$

Note that the right hand side of both equations are negative because d is the distance from the load and it must be positive.
It is worth noting that maxima and minima are spaced by $\lambda/2$ and the distance between a minimum and a maximum is $\lambda/4$.

Looking at the current, where the voltage is maximum we have:

$$|\bar{I}(z)| = \frac{|\bar{V}_0^+|}{Z_0}\left(1-|\Gamma_L|\right) = I_{min} \tag{3.83}$$

where the voltage is minimum we have:

$$|\bar{I}(z)| = \frac{|\bar{V}_0^+|}{Z_0}\left(1+|\Gamma_L|\right) = I_{max} \tag{3.84}$$

This means that where the voltage is maximum the current is minimum and viceversa. The corresponding input impedances are:

$$Z_{max} = \frac{V_{max}}{I_{min}} = \frac{|\bar{V}_0^+|\left(1+|\Gamma_L|\right)}{\frac{|\bar{V}_0^+|}{Z_0}\left(1-|\Gamma_L|\right)} = Z_0 \cdot VSWR \tag{3.85}$$

and

$$Z_{min} = \frac{V_{min}}{I_{max}} = \frac{|\bar{V}_0^+|\left(1-|\Gamma_L|\right)}{\frac{|\bar{V}_0^+|}{Z_0}\left(1+|\Gamma_L|\right)} = \frac{Z_0}{VSWR} \tag{3.86}$$

Note that both Z_{max} and Z_{min} are real.

Exercises

3.3. It is given a transmission line of characteristic impedance equal to $Z_0 = 50\Omega$ closed to a load impedance $Z_L = 75 + j100\Omega$. Compute the standing wave ratio.

Solution

The reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{75 + j100 - 50}{75 + j100 + 50} = 0.512 + j0.39$$

and its amplitude is:

$$|\Gamma_L| = \sqrt{0.512^2 + 0.39^2} = 0.644$$

Consequently, the standing wave ratio is:

$$VSWR = \frac{1+|\Gamma_L|}{1-|\Gamma_L|} \approx 4.62$$

3.4. Considering that the (voltage) Return Loss (RL) is defined as:

$$RL = 20\log_{10}\left(\frac{1}{|\Gamma_L|}\right) \qquad dB$$

fill the following table for every value of the amplitude of the reflection coefficient:

$\lvert\Gamma_L\rvert$	RL (dB)	VSWR
0		
0.1		
0.2		
0.3		
0.4		
0.5		
0.6		
0.7		
0.8		
0.9		
1		

I.7 Some interesting cases

Matched line

We have seen that a transmission line is closed to its matched load when $Z_L = Z_0$. In this case there is no reflection on the line ($\Gamma_L = 0$). There is only the progressive wave propagating along the line, then the standing wave ratio is equal to 1.

Transmission line closed to a short circuit

A transmission line closed to a short circuit ($Z_L = 0$) is also called a *shorted stub*. The reflection coefficient is:

$$\Gamma_L = -\frac{0-Z_0}{0+Z_0} = -1 \tag{3.87}$$

The amplitude of the reflected wave is equal in magnitude to the amplitude of the incident wave and its phase is shifted by 180°, then $\bar{V}_0^- = -\bar{V}_0^+$.
The voltage and current along the line are then:

$$\begin{aligned} \bar{V}(z) &= \bar{V}_0^+ \left(e^{-j\beta z} - e^{j\beta z}\right) = -2j\bar{V}_0^+ \sin(\beta z) \\ \bar{I}(z) &= \frac{\bar{V}_0^+}{Z_0} \left(e^{-j\beta z} + e^{j\beta z}\right) = 2\frac{\bar{V}_0^+}{Z_0} \cos(\beta z) \end{aligned} \tag{3.88}$$

Looking at the equation, we can see that along the line there is only a standing wave, without a progressive wave carrying power. This is not a surprise, since the resistive part of the load is zero, there is no dissipated power.
The minima and maxima of voltage are separated by $\lambda/4$ but in this particular case the first minimum is placed on the load. The maximum and minumim voltages are:

$$\begin{aligned} V_{MAX} &= 2|\bar{V}_0^+| \\ V_{MIN} &= 0 \end{aligned} \tag{3.89}$$

the standing wave ratio is equal to ∞.

The impedance along the line is:

$$\begin{aligned} Z_{in}^{sc}(-d) = \frac{\bar{V}(-d)}{\bar{I}(-d)} &= Z_0 \frac{-2j\bar{V}_0^+ \sin(-\beta d)}{2\frac{\bar{V}_0^+}{Z_0} \cos(-\beta d)} \\ &= jZ_0 \frac{\sin(\beta d)}{\cos(\beta d)} = jZ_0 \tan(\beta d) \end{aligned} \tag{3.90}$$

The input impedance is purely imaginary. It can be positive or negative depending on the distance d from the load. If it is positive the shorted line behaves as an inductor, with an equivalent inductance:

$$j\omega L_{eq} = jZ_0 \tan(\beta d) \qquad \Rightarrow \qquad L_{eq} = \frac{Z_0 \tan(\beta d)}{\omega} \tag{3.91}$$

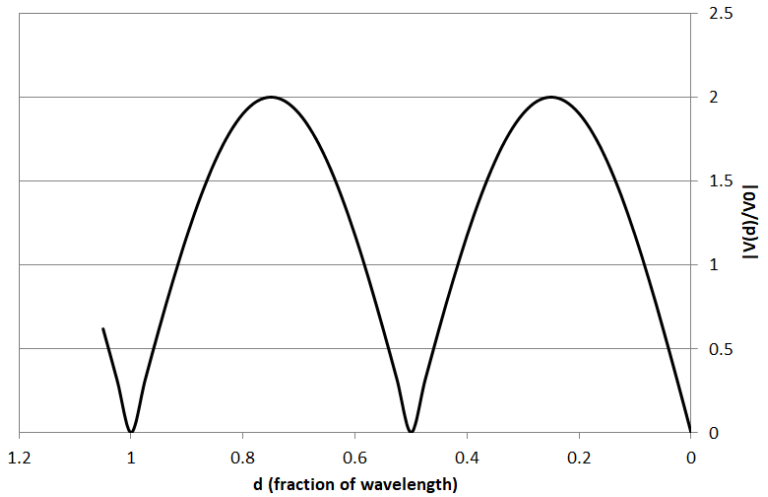

Fig. 3.7. Voltage along the line closed to a short circuit

If the input impedance is negative, then the shorted line behaves as a capacitor with a capacitance equal to:

$$\frac{1}{j\omega C_{eq}} = jZ_0\tan(\beta d) \qquad \Rightarrow \qquad C_{eq} = -\frac{1}{Z_0\omega\tan(\beta d)} \tag{3.92}$$

In this way, the shorted stub can be used to create inductors and capacitors, at a given frequency, in a distributed circuit simply by adjusting the length of the stub.

Transmission line closed to an open circuit

In this case the line closed to an open circuit is also called an *open stub*.
The reflection coefficient is

$$\Gamma_L = -\frac{\infty - Z_0}{\infty + Z_0} = 1 \tag{3.93}$$

In this case, the amplitude of the reflected wave is equal in magnitude and phase to the incident wave, then $\bar{V}_0^- = \bar{V}_0^+$.

The voltage and current along the line is then:

$$\begin{aligned}\bar{V}(z) &= \bar{V}_0^+\left(e^{-j\beta z}+e^{j\beta z}\right) = 2\bar{V}_0^+\cos(\beta z)\\ \bar{I}(z) &= \frac{\bar{V}_0^+}{Z_0}\left(e^{-j\beta z}-e^{j\beta z}\right) = -2j\frac{\bar{V}_0^+}{Z_0}\sin(\beta z)\end{aligned} \tag{3.94}$$

Like the shorted stub, along the line there is only a standing wave, without a progressive wave carrying power.

The minima and maxima of voltage are separated by $\lambda/4$, however, because of the open circuit at the load section, the maximum of voltage is placed at $z = 0$. As in the case of the shorted stub:

$$\begin{aligned}V_{MAX} &= 2|\bar{V}_0^+|\\ V_{MIN} &= 0\end{aligned} \tag{3.95}$$

so the standing wave ratio is equal to ∞.

The impedance along the line is:

$$\begin{aligned}Z_{in}^{sc}(-d) = \frac{\bar{V}(-d)}{\bar{I}(-d)} &= Z_0\frac{2\bar{V}_0^+\cos(-\beta d)}{-2j\frac{\bar{V}_0^+}{Z_0}\sin(-\beta d)}\\ &= -jZ_0\frac{\cos(\beta d)}{\sin(\beta d)} = -jZ_0\mathrm{cotg}(\beta d)\end{aligned} \tag{3.96}$$

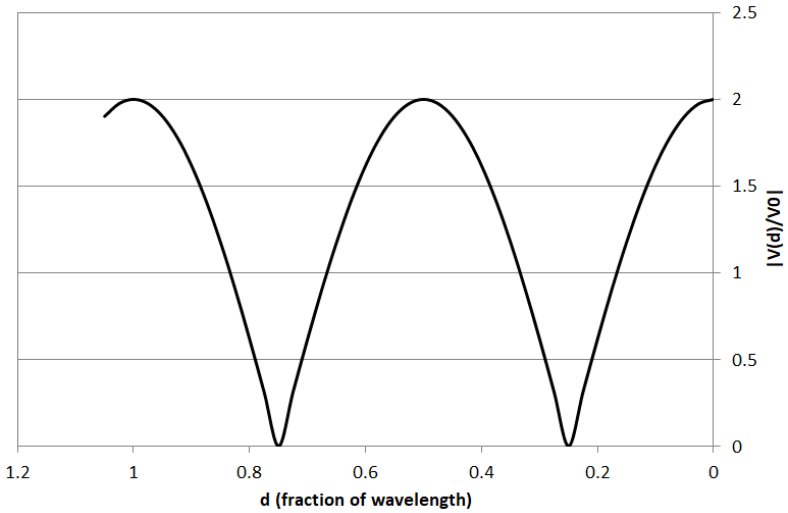

Fig. 3.8. Voltage along the line closed to an open circuit

$n\lambda/2$ long transmission line

Let us consider now a line of length $n\lambda/2$ terminated with a load Z_L. From (3.62) we have:

$$Z_{in}(-d) = Z_0 \left(\frac{1+\Gamma_L e^{-2j\beta d}}{1-\Gamma_L e^{-2j\beta d}} \right) \tag{3.97}$$

Since $d = n\lambda/2$ the argument of the exponentials is $2n\pi$, then the exponentials are equal to 1:

$$Z_{in}(-d) = Z_0 \left(\frac{1+\Gamma_L}{1-\Gamma_L} \right) = Z_L \tag{3.98}$$

For every load, if we move along the line at a distance which is a multiple integer then half wavelength, the input impedance is equal to the load impedance. This is a strong property of trasmission lines and it means that trasmission lines are periodic by $\lambda/2$, so that:

$$\Gamma \left(\frac{-n\lambda}{2} \right) = \Gamma_L \tag{3.99}$$

$\lambda/4$ long transmission line

If the line if $\lambda/4$ long, the input impedance is:

$$Z_{in}(-d) = Z_0 \frac{Z_L \cos(\beta d) + jZ_0 \sin(\beta d)}{Z_0 \cos(\beta d) + jZ_L \sin(\beta d)} \tag{3.100}$$

since $d = \lambda/4$, we have:

$$\beta \frac{\lambda}{4} = \frac{\pi}{4}$$

then:

$$\begin{aligned} \sin(\beta d) &= 1 \\ \cos(\beta d) &= 0 \end{aligned} \tag{3.101}$$

substituting into 3.100:

$$Z_{in}(-d) = Z_0 \frac{jZ_0}{jZ_L} = \frac{Z_0^2}{Z_L} \tag{3.102}$$

From the last equation we can see that the input impedance depends on the reciprocal of the load impedance Z_L. For this reason this component is sometimes called *impedance inverter*.
This is even clearer if we consider extreme cases. For example if the load impedance is zero (i.e. a short circuit), the input impedance at a distance $d = \lambda/4$ is infinite (i.e. it is an open circuit). On the contrary, if the load impedance is infinite (i.e. an open circuit), the input impedance at a distance $d = \lambda/4$ is zero (short circuit). This circuit can then be used to transform a short circuit into an open and viceversa.

Reactive load

If the load is purely imaginary we have, for example:

$$Z_L = jX_L \tag{3.103}$$

when computing the reflection coefficient at the load, we obtain:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{jX_L - Z_0}{jX_L + Z_0} = -\frac{Z_0 - jX_L}{jX_L + Z_0} \tag{3.104}$$

rationalizing this equation:

$$\Gamma = -\frac{\sqrt{Z_0^2 + X_L^2}e^{-j\phi}}{\sqrt{Z_0^2 + X_L^2}e^{j\phi}} = -e^{-2j\phi} \tag{3.105}$$

where

$$\phi = \arctan\frac{X_L}{Z_0}$$

This means that the amplitude of the reflection coefficient is:

$$|\Gamma| = |e^{-2j\phi}| = 1 \tag{3.106}$$

for every load reactance X_L.

II Smith chart

The Smith chart is a graphic tool that is useful to solve many transmission line problems. It consists of loci of constant resistance and reactance plotted on the complex plane where axis are the real and imaginary part of the reflection coefficient (Fig. 3.9).

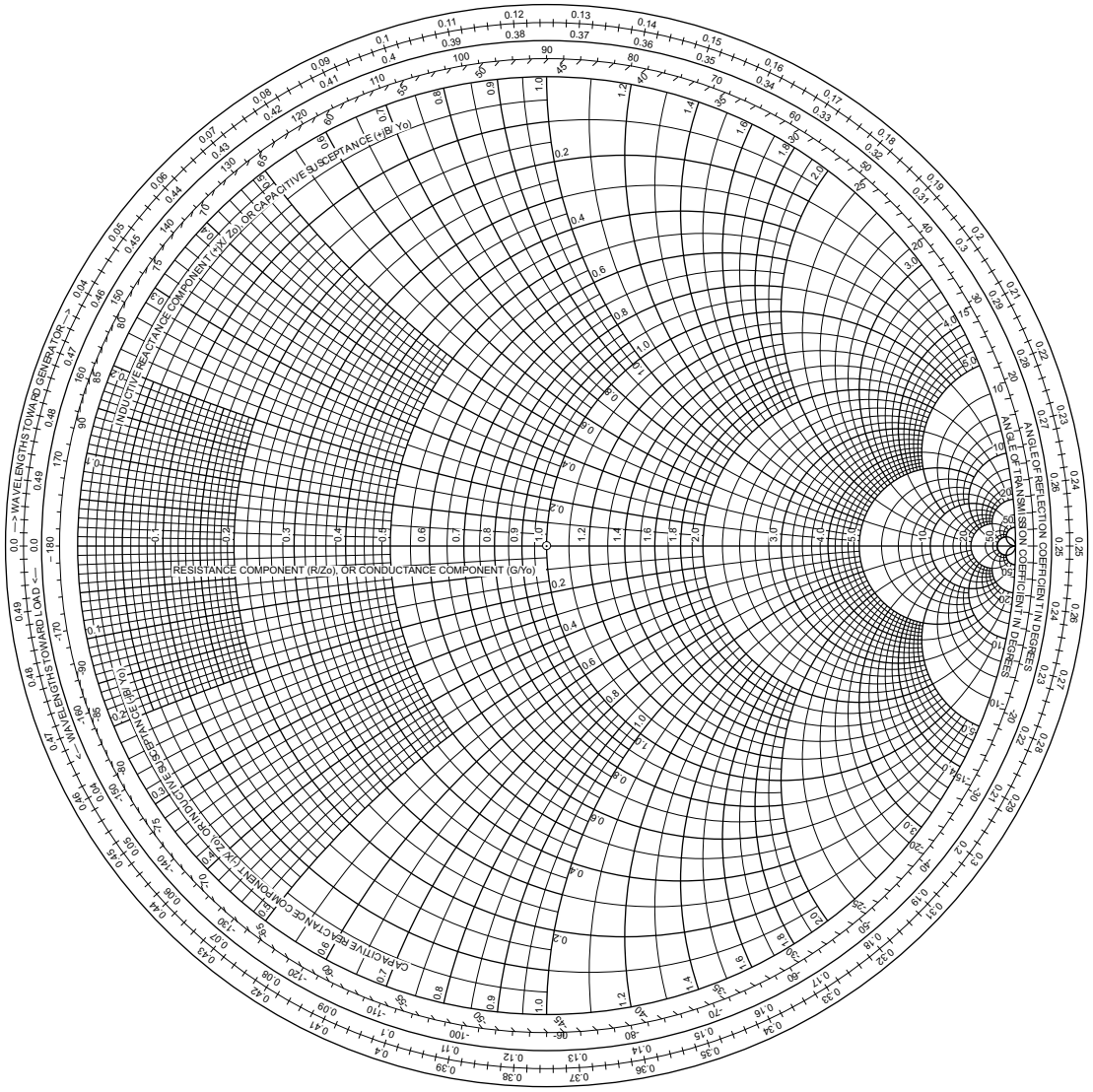

Fig. 3.9. The Smith chart

The main purpose is then to switch between the load impedance and the corresponding reflection coefficient and viceversa. However, it can be used to better understand the transformation of impedance along the line and to solve many problems like impedance matching.
We know that the reflection coefficient is equal to:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{3.107}$$

Because we want a single chart that can be used with every characteristic impedance of the transmission line, we define a normalized load impedance

$$z_L = \frac{Z_L}{Z_0} = r_L + jx_L \tag{3.108}$$

In this way the reflection coefficient can be written as

$$\Gamma_L = \frac{z_L - 1}{z_L + 1} \tag{3.109}$$

and inverting this equation, the normalized load impedance is

$$z_L = \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{3.110}$$

Because we are working on the complex plane where the reflection coefficient is represented, substituting (3.108) and explicitating the real and imaginary parts of the reflection coefficient we obtain

$$r_L + jx_L = \frac{1 + \Gamma_r + j\Gamma_i}{1 - \Gamma_r - j\Gamma_i} \tag{3.111}$$

separating the real and imaginary parts of the right side

$$r_L + jx_L = \frac{1 - \Gamma_r^2 - \Gamma_i^2 + j2\Gamma_i}{(1 - \Gamma_r)^2 + \Gamma_i^2} \tag{3.112}$$

then

$$\begin{aligned} r_L &= \frac{1-\Gamma_r^2-\Gamma_i^2}{(1-\Gamma_r)^2+\Gamma_i^2} \\ x_L &= \frac{2\Gamma_i}{(1-\Gamma_r)^2+\Gamma_i^2} \end{aligned} \tag{3.113}$$

Looking at this equations it is clear that for a given value of r_L or x_L there are infinite values of Γ_r, Γ_i that satisfy (3.113). Then, we can draw the loci of constant resistance and reactance on the complex plane by simply reworking the equations:

$$\left(\Gamma_r - \frac{r_L}{1+r_L}\right)^2 + \Gamma_i^2 = \frac{1}{(1+r_L)^2} \tag{3.114}$$

$$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x_L}\right)^2 = \frac{1}{x_L^2} \tag{3.115}$$

Both equations represent circles in the complex plane parametrized with r_L for the first equation and x_L for the second.

In particular, looking at equation (3.114) we see that the center of the circle is at $(r_L/(1+r_L), 0)$ and the radius is equal to $1/(1+r_L)$. Looking at equation (3.115) instead, we can see that the center of the circle is placed at $(1, 1/x_L)$ ad the radius is $1/|x_L|$.

In Fig. 3.10 it is shown a simplified chart with few curves. For example the set of curves for r_L are the following:

- for $r_L = 0$ the center is $(0,0)$ and the radius is 1;
- for $r_L = 1$ the center is $(1/2, 0)$ and the radius is $1/2$;
- for $r_L = 2$ the center is $(2/3, 0)$ and the radius is $1/3$;
- for $r_L \to \infty$ the center $\to (1,0)$ and the radius $\to 0$, that is the open circut position.

There are also some curves for x_L:

- for $x_L = 0.5$ the center is $(1,2)$ and the radius is 2;
- for $x_L = 1$ the center is $(1,1)$ and the radius is 1;
- for $x_L = 2$ the center is $(1, 1/2)$ and the radius is $1/2$;

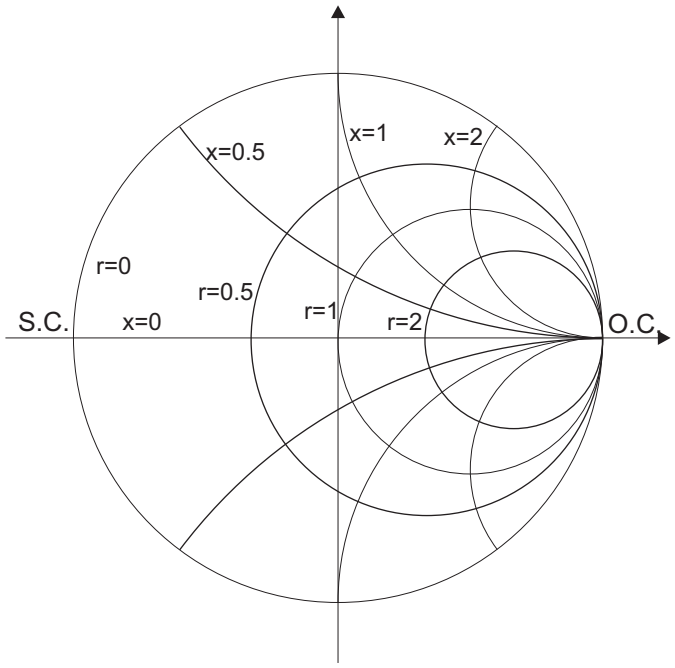

Fig. 3.10. Simplified version of the Smith chart

- for $x_L = -1$ the center is $(1,-1)$ and the radius is 1 (curves with negative reactance are symmetric with respect to the x-axis);
- for $x_L \to 0$ the center $\to (1,\infty)$ and the radius $\to \infty$, that is x-axis;
- for $x_L \to 0$ the center $\to (1,0)$ and the radius $\to 0$, that is open circuit point.

In the chart some other important points coud be identified:

- the point $(0,0)$ (the origin) is for $r_L = 1$ and $x_L = 0$, that is the matched load and, in fact, the reflection coefficient $\Gamma = 0$;
- the point $(-1,0)$ is for $r_L = 0$ and $x_L = 0$ which corresponds to the short circuit. This is often called the short circuit point (S.C.)
- the point $(1,0)$ is for $r_L = \infty$ or $x_L = \infty$ which corresponds to the open circuit (O.C.)

In this view, the chart in Fig. 3.9 shows only more curves in order to be useful for calculations. It is worth noting that for passive loads ($r_L \geq 0$) the useful area of the chart is limited by the circle centered in the origin and unitary radius.

II.1 Basic operations with the Smith chart

The simplest use of the Smith chart is the conversion between load impedance and reflection coefficien, since the chart is derived from the equation (3.109).

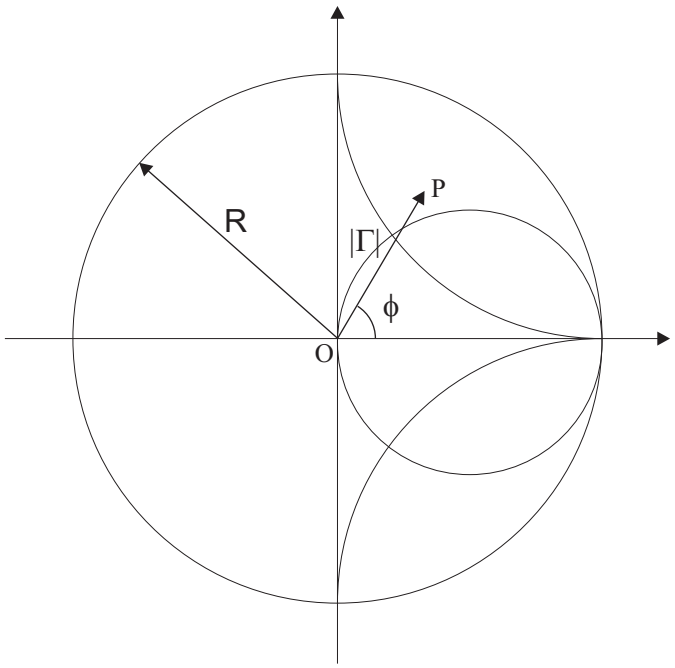

Fig. 3.11. Trasformation between impedance and reflection coefficient

Looking at the Fig. 3.11, if we know the value of the normalized load impedance, the point **P** in the chart can be found by searching the intersection of the resistance and reactance curves. From this point the magnitude of the reflection coefficient ($|\Gamma|$) is the ratio of the lenght of the vector $\bar{OP}$ divided by the radius (R) of the chart. The phase of the reflection coefficient can be found by measuring the inclination of the vector $\bar{OP}$ with respect to the horizontal axis (ϕ).
On the contraty, starting from the reflection coefficient we can draw the vector $|\Gamma|e^{j\phi}$ and find the load impedance by looking at the resistance and reactance curves that cross at position **P**.

Impedance transformation along the line

In a previous section, we have seen that the reflection coefficient at a distance d from the load section can be computed by eq. (3.66):

$$\Gamma(-d) = |\Gamma_L|e^{j\phi_r}e^{-2j\beta d} \tag{3.116}$$

The equation is valid if the line is lossless and shows that only the phase of the reflection coefficient is affected. If we visualize this in the Smith chart, the effect is a rotation of the Γ vector. In particular, because of the negative sign in the exponent, if we move from the load to

the generator the rotation is clockwise. On the contrary if we move from the generator to the load, the rotation is anti-clockwise. Once the Γ vector is rotated, the normalied input impedance in the current line section can be easily found as described in the previous section.
Usually the distance d is expressed as a function of the wavelength λ, in fact in the outer part of the complete chart of Fig. 3.9, there are two scales that express the shift in function of the wavelength. The scales are two, one for the shift toward the generator and the other for the shift toward the load. Because of the factor 2 in equation (3.116), a complete rotation (360°) of the vector can be achieved by a spatial shift equal to $\lambda/2$.

Voltage standing wave ratio

We have seen that for a non macthed trasmission line, there is a standing wave along the line. We can identify some positions where the voltage is maximum and some position were the voltage is minimum. The positions of these points from the load depend on the phase of the reflection coefficient, while the reciprocal distance between them is $\lambda/4$. Moreover, where the voltage is maximum the corresponding input impedance is also maximum and real and where the voltage is minimum the input impedance is minimum and real.
If we consider the Smith chart, during the rotation we can find two points where the input impedance is real. They are the intersection of the circle described by the rotation of the Γ vector, called the *VSWR circle* (Fig. 3.12), and the horizontal axis. In particular the intersectio for $r_L > 1$ corresponds to the maximum while the intersection for $r_L < 1$ corresponds to the minimum. Moreover, since the maximum input impedance is $Z_{MAX} = Z_0 \cdot VSWR$ the value read from the intersection, being the normalized maximum impedance, is directly the standing wave ratio. On the intersection corresponding to the minimum impedance, the value read is the reciprocal of the standing wave ratio.

II.2 Admittance chart

The equation (3.44) can be written in terms of the load admittance and the characteristic admittance:

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = -\frac{Y_L - Y_0}{Y_L + Y_0} \tag{3.117}$$

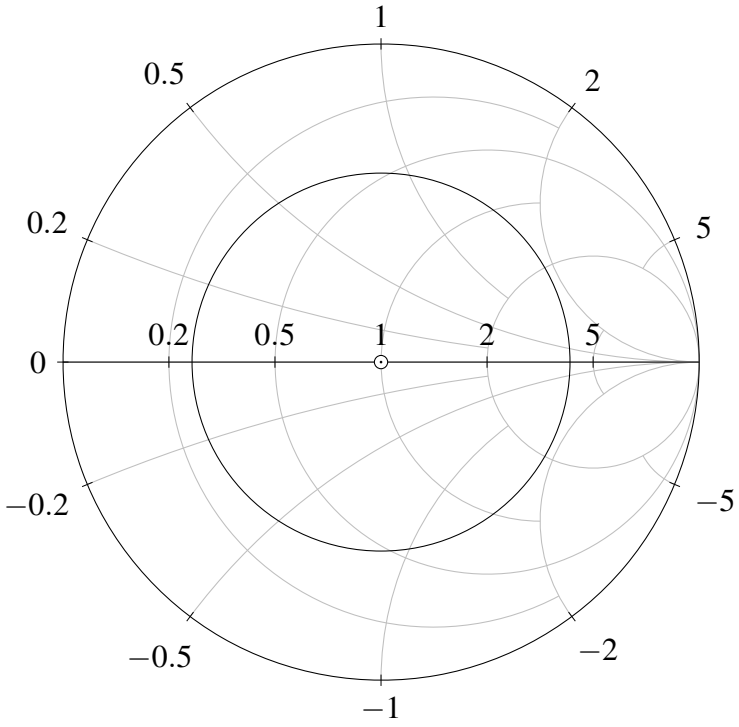

Fig. 3.12. VSWR circle in the Smith chart

or in the normalized form:

$$\Gamma_L = -\frac{Y_L/Y_0 - 1}{Y_L/Y_0 + 1} = -\frac{y_L - 1}{y_L + 1} \tag{3.118}$$

The equation (3.118) is formally identical to the one with impedances with the exception of the minus sign. This means that the Smith chart for admittances can be derived from the original chart by a simple 180° rotation. In theory the curves for normalized conductance and susceptance can be drawn overlapped to the impedance chart, however it is not very useful. Instead, we can pass from one chart to another by simply rotating the reflection coefficient by 180° or, equivalently, by mirroring it with respect to the origin of the complex plane. In this way we can use exactly the same chart for both cases, and in fact the curves in Fig. 3.9 are parametrized for impedances and admittances.

In Fig. 3.13 there is an example of the admittance transformation. Let the normlized load impedance be $z_L = 0.8 + j1.5$, the corresponding point in the Smith chart is P. By rotating the vector $O\bar{z}_L$ by 180° we obtain the point y_L, which corresponds to the normalized load ad-

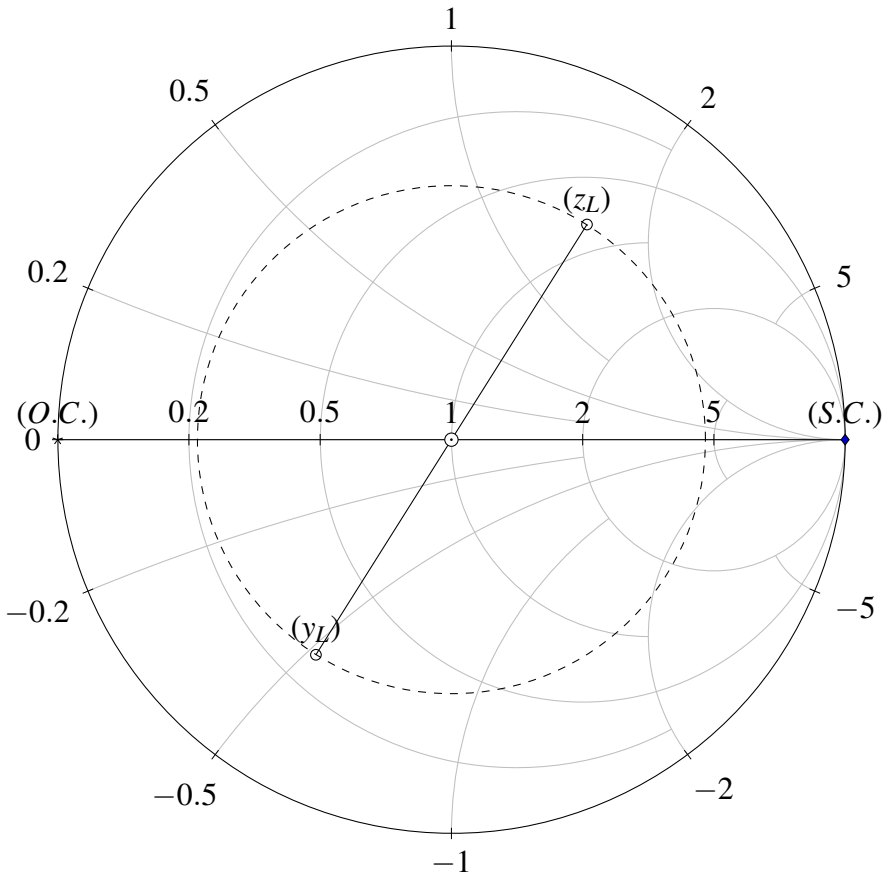

Fig. 3.13. Admittance transformation

mittance $y_L = 1/z_L = 0.277 - j0.519$. This point is the intersection between the rect crossing the origin and point z_L and the *SWR* circle (dashed in Fig. 3.13).

Note that, since the rotation of 180°, the short circuit (S.C.) and open circuit (O.C.) points are swapped with respect to the impedance chart.

III Matching

A transmission line is used to connect a generator to a load. The latter can be for example an antenna or any circuit with an input impedance Z_L. We have seen that a if a lossless transmission line is closed to its matched load ($Z_L = Z_0$) the power delivered to the line by the generator is entirely trasferrred to the load. The design of a device with such input impedance is not always possible and often the load to be connected to the line is not matched. However, it is possible to design a *matching network* with the aim to match the load to the characteristic impedance of the line.

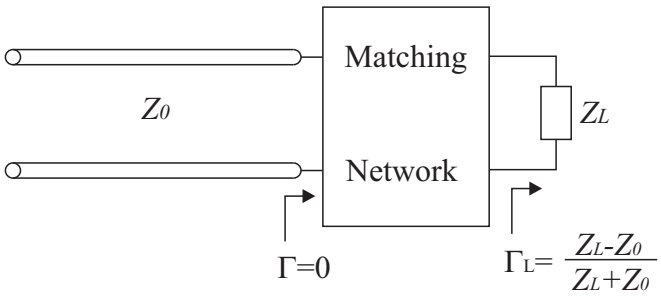

Fig. 3.14. Matching network

This two-port network is interposed between the line and the load (Fig. 3.14) in such a way to have an input impedance equal to Z_0. It is intended to match a load $Z_L = R_L + jX_L$ with $R_L > 0$. The purpose is to eliminate the reflection at the input port of the network so that there is only the propagating wave along the line and all the power is transferred to the load. The matching network itself is lossless, at least ideally, in order not to dissipate active power. It can be made by lumped elements, such as inductors and capacitors, or distributed elements, such as section of transmission lines.

III.1 $\lambda/4$ adapter

One of the simplest distributed line matching network is the $\lambda/4$ adapter. Let us consider a transmissione line with a characteristic impedance Z_0 closed to a purely real load $Z_L = R_L$. The corresponding reflection cofficient is:

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{3.119}$$

which is also purely real.

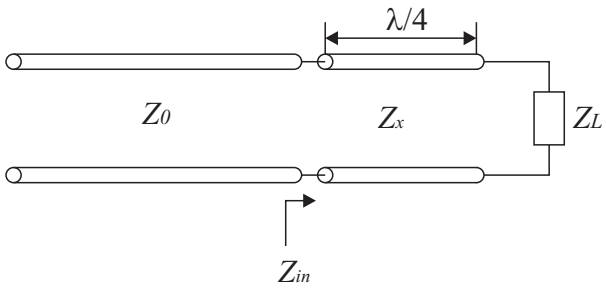

Fig. 3.15. $\lambda/4$ matching network

In this case, the matching network can be a simple trasmission line of length $\lambda/4$ with a suitable characteristic impedance Z_x (Fig. 3.15). The input impedance in the figure is:

$$Z_{in} = Z_x \frac{Z_L + jZ_x \tan\beta l}{Z_x + jZ_L \tan\beta l} \tag{3.120}$$

Since

$$\beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2} \tag{3.121}$$

the $\tan\beta l \to \infty$. The input impedance is:

$$Z_{in} = \frac{Z_x^2}{Z_L} \tag{3.122}$$

In order to obtain the matching, this impedance must be equal to the characteristic impedance of the feeding line $Z_{in} = Z_0$, then:

$$Z_x = \sqrt{Z_0 Z_L} \tag{3.123}$$

Looking at the last equation, because the characteristic impedance of the $\lambda/4$ section must be real (the line is lossless), the load must be also real. This means that such matching network works only for real load.

However, with some modifications, it can be used also for complex load. In fact, remembering the behaviour of the impedance along the line, there are some points where the input impedance is purely real. Such points are where the voltage is maximum and minimum and the corresponding input impedances are $Z_0 SWR$ and Z_0/SWR respectively. These are the intersections of the SWR circle with the horizontal axis in the Smith chart. Then, placing the $\lambda/4$ adapter into one of these points we can solve the matching problem (Fig. 3.16). Usually the best choice is to

place the adapter as close as possible to the load, so it is better to choose the closest point to the load where the input impedance is real.

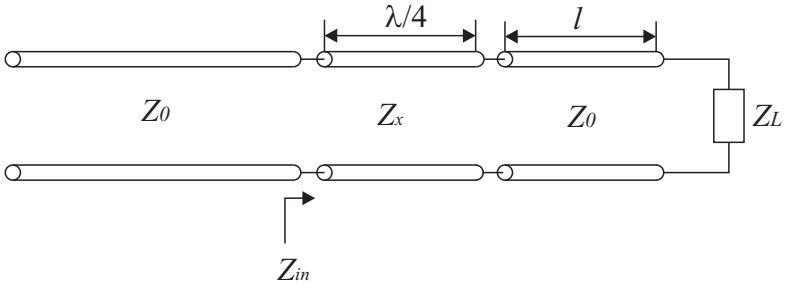

Fig. 3.16. $\lambda/4$ matching network for complex load

In order to find such points we can use either the Smith chart or equations (3.81) and (3.82). If we choose the maximum the distance l is

$$l = \frac{2n\pi + \phi_r}{2\beta} \tag{3.124}$$

and the characteristic impedance of the $\lambda/4$ section is

$$Z_x = \sqrt{Z_0 Z_0 SWR} = Z_0\sqrt{SWR} \tag{3.125}$$

on the other hand, if we choose the minimum the distance l is

$$l = \frac{2(n+1)\pi + \phi_r}{2\beta} \tag{3.126}$$

and the characteristic impedance of the $\lambda/4$ section is

$$Z_x = \sqrt{Z_0 \frac{Z_0}{SWR}} = \frac{Z_0}{\sqrt{SWR}} \tag{3.127}$$

It is worth noting that in the first case (real load) we need only one section of line whereas in the second case (complex) we need two line sections. The reason is that in the latter we need two degrees of freedom in order to adjust the real part and cancel the imaginary part of the load. This is a general concept and in the following we will see other adapters that have two adjustable parameters.

III.2 Stub adapter

Depending on the technology, it is not simple to make transmission line with different characteristic impedances, so it is preferable a solution that uses the same impedance for all line sections of the adapter. A possible solution is shown in Fig. 3.17.

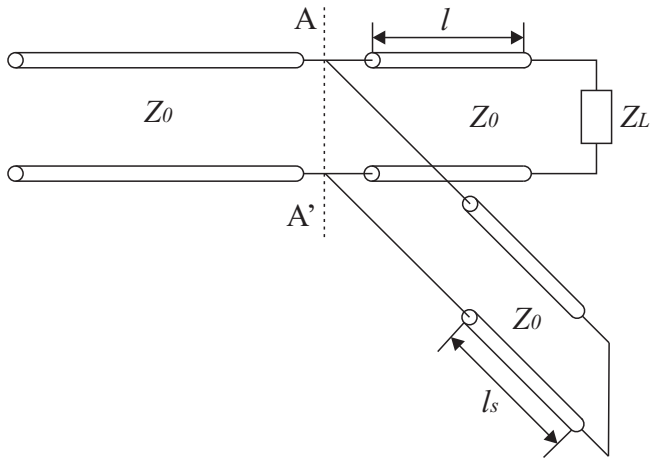

Fig. 3.17. Shorted stub adapter

The adapter is composed of a shorted stub of length l_s connected at a suitable distance l from the load. All the transmission lines have the same impedance Z_0. Since the stub is connected in parallel, it is convenient to use the admittances instead of impedances. Moreover, since this problem can be easily solved by using the Smith chart, it is also convenient to normalize all the admittances (that is $y_L = Y_L/Y_0$, where $Y_0 = 1/Z_0$).

If the input admittance of the line section of length l closed to the load is y'_L, looking at the section AA', the input admittance is the sum of the input admittance of the stub and y'_L:

$$y_{AA'} = y_s + y'_L = jb_s + y'_L \tag{3.128}$$

because the input admittance of the stub is purely imaginary, where:

$$jb_s = -j\text{cotg}(\beta l_s) \tag{3.129}$$

Now, in order to have a matched load, the admittance seen at the section AA' must be equal to 1:

$$1 = jb_s + y'_L \tag{3.130}$$

This means that the length l is chosen in such a way to transform the load admittance into a new admittance in the form $y'_L = 1 - jb_s$, (that is the real part is equal to 1). The shorted stub is then used to cancel the imaginary part of this impedance.
The procedure is simplified using the Smith chart (Fig. 3.18) and can be better explained by a numerical example. Let us consider a load impedance $Z_L = 75 + j50$ Ohm and a transmission line with characteristic impedance of 50 Ohm. The normalized load impedance is:

$$z_L = \frac{Z_L}{Z0} = 1.5 + j1 \tag{3.131}$$

Starting from the z_L point in the Smith chart the procedure is:

1. Since we want to work with admittances, this impedance is converted by rotating by 180° the z_L point in the chart along the SWR circle (dashed in Fig. 3.18) to find the y_L point. In the example $y_L = 0.46 - j0.31$;
2. Rotate toward the generator (clockwise) along the SWR cirle up to the intersection with the circle $g_L = 1$. This point is the input admittance y'_L. In the example $y'_L = 1 + j0.91$;
3. The length of the piece of line needed to do the admittance transformation at the previous point can be computed by tracing the two segments starting from the origin and crossing y_L and y'_L respectively. The length of the transmission line, in terms of λ can be derived from the length scales. In the example $l = (0.159 - 0.058)\lambda = 0.101\lambda$;

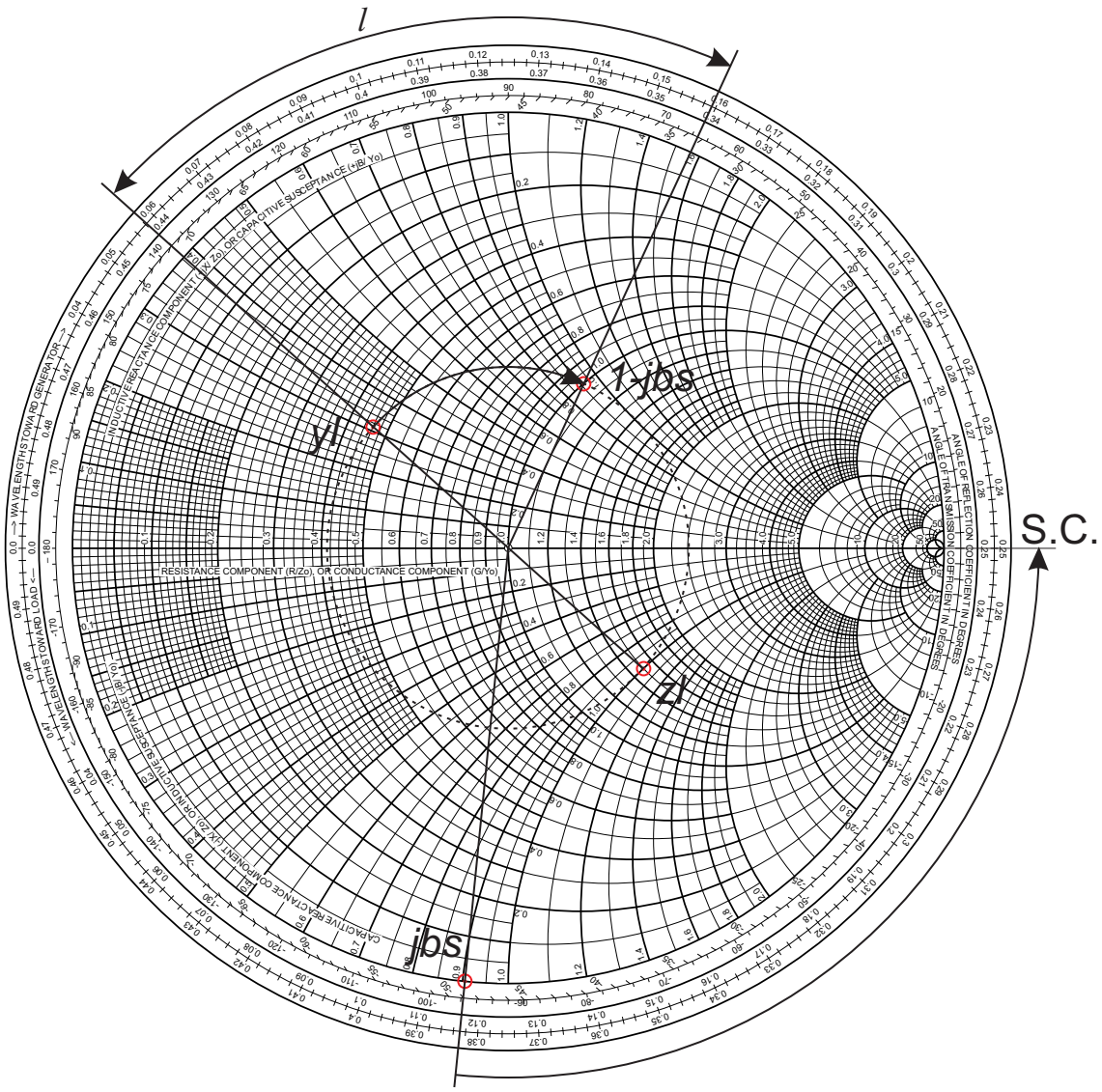

Fig. 3.18. Design of a shorted stub adapter

4. Take the imaginary part of the $y'_L = 1 - jb_s$ and change the sign, this is the input admittance of the shorted stub jb_s. In the example, the corresponding susceptance is $b_s = -0.91$;
5. Find the point jb_s in the chart and rotate toward the load (counter-clockwise) up to the short circuit point (S.C.) in the chart. The rotation gives the length of the stub in terms of wavelength. In the example $l_s = (0.25 - 0.117)\lambda = 0.133\lambda$;

With a similar procedure it is possible to design an adapter with an open stub connected in parallel (Fig. 3.19)

The only difference is the point 5 of the procedure. In fact, with open stub we have to rotate counter-clockwise up to the open circuit point (O.C.). Then, in the example the open stub is

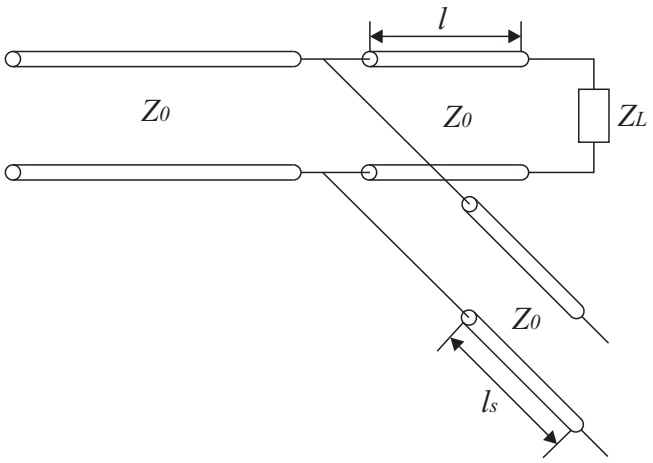

Fig. 3.19. Open stub adapter

0.25λ longer than the shorted one (Fig. 3.20). On the other hend, the same length can be computed considering that the input admittance of an open stub is:

$$jb_o = j\tan(\beta l_s) \tag{3.132}$$

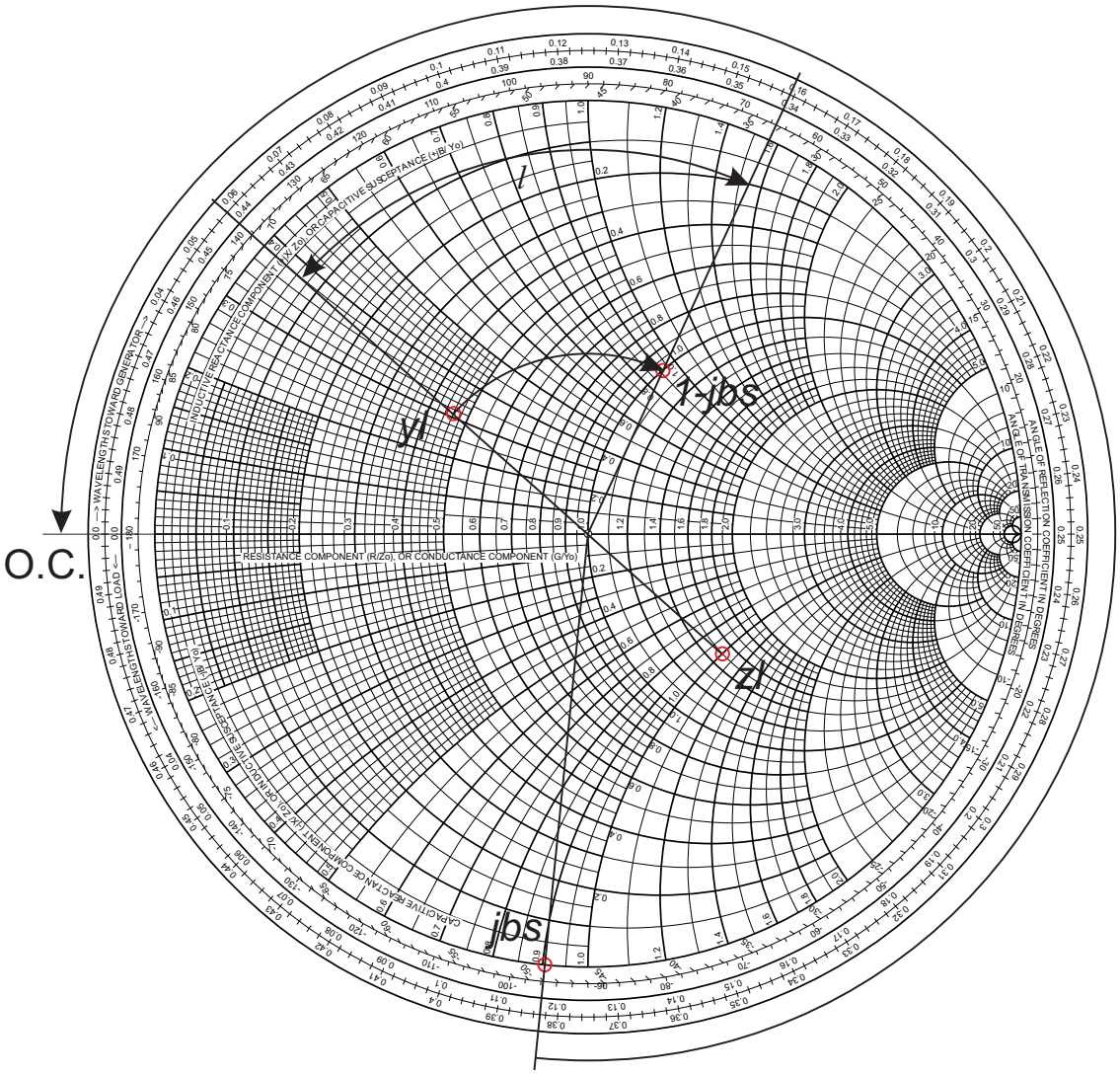

Fig. 3.20. Design of an open stub adapter

4

Representations of Electromagnetic Fields

I Maxwell's Equations

Equations linking electromagnetic field quantities have been introduced by James Clerk Maxwell in an elegant treatise first published in 1873 and then inserted into [1] (see also [2] for more historical information). We shall assume that a student reader is familiar with these equations, which are usually introduced in preliminary courses, and that he/she has a general knowledge of the relevant experimental facts and their theoretical interpretation. In what follows, we summarize Maxwell's equations in time and frequency domains.

I.1 Maxwell's Equations in Time–Dependent Form

It is customary to write Maxwell's equations in either local or in global form; we shall first consider their local form. We also note that, unfortunately, it is customary to designate the local form as differential form and this generates some confusion with the general meaning that differential forms have. In the following of this book, since differential forms are not used, the ambiguity is resolved.

Local Form of Maxwell's Equations

In three–dimensional vector notation, with vector $\mathbf{r}$ a indicating a position in space and t the time variable, Maxwell's equations are

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}, \qquad \text{Faraday's law} \tag{4.1a}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \mathbf{J}(\mathbf{r},t), \qquad \text{Ampère's law} \tag{4.1b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho_e(\mathbf{r},t), \qquad \text{Gauss' law} \tag{4.1c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r},t) = 0, \qquad \text{Magnetic flux continuity} \tag{4.1d}$$

where bold face symbols denote vector quantities. The quantities are defined as

$\mathbf{E}(\mathbf{r},t)$	electric field strength
$\mathbf{D}(\mathbf{r},t)$	electric displacement
$\mathbf{B}(\mathbf{r},t)$	magnetic flux density
$\mathbf{H}(\mathbf{r},t)$	magnetic field strength
$\mathbf{J}(\mathbf{r},t)$	electric current density
$\rho_e(\mathbf{r},t)$	electric charge density

Equations (4.1a)–(4.1d) are not independent since, for example, we may derive (4.1d) by taking the divergence of (4.1a). Another fundamental relationship can be derived by introducing (4.1c) into the divergence of (4.1b)

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial \rho_e(\mathbf{r},t)}{\partial t} \tag{4.2}$$

which provides the conservation law for electric charge and current densities.
Actually, the set of three equations (4.1a), (4.1b) and (4.2) may be considered as the independent equations describing macroscopic electromagnetic fields, since the two Gauss equations (4.1c) and (4.1d) can be derived from this set.

Static case

Note that in the static case $\frac{\partial}{\partial t} = 0$ the electric and magnetic fields are not any more interdependent and the equations (4.1a) – (4.1d) become

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0\,, \tag{4.3a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r})\,, \tag{4.3b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r})\,, \tag{4.3c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0\,. \tag{4.3d}$$

Finally also note that, if we assign the electric current density $\mathbf{J}(\mathbf{r})$ and the electric charge density $\rho_e(\mathbf{r})$, we have, from (4.1a) and (4.1b), two vector equations (i.e. six scalar equations) while we have four unknown vectors (i.e. twelve scalar quantities). To complete the number of equations we have to account for the media properties expressed by the constitutive relations.

Some identities for expressing Maxwell's Equations in GA

Let us consider a generic vector $\mathbf{B}$ expressed in terms of the Pauli representation $\tilde{B}$ as

$$\tilde{B} = \sigma_1 B_x + \sigma_2 B_y + \sigma_3 B_z \tag{4.4}$$

where the elements σ_i constitutes the basis elements. The bivector $\hat{B} = i\sigma_0 \tilde{B}$ is expressed, using the identity $i\sigma_0 = \sigma_1 \sigma_2 \sigma_3$, as

$$\begin{aligned} \hat{B} &= i\sigma_0 \left(\sigma_1 B_x + \sigma_2 B_y + \sigma_3 B_z\right) \\ &= \sigma_2 \sigma_3 B_x + \sigma_3 \sigma_1 B_y + \sigma_1 \sigma_2 B_z \end{aligned} \tag{4.5}$$

The ∇ operator, in cartesian coordinates, is given by:

$$\nabla = \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z\,. \tag{4.6}$$

Although we are referring to cartesian coordinates, the results derived next are valid in general. The divergence is readily expressed as usual as:

$$\nabla \cdot \mathbf{B} = \partial_x B_x + \partial_y B_y \; + \partial_z B_z\,. \tag{4.7}$$

The external product of nabla with the bivector $\hat{B}$ is

$$\begin{aligned}\nabla \wedge \hat{B} &= \left(\sigma_1\,\partial_x + \sigma_2\,\partial_y + \sigma_3\,\partial_z\right) \wedge \left(\sigma_2\,\sigma_3\,B_x + \sigma_3\,\sigma_1\,B_y + \sigma_1\,\sigma_2\,B_z\right) \\ &= \sigma_1\,\sigma_2\,\sigma_3\left(\partial_x B_x + \partial_y B_y\ + \partial_z B_z\right) \\ &= i\nabla\cdot\mathbf{B}\,. \end{aligned} \tag{4.8}$$

Note that the divergence of $\mathbf{B}$ is a scalar; when multiplied by i it becomes a pseudoscalar. Also, when performing the external product of nabla with the bivector $\hat{B}$ a pseudoscalar is obtained. The divergence of the bivector $\hat{B}$ is

$$\begin{aligned}\nabla \cdot \hat{B} &= \left(\sigma_1\,\partial_x + \sigma_2\,\partial_y + \sigma_3\,\partial_z\right) \cdot \left(\sigma_2\,\sigma_3\,B_x + \sigma_3\,\sigma_1\,B_y + \sigma_1\,\sigma_2\,B_z\right) \\ &= -\sigma_3\,\partial_x B_y + \sigma_2\,\partial_x B_z + \sigma_3\,\partial_y B_x - \sigma_1\,\partial_y B_z - \sigma_2\,\partial_z B_x + \sigma_1\,\partial_z B_y \\ &= -\nabla\times\mathbf{B} = i\nabla\wedge\mathbf{B}\,. \end{aligned} \tag{4.9}$$

In summary we have derived the following important identities:

$$\nabla \wedge \hat{B} = i\nabla\cdot\mathbf{B} \tag{4.10}$$

$$\nabla \cdot \hat{B} = -\nabla\times\mathbf{B} = i\,\nabla\wedge\mathbf{B} \tag{4.11}$$

$$\nabla \hat{B} = i\,\nabla\mathbf{B} \tag{4.12}$$

Geometric Algebra form of Maxwell's Equations

Faraday's law

By multiplying both sides of (4.1a) times i one obtains:

$$\nabla\wedge\mathbf{E} = -\partial_t\hat{B}\,. \tag{4.13}$$

It is noted that is an equation of grade 2, i.e. between bivectors.

Ampere's law

Let us now consider (4.1b) and make use of (4.11):

$$\nabla\cdot\hat{H} = -\partial_t\mathbf{D} - \mathbf{J}\,. \tag{4.14}$$

This is an equation of grade 1, i.e. between vectors.

Gauss' law

This equation remains unchanged: in fact, by considering (4.1c):

$$\nabla \cdot \mathbf{D} = \rho_e \tag{4.15}$$

we have an equation of grade 0, i.e. a scalar equation.

Magnetic flux continuity

By considering (4.1d), multiplying by i and using (4.10):

$$\nabla \wedge \hat{B} = 0 \tag{4.16}$$

we have an equation of grade 3, i.e. a pseudoscalar equation.
In summary the following form are the GA equivalent of Maxwell's equations local form:

$$\nabla \wedge \mathbf{E} = -\partial_t \hat{B}\,, \qquad \text{grade 2} \tag{4.17a}$$

$$\nabla \cdot \hat{H} = -\partial_t \mathbf{D} - \mathbf{J}\,, \qquad \text{grade 1} \tag{4.17b}$$

$$\nabla \cdot \mathbf{D} = \rho_e\,, \qquad \text{grade 0} \tag{4.17c}$$

$$\nabla \wedge \hat{B} = 0\,, \qquad \text{grade 3} \tag{4.17d}$$

Sometimes it is convenient to consider also magnetic sources obtaining the following local form of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\varepsilon}\,, \qquad \text{grade 0} \tag{4.18a}$$

$$\nabla \cdot (i\eta \mathbf{H}) = -\frac{1}{c}\,\partial_t \mathbf{E} - \eta\,\mathbf{J}\,, \qquad \text{grade 1} \tag{4.18b}$$

$$\nabla \wedge \mathbf{E} = -\frac{1}{c}\,\partial_t\,(i\eta \mathbf{H}) - i\mathbf{M}, \qquad \text{grade 2} \tag{4.18c}$$

$$\nabla \wedge (i\eta \mathbf{H}) = -ic\,\rho_m\,, \qquad \text{grade 3} \tag{4.18d}$$

Integral (global) Form of Maxwell's Equations

The properties of an electromagnetic field may also be expressed globally by an equivalent system of integral relations through use of the two fundamental theorems of vector analysis: the divergence theorem and Stokes' theorem [3].

Divergence or Gauss' Theorem

Let $\mathbf{U}(\mathbf{r})$ be any vector function of position, continuous together with its first derivative throughout a volume V bounded by a surface S. The divergence theorem states that

$$\oint_S \mathbf{U}(\mathbf{r}) \cdot \mathbf{n}\, dS = \int_V \nabla \cdot \mathbf{U}(\mathbf{r})\, dV\,, \tag{4.19}$$

where $\mathbf{n}$ is the outward unit vector normal to S. In fact, Gauss's theorem may also be used to *define* the divergence.

Stokes' Theorem

Let $\mathbf{U}(\mathbf{r})$ be any vector function of position, continuous together with its first derivatives throughout an arbitrary surface S bounded by a contour C, and assumed to be resolvable into a finite number of regular arcs. Stokes' theorem (also called curl theorem) states that

$$\oint_C \mathbf{U}(\mathbf{r}) \cdot d\mathbf{l} = \int_S \big[\nabla \times \mathbf{U}(\mathbf{r})\big] \cdot \mathbf{n}\, dS\,, \tag{4.20}$$

where $d\mathbf{l}$ is an element of length along C, and $\mathbf{n}$ is a unit vector normal to the positive side of the element area dS as defined by the right–hand thumb rule. This relationship may also be considered as an equation defining the *curl* or *circulation*.

By applying the curl theorem to (4.1a) and (4.1b), and the divergence theorem to (4.1c) and (4.1d), we get

$$\oint_C \mathbf{E}(\mathbf{r},t)\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\cdot \mathbf{n}\, dS, \tag{4.21a}$$

$$\oint_C \mathbf{H}(\mathbf{r},t)\cdot d\mathbf{l} = \int_S \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}\cdot \mathbf{n}\, dS + \int_S \mathbf{J}(\mathbf{r},t)\cdot \mathbf{n}\, dS, \tag{4.21b}$$

$$\int_V \nabla\cdot\mathbf{D}(\mathbf{r},t)dv = \oint_S \mathbf{D}(\mathbf{r},t)\cdot \mathbf{n}\, dS = \int_V \rho_e(\mathbf{r},t)\, dv, \tag{4.21c}$$

$$\int_V \nabla\cdot\mathbf{B}(\mathbf{r},t)dv = \oint_S \mathbf{B}(\mathbf{r},t)\cdot \mathbf{n}\, dS = 0. \tag{4.21d}$$

By defining the current $I(t)$ as

$$I(t) = \int_S \mathbf{J}(\mathbf{r},t)\cdot \mathbf{n}\, dS, \tag{4.22}$$

the charge $Q(t)$ as

$$Q(t) = \int_V \rho_e\, dv, \tag{4.23}$$

and the flux of the magnetic induction as

$$\Phi_m(t) = \int_S \mathbf{B}(\mathbf{r},t)\cdot \mathbf{n}\, dS, \tag{4.24}$$

we may write the previous equations as

$$\oint_C \mathbf{E}(\mathbf{r},t)\cdot d\ell = -\frac{\partial \Phi_m(t)}{\partial t}, \tag{4.25a}$$

$$\oint_C \mathbf{H}(\mathbf{r},t)\cdot d\ell = \int_S \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}\cdot \mathbf{n}\, dS + I(t), \tag{4.25b}$$

$$\int_V \nabla\cdot\mathbf{D}(\mathbf{r},t)\, dv = Q(t). \tag{4.25c}$$

Geometric Algebra Global form of Maxwell's Equations

Let us refer to the equations (4.21a)–(4.21d). Instead of dealing with the term $\mathbf{n}dS$ we can now consider the bivector $d\hat{A}$ denoting the oriented surface area. In addition note that when a volume term dv is considered, it corresponds to a pseudoscalar as $dv = dx\,dy\,dz$.

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \hat{B}}{\partial t}\cdot d\hat{A}\,, \tag{4.26a}$$

$$\oint_C \hat{H}\wedge d\mathbf{l} = \int_S \left(\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}\right)\wedge d\hat{A} \tag{4.26b}$$

$$\int_V \nabla\cdot\mathbf{D}\,dv = \oint_S \mathbf{D}\wedge d\hat{A} = \int_V \rho_e\,dv\,, \tag{4.26c}$$

$$\int_V \nabla\cdot\mathbf{B}\,dv = \oint_S \hat{B}\cdot d\,\hat{A} = 0\,. \tag{4.26d}$$

It is noted that, in the above equations, the dot product between two bivectors give rise to a scalar, while the volume integral and the external product of a vector with a bivector produce a pseudoscalar.

I.2 Maxwell's Equations in the Frequency Domain

Electromagnetic fields operating at a particular frequency are known as time–harmonic steady–state or monochromatic fields. By adopting the time dependence $e^{j\omega t}$ to denote a time–harmonic field with angular frequency ω, we write

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{\mathbf{E}(\mathbf{r})e^{j\omega t}\right\}, \tag{4.27}$$

where Re denotes the mathematical operator which selects the real part of a complex quantity. The complex quantity is $\mathbf{E}(\mathbf{r})$ is called a *vector phasor*. In (4.27) we have used the same symbol to denote both the real quantity in the time domain, $\mathbf{E}(\mathbf{r},t)$, and the complex quantity, $\mathbf{E}(\mathbf{r})$, in the frequency domain. In what follows we shall generally refer to complex quantities unless otherwise explicitly stated.

By applying (4.27) to the field quantities appearing in (4.1a), (4.1b), (4.1c) and (4.1d) we obtain Maxwell's equations in the frequency domain. As an example, let us consider (4.1a) for which we have

$$\mathrm{Re}\left\{\left[\nabla\times\mathbf{E}(\mathbf{r})+j\omega\mathbf{B}(\mathbf{r})\right]e^{j\omega t}\right\} = 0\,. \tag{4.28}$$

Since this equation is valid for *all times* t, we may make use of the above lemma and state that the quantity inside the square bracket must be equal to zero. By applying the same reasoning also to the other equations (4.1b), (4.1c) and (4.1d) we get

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mathbf{B}(\mathbf{r}), \tag{4.29a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \tag{4.29b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r}), \tag{4.29c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0. \tag{4.29d}$$

In the following, we make use of equivalence theorems which introduce magnetic current density, $\mathbf{M}(\mathbf{r})$, and magnetic charge distributions, $\rho_m(\mathbf{r})$. These quantities, although not physically present, help in the solution of several boundary value problems. When considering also magnetic currents and charges, the frequency–domain Maxwell's equations become (citare Jackson pag. 273, eq. 6.150 Classical Electrodynamics, third edition, John Wiley and sons)

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}), \tag{4.30a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \tag{4.30b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r}), \tag{4.30c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \rho_m(\mathbf{r}). \tag{4.30d}$$

I.3 Constitutive Relations

As already pointed out Maxwell's equations cannot be solved unless the relationships between the field vectors **D** and **B** with **E** and **H** are specified. The type of field generated by given sources depends on the medium characteristics, which are accounted for by *constitutive relations*; they may be written as

$$\mathbf{D} = \mathscr{F}_d(\mathbf{E}, \mathbf{H}), \tag{4.31a}$$

$$\mathbf{B} = \mathscr{F}_b(\mathbf{E}, \mathbf{H}). \tag{4.31b}$$

Here, $\mathscr{F}_d$ and $\mathscr{F}_d$ are suitable functionals dependent on the medium considered; they may be classified as:

- *nonlinear*, when functionals depend on the electromagnetic field;

- *inhomogeneous*, when functionals depend on *space coordinates*; they are called *spatially–dispersive* when functionals also depend on *spatial derivatives*;
- *nonstationary*, if functionals depend on *time* or *temporally–dispersive* when functionals depend on *time derivatives*.

We shall deal only with linear, stationary media; however, inhomogeneous media are included because of their practical importance.

Another classification of media is provided by the vector form of the constitutive relations. The simplest possibility arises when considering *isotropic media*, where the constitutive relations are given by

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{4.32a}$$

$$\mathbf{B} = \mu \mathbf{H}. \tag{4.32b}$$

with ε denoting permittivity and μ permeability. In this case $\mathbf{E}$ is parallel to $\mathbf{D}$ and $\mathbf{B}$ is parallel to $\mathbf{H}$. In particular, in free space, the above equations are rewritten by using the vacuum constitutive parameters, i.e. permittivity ε_0 and permeability μ_0, as

$$\mathbf{D} = \varepsilon_0 \mathbf{E}, \tag{4.33a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{4.33b}$$

with

$$\varepsilon_0 = 8.854 \cdot 10^{-12}\,\mathrm{Fm}^{-1} \cong \frac{1}{36\pi} 10^{-9}\,\mathrm{Fm}^{-1}, \tag{4.34a}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{Hm}^{-1}. \tag{4.34b}$$

Anisotropic media are characterized by constitutive relations of the type

$$\mathbf{D} = \underline{\underline{\varepsilon}}\, \mathbf{E}, \tag{4.35a}$$

$$\mathbf{B} = \underline{\underline{\mu}}\, \mathbf{H}, \tag{4.35b}$$

where $\underline{\underline{\mu}}$ is the permeability tensor and $\underline{\underline{\varepsilon}}$ is the permittivity tensor. The medium is called *electrically anisotropic* if it is described by the permittivity tensor $\underline{\underline{\varepsilon}}$, and *magnetically anisotropic* when it is described by the permeability tensor $\underline{\underline{\mu}}$. A medium can be both electrically and magnetically anisotropic. An interesting particular case is that of *biaxial* crystals, which may be described, by choosing a suitable particular coordinate system, the so–called principal system, in terms of a tensor of the type:

$$\underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} . \tag{4.36}$$

Cubic crystals, where $\varepsilon_x = \varepsilon_y = \varepsilon_z$, are isotropic; tetragonal, hexagonal and rhombohedral crystals have two parameters equal and the medium is called *uniaxial*. The *principal axis* that exhibits this anisotropy is also referred to as the *optic axis*. When all the three parameters are different, as in orthorhombic crystals, the medium is referred to as *biaxial*.

When the medium has elements possessing permanent electric and magnetic dipoles parallel or antiparallel to each other, an applied electric field simultaneously aligns electric *and* magnetic dipoles; analogously, an applied magnetic field that aligns the magnetic dipoles simultaneously aligns the electric dipoles [4, pag.8]. In order to describe such media Tellegen, in 1948, introduced a new element, the *gyrator*, in addition to the resistor, the capacitor, the inductor and the ideal transformer. These media, when placed in an electric field or a magnetic field, become both polarized and magnetized, and they are referred to as *bianisotropic*, being characterized by constitutive relations of the type

$$\mathbf{D} = \underline{\underline{\varepsilon}}\,\mathbf{E} + \underline{\underline{\xi}}\,\mathbf{H}, \tag{4.37a}$$

$$\mathbf{B} = \underline{\underline{\xi}}\,\mathbf{E} + \underline{\underline{\mu}}\,\mathbf{H}. \tag{4.37b}$$

Examples of hypothetic materials which directly relate electric and magnetic fields are the perfect electromagnetic conductors (PEMCs) as discussed by Sihvola and Lindell [5]. In a PEMC electric and magnetic fields on a material response level both cause electric and magnetic polarizations, however the medium response is not sensitive to the vector orientation of the electric

and magnetic fields. P. Russer has introduced the field theoretical analogon to the gyrator circuit of network theory by boundary surfaces with gyrator properties [6].

II Maxwell–Dirac equation

We will introduce the field equations using Geometric Algebra (GA) in a three–dimensional space. While the reader is assumed to be familiar with Clifford and GA, only very few rules are necessary for the following derivation. These rules are summarized next. In GA, for two vectors $\mathbf{a},\mathbf{b}$ we have:

$$\mathbf{a}\,\mathbf{b} = \mathbf{a}\cdot\mathbf{b} + \mathbf{a}\wedge\mathbf{b} \tag{4.38}$$

$$\mathbf{a}\wedge\mathbf{b} = i\,\mathbf{a}\times\mathbf{b} \tag{4.39}$$

where (4.38) is the fundamental identity which introduces the geometric product, while (4.39) relates the external product to the conventional cross product.
By replacing the vector $\mathbf{a}$ with the nabla operator ∇, we have also the the following identities:

$$\nabla\,\mathbf{b} = \nabla\cdot\mathbf{b} + \nabla\wedge\mathbf{b} \tag{4.40}$$

$$\nabla\wedge\mathbf{b} = i\nabla\times\mathbf{b}\,. \tag{4.41}$$

It is also noted that the external product introduces the bivector (here denoted with an hat) and that a bivector $\hat{\mathbf{H}}$ can be related to a conventional vector $\mathbf{H}$ by

$$\hat{\mathbf{H}} = i\mathbf{H} \tag{4.42}$$

In addition we have derived the following important identities:

$$\nabla\wedge\hat{H} = i\nabla\cdot\mathbf{H} \tag{4.43}$$

$$\nabla\cdot\hat{H} = -\nabla\times\mathbf{H} = i\,\nabla\wedge\mathbf{H} \tag{4.44}$$

$$\nabla\hat{H} = i\nabla\mathbf{H} \tag{4.45}$$

These are all the preliminary informations needed in the following derivations.

II.1 Time–domain Maxwell's Equation in compact form

It is convenient to express the above equations making use of the light velocity in the medium v and of the medium impedance η, recalling that:

$$v = \frac{1}{\sqrt{\mu\varepsilon}} \tag{4.46}$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon}} \tag{4.47}$$

$$\mu = \frac{\eta}{v} \tag{4.48}$$

$$\varepsilon = \frac{1}{v\eta}\,. \tag{4.49}$$

Time–domain Maxwell's equations are commonly expressed as:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.50}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{4.51}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{4.52}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{4.53}$$

Rearranging the equations

With a few superficial changes we can make more evident the symmetries in Maxwell equations as,

$$\nabla \times \mathbf{E} + \frac{\partial\, v\mathbf{B}}{\partial vt} = 0 \tag{4.54}$$

$$\nabla \times v\mathbf{B} - \frac{\partial \mathbf{E}}{\partial vt} = \eta\mathbf{J} \tag{4.55}$$

$$\nabla \cdot \mathbf{E} = \frac{v\rho}{v\varepsilon} = \eta v\rho \tag{4.56}$$

$$\nabla \cdot v\mathbf{B} = 0 \tag{4.57}$$

It is also noted that

$$v\mathbf{B} = \eta\,\mathbf{H} \tag{4.58}$$

so that one can change the expression containing $v\mathbf{B}$ in $\eta\,\mathbf{H}$ or viceversa. Equation (4.54) can be multiplied by i and (4.55) can be multiplied by i^2, obtaining

$$i\nabla\times\mathbf{E}+\frac{\partial\, i\eta\mathbf{H}}{\partial vt}=0 \tag{4.59}$$

$$i\nabla\times i\,\eta\mathbf{H}+\frac{\partial\,\mathbf{E}}{\partial vt}=-\eta\mathbf{J} \tag{4.60}$$

$$\nabla\cdot\mathbf{E}=\eta\, v\rho \tag{4.61}$$

$$\nabla\cdot i\,\eta\mathbf{H}=0 \tag{4.62}$$

By using the Pauli identity for a generic vector $\mathbf{A}$

$$\nabla\mathbf{A}=\nabla\cdot\mathbf{A}+i\nabla\times\mathbf{A} \tag{4.63}$$

and by introducing the notation

$$\partial_0=\frac{\partial}{\partial vt} \tag{4.64}$$

we can write compactly (4.59)–(4.62) as

$$\nabla\,\mathbf{E}+\partial_0\, i\,\eta\mathbf{H}=\eta\, v\rho \tag{4.65}$$

$$\nabla\,(i\,\eta\mathbf{H})+\partial_0\,\mathbf{E}=-\eta\mathbf{J}\,. \tag{4.66}$$

or, by changing sign at (4.65) and arranging in a different order

$$\partial_0\,\mathbf{E}+\nabla\,(i\,\eta\mathbf{H})=-\eta\mathbf{J} \tag{4.67}$$

$$-\nabla\,\mathbf{E}-\partial_0\, i\,\eta\mathbf{H}=-\eta\, v\rho \tag{4.68}$$

$$\tag{4.69}$$

These two equations may be rewritten in terms of Pauli matrices as

$$\sigma_0\partial_0\,\tilde{E}+\tilde{\nabla}\left(i\,\eta\tilde{H}\right)=-\eta\tilde{J} \tag{4.70}$$

$$-\tilde{\nabla}\,\tilde{E}-\sigma_0\partial_0\, i\,\eta\tilde{H}=-\eta\, v\tilde{\rho}\,. \tag{4.71}$$

By using matrix notation we can write

$$\begin{pmatrix} \partial_0 \sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\partial_0 \sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\eta \tilde{H} \end{pmatrix} = -\eta \begin{pmatrix} \tilde{J} \\ v\tilde{\rho} \end{pmatrix} \tag{4.72}$$

where we have changed sign at (4.71). Equation (4.72) is ready to be cast in Dirac form.
A few observations are in order:

- In every place where t appears, we have arranged things so that vt appears, rather than t alone. The rationale is that vt has the same dimensions as x, y, and z. To say the same thing another way, in spacetime, the partner to x, y, and z is not t but rather vt.
- Similarly, the partner to $\mathbf{J}$ is not ρ but rather $v\rho$. In spacetime, $v\rho$ represents a certain amount of charge that sits at one spatial location and flows toward the future, whereas $\mathbf{J}$ represents charge flowing from one spatial location to another.
- Last but not least, the proper partner for $\mathbf{E}$ is not $\mathbf{H}$ but rather $\eta\mathbf{H}$. In every place where $\mathbf{H}$ appears, we have arranged things so the combination $\eta\mathbf{H}$ appears, rather than $\mathbf{H}$ alone. This is just an exercise in algebraic re–arrangement, and does not change the meaning of the equations. The rationale is that $\eta\mathbf{H}$ has the same dimensions as $\mathbf{E}$, and arranging things this way makes the equations more symmetric. It is also noted that since $\eta\mathbf{H}$ is always multiplied by i it is a bivector while $\mathbf{E}$ is a vector.

The field multivector

The multivector $\mathscr{F}$, composed by a vector and a bivector part, is now introduced with the following definition:

$$\mathscr{F} = \mathbf{E} + i\eta\mathbf{H}. \tag{4.73}$$

By summing together (4.65) and (4.66) and using (4.73), the well known results that allows to express the four Maxwell equation as a single one is recovered:

$$\left(\nabla + \frac{1}{v}\partial_t\right)\mathscr{F} = \eta\left(v\rho - \mathbf{J}\right). \tag{4.74}$$

It is convenient to introduce the operator ∂^+ as

$$\partial^+ = \nabla + \frac{1}{v}\partial_t \tag{4.75}$$

and the source multivector $\mathscr{J}$

$$\mathscr{J} = \eta\ (v\rho - \mathbf{J}) \tag{4.76}$$

which allows to write (4.74) as

$$\partial^{+}\mathscr{F} = \mathscr{J}\ . \tag{4.77}$$

It is seen that Maxwells equations are reduced to a single equation and, in particular, to one of the simplest possible first–order differential equations. This allows an intuitive interpretation of Maxwells equations, namely that the gradient of the field $\mathscr{F}$ is proportional to the electromagnetic sources $\mathscr{J}$ that are present.

II.2 Maxwell's equations in Dirac form

The Gamma matrices, also known as Dirac matrices, are often used. It is sufficient to observe that in (4.72) we have the following matrices defined for $i = 1,2,3$ as

$$\begin{aligned}
\gamma^0 = \gamma^4 &= \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \\
\gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \\
\gamma^5 &= -\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}
\end{aligned} \tag{4.78}$$

Note that we have used the superscript in order to differentiate from the gamma matrices previously defined. For the gamma matrices the following relations hold for $i = 1,2,3$:

$$
\begin{aligned}
\gamma^0\gamma^0 &= \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} = I_4 \\
\gamma^i\gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = -I_4 \\
\gamma^0\gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
\gamma^i\gamma^0 &= -\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \\
\gamma^i\gamma^j &= -\begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix} \\
\gamma^j\gamma^i &= \begin{pmatrix} \sigma_i\sigma_j & 0 \\ 0 & \sigma_i\sigma_j \end{pmatrix}.
\end{aligned} \tag{4.79}
$$

From the first relation in (4.79) we note that γ^0 squares to plus one, while from the second relation we see that all the γ^i with $i = 1,2,3$ square to minus one. The remaining four relationships shows that they anti-commute. As a consequence the set of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ form a Clifford basis $Cl(1,-3)$.

It is convenient to introduce also the following notation:

$$
\begin{aligned}
x_0 &= vt \\
x_1 &= x \\
x_2 &= y \\
x_3 &= z
\end{aligned} \tag{4.80}
$$

which is valid for the cartesian coordinate system. Similarly, we use for the derivatives the symbol

$$
\partial_i = \frac{\partial}{\partial x_i}. \tag{4.81}
$$

When dealing with frequency domain formulations the considered quantities are multiplied by $exp(j\omega t)$. With the above assumptions we have for the generic quantity Ψ

$$\Psi e^{j\omega t} = \Psi e^{j\frac{\omega}{v}vt} = \Psi e^{jkx_0} \tag{4.82}$$

where we have used $k = \omega/v$. It is apparent that performing the derivatives with respect to ∂_0 gives

$$\partial_0(\Psi e^{jkx_0}) = jk\,\Psi e^{jkx_0}\,. \tag{4.83}$$

As a consequence the substitution of ∂_0 with jk allows to retrieve the results in the frequency domain.

We have already seen eq. (4.72), here repeated for convenience:

$$\begin{pmatrix} \partial_0\,\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\partial_0\,\sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\,\eta\tilde{H} \end{pmatrix} = -\eta \begin{pmatrix} \tilde{J} \\ v\tilde{\rho} \end{pmatrix} \tag{4.84}$$

with $\tilde{\nabla}$ being

$$\tilde{\nabla} = \sigma\cdot\nabla = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3\,.$$

Therefore, if we write explicitly eq. (4.84) we have

$$\begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\,\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} E_z \\ iE_y + E_x \\ \eta\,iH_z \\ \eta\,\left(iH_x - H_y\right) \end{pmatrix} = -\eta \begin{pmatrix} \tilde{J} \\ v\tilde{\rho} \end{pmatrix} \tag{4.85}$$

In (4.85) we have used just the first column of the Pauli matrices representing the fields $\tilde{E}$ and $i\,\eta\tilde{H}$ and of the source quadrivector, the second column not carrying further information. By using the Dirac gamma matrices we have

$$\left(\gamma^0\partial_0+\gamma^1\partial_1+\gamma^2\partial_2+\gamma^3\partial_3\right)\begin{pmatrix} E_z \\ iE_y+E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x-H_y\right)\end{pmatrix}=-\eta\begin{pmatrix}\tilde{J} \\ v\tilde{\rho}\end{pmatrix} \tag{4.86}$$

It is therefore possible to write the Maxwell equations in a Dirac like notation in terms of the gamma matrices as

$$\sum_{i=0}^{3}\gamma^i\partial_i\begin{pmatrix} E_z \\ iE_y+E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x-H_y\right)\end{pmatrix}=-\eta\begin{pmatrix} J_z \\ iJ_y+J_x \\ v\rho_e \\ 0\end{pmatrix} \tag{4.87}$$

and, by introducing the Feynman slash notation

$$\not{\partial}=\sum_{i=0}^{3}\gamma^i\partial_i \tag{4.88}$$

and the shorthand notation for the quadrivectors

$$\begin{aligned}\bar{F}&=\begin{pmatrix} E_z \\ iE_y+E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x-H_y\right)\end{pmatrix}\\ \bar{J}&=\begin{pmatrix} J_z \\ iJ_y+J_x \\ v\rho_e \\ 0\end{pmatrix}\end{aligned} \tag{4.89}$$

we simply have

$$\not{\partial}\bar{F}=-\eta\bar{J} \tag{4.90}$$

which presents a form similar to the Dirac equation for null mass.

The representation of $\not{\partial}\bar{F}$ in matrix terms is:

$$\not{\partial}\bar{F} = \begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\,\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} E_z \\ iE_y + E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x - H_y\right) \end{pmatrix}$$

It is possible to note that the four equations are coupled: we need to solve 4 coupled equations.

Sourceless case: Wave and Helmholtz equations

When no sources are present (4.90) becomes

$$\not{\partial}\bar{F} = 0\,. \tag{4.91}$$

Since $\not{\partial} = \sum_{i=0}^{3} \gamma^i\,\partial_i$ we have that

$$\begin{aligned} \not{\partial}^2 &= \left(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right)\left(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) \\ &= -\left(\partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_0^2\right) I_4 = -\left(\nabla^2 - \partial_0^2\right) I_4 \end{aligned} \tag{4.92}$$

and therefore is a diagonalized operator. The same result can be obtained by considering the following expression explicitly

$$\not{\partial}^2 = \begin{pmatrix} \sigma_0\partial_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\sigma_0\partial_0 \end{pmatrix} \begin{pmatrix} \sigma_0\partial_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\sigma_0\partial_0 \end{pmatrix} = -\begin{pmatrix} \tilde{\nabla}^2 - \sigma_0\partial_0^2 & 0 \\ 0 & \tilde{\nabla}^2 - \sigma_0\partial_0^2 \end{pmatrix} \tag{4.93}$$

where we have made use of the substitution in (4.80).
Thus, by pre–multiplying (4.91) with $\not{\partial}$ we obtain

$$\not{\partial}^2\bar{F} = \left(\partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_0^2\right)\bar{F} = 0 \tag{4.94}$$

By separating also the real and imaginary parts we note that each component of the electromagnetic field, say Ψ, must satisfy the wave equation in time domain

$$\left(\nabla^2 - \partial_0^2\right)\Psi = 0 \tag{4.95}$$

or the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\Psi = 0 \tag{4.96}$$

in the frequency domain.

Note that in this derivation of the wave equation only matrix multiplication has been used without requiring operations such as curl curl or grad div.

II.3 Sourceless case: Weyl decomposition

Let us consider the case without sources, which applies e.g. to propagation problems. The four eqs. (4.90) are coupled, but it is possible to separate them in two independent sets of two equations of the first order. We recall that

$$\begin{pmatrix} \partial_0 \sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\partial_0 \sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\eta\tilde{H} \end{pmatrix} = -\eta \begin{pmatrix} \tilde{J} \\ v\tilde{\rho} \end{pmatrix} \tag{4.97}$$

and we introduce the column vectors $\mathbf{u}, \mathbf{w}$ defined as

$$\begin{aligned} \mathbf{u} &= \begin{pmatrix} E_z \\ iE_y + E_x \end{pmatrix} = \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \end{pmatrix} \\ \mathbf{w} &= \begin{pmatrix} \eta\, iH_z \\ \eta\,\left(iH_x - H_y\right) \end{pmatrix} = \begin{pmatrix} a_0 - b_0 \\ a_1 - b_1 \end{pmatrix} \end{aligned} \tag{4.98}$$

we have

$$\begin{aligned} \tilde{\nabla}\mathbf{u} + \sigma_0\,\partial_0\,\mathbf{w} &= 0 \\ \tilde{\nabla}\mathbf{w} + \sigma_0\,\partial_0\,\mathbf{u} &= 0\,. \end{aligned} \tag{4.99}$$

By summing and subtracting the above eqs. we have:

$$\begin{aligned} \tilde{\nabla}\,(\mathbf{u}+\mathbf{w}) + \sigma_0\,\partial_0\,(\mathbf{u}+\mathbf{w}) &= 0 \\ \tilde{\nabla}\,(\mathbf{u}-\mathbf{w}) - \sigma_0\,\partial_0\,(\mathbf{u}-\mathbf{w}) &= 0 \end{aligned} \tag{4.100}$$

and since

$$\begin{aligned}\mathbf{u}+\mathbf{w} &= 2\,\mathbf{a} = 2\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ \mathbf{u}-\mathbf{w} &= 2\,\mathbf{b} = 2\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}\end{aligned} \tag{4.101}$$

i.e. the problem is decomposed in two systems as following:

$$\begin{aligned}\left(\tilde{\nabla}+\sigma_0\partial_0\right)\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} &= \tilde{\partial}^{+}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \tilde{\partial}^{+}\mathbf{a} = 0 \\ \left(\tilde{\nabla}-\sigma_0\partial_0\right)\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} &= \tilde{\partial}^{-}\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = \tilde{\partial}^{-}\mathbf{b} = 0\,.\end{aligned} \tag{4.102}$$

This result can be derived in another manner by introducing the two matrices:

$$\begin{aligned}A^{-} &= \frac{1}{\sqrt{2}}\left(\gamma^4-\gamma^5\right) \\ A^{+} &= \frac{1}{\sqrt{2}}\left(\gamma^4+\gamma^5\right)\,.\end{aligned} \tag{4.103}$$

These matrices square to the identity matrix thus being equal to their inverse. It is also noted that

$$\begin{pmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{pmatrix} = \frac{1}{\sqrt{2}}A^{-}\begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_0-b_0 \\ a_1-b_1 \end{pmatrix}\,. \tag{4.104}$$

We can therefore transform eq. (4.90) as

$$A^{+}\not{\partial}A^{-}\,A^{-}\bar{F} = 0 \tag{4.105}$$

or, explicitly, in cartesian coordinates, we have

$$A^+ \not{\partial} A^- = \begin{pmatrix} \partial_3+\partial_0 & \partial_1 - i\,\partial_2 & 0 & 0 \\ i\,\partial_2+\partial_1 & \partial_0-\partial_3 & 0 & 0 \\ 0 & 0 & \partial_3-\partial_0 & \partial_1-i\,\partial_2 \\ 0 & 0 & i\,\partial_2+\partial_1 & -\partial_3-\partial_0 \end{pmatrix}$$

i.e. the problem is decomposed in two systems as following:

$$\begin{aligned} \left(\tilde{\nabla}+\sigma_0\partial_0\right)\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} &= \tilde{\partial}^+ \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0 \\ \left(\tilde{\nabla}-\sigma_0\partial_0\right)\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} &= \tilde{\partial}^- \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0\,. \end{aligned} \tag{4.106}$$

When the sources

$$\bar{S} = \eta \begin{pmatrix} J_z \\ iJ_y + J_x \\ v\,\rho_e \\ 0 \end{pmatrix} \tag{4.107}$$

are present we have

$$\begin{aligned} \not{\partial}\bar{F} &= -S \\ A^+ \not{\partial} A^-\, A^-\bar{F} &= -A^+\bar{S} \end{aligned} \tag{4.108}$$

or, explicitly,

$$\begin{pmatrix} \partial_3+\partial_0 & \partial_1 - i\,\partial_2 & 0 & 0 \\ i\,\partial_2+\partial_1 & \partial_0-\partial_3 & 0 & 0 \\ 0 & 0 & \partial_3-\partial_0 & \partial_1-i\,\partial_2 \\ 0 & 0 & i\,\partial_2+\partial_1 & -\partial_3-\partial_0 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{pmatrix} = -\frac{\eta}{2} \begin{pmatrix} J_z - \rho\,v \\ iJ_y + J_x \\ -\rho\,v - J_z \\ -iJ_y - J_x \end{pmatrix} \tag{4.109}$$

As before, the problem is decomposed in two systems as following:

$$\left(\tilde{\nabla}+\sigma_0\partial_0\right)\begin{pmatrix}a_0\\a_1\end{pmatrix}=\tilde{\partial}^{+}\begin{pmatrix}a_0\\a_1\end{pmatrix}=-\frac{\eta}{2}\begin{pmatrix}J_z-\rho\,v\\iJ_y+J_x\end{pmatrix}$$
$$\left(\tilde{\nabla}-\sigma_0\partial_0\right)\begin{pmatrix}b_0\\b_1\end{pmatrix}=\tilde{\partial}^{-}\begin{pmatrix}b_0\\b_1\end{pmatrix}=-\frac{\eta}{2}\begin{pmatrix}-\rho\,v-J_z\\-iJ_y-J_x\end{pmatrix}. \tag{4.110}$$

These computations are developed in the code:

```
wxm/Maxwell_Weyl_v01.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

/* Maxwell_Weyl_v01 */

/* Define Dirac Gammas */
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

g0 :  kronecker_product(%sigma[3],%sigma[0])$
g1 :  kronecker_product(%i*%sigma[2],%sigma[1])$
g2 :  kronecker_product(%i*%sigma[2],%sigma[2])$
g3 :  kronecker_product(%i*%sigma[2],%sigma[3])$
g4 :  g0$
g5 :  - kronecker_product(%sigma[1],%sigma[0])$

/*
The block Dslash perform the d-slash operation
in rectangular coordinates on a quadrivector
*/
/********** Block start **********************/
Dslash(Ap) :=
block([],
print("----------------------------------------"),
print("d-slash result"),
print("----------------------------------------"),
DAp :  factor(j*k*g0.Ap+diff(g1.Ap,x)+diff(g2.Ap,y)+diff(g3.Ap,z))
)$
/********** Block end **********************/

print("----------------------------------------")$
print("The vector E and H in general depend on x,y,z,")$
```

```
depends([Ex,Ey,Ez],[x,y,z]);
depends([Hx,Hy,Hz],[x,y,z]);

/* 2D—case no y dependence
depends([Ex,Ey,Ez],[x,z]);
depends([Hx,Hy,Hz],[x,z]);
*/

/* 1D—case no x,y dependence
depends([Ex,Ey,Ez],[z]);
depends([Hx,Hy,Hz],[z]);
*/

print("compose the Field Quadrivector")$
F: matrix([Ez],[Ex+%i * Ey],[%i * %eta *Hz],[%i*%eta*(Hx + %i*Hy)]);

print("compose the Source Quadrivector")$
J: matrix([Jz],[Jx+%i * Jy],[%rho*v],[0]);

print("Fictitious d-slash")$
dslash :    [0]*g0+   [1]*g1+   [2]*g2+   [3]*g3;

print("Compose the matrices Am,Ap")$
print("Am * sqrt(2)")$
Am : 1/sqrt(2)*(g4-g5)$
Am*sqrt(2);

print("Ap * sqrt(2)")$
Ap : 1/sqrt(2)*(g4+g5)$
Ap*sqrt(2);

print("The matrices are equal to their inverses")$
print("Am.Am = I, Ap.Ap = I")$
Am.Am$
Ap.Ap$

print("ab vector")$
```

```
ab : matrix([a[0]+b[0]], [a[1]+b[1]],[a[0]-b[0]], [a[1]-b[1]]);
print("Am F/sqrt(2)")$
abp : Am.ab, ratsimp$
 abp: abp/sqrt(2);

print("Ap . dslash . Am ")$
sdslash : Ap . dslash . Am, ratsimp;

print("Ap . J /sqrt(2) ")$
Jp : Ap . J, ratsimp$
Jp : Jp/sqrt(2),ratsimp;

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

In the following several operators will find use. Their definition is summarized in Table 4.1.

Quadrivectors grade structure

Let us consider the frequency domain case in cartesian coordinates. By explicitly writing the equations we have:

$$
\begin{aligned}
\not{\partial}\bar{F} &= \begin{pmatrix} jk & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & jk & \partial_x + i\partial_y & -\partial_z \\ -\partial_z & -\partial_x + i\partial_y & -jk & 0 \\ -\partial_x - i\partial_y 2 & \partial_z & 0 & -jk \end{pmatrix} \begin{pmatrix} E_z \\ iE_y + E_x \\ \eta\, iH_z \\ \eta\left(iH_x - H_y\right) \end{pmatrix} \\
&= \begin{pmatrix} jkE_z - \eta\left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right) \\ jkE_x - \eta\left(\frac{\partial}{\partial y}H_z + \frac{\partial}{\partial z}H_y\right) \\ -\nabla\cdot\mathbf{E} \\ jk\eta H_y - \frac{\partial}{\partial x}E_z + \frac{\partial}{\partial z}E_x \end{pmatrix} \\
&\quad + i\begin{pmatrix} \eta\,\nabla\cdot\mathbf{H} \\ jkE_y + \eta\left(\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_x\right) \\ -jk\eta H_z - \frac{\partial}{\partial x}E_y + \frac{\partial}{\partial y}E_x \\ -jk\eta H_x - \frac{\partial}{\partial y}E_z + \frac{\partial}{\partial z}E_y \end{pmatrix} \\
&= -\eta\begin{pmatrix} J_z \\ iJ_y + J_x \\ v\rho_e \\ 0 \end{pmatrix}
\end{aligned}
\tag{4.111}
$$

It is seen that:

- the first row of a quadrivector contains the z–component of the vector part plus the pseudoscalar (trivector) part. In particular it is seen that it contains the $\nabla\cdot\mathbf{H} = 0$ equation
- the second row of the quadrivector contains the x (real part) component and the y (imaginary part) component of a vector.
- The third row of a quadrivector contains the bivector component iz and the scalar part. The scalar part, in particular, gives the equation $\nabla\cdot\mathbf{E} = \eta v\rho_e$.

- the fourth row contains the ix, iy components of the bivector. More precisely the real part of the fourth raw gives the H_y component while the imaginary part provides the H_x component.

It is therefore apparent that the Maxwell–Dirac expression contains both the divergence equations and the curl equations.

A code fraction reporting part of the above theory is given next.

```
wxm/Maxwell_Dirac_v03.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

/* Maxwell_Dirac_v03 */

/* Define Dirac Gammas */
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

g0 :  kronecker_product(%sigma[3],%sigma[0])$
g1 :  kronecker_product(%i*%sigma[2],%sigma[1])$
g2 :  kronecker_product(%i*%sigma[2],%sigma[2])$
g3 :  kronecker_product(%i*%sigma[2],%sigma[3])$
g4 :  g0$
g5 :  - kronecker_product(%sigma[1],%sigma[0])$

/*
The block Dslash perform the d-slash operation
in rectangular coordinates on a quadrivector
*/
/********** Block start **********************/
Dslash(Ap) :=
block([],
print("----------------------------------------"),
print("d-slash result"),
print("----------------------------------------"),
DAp :  factor(j*k*g0.Ap+diff(g1.Ap,x)+diff(g2.Ap,y)+diff(g3.Ap,z))
)$
/********** Block end **********************/

print("----------------------------------------")$
print("The vector E and H in general depend on x,y,z,")$
```

```
depends([Ex,Ey,Ez],[x,y,z]);
depends([Hx,Hy,Hz],[x,y,z]);

/* 2D—case no y dependence
depends([Ex,Ey,Ez],[x,z]);
depends([Hx,Hy,Hz],[x,z]);
*/

/* 1D—case no x,y dependence
depends([Ex,Ey,Ez],[z]);
depends([Hx,Hy,Hz],[z]);
*/

print("compose the Field Quadrivector")$
F: matrix([Ez],[Ex+%i * Ey],[%i * %eta *Hz],[%i*%eta*(Hx + %i*Hy)]);

print("compose the Source Quadrivector")$
J: matrix([Jz],[Jx+%i * Jy],[%rho*v],[0]);

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the Field gives")$
Ap : rectform(F)$
Dslash(Ap)$
DAp1 : DAp;

print("----------------- MAXWELL ----------------------------------------")$
print("Maxwell's equations are recovered by equating to the sources")$
Maxwell : rectform(DAp1)$

print("Real part of Maxwell's equations gives")$
Mr : realpart(Maxwell);

print("Imaginary part of Maxwell's equations gives")$
Mi : imagpart(Maxwell);

print("----------------- HELMHOLTZ ----------------------------------------")$
print("In the sourceless case the wave equation can be obtained as")$
```

```
print("second order")$
Dslash(DAp1)$
DAp2 : DAp$
DAp2 :subst(-1,j^2,DAp2);

print("This is the Helmholtz equation")$
Helm : DAp2$

print("Real part of Helmholtz equations gives")$
Hr : realpart(Helm);

print("Imaginary part of Helmholtz equations gives")$
Hi : imagpart(Helm);

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Table 4.1. Table summarizing operator definitions. The underlined symbols refer to frequency domain operators.

$\not{\partial}$	$= \sum_{i=0}^{3} \gamma^i \partial_i$
$\underline{\not{\partial}}$	$= jk\gamma^0 + \sum_{i=1}^{3} \gamma^i \partial_i$
$-\not{\partial}^2$	$= \left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right) I_4$
$-\underline{\not{\partial}}^2$	$= \left(\nabla^2 + k^2\right) I_4$

II.4 Boundary Conditions

In order to obtain a unique solution of the Maxwell field equations, one must impose appropriate boundary, radiation and edge conditions. Radiation and edge conditions formalize, respectively, the outgoing wave requirement on fields in an infinite region excited by sources in a bounded domain and by conservation of energy in the possibly singular fields induced in the vicinity of edges and corners (tips) on obstacle scatterers. We shall deal here only with the boundary conditions arising at the interface between two different media.

Consider a regular surface S of a medium discontinuity, as shown in Figure 4.1, where the subscripts 1 and 2 distinguish quantities in regions 1 and 2, respectively. From (4.21a) and (4.21b), as a consequence of a limiting process, one obtains the following conditions:

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}\,, \tag{4.112a}$$

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = -\mathbf{M}\,, \tag{4.112b}$$

where $\mathbf{J}$ and $\mathbf{M}$ are, respectively, the electric and magnetic surface current density distributions at the interface. Similarly, from (4.21c) and (4.21d), for a small volume at the interface, a limiting process yields,

$$\mathbf{n} \cdot (\mathbf{B}_2 - \mathbf{B}_1) = -\rho_m\,, \tag{4.113a}$$

$$\mathbf{n} \cdot (\mathbf{D}_2 - \mathbf{D}_1) = \rho_e\,, \tag{4.113b}$$

where ρ_e and ρ_m are, respectively, the electric and magnetic surface charge density distributions on the interface.

If neither medium is perfectly conducting, the tangential component of the fields $\mathbf{E}$ and $\mathbf{H}$ are continuous while their normal components undergo a jump due to the discontinuity in the permittivity and permeability.

When medium 1 is a perfect electric conductor, the field inside the medium vanishes everywhere and induced electric charges and currents exist on the surface. In this case we have:

$$\mathbf{n} \times \mathbf{H}_2 = \mathbf{J}\,, \tag{4.114a}$$

$$\mathbf{n} \times \mathbf{E}_2 = 0\,, \tag{4.114b}$$

$$\mathbf{n} \cdot \mathbf{B}_2 = 0\,, \tag{4.114c}$$

$$\mathbf{n} \cdot \mathbf{D}_2 = \rho_e\,, \tag{4.114d}$$

which states the vanishing, at the metal surface, of the tangential components of **E** and of the normal component of **H**.

In certain cases, it is convenient to include fields generated from equivalent magnetic currents. Accordingly, the field generated by a magnetic current distribution in the immediate vicinity of a perfectly (electrically) conducting surface is given by

$$\mathbf{n} \times \mathbf{E}_2 = -\mathbf{M}\,. \tag{4.115}$$

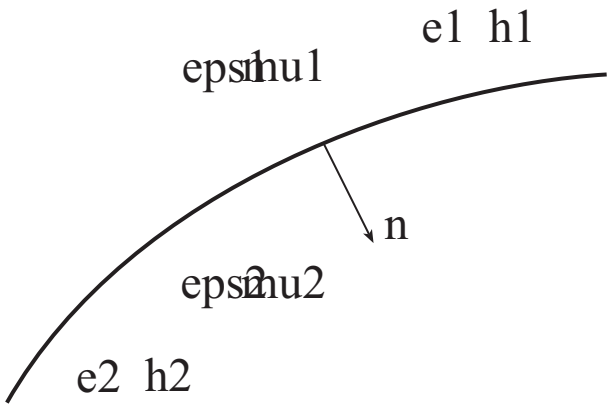

Fig. 4.1. Interface between two media.

When considering the normal **n** directed along z, the boundary conditions can also be stated in the following way

$$\begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu_1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{1z} \\ iE_{1y} + E_{1x} \\ H_{1z} \\ H_{1x} + iH_{1y} \end{pmatrix} = \begin{pmatrix} \varepsilon_2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{2z} \\ iE_{2y} + E_{2x} \\ H_{2z} \\ H_{2x} + iH_{2y} \end{pmatrix} \tag{4.116}$$

where we have assumed that no superficial currents or charges are present. It is immediate to see that

$$\begin{pmatrix} E_{2z} \\ iE_{2y}+E_{2x} \\ H_{2z} \\ H_{2x}+iH_{2y} \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon_1}{\varepsilon_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\mu_1}{\mu_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_{1z} \\ iE_{1y}+E_{1x} \\ H_{1z} \\ H_{1x}+iH_{1y} \end{pmatrix} \tag{4.117}$$

also holds. However, we recognize that

$$\begin{pmatrix} E_{1z} \\ iE_{1y}+E_{1x} \\ H_{1z} \\ H_{1x}+iH_{1y} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{i\eta_1} & 0 \\ 0 & 0 & 0 & \frac{1}{i\eta_1} \end{pmatrix} \begin{pmatrix} E_{1z} \\ iE_{1y}+E_{1x} \\ \eta_1 iH_z \\ \eta_1 \left(iH_x - H_y\right) \end{pmatrix} \tag{4.118}$$

and

$$\bar{F}_2 = \begin{pmatrix} E_{2z} \\ iE_{2y}+E_{2x} \\ \eta_2 iH_{2z} \\ \eta_2 \left(iH_{2x} - H_{2y}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & i\eta_2 & 0 \\ 0 & 0 & 0 & i\eta_2 \end{pmatrix} \begin{pmatrix} E_{2z} \\ iE_{2y}+E_{2x} \\ H_{2z} \\ H_{2x}+iH_{2y} \end{pmatrix}. \tag{4.119}$$

We can relate the field multivector in the two regions of space as

$$\bar{F}_2 = \begin{pmatrix} \frac{\varepsilon_1}{\varepsilon_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\eta_2\mu_1}{\eta_1\mu_2} & 0 \\ 0 & 0 & 0 & \frac{\eta_2}{\eta_1} \end{pmatrix} \bar{F}_1 \tag{4.120}$$

and by noting that

$$\frac{\eta_2\mu_1}{\eta_1\mu_2} = \frac{v_2}{v_1} \tag{4.121}$$

we obtain the form

$$\bar{F}_2 = \begin{pmatrix} \frac{\varepsilon_1}{\varepsilon_2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{v_2}{v_1} & 0 \\ 0 & 0 & 0 & \frac{\eta_2}{\eta_1} \end{pmatrix} \bar{F}_1 \,. \tag{4.122}$$

In many cases it is possible to assume that $\mu_1 = \mu_2 = \mu$; by introducing the refractive index

$$n_{12} = \sqrt{\varepsilon_1/\varepsilon_2} \tag{4.123}$$

we can write the boundary conditions as

$$\bar{F}_2 = \begin{pmatrix} n_{12}^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n_{12} & 0 \\ 0 & 0 & 0 & n_{12} \end{pmatrix} \bar{F}_1 \tag{4.124}$$

which is an algebraic version of the boundary conditions.

Equation (4.124) also allows to recover the condition on the incident and reflected waves previously introduced. In fact, by defining

$$\begin{pmatrix} a'_0 + b'_0 \\ a'_1 + b'_1 \\ a'_0 - b'_0 \\ a'_1 - b'_1 \end{pmatrix} = \begin{pmatrix} E_{1z} \\ iE_{1y} + E_{1x} \\ \eta_1 \, iH_{1z} \\ \eta_1 \left(iH_{1x} - H_{1y} \right) \end{pmatrix} = \bar{F}_1 \tag{4.125}$$

and

$$\begin{pmatrix} a''_0 + b''_0 \\ a''_1 + b''_1 \\ a''_0 - b''_0 \\ a''_1 - b''_1 \end{pmatrix} = \begin{pmatrix} E_{2z} \\ iE_{2y} + E_{2x} \\ \eta_2 \, iH_{2z} \\ \eta_2 \left(iH_{2x} - H_{2y} \right) \end{pmatrix} = \bar{F}_2 \tag{4.126}$$

we can obtain the following relationship

$$\begin{pmatrix} a_0'' + b_0'' \\ a_1'' + b_1'' \\ a_0'' - b_0'' \\ a_1'' - b_1'' \end{pmatrix} = \begin{pmatrix} n_{12}^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & n_{12} & 0 \\ 0 & 0 & 0 & n_{12} \end{pmatrix} \begin{pmatrix} a_0' + b_0' \\ a_1' + b_1' \\ a_0' - b_0' \\ a_1' - b_1' \end{pmatrix} . \tag{4.127}$$

which can be solved for expressing the reflected waves in terms of the incident ones. By performing the solution we obtain

$$\begin{pmatrix} b_0' \\ b_1' \\ b_0'' \\ b_1'' \end{pmatrix} = \begin{pmatrix} -r & 0 & \frac{t}{n_{12}} & 0 \\ 0 & r & 0 & -t \\ -n_{12}^2 t & 0 & r & 0 \\ 0 & n_{12} t & 0 & -r \end{pmatrix} \begin{pmatrix} a_0' \\ a_1' \\ a_0'' \\ a_1'' \end{pmatrix} \tag{4.128}$$

where we have made use of the symbols

$$r = \frac{n_{12} + 1}{n_{12} - 1} \tag{4.129}$$

$$t = \frac{2}{n_{12} - 1} . \tag{4.130}$$

In conclusion we have (either in region 1 or region 2)

$$\begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_0 - b_0 \\ a_1 - b_1 \end{pmatrix} = \begin{pmatrix} E_z \\ i E_y + E_x \\ \eta\, i H_z \\ \eta \left(i H_x - H_y \right) \end{pmatrix} . \tag{4.131}$$

which corresponds to:

$$\begin{aligned} a_0 &= \frac{1}{2} \left(i \eta H_z + E_z \right) \\ a_1 &= \frac{1}{2} \left(-\eta H_y + i E_y + i \eta H_x + E_x \right) \\ b_0 &= \frac{1}{2} \left(E_z - i \eta H_z \right) \\ b_1 &= \frac{1}{2} \left(\eta H_y + i E_y - i \eta H_x + E_x \right) . \end{aligned} \tag{4.132}$$

The reason for introducing the wave parameters is that with them we have two sets of two equations which are decoupled. One set represent progressive waves and is expressed by:

$$\begin{pmatrix} \partial_z + \frac{\partial_t}{v} & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & \frac{\partial_t}{v} - \partial_z \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0 \tag{4.133}$$

while the regressive waves are given by:

$$\begin{pmatrix} \partial_z - \frac{\partial_t}{v} & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & -\partial_z - \frac{\partial_t}{v} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0 \tag{4.134}$$

At an interface perpendicular to the z direction the wave parameters are mixed according to (4.128).

III Field Potentials

Field potentials can be introduced by using either Dirac matrices or Pauli matrices. In the following we will discuss their introduction in Dirac form, i.e. by using quadri–vectors. For the reader interested in their introduction in Pauli form, i.e. by using 2 by 2 matrices, please refer to section III.8.

III.1 Field Potential in Dirac form

By introducing the quadrivector for the potential

$$\bar{A} = \begin{pmatrix} vA_z \\ v\left(iA_y + A_x\right) \\ \phi \\ 0 \end{pmatrix} \tag{4.135}$$

we try to recover the field as

$$\bar{F} = -\partial\!\!\!/\,\bar{A} \tag{4.136}$$

i.e. with the same type of equation linking the field to the sources.

Let us first introduce the following notation:

$$\mathbf{e} = \begin{pmatrix} E_z \\ iE_y + E_x \end{pmatrix}$$
$$\mathbf{h} = \begin{pmatrix} H_z \\ (H_x + iH_y) \end{pmatrix} \tag{4.137}$$

with the position

$$\mathbf{A} = \begin{pmatrix} A_z \\ A_x + iA_y \end{pmatrix}$$
$$\Phi = \begin{pmatrix} \phi \\ 0 \end{pmatrix} \tag{4.138}$$

According to (4.136) we have

$$\begin{pmatrix} \mathbf{e} \\ i\eta\,\mathbf{h} \end{pmatrix} = -\begin{pmatrix} \partial_0\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\partial_0\sigma_0 \end{pmatrix} \begin{pmatrix} v\mathbf{A} \\ \Phi \end{pmatrix} \tag{4.139}$$

III.2 Potentials' Equation

By further multiplication of (4.139) with $\partial\!\!\!/$ one obtains the equation containing the sources the field and the potential as

$$-\partial\!\!\!/^2\bar{A} = \partial\!\!\!/\bar{F} = -\eta\bar{J}. \tag{4.140}$$

Since we have seen that

$$\partial\!\!\!/^2 = -\left(\nabla^2 - \partial_0^2\right) I_4 \tag{4.141}$$

we recover the following equations for the potentials in terms of the sources

$$\left(\nabla^2 - \partial_0^2\right)\mathbf{A} = -\frac{\eta}{v}\mathbf{J} = -\mu\,\mathbf{J} \tag{4.142}$$
$$\left(\nabla^2 - \partial_0^2\right)\phi = -\eta\,v\,\rho = -\frac{\rho}{\varepsilon}. \tag{4.143}$$

Expressed in frequency domain the equations for the potentials are:

$$\left(\nabla^2 + k^2\right)\mathbf{A} = -\frac{\eta}{v}\mathbf{J} = -\mu\,\mathbf{J} \tag{4.144}$$

$$\left(\nabla^2 + k^2\right)\phi = -\eta\, v\,\rho = -\frac{\rho}{\varepsilon}\,. \tag{4.145}$$

III.3 Lorenz Gauge

By writing (4.139) in geometric algebra form we have

$$\mathbf{e} = -\,v\,\partial_0\mathbf{A} - \nabla\Phi \tag{4.146}$$

$$i\,\eta\,\mathbf{h} = v\,\nabla\mathbf{A} + \partial_0\,\Phi \tag{4.147}$$

By expanding (4.147) we get

$$v\,(\nabla\cdot\mathbf{A} + \nabla\wedge\mathbf{A}) + \partial_0\,\Phi = i\,\eta\,\mathbf{h} \tag{4.148}$$

and, by separating real and imaginary parts,

$$i\,v\,\nabla\times\mathbf{A} = i\,\eta\,\mathbf{H} \tag{4.149}$$

$$v\,\nabla\cdot\mathbf{A} + \partial_0\,\phi = 0\,. \tag{4.150}$$

Note that in (4.149) we have substituted **h** with **H**, i.e. a three–components vector.
Equation (4.150) provides the Lorenz gauge in the time domain. The same condition in the frequency domain is obtained by substituting ∂_0 with jk and becomes:

$$\phi = \frac{j\,v}{k}\nabla\cdot\mathbf{A} \tag{4.151}$$

III.4 Recovering conventional field expressions in terms of potentials

By considering (4.149) we recover:

$$\mathbf{H} = \frac{v}{\eta}\nabla\times\mathbf{A} = \frac{1}{\mu}\nabla\times\mathbf{A}\,. \tag{4.152}$$

On the other hand, by writing (4.146) in the frequency domain we get

$$\mathbf{E} = -jkv\mathbf{A} - \nabla\Phi = -jkv\mathbf{A} - \frac{jv}{k}\nabla\nabla\cdot\mathbf{A} = -jkv\left(\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{k^2}\right). \tag{4.153}$$

Therefore in the conventional form we have the following expressions for the time–harmonic potentials

$$\begin{aligned}\mathbf{E} &= -jkv\left(\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{k^2}\right)\\ \mathbf{H} &= \frac{1}{\mu}\nabla\times\mathbf{A}.\end{aligned} \tag{4.154}$$

However, it is noted that (4.136) is far simpler to use than (4.154).

Written by components the field equations are:

$$\begin{aligned}E_x &= -\frac{jv}{k}\left(A_x k^2 + \frac{\partial^2}{\partial x\partial z}A_z + \frac{\partial^2}{\partial x\partial y}A_y + \frac{\partial^2}{\partial x^2}A_x\right)\\ E_y &= -\frac{jv}{k}\left(A_y k^2 + \frac{\partial^2}{\partial y\partial_z}A_z + \frac{\partial^2}{\partial y^2}A_y + \frac{\partial^2}{\partial x\partial y}A_x\right)\\ E_z &= -\frac{jv}{k}\left(A_z k^2 + \frac{\partial^2}{\partial z^2}A_z + \frac{\partial^2}{\partial y\partial z}A_y + \frac{\partial^2}{\partial x\partial z}A_x\right)\\ H_x &= \frac{v}{\eta}\left(\frac{\partial}{\partial y}A_z - \frac{\partial}{\partial z}A_y\right)\\ H_y &= -\frac{v}{\eta}\left(\frac{\partial}{\partial x}A_z - \frac{\partial}{\partial z}A_x\right)\\ H_z &= \frac{v}{\eta}\left(\frac{\partial}{\partial x}A_y - \frac{\partial}{\partial y}A_x\right).\end{aligned} \tag{4.155}$$

It is noted that, when one only one potential component is considered say e.g. A_z, we have that corresponding electric field component is proportional, i.e. E_z shares, apart for a scale factor, the same dependence as A_z.

A very important fact follows from the above consideration. If we know the field dependence of e.g. E_z due to boundary conditions, then we can take that form as the potential A_z and derive all field expressions as in (4.155). Note also that when only A_z is present the corresponding magnetic field expression H_z is zero and we have a Transverse Magnetic (TM) field.

A code fraction reporting part of the above theory is given next.

wxm/Potentials_v01.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
/* Potentials_v01 */

/* Define Dirac Gammas */
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

g0 :  kronecker_product (%sigma[3],%sigma[0])$
g1 :  kronecker_product (%i*%sigma[2],%sigma[1])$
g2 :  kronecker_product (%i*%sigma[2],%sigma[2])$
g3 :  kronecker_product (%i*%sigma[2],%sigma[3])$
g4 :  g0$
g5 :  - kronecker_product (%sigma[1],%sigma[0])$

/*
The block Dslash perform the d-slash operation
in rectangular coordinates on a quadrivector
*/
/********** Block start **********************/
Dslash(Ap) :=
block([],
print("----------------------------------------"),
print("d-slash result"),
print("----------------------------------------"),
DAp :  factor(j*k*g0.Ap+diff(g1.Ap,x)+diff(g2.Ap,y)+diff(g3.Ap,z))
)$
/********** Block end **********************/

print("----------------------------------------")$
print("The vector E and H in general depend on x,y,z,")$
```

```
/* 3D case */
depends([Ex,Ey,Ez],[x,y,z]);
depends([Hx,Hy,Hz],[x,y,z]);

depends([Ax,Ay,Az],[x,y,z]);
depends([%Phi],[x,y,z]);

/* For recovering the Hertzian potential
depends(Psie,[x,y,z]);
Ax : 0$
Ay : 0$
Az : j * k/(v)*Psie$
*/

print("----------------------------------------")$
print("compose the Potential Quadrivector")$
A:  matrix([v* Az],[v* Ax+%i * Ay*v],[%Phi],[0]);

print("compose the Field Quadrivector")$
F: matrix([Ez],[Ex+%i * Ey],[%i * %eta *Hz],[%i*%eta*(Hx + %i*Hy)]);

print("compose the Source Quadrivector")$
J: matrix([Jz],[Jx+%i * Jy],[%rho*v],[0]);

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the potential gives")$
Apot : rectform(A)$
Dslash(Apot)$
DslApot : rectform(DAp);

print("further application gives")$
Dslash(DslApot)$
DAp2 : DAp$
DAp2 :subst(-1,j^2,DAp2);
print("Potentials equations")$
DAp2 - %eta * J;
```

```
print("setting to zero the scalar part gives Lorenz condition")$
re3 :realpart(DslApot[3,1]);
solphi : solve(re3,%Phi);
%Phi : rhs(solphi[1]);
print("compose again the Potential Quadrivector satisfying Lorenz gauge")$
A: matrix([v* Az],[v* Ax+%i * Ay*v],[%Phi],[0]);

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the Potential gives")$
Ap : rectform(A)$
Dslash(Ap)$
DAp1 : - rectform(DAp)$
DAp1 :subst(-1,j^2,DAp1)$
DAp1 :subst(-1/j,j,DAp1);

Ex : factor(realpart(DAp1[2][1]))$
Ey : factor(imagpart(DAp1[2][1]))$
Ez : factor(DAp1[1][1])$
Hy : -1/%eta *factor(realpart(DAp1[4][1]))$
Hx : 1/%eta *factor(imagpart(DAp1[4][1]))$
Hz : 1/%eta * factor(imagpart(DAp1[3][1]))$

print("Which provides the field equations")$
print('Ex," = ",Ex)$
print('Ey," = ",Ey)$
print('Ez," = ",Ez)$
print('Hx," = ",Hx)$
print('Hy," = ",Hy)$
print('Hz," = ",Hz)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

III.5 Magnetic sources

When considering also magnetic currents and charges, the frequency–domain Maxwell's equations become

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}), \tag{4.156a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}), \tag{4.156b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r}), \tag{4.156c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = \rho_m(\mathbf{r}). \tag{4.156d}$$

By using the conventional substitutions one obtains the following equations

$$\nabla \wedge \mathbf{E} = -jk\,(i\eta\mathbf{H}) - i\mathbf{M}, \tag{4.157a}$$

$$\nabla \wedge (i\eta\mathbf{H}) = -jk\mathbf{E} - \eta\mathbf{J}, \tag{4.157b}$$

$$\nabla \cdot \mathbf{E} = \eta\, v\rho_e, \tag{4.157c}$$

$$\nabla \cdot (i\eta\mathbf{H}) = i\, v\rho_m. \tag{4.157d}$$

which can be condensed into the two equations geometric algebra form

$$\nabla\mathbf{E} = -jk\,(i\eta\mathbf{H}) - i\mathbf{M} + \eta\, v\rho_e, \tag{4.158a}$$

$$\nabla(i\eta\mathbf{H}) = -jk\mathbf{E} + i\, v\rho_m - \eta\mathbf{J}. \tag{4.158b}$$

By writing the above equations in matrix form we get

$$\begin{pmatrix} jk\,\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -jk\,\sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\,\eta\tilde{H} \end{pmatrix} = -\eta \begin{pmatrix} \tilde{J} \\ v\,\tilde{\rho} \end{pmatrix} + i \begin{pmatrix} v\tilde{\rho}_m \\ \tilde{M} \end{pmatrix} \tag{4.159}$$

Note that, as already done previously, we have changed the sign in (4.158a), and so the sources $\mathbf{M}$ and ρ_e have now opposite sign. By considering the spinor form we get

$$\underline{\partial}\bar{F} = -\begin{pmatrix} \eta J_z - i\,v\,\rho_m \\ \eta(J_x + iJ_y) \\ -iM_z + \eta\,v\,\rho_e \\ -i(M_x + iM_y) \end{pmatrix} = -\bar{S} \tag{4.160}$$

with $\bar{S}$ denoting the sources, both of electric and magnetic type.

When only magnetic sources are present the spinor of the sources is

$$\underline{\partial}\bar{F} = i\begin{pmatrix} v\,\rho_m \\ 0 \\ M_z \\ M_x + iM_y \end{pmatrix} = i\bar{J}_m \tag{4.161}$$

We introduce the quadrivector for the potential

$$\bar{C} = -i\,\eta\begin{pmatrix} \psi \\ 0 \\ vC_z \\ v\left(iC_y + C_x\right) \end{pmatrix} = -i\,\eta\begin{pmatrix} \Psi \\ v\mathbf{C} \end{pmatrix} \tag{4.162}$$

with the position

$$\mathbf{C} = \begin{pmatrix} C_z \\ C_x + iC_y \end{pmatrix}$$

$$\Psi = \begin{pmatrix} \psi \\ 0 \end{pmatrix}. \tag{4.163}$$

The field is recovered as

$$\bar{F} = -\partial\bar{C} \tag{4.164}$$

i.e. with the same type of equation linking the field to the sources.

According to (4.164) we have

$$\begin{pmatrix} \mathbf{e} \\ i\eta\,\mathbf{h} \end{pmatrix} = i\eta \begin{pmatrix} jk\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -jk\sigma_0 \end{pmatrix} \begin{pmatrix} \Psi \\ v\,\mathbf{C} \end{pmatrix} \tag{4.165}$$

III.6 Potentials' Equation

By further multiplication of (4.164) with $\not{\partial}$ one obtains the equation containing the sources the field and the potential as

$$-\not{\partial}^2 \bar{C} = \not{\partial}\bar{F} = i\bar{J}_m\,. \tag{4.166}$$

Since we have seen that

$$\not{\partial}^2 = -\left(\nabla^2 + k^2\right) I_4 \tag{4.167}$$

we recover the following equations for the potentials in terms of the sources

$$\left(\nabla^2 + k^2\right)\mathbf{C} = -\frac{1}{\eta\,v}\mathbf{M} = -\varepsilon\,\mathbf{M} \tag{4.168}$$

$$\left(\nabla^2 + k^2\right)\psi = -\frac{v}{\eta}\,\rho_m = -\frac{\rho_m}{\mu}\,. \tag{4.169}$$

III.7 Lorenz Gauge and conventional field expressions

By writing (4.165) in geometric algebra form we have

$$\begin{aligned} \mathbf{E} &= i\eta\,[\,jk\,\psi + v\nabla\mathbf{C}] \\ &= i\eta\,\left[jk\,\psi + v\,(\nabla\cdot\mathbf{C} + i\nabla\times\mathbf{C})\right] \end{aligned} \tag{4.170}$$

$$i\eta\,\mathbf{H} = -i\eta\,[\nabla\psi + jk\,v\mathbf{C}] \tag{4.171}$$

By separating real and imaginary parts in (4.170), one obtains

$$\mathbf{E} = -\eta v\nabla\times\mathbf{C} = -\frac{1}{\varepsilon}\nabla\times\mathbf{C} \tag{4.172}$$

$$\psi = \frac{jv}{k}\nabla\cdot\mathbf{C}\,. \tag{4.173}$$

The electric field is provided by (4.172), while (4.173) is the Lorenz gauge in the frequency–domain for the electric potential $\mathbf{C}$. The magnetic field is recovered as

$$\begin{aligned}\mathbf{H} &= -\left[\nabla\psi + jkv\mathbf{C}\right] \\ &= -\left[\frac{jv}{k}\nabla\nabla\cdot\mathbf{C} + jkv\mathbf{C}\right] \\ &= -jkv\left[\mathbf{C} + \frac{\nabla\nabla\cdot\mathbf{C}}{k^2}\right].\end{aligned} \tag{4.174}$$

Therefore in the conventional form we have the following expressions for the time–harmonic potentials

$$\begin{aligned}\mathbf{E} &= -\frac{1}{\varepsilon}\nabla\times\mathbf{C} \\ \mathbf{H} &= -j\omega\left[\mathbf{C} + \frac{\nabla\nabla\cdot\mathbf{C}}{k^2}\right].\end{aligned} \tag{4.175}$$

In component form we have:

$$\begin{aligned}E_x &= -\eta\, v\left(\frac{\partial}{\partial y}C_z - \frac{\partial}{\partial z}C_y\right) \\ E_y &= \eta\, v\left(\frac{\partial}{\partial x}C_z - \frac{\partial}{\partial z}C_x\right) \\ E_z &= -\eta\, v\left(\frac{\partial}{\partial x}C_y - \frac{\partial}{\partial y}C_x\right) \\ H_x &= -\frac{j\,v}{k}\left(C_x k^2 + \frac{\partial^2}{\partial x\partial z}C_z + \frac{\partial^2}{\partial x\,\partial y}C_y + \frac{\partial^2}{\partial x^2}C_x\right) \\ H_y &= -\frac{j\,v}{k}\left(C_y k^2 + \frac{\partial^2}{\partial y\,\partial z}C_z + \frac{\partial^2}{\partial y^2}C_y + \frac{\partial^2}{\partial x\,\partial y}C_x\right) \\ H_z &= -\frac{j\,v}{k}\left(C_z k^2 + \frac{\partial^2}{\partial z^2}C_z + \frac{\partial^2}{\partial y\,\partial z}C_y + \frac{\partial^2}{\partial x\,\partial z}C_x\right).\end{aligned} \tag{4.176}$$

We can repeat the same considerations done before this time considering e.g. a potential C_z as the only potential component different from zero. This results in a TE field (no E_z component) and the C_z expression should satisfy the same expressions as the H_z field.

`wxm/Mag_Potentials_v01.wxm`

Table 4.2. Table summarizing spinors definitions. The first column refers to sources, $\bar{S}$, the second to the field, $\bar{F}$, the third to the potential $\bar{P}$. The last two row reports the relevant equations.

$\bar{S}$	$\bar{F}$	$\bar{P}$
$\begin{pmatrix} \eta J_z - i v \rho_m \\ \eta (J_x + i J_y) \\ -i M_z + \eta v \rho_e \\ -i (M_x + i M_y) \end{pmatrix}$	$\begin{pmatrix} E_z \\ i E_y + E_x \\ \eta i H_z \\ \eta \left(i H_x - H_y \right) \end{pmatrix}$	$\begin{pmatrix} v A_z - i \eta \psi \\ v (A_x + i A_y) \\ -i \eta v C_z + \phi \\ -i \eta v (C_x + i C_y) \end{pmatrix}$
$\not{\partial} \bar{F} = -\bar{S}$	$\not{\partial} \bar{P} = -\bar{F}$	$-\not{\partial}^2 \bar{P} = \not{\partial} \bar{F} = -\bar{S}$
Lorenz	$\phi = \frac{jv}{k} \nabla \cdot \mathbf{A}$	$\psi = \frac{jv}{k} \nabla \cdot \mathbf{C}$

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

/* Mag_Potentials_v01 */

/* Define Dirac Gammas */
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

g0 :  kronecker_product(%sigma[3],%sigma[0])$
g1 :  kronecker_product(%i*%sigma[2],%sigma[1])$
g2 :  kronecker_product(%i*%sigma[2],%sigma[2])$
g3 :  kronecker_product(%i*%sigma[2],%sigma[3])$
g4 :  g0$
g5 :  - kronecker_product(%sigma[1],%sigma[0])$

/*
The block Dslash perform the d-slash operation
in rectangular coordinates on a quadrivector
*/
/********** Block start **********/
```

```
Dslash(Ap) :=
block([],
print("---------------------------------------"),
print("d-slash result"),
print("---------------------------------------"),
DAp :  factor(j*k*g0.Ap+diff(g1.Ap,x)+diff(g2.Ap,y)+diff(g3.Ap,z))
)$
/********** Block end *********************/

print("---------------------------------------")$
print("The vector E and H in general depend on x,y,z,")$

/* 3D case */
depends([Ex,Ey,Ez],[x,y,z]);
depends([Hx,Hy,Hz],[x,y,z]);

depends([Cx,Cy,Cz],[x,y,z]);
depends([%psi],[x,y,z]);

/* For recovering the Hertzian potential
depends(Psih,[x,y,z]);
Cx : 0$
Cy : 0$
Cz : j * k/v*Psih$
*/

print("---------------------------------------")$
print("compose the Field Quadrivector")$
C:  - %i * %eta * matrix([%psi],[0],[v* Cz],[v* Cx+%i * Cy*v]);

print("compose the Field Quadrivector")$
F: matrix([Ez],[Ex+%i * Ey],[%i * %eta *Hz],[%i*%eta*(Hx + %i*Hy)]);

print("compose the Source Quadrivector")$
Jm: -%i * matrix([%rho[m]*v],[0],[Mz],[Mx+%i * My]);

print("----------------- D-slah ----------------------------------------")$
```

```
print("Application of the d-slash operator on the potential gives")$
Cpot : rectform(C)$
Dslash(Cpot)$
DslCpot : rectform(DAp);

print("further application gives")$
Dslash(DslCpot)$
DCp2 : DAp$
DCp2 :subst(-1,j^2,DCp2);
print("Potentials equations")$
DCp2 +%i * Jm;

print("setting to zero the imaginary part gives Lorenz condition")$
ie1 :imagpart(DslCpot[1,1]);
solpsi : solve(ie1,%psi);
%psi : rhs(solpsi[1]);
print("compose again the Potential Quadrivector satisfying Lorenz gauge")$
C: -%i * %eta * matrix([%psi],[0],[v* Cz],[v* Cx+%i * Cy*v]);

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the Potential gives")$
Cp : rectform(C)$
Dslash(Cp)$
DCp1 : - rectform(DAp)$
DCp1 :subst(-1,j^2,DCp1)$
DCp1 :subst(-1/j,j,DCp1);

Ex : factor(realpart(DCp1[2][1]))$
Ey : factor(imagpart(DCp1[2][1]))$
Ez : factor(DCp1[1][1])$
Hy : -1/%eta *ratsimp(realpart(DCp1[4][1]))$
Hx : 1/%eta *ratsimp(imagpart(DCp1[4][1]))$
Hz : 1/%eta * ratsimp(imagpart(DCp1[3][1]))$

Hx:factor(Hx)$
Hy:factor(Hy)$
Hz:factor(Hz)$
```

```
print("Which provides the field equations")$
print('Ex," = ",Ex)$
print('Ey," = ",Ey)$
print('Ez," = ",Ez)$
print('Hx," = ",Hx)$
print('Hy," = ",Hy)$
print('Hz," = ",Hz)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

III.8 Potentials with Geometric Algebra and Pauli matrices

Pauli matrices are a means to write, in matrix form form, the geometric algebra expressions for the three–dimensional space. We have seen in (4.74) that the four Maxwell equation can be expressed as a single one:

$$\left(\nabla + \frac{1}{v}\partial_t\right)\mathscr{F} = \eta\left(v\rho - \mathbf{J}\right). \tag{4.177}$$

or, by using the notation already introduced,

$$\partial_0 = \frac{1}{v}\partial_t \tag{4.178}$$

and, translating into Pauli matrices, we have

$$\left(\tilde{\nabla} + \partial_0\sigma_0\right)\tilde{F} = \eta\left(v\rho\sigma_0 - \tilde{J}\right). \tag{4.179}$$

Therefore, the following formulas can be written in terms of Pauli matrices or, equivalently, in geometric algebra form.

Potentials

It is assumed that the field $\mathscr{F}$ can be recovered from a scalar potential ϕ and a vector potential $\mathbf{A}$, using the following expression:

$$\mathscr{F} = \left(\nabla - \frac{1}{v}\partial_t\right)(-\phi + v\mathbf{A}) \tag{4.180}$$

or, equivalently, by introducing the multivector $\mathscr{A}$ defined as

$$\mathscr{A} = -\phi + v\mathbf{A} \tag{4.181}$$

we can write

$$\mathscr{F} = \left(\nabla - \frac{1}{v}\partial_t\right)\mathscr{A}\,. \tag{4.182}$$

When (4.180) is inserted into (4.177) we recover the following equation:

$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)(-\phi + v\mathbf{A}) = \frac{\rho}{\varepsilon} - \eta\,\mathbf{J}\,. \tag{4.183}$$

By separating the scalar and the vector part, we recover respectively the scalar and vector wave equations

$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\phi = -\frac{\rho}{\varepsilon} \tag{4.184}$$

$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\mathbf{A} = -\mu\mathbf{J}\,. \tag{4.185}$$

Finally, by equating (4.180) and (4.73) we get:

$$\begin{aligned}\mathscr{F} &= \mathbf{E} + \eta\,\hat{\mathbf{H}} = \partial^-\mathscr{A}\\ &= \left(\nabla - \frac{1}{v}\partial_t\right)(-\phi + v\mathbf{A})\\ &= \frac{1}{v}\partial_t\phi + \nabla\cdot(v\mathbf{A}) - \nabla\wedge\phi - \frac{1}{v}\partial_t(v\mathbf{A}) + \nabla\wedge(v\mathbf{A})\,.\end{aligned} \tag{4.186}$$

Considering the various grades one gets

$$0 = \frac{1}{v}\partial_t \phi + v\nabla \cdot \mathbf{A} \tag{4.187}$$

$$\mathbf{E} = -\nabla \wedge \phi - \partial_t \mathbf{A} \tag{4.188}$$

$$\eta \hat{\mathbf{H}} = v\nabla \wedge \mathbf{A} \tag{4.189}$$

with the scalar part which should be null thus giving

$$0 = \frac{1}{v}\partial_t \phi + v\nabla \cdot \mathbf{A} \tag{4.190}$$

This is the Lorenz gauge and has been obtained simply by equating the grades. The other two equations provide the electric and magnetic fields in terms of the potentials.

Frequency–domain potentials: GA approach

In frequency domain the field $\mathscr{F}$ satisfies the following equation

$$(\nabla + jk)\mathscr{F} = \frac{\rho}{\varepsilon} - \eta \mathbf{J} \tag{4.191}$$

It is assumed that the field $\mathscr{F}$ can be recovered from a scalar potential ϕ and a vector potential $\mathbf{A}$, using the following expression:

$$\mathscr{F} = (\nabla - jk)(v\mathbf{A} - \phi)\,. \tag{4.192}$$

When (4.192) is inserted into (4.191) we recover the following equation:

$$\left(\nabla^2 + k^2\right)(v\mathbf{A} - \phi) = \frac{\rho}{\varepsilon} - \eta \mathbf{J}\,. \tag{4.193}$$

and, by separating the scalar and the vector part, we recover respectively the scalar and vector wave equations

$$\left(\nabla^2 + k^2\right)\phi = -\frac{\rho}{\varepsilon} \tag{4.194}$$

$$\left(\nabla^2 + k^2\right)\mathbf{A} = -\mu \mathbf{J}\,. \tag{4.195}$$

Finally, by equating (4.192) and (4.73) we get:

$$\mathscr{F} = \mathbf{E} + \eta\,\hat{\mathbf{H}} \tag{4.196}$$

$$= (\nabla - jk)\,(v\mathbf{A} - \phi) \tag{4.197}$$

$$= jk\phi + v\nabla\cdot\mathbf{A} - \nabla\wedge\phi - jkv\mathbf{A} + v\nabla\wedge\mathbf{A}\,. \tag{4.198}$$

Considering the scalar, vector and bivectors parts one gets

$$0 = jk\phi + v\nabla\cdot\mathbf{A} \tag{4.199}$$

$$\mathbf{E} = -\nabla\wedge\phi - jkv\mathbf{A} \tag{4.200}$$

$$\eta\,\hat{\mathbf{H}} = v\nabla\wedge\mathbf{A} \tag{4.201}$$

and therefore

$$\phi = \frac{jv}{k}\nabla\cdot\mathbf{A} \tag{4.202}$$

$$\mathbf{E} = -jkv\mathbf{A} - \nabla\phi \tag{4.203}$$

$$\hat{\mathbf{H}} = \frac{1}{\mu}\nabla\wedge\mathbf{A} \tag{4.204}$$

Note that the Lorenz condition is not imposed, but it is derived directly from the assumption in (4.192).

Potentials with Pauli matrices

We have seen that Dirac matrices allows a very compact formulation of potential and field equations. Nevertheless, in some instances, it may be convenient to refer to a formulation in terms of the Pauli matrices. Equation (4.192) may be written in terms of Pauli matrices as:

$$\tilde{F} = \left(\tilde{\nabla} - jk\sigma_0\right)\left(v\tilde{A} - \phi\sigma_0\right) \tag{4.205}$$

with ϕ satisfying Lorenz condition (4.202).

Magnetic sources

By application of superposition it is convenient to consider the case when only magnetic sources are present. In the frequency domain we have

$$\begin{aligned}\nabla\cdot\mathbf{E} &= 0\\ \nabla\cdot(i\eta\mathbf{H}) &= iv\rho_m\\ \nabla\wedge\mathbf{E} &= -jk(i\eta\mathbf{H}) - i\mathbf{M}\\ \nabla\wedge(i\eta\mathbf{H}) &= -jk\mathbf{E}\end{aligned} \tag{4.206}$$

which can be condensed as

$$\begin{aligned}\nabla\mathbf{E} &= -jk(i\eta\mathbf{H}) - i\mathbf{M}\\ \nabla(i\eta\mathbf{H}) &= -jk\mathbf{E} + iv\rho_m\end{aligned} \tag{4.207}$$

or, by using the multivector form, yields:

$$(\nabla + jk)\mathscr{F} = -i(\mathbf{M} - v\rho_m)\,. \tag{4.208}$$

It is assumed that the field $\mathscr{F}$ can be recovered from a scalar potential ψ and a vector potential $\mathbf{C}$, using the following expression:

$$\mathscr{F} = -i\eta(\nabla - jk)(\psi - v\mathbf{C})\,. \tag{4.209}$$

When (4.209) is inserted into (4.208) we recover the following equation:

$$\eta\left(\nabla^2 + k^2\right)(\psi - v\mathbf{C}) = (\mathbf{M} - v\rho_m)\,. \tag{4.210}$$

and, by separating the scalar and the vector part, we recover, respectively, the scalar and vector wave equations:

$$\begin{aligned}\left(\nabla^2 + k^2\right)\psi &= -\frac{v}{\eta}\rho_m = -\frac{\rho_m}{\mu}\\ \left(\nabla^2 + k^2\right)\mathbf{C} &= -\frac{1}{\eta v}\mathbf{M} = -\varepsilon\mathbf{M}.\end{aligned} \tag{4.211}$$

Finally, we get:

$$\begin{aligned}\mathscr{F} &= \mathbf{E} + i\eta\,\mathbf{H} \\ &= -i\eta\,(\nabla - jk)\,(\psi - v\,\mathbf{C}) \\ &= -i\eta\,(-jk\,\psi - v\nabla\cdot\mathbf{C} + \nabla\psi + jkv\,\mathbf{C} - v\nabla\wedge\mathbf{C})\,. \end{aligned} \tag{4.212}$$

Considering the scalar, vector and bivectors parts one gets

$$\begin{aligned} 0 &= -jk\,\psi - v\nabla\cdot\mathbf{C} \\ \mathbf{E} &= i\eta\,v\nabla\wedge\mathbf{C} = -\frac{1}{\varepsilon}\nabla\times\mathbf{C} \\ \mathbf{H} &= -\nabla\psi + jkv\,\mathbf{C} \end{aligned} \tag{4.213}$$

The Lorenz condition (4.214)

$$\psi = \frac{jv}{k}\nabla\cdot\mathbf{C} \tag{4.214}$$

is derived directly from the assumption in (4.209). In terms of Pauli matrices the field can therefore be recovered as

$$\tilde{F} = -i\eta\left(\tilde{\nabla} - jk\sigma_0\right)\left(\psi\sigma_0 - v\tilde{C}\right)\,. \tag{4.215}$$

III.9 Potentials with only two scalar components

In a region of space without sources, the field can be represented by using only two scalar components. A common choice is to select potentials with only the z components. In this case, by setting:

$$\begin{aligned} A_z &= j\frac{k}{v}\,\psi_e \\ C_z &= j\frac{k}{v}\,\psi_h \end{aligned} \tag{4.216}$$

we recover the following components for the fields

$$
\begin{aligned}
E_x &= -j\,k\,\eta\,\partial_y\,\psi_h + \partial_x\,\partial_z\,\psi_e \\
E_y &= \partial_y\,\partial_z\,\psi_e + j\,k\,\eta\,\partial_x\,\psi_h \\
E_z &= \left(k^2 + \partial_z^2\right)\psi_e \\
H_x &= j\,\frac{k}{\eta}\,\partial_y\,\psi_e + \partial_x\,\partial_z\,\psi_h \\
H_y &= \partial_y\,\partial_z\,\psi_h - j\,\frac{k}{\eta}\,\partial_x\,\psi_e \\
H_z &= \left(k^2 + \partial_z^2\right)\psi_h
\end{aligned}
\tag{4.217}
$$

It will be shown next that the choice in (4.216) corresponds to the selection of Hertzian potentials.

III.10 Field Potentials: conventional approach

Auxiliary potentials are conventionally introduced to simplify the solution of the vector field equations [3, 7, 8]. When only *electric* sources are present in a homogeneous region, the two curl equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}\,, \tag{4.218a}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{4.218b}$$

provide six scalar equations. The divergence equations

$$\nabla \cdot \mathbf{D} = \rho_e\,, \tag{4.219a}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{4.219b}$$

provide two additional scalar equations, which need to be complemented by the constitutive relations

$$\mathbf{D} = \varepsilon\mathbf{E}\,, \tag{4.220a}$$

$$\mathbf{B} = \mu\mathbf{H}\,, \tag{4.220b}$$

and the relevant boundary conditions. The use of potential functions can systematize the solution of this large set of equations.

Magnetic Vector and Electric Scalar Potentials

The *vector* and *scalar potential* functions $\mathbf{A}$ and ϕ, respectively, represent the electrodynamic extensions of the static magnetic vector potential and electrical scalar potential, respectively. While potential theory is generally developed for the time–dependent form of Maxwell's equations [3, 7, 8], we shall deal directly with the time–harmonic potentials (an $\exp(j\omega t)$ time–dependence is assumed and suppressed). By taking the divergence of (4.218a) we see that

$$\nabla \cdot \mathbf{H} = 0, \tag{4.221}$$

i.e. the divergence equation for $\mathbf{H}$ is automatically satisfied. This suggests expressing $\mathbf{H}$ as

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A}\,, \tag{4.222}$$

where $\mathbf{A}$ is referred to as the *magnetic vector potential*. By inserting (4.222) into (4.218a) we note that

$$\nabla \times (\mathbf{E} + j\omega \mathbf{A}) = 0 \tag{4.223}$$

and since

$$\nabla \times \nabla \phi = 0 \tag{4.224}$$

the vector $\mathbf{E}$ can be expressed as

$$\mathbf{E} = -j\omega \mathbf{A} - \nabla \Phi \tag{4.225}$$

with Φ denoting the *electric scalar potential*. By substitution of (4.222), (4.225) into (4.218b) and recalling the vector identity

$$\nabla \times \nabla \times \mathbf{A} = \nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} \tag{4.226}$$

we obtain

$$\nabla \nabla \cdot \mathbf{A} - \nabla^2 \mathbf{A} = k^2 \mathbf{A} - j\omega \mu \varepsilon \nabla \phi + \mu \mathbf{J}\,. \tag{4.227}$$

Lorenz Gauge

Equation (4.227) can be phrased in a different manner by selecting the as yet unspecified divergence (lamellar part) of $\mathbf{A}$. One possible choice is to satisfy the *Lorenz* or *gauge condition*,

$$\nabla \cdot \mathbf{A} = -j\omega\mu\varepsilon\phi \tag{4.228}$$

which reduces (4.227) to the *vector Helmholtz equation*

$$\nabla^2\mathbf{A} + k^2\mathbf{A} = -\mu\mathbf{J}. \tag{4.229}$$

Taking the divergence of (4.225) and using (4.228) it follows that the scalar potential Φ satisfies the scalar Helmholtz equation

$$\nabla^2\phi + k^2\phi = -\frac{\rho_e}{\varepsilon} \tag{4.230}$$

The electric and magnetic fields, when using the Lorentz condition (4.228), are

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\mu\varepsilon}, \tag{4.231a}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A}. \tag{4.231b}$$

Electric Vector and Magnetic Scalar Potentials

When only *magnetic* sources are present, the vector $\mathbf{E}$ has zero divergence. By duality, introducing an *electric vector potential* $\mathbf{C}$ and a *scalar potential* ψ, we obtain [9, p.129]

$$\mathbf{E} = -\frac{1}{\varepsilon}\nabla\times\mathbf{C}, \tag{4.232a}$$

$$\mathbf{H} = -j\omega\mathbf{C} + \frac{\nabla\nabla\cdot\mathbf{C}}{j\omega\varepsilon\mu} \tag{4.232b}$$

with

$$\nabla^2\mathbf{C} + k^2\mathbf{C} = -\varepsilon\mathbf{M}, \tag{4.233a}$$

$$\nabla^2\psi + k^2\psi = -\frac{\rho_m}{\mu}, \tag{4.233b}$$

where $\mathbf{M}$ denotes magnetic currents and ρ_m magnetic charges. When *electric and magnetic* currents are present simultaneously in a linear system, we make use of superposition to obtain:

$$\mathbf{E} = -j\omega\mathbf{A} + \frac{\nabla\nabla\cdot\mathbf{A}}{j\omega\mu\varepsilon} - \frac{1}{\varepsilon}\nabla\times\mathbf{C} \tag{4.234a}$$

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} - j\omega\mathbf{C} + \frac{\nabla\nabla\cdot\mathbf{C}}{j\omega\varepsilon\mu} \tag{4.234b}$$

Hertz Potentials

Hertz vector potentials for electric and magnetic time-harmonic fields are simply related to the electric and magnetic vectors potentials via

$$\mathbf{A} = j\omega\varepsilon\mu\Pi_e\,, \tag{4.235a}$$

$$\mathbf{C} = j\omega\varepsilon\mu\Pi_h\,. \tag{4.235b}$$

The general field expression in terms of the Hertz vector potentials is:

$$\mathbf{E} = k^2\Pi_e + \nabla\nabla\cdot\Pi_e - j\omega\mu\nabla\times\Pi_h\,, \tag{4.236a}$$

$$\mathbf{H} = j\omega\varepsilon\nabla\times\Pi_e + k^2\Pi_h + \nabla\nabla\cdot\Pi_h\,. \tag{4.236b}$$

For Hertz potentials related to time-dependent fields see [8].

III.11 Hertzian Potentials

The general field expression in terms of the Hertz vector potentials in frequency domain is:

$$\mathbf{E} = k^2\Pi_e + \nabla\nabla\cdot\Pi_e - j\omega\mu\nabla\times\Pi_h\,, \tag{4.237a}$$

$$\mathbf{H} = j\omega\varepsilon\nabla\times\Pi_e + k^2\Pi_h + \nabla\nabla\cdot\Pi_h\,. \tag{4.237b}$$

Let us assume $z-$directed potentials:

$$\Pi_e = \psi_e(x,y,z)\,\mathbf{z}_0$$
$$\Pi_h = \psi_h(x,y,z)\,\mathbf{z}_0 \tag{4.238}$$

By performing the conventional vector operations required in (4.237a), (4.237b) provides the following components for the field:

$$\begin{aligned}
E_x &= -j\,k\,\eta\,\partial_y\,\psi_h + \partial_x\,\partial_z\,\psi_e \\
E_y &= \partial_y\,\partial_z\,\psi_e + j\,k\,\eta\,\partial_x\,\psi_h \\
E_z &= \left(k^2 + \partial_z^2\right)\psi_e \\
H_x &= j\,\frac{k}{\eta}\,\partial_y\,\psi_e + \partial_x\,\partial_z\,\psi_h \\
H_y &= \partial_y\,\partial_z\,\psi_h - j\,\frac{k}{\eta}\,\partial_x\,\psi_e \\
H_z &= \left(k^2 + \partial_z^2\right)\psi_h
\end{aligned} \tag{4.239}$$

which are coincident with (4.217).

IV Theorems and Concepts for Electromagnetic Field Computation

Certain theorems and concepts of electromagnetic theory are of fundamental importance for efficient and systematic electromagnetic field computation. Their short description follows.

IV.1 Energy and Power

The field concept is based upon the hypothesis that the electromagnetic energy is distributed over the space. We introduce the electric energy density

$$w_e(\mathbf{r},t) = \frac{\varepsilon}{2}\,\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t) \tag{4.240}$$

and the magnetic energy density

$$w_m(\mathbf{r},t) = \frac{\mu}{2}\,\mathbf{H}(\mathbf{r},t)\cdot\mathbf{H}(\mathbf{r},t) \tag{4.241}$$

We have already defined the field multivector $\mathscr{F}$, which is here repeated for convenience

$$\mathscr{F} = \mathbf{E} + i\eta\mathbf{H}$$
$$\mathscr{F}^\dagger = \mathbf{E} - i\eta\mathbf{H}. \tag{4.242}$$

In (4.242) the quantity $\mathscr{F}^\dagger$, i.e. the *reverse* of $\mathscr{F}$ has been introduced. The reverse is obtained by taking $-i$ instead of i, thus making the conjugate w.r.t. i. By using the rules of geometric algebra we can compute

$$U = \frac{1}{2}\varepsilon\mathscr{F}\mathscr{F}^\dagger = \frac{1}{2}\varepsilon\mathbf{E}^2 + \frac{1}{2}\mu\mathbf{H}^2 - i\eta\varepsilon\mathbf{E}\wedge\mathbf{H}. \tag{4.243}$$

By noting that

$$\mathbf{S} = -i\mathbf{E}\wedge\mathbf{H} \tag{4.244}$$

we can rewrite U as

$$U = w_e + w_m + \eta\varepsilon\mathbf{S} = \mathscr{E} + \frac{1}{v}\mathbf{S} \tag{4.245}$$

where we have introduced the total energy $\mathscr{E}$defined as:

$$\mathscr{E} = w_e + w_m. \tag{4.246}$$

Considering the product

$$V = \frac{1}{2}\varepsilon\mathscr{F}^\dagger\mathscr{F} = \mathscr{E} - \frac{1}{v}\mathbf{S}. \tag{4.247}$$

it is possible to write

$$\mathscr{E} = \frac{1}{2}(U+V)$$
$$\mathbf{S} = \frac{v}{2}(U-V) \tag{4.248}$$

Poynting vector using Pauli matrices and spinors

Let us first try to express the Poynting vector using Pauli matrices. We have for the electric and magnetic field

$$\tilde{E} = \begin{pmatrix} E_z & E_x - iE_y \\ iE_y + E_x & -E_z \end{pmatrix}$$
$$\tilde{H} = \begin{pmatrix} H_z & H_x - iH_y \\ iH_y + H_x & -H_z \end{pmatrix} \tag{4.249}$$

We can for the geometric product of $\mathbf{E}$ and $\mathbf{H}$ which is

$$\mathbf{E}\mathbf{H} = \mathbf{E}\cdot\mathbf{H} + \mathbf{E}\wedge\mathbf{H}. \tag{4.250}$$

By writing this product in terms of Pauli matrices and, introducing the matrix $EH = \tilde{E}\tilde{H}$ obtained as the product of $\tilde{E}$ and $\tilde{H}$, we have

$$EH = \begin{pmatrix} E_z H_z + \left(E_y + iE_x\right) H_y - iH_x E_y + E_x H_x & \left(iE_y - E_x\right) H_z + \left(H_x - iH_y\right) E_z \\ \left(iE_y + E_x\right) H_z + \left(-iH_y - H_x\right) E_z & E_z H_z + \left(E_y - iE_x\right) H_y + iH_x E_y + E_x H_x \end{pmatrix}.$$

It is apparent that the trace of this matrix divided by 2 gives the scalar term

$$\mathbf{E}\cdot\mathbf{H} = E_z H_z + E_y H_y + E_x H_x. \tag{4.251}$$

The wedge product is obtained as

$$\mathbf{E}\wedge\mathbf{H} = EH - \sigma_0 \mathbf{E}\cdot\mathbf{H} \tag{4.252}$$

giving

$$\mathbf{E}\wedge\mathbf{H} = \begin{pmatrix} iE_x H_y - iH_x E_y & \left(iE_y - E_x\right) H_z + \left(H_x - iH_y\right) E_z \\ \left(iE_y + E_x\right) H_z + \left(-iH_y - H_x\right) E_z & iH_x E_y - iE_x H_y \end{pmatrix}.$$

Finally, by dividing by i, the expression for $\mathbf{E}\times\mathbf{H}$ is recovered

$$\mathbf{E}\times\mathbf{H} = \begin{pmatrix} E_x H_y - H_x E_y & \left(E_y + iE_x\right) H_z + \left(-H_y - iH_x\right) E_z \\ \left(E_y - iE_x\right) H_z + \left(iH_x - H_y\right) E_z & H_x E_y - E_x H_y \end{pmatrix}.$$

It is also instructive to express the Poynting vector in terms of the bivector $i\eta\,\mathbf{H}$. Note that by taking the geometric product of the vector $\mathbf{E}$ and the bivector $i\eta\,\mathbf{H}$, we have

$$\mathbf{E}\,(i\eta\,\mathbf{H}) = \mathbf{E}\cdot(i\eta\,\mathbf{H}) + \mathbf{E}\wedge(i\eta\,\mathbf{H})\,. \tag{4.253}$$

The reader can easily recognize that the dot product of a bivector and a vector gives a vector and therefore we have

$$\eta\mathbf{S} = \mathbf{E}\cdot(i\eta\,\mathbf{H})\,. \tag{4.254}$$

The external product of the vector and the bivector instead gives a pseudo scalar. It is instructive, as an exercise, to consider the Pauli matrix corresponding to the bivector $i\eta\,\mathbf{H}$ as

$$i\eta\tilde{H} = \eta\begin{pmatrix} iH_z & H_y + iH_x \\ iH_x - H_y & -iH_z \end{pmatrix} \tag{4.255}$$

and multiply $\tilde{E}$ with the above matrix, obtain from half of the trace the term

$$i\eta\left(E_zH_z + E_yH_y + E_xH_x\right) \tag{4.256}$$

which corresponds to a pseudo–scalar. The remaining terms in the Pauli matrix give the vector term as

$$\begin{aligned}\eta\mathbf{S} &= \mathbf{E}\cdot(i\eta\,\mathbf{H}) \\ &= \eta\begin{pmatrix} H_xE_y - E_xH_y & \left(-E_y - iE_x\right)H_z + \left(H_y + iH_x\right)E_z \\ \left(iE_x - E_y\right)H_z + \left(H_y - iH_x\right)E_z & E_xH_y - H_xE_y \end{pmatrix}\end{aligned}$$

IV.2 Spinor representation

We have already introduced the following definitions

$$\begin{pmatrix} a_0+b_0 \\ a_1+b_1 \\ a_0-b_0 \\ a_1-b_1 \end{pmatrix} = \begin{pmatrix} E_z \\ iE_y+E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x-H_y\right) \end{pmatrix}. \tag{4.257}$$

which corresponds to:

$$\begin{aligned} a_0 &= \frac{1}{2}\left(i\eta H_z+E_z\right) \\ a_1 &= \frac{1}{2}\left(-\eta H_y+iE_y+i\eta H_x+E_x\right) \\ b_0 &= \frac{1}{2}\left(E_z-i\eta H_z\right) \\ b_1 &= \frac{1}{2}\left(\eta H_y+iE_y-i\eta H_x+E_x\right). \end{aligned} \tag{4.258}$$

The reason for introducing the wave parameters is that with them we have two sets of two equations which are decoupled. One set represent progressive waves and is expressed by:

$$\begin{pmatrix} \partial_z+\frac{\partial_t}{v} & \partial_x-i\partial_y \\ i\partial_y+\partial_x & \frac{\partial_t}{v}-\partial_z \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0 \tag{4.259}$$

while the regressive waves are given by:

$$\begin{pmatrix} \partial_z-\frac{\partial_t}{v} & \partial_x-i\partial_y \\ i\partial_y+\partial_x & -\partial_z-\frac{\partial_t}{v} \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0 \tag{4.260}$$

It is convenient to introduce the matrices

$$\mathbf{a} = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$$

$$\mathbf{b} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}.$$

The Poynting vector can be expressed in terms of the above parameters. Let us consider e.g. the component of power flowing in the z direction, given by

$$S_z = \mathbf{E} \times \mathbf{H} \cdot \mathbf{z}_0 = H_x E_y - E_x H_y \tag{4.261}$$

We can write the Poynting vector as

$$S_z = \frac{1}{\eta} \left(\mathbf{a} \dagger \, \sigma_3 \, \mathbf{a} - \mathbf{b} \dagger \, \sigma_3 \, \mathbf{b} \right) \tag{4.262}$$

where the dagger, as before, denotes the reverse. This can be readily proved by constructing the vectors

$$\mathbf{a} = \begin{pmatrix} \frac{i\eta H_z + E_z}{2} \\ \frac{-\eta H_y + i E_y + i\eta H_x + E_x}{2} \end{pmatrix}$$
$$\mathbf{b} = \begin{pmatrix} \frac{E_z - i\eta H_z}{2} \\ \frac{\eta H_y + i E_y - i\eta H_x + E_x}{2} \end{pmatrix}.$$

and performing the matrix product. The results in this section are obtained with the following code, while the identity in (4.272) is proved with the next code.

wxm/GA_Poynting_v02.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
/* code name : GA_Poynting_v02 */

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$
/*
print("Nabla as Pauli matrix")$
Np : %delta[x] * %sigma[1] + %delta[y] * %sigma[2] + %delta[z] * %sigma[3];
*/
print("vector H as Pauli matrix")$
Hp : H[x] * %sigma[1] + H[y] * %sigma[2] + H[z] * %sigma[3];
print("vector E as Pauli matrix")$
Ep : E[x] * %sigma[1] + E[y] * %sigma[2] + E[z] * %sigma[3];

print("------------------------------------------------")$
print("Compute  (E x H)")$
print("This is  E H ")$
ErotH : ratsimp(Ep . Hp);
print("This is E cdot H")$
divEH : ratsimp((ErotH[1,1]+ ErotH[2,2])/2);
print("E wedge H")$
EwH : ratsimp(ErotH - divEH*%sigma[0]);
ExH : ratsimp(EwH/%i);

print(" %i * H")$
iHp : ratsimp(%i * Hp);
print("Compute  (E H)")$
EiH : ratsimp(Ep . iHp);

print("div EiH")$
divEiH : ratsimp((EiH[1,1]+ EiH[2,2])/2);
```

```
print("E wedge %i H")$
EwiH : ratsimp(EiH - divEiH*%sigma[0]);

print("Taking the dot product with az")$
EwiHz : EwiH . %sigma[3];
Pz : ratsimp((EwiHz[1,1]+ EwiHz[2,2])/2);

print("Here we insert the relationship with the waves a,b")$

eq1: a[0]+b[0]-E[z];
eq2: a[1]+b[1]-E[x] - %i * E[y];
eq3: a[0]-b[0]-%i * %eta * H[z];
eq4: a[1]-b[1]-%i * %eta * H[x] +  %eta * H[y];

absol :solve([eq1,eq2,eq3,eq4],[a[0],a[1],b[0],b[1]]);

a0 : rhs(absol[1][1]);
a1 : rhs(absol[1][2]);
b0 : rhs(absol[1][3]);
b1 : rhs(absol[1][4]);

print("construct the vector a")$
av : matrix([a0],[a1]);

print("construct the vector b")$
bv : matrix([b0],[b1]);

print("perform conjugate a %sigma[3]  a")$
asa : conjugate(av) . %sigma[3] . av;

bsb : conjugate(bv) . %sigma[3] . bv;

Pzab : ratsimp(1/%eta * (asa-bsb));

print("Bye")$
/* [wxMaxima: input   end   ] */
```

```
/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

wxm/GA_identity_v03.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
/* code name : GA_identity_v03 */

%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

print("Nabla as Pauli matrix")$
Np : %delta[x] * %sigma[1] + %delta[y] * %sigma[2] + %delta[z] * %sigma[3];

print("vector U as Pauli matrix")$
Up : U[x] * %sigma[1] + U[y] * %sigma[2] + U[z] * %sigma[3];
print("vector V as Pauli matrix")$
Vp : V[x] * %sigma[1] + V[y] * %sigma[2] + V[z] * %sigma[3];

print("------------------------------------------------")$
print("Compute div (U wedge V)")$
UV : ratsimp(Up . Vp)$
divUV : ratsimp((UV[1,1]+ UV[2,2])/2)$
print("U wedge V")$
UwV : ratsimp(UV - divUV*%sigma[0]);
print("Nabla UwV")$
NUwV: ratsimp(Np.UwV);
NdivUwV : ratsimp((NUwV[1,1]+ NUwV[2,2])/2);
print("div U wedge V")$
NdUwV : ratsimp(NUwV - NdivUwV*%sigma[0]);

print("------------------------------------------------")$
print("Compute V dot (nabla wedge U)")$
print("Nabla U")$
NpUp : (Np . Up);
divNU : ratsimp((NpUp[1,1]+ NpUp[2,2])/2)$
```

```
print("Nabla wedge U")$
NwU : ratsimp(NpUp - divNU*%sigma[0]);
print("V Nabla wedge U")$
VNwU : ratsimp(Vp . NwU);
VdNwU : ratsimp((VNwU[1,1]+ VNwU[2,2])/2)$
print("V dot (Nabla wedge U)")$
VdotNwU : ratsimp(VNwU-VdNwU*%sigma[0]);

print("------------------------------------------------")$
print("Compute U dot (nabla wedge V)")$
print("Compute nabla wedge V")$
NpVp : (Np . Vp)$
divNV : ratsimp((NpVp[1,1]+ NpVp[2,2])/2)$
NwV : ratsimp(NpVp - divNV*%sigma[0]);

print("U Nabla wedge V")$
UNwV : ratsimp(Up . NwV);

UdNwV : ratsimp((UNwV[1,1]+ UNwV[2,2])/2)$

print("U dot (Nabla wedge V)")$
UdotNwV : ratsimp(UNwV - UdNwV*%sigma[0]);



print("------------------------------------------------")$
ratsimp(NdUwV - (-VdotNwU + UdotNwV));
/*
ratsimp(NdUwV - (VdotNwU ));
ratsimp(NdUwV - ( - UdotNwV));
*/

print("Bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

IV.3 Poynting's theorem: conventional approach

In order to investigate energy storage and power flow in the electromagnetic field, we start again with Maxwell's equations. By scalar multiplication of Ampére's law with $-\mathbf{E}$ and Faraday's law with $\mathbf{H}$, we obtain

$$\begin{aligned} \nabla \times \mathbf{H} &= \tfrac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad | \quad \cdot(-\mathbf{E}) \,, \\ \nabla \times \mathbf{E} &= \quad -\tfrac{\partial \mathbf{B}}{\partial t} \quad | \quad \cdot \mathbf{H} \end{aligned} \tag{4.263}$$

and, by summing, yields:

$$\mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H} = -\mu \mathbf{H} \cdot \frac{\partial \mathbf{H}}{\partial t} - \varepsilon \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \mathbf{E} \cdot \mathbf{J}. \tag{4.264}$$

Using the relation

$$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot \nabla \times \mathbf{U} - \mathbf{U} \cdot \nabla \times \mathbf{V}, \tag{4.265}$$

we transform the left side of (4.264) and obtain the differential form of *Poynting's theorem*

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \frac{\partial}{\partial t} \left(\frac{\mu}{2} \mathbf{H} \cdot \mathbf{H} + \frac{\varepsilon}{2} \mathbf{E} \cdot \mathbf{E} \right) + \sigma \mathbf{E} \cdot \mathbf{E} + \mathbf{E} \cdot \mathbf{J}_0 \,. \tag{4.266}$$

On the right side of (4.266), we have the time derivative of the electric and magnetic energy densities corresponding to (4.240) and (4.241). The third term is the power loss density

$$p_v(\mathbf{r},t) = \sigma(\mathbf{r}) \mathbf{E} \cdot \mathbf{E}. \tag{4.267}$$

Due to the impressed current density $\mathbf{J}_0$, a power

$$p_0(\mathbf{r},t) = -\mathbf{E}(\mathbf{r},t) \cdot \mathbf{J}_0(\mathbf{r},t) \tag{4.268}$$

is added to the electromagnetic field per unit of volume. Introducing the *Poynting's* vector

$$\mathbf{S}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t) \tag{4.269}$$

allows to write down Poynting's Theorem in the following form:

$$\nabla \cdot \mathbf{S} = -\frac{\partial w_m}{\partial t} - \frac{\partial w_e}{\partial t} - p_v + p_0. \tag{4.270}$$

Integrating (4.270) over a volume V and transforming the integral over S into a surface integral over the boundary ∂V, we obtain the integral form of Poynting's Theorem:

$$\oint_{\partial V} \mathbf{S} \cdot \mathrm{d}A = \int_V p_0 \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t} \int_V w_m \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t} \int_V w_e \mathrm{d}V - \int_V p_v \mathrm{d}V\,. \tag{4.271}$$

The first term on the right side of equation (4.271) describes the power added into the volume V via impressed currents. The second and the third term, respectively, describe time variation of the magnetic and electric energy stored in the volume. The last term describes the conductive losses occurring inside the volume V. The right side of the equation comprises the total electromagnetic power generated within the volume V minus the power losses in the volume minus the increase of electric and magnetic power stored in the volume. This net power must be equal to the power, which is flowing out from the volume V through the boundary ∂V. Therefore we may interpret the surface integral over the pointing vector on the left side of (4.271) as the total power flowing from inside the volume V to the outside. Since this is valid for an arbitrary choice of volume V, it follows that the Poynting's vector describes the energy flowing by unit of time through an unit area oriented perpendicular to S.

Poyinting vector according to geometric algebra

It is possible to prove the following identity

$$\nabla \cdot (\mathbf{U} \wedge \mathbf{V}) = \mathbf{U} \cdot \nabla \wedge \mathbf{V} - \mathbf{V} \cdot \nabla \wedge \mathbf{U}, \tag{4.272}$$

which is slightly different with respect to (4.265). The listing for proving this identity is reported at the end of the subsection. By starting from (4.263) and multiplying both sides of both equations by i we recover

$$\begin{aligned} \nabla \wedge \mathbf{E} &= -i\mu\, \partial_t \mathbf{H} \\ \nabla \wedge \mathbf{H} &= i\varepsilon\, \partial_t \mathbf{E} + i\mathbf{J}\,. \end{aligned} \tag{4.273}$$

Proceeding as before, i.e. by scalar multiplying the first equation of (4.273) times the magnetic field and the second times the electric field, one obtains

$$\begin{aligned}\mathbf{H}\cdot\nabla\wedge\mathbf{E} &= -i\mu\,\mathbf{H}\cdot\partial_t\mathbf{H}\\ \mathbf{E}\cdot\nabla\wedge\mathbf{H} &= i\varepsilon\,\mathbf{E}\cdot\partial_t\mathbf{E}+i\mathbf{J}\cdot\mathbf{E}\end{aligned}\tag{4.274}$$

which, using (4.272) and assuming $\sigma = 0$, gives

$$\begin{aligned}\nabla\cdot(\mathbf{E}\wedge\mathbf{H}) &= \mathbf{E}\cdot\nabla\wedge\mathbf{H}-\mathbf{H}\cdot\nabla\wedge\mathbf{E}\\ &= i\,(\varepsilon\,\mathbf{E}\cdot\partial_t\mathbf{E}+\mu\,\mathbf{H}\cdot\partial_t\mathbf{H}+\mathbf{J}\cdot\mathbf{E})\\ &= i\left[\frac{\partial}{\partial t}\left(\frac{\mu}{2}\mathbf{H}\cdot\mathbf{H}+\frac{\varepsilon}{2}\mathbf{E}\cdot\mathbf{E}\right)+\mathbf{E}\cdot\mathbf{J}\right].\end{aligned}\tag{4.275}$$

It is noted that in the last row all the terms are multiplied by i thus having the grade of a pseudoscalar, i.e. of a volume element. By proceeding as before, integrating over the volume V and applying the divergence theorem, one gets

$$\begin{aligned}\int_V \nabla\cdot(\mathbf{E}\wedge\mathbf{H})\,dV &= \oint_{\partial V}\mathbf{E}\wedge\mathbf{H}\,\mathrm{d}A\\ &= i\int_V \varepsilon\,\mathbf{E}\cdot\partial_t\mathbf{E}+\mu\,\mathbf{H}\cdot\partial_t\mathbf{H}+\mathbf{J}\cdot\mathbf{E}dV\,.\end{aligned}\tag{4.276}$$

So far the geometric algebra approach does not provide a very different insight.

IV.4 Poyinting vector for harmonic fields

For harmonic electromagnetic fields, the introduction of a complex Poynting's vector is useful. For this we construct

$$\begin{aligned}\nabla\times\mathbf{H}^* &= -j\omega\varepsilon^*\mathbf{E}^*+\mathbf{J}_0^* \;\;|\quad\cdot(-\mathbf{E}),\\ \nabla\times\mathbf{E} &= \quad -j\omega\mu\mathbf{H}\quad\;\; |\quad\cdot\mathbf{H}^*.\end{aligned}\tag{4.277}$$

Summing both equations, we obtain

$$\nabla\times\mathbf{H}^*\cdot\mathbf{E}-\mathbf{E}\cdot\nabla\times\mathbf{H}^* = -j\omega(\mu\mid\mathbf{H}\mid)^2-\varepsilon^*\mid\mathbf{E}\mid^2)-\mathbf{E}\cdot\mathbf{J}_0^*.\tag{4.278}$$

With the relation

$$\nabla \cdot (\mathbf{U} \times \mathbf{V}) = \mathbf{V} \cdot \nabla \times \mathbf{U} - \mathbf{U} \cdot \nabla \times \mathbf{V}, \tag{4.279}$$

we can transform (4.278) into the differential form of the *complex Poynting's theorem*

$$\mathrm{div} \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*) = -2j\omega \left(\frac{\mu}{4} \mid \mathbf{H} \mid^2 - \frac{\varepsilon^*}{4} \mid \mathbf{E} \mid^2 \right) - \frac{1}{2}\mathbf{E} \cdot \mathbf{J}_0^*. \tag{4.280}$$

We now introduce the *complex Poynting's vector* $\mathbf{T}$:

$$\mathbf{T}(\mathbf{r}) = \frac{1}{2} \left(\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}) \right). \tag{4.281}$$

We have to note that $\mathbf{T}$ is not the phasor corresponding to $\mathbf{S}$. Therefore we have used a different character to distinguish between the complex Poynting's vector and the real Poynting's vector. In order to give an interpretation of the complex Poynting's vector $\mathbf{T}$, we compute first the time-dependent Poynting's vector $\mathbf{S}$ for a harmonic electromagnetic field

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left\{ \mathbf{E}(\mathbf{r}) \mathrm{e}^{\mathrm{j}\omega t} \right\} = \frac{1}{2} \left(\mathbf{E}(\mathbf{r}) \mathrm{e}^{\mathrm{j}\omega t} + \mathbf{E}^*(\mathbf{r}) \mathrm{e}^{-\mathrm{j}\omega t} \right), \tag{4.282a}$$

$$\mathbf{H}(\mathbf{r},t) = \mathrm{Re}\left\{ \mathbf{H}(\mathbf{r}) \mathrm{e}^{\mathrm{j}\omega t} \right\} = \frac{1}{2} \left(\mathbf{H}(\mathbf{r}) \mathrm{e}^{\mathrm{j}\omega t} + \mathbf{H}^*(\mathbf{r}) \mathrm{e}^{-\mathrm{j}\omega t} \right) \tag{4.282b}$$

we obtain

$$\mathbf{S}(\mathbf{r},t) = \frac{1}{2} \mathrm{Re}\left\{ \mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r}) \right\} + \frac{1}{2} \mathrm{Re}\left\{ \mathbf{E}(\mathbf{r}) \times \mathbf{H}(\mathbf{r}) \mathrm{e}^{2\mathrm{j}\omega t} \right\}. \tag{4.283}$$

The first term on the right side of (4.283) is equal to the real part of the complex Poynting's vector $\mathbf{T}$ after equation (4.281). This term is independent of time. The second on the right-hand side of (4.283) oscillates with twice the frequency of the alternating electromagnetic field. The time average of this part vanishes. Therefore the real part of the complex Poynting's vector $\mathbf{T}$ is the time average of the Poynting's vector $\mathbf{S}$.

$$\overline{\mathbf{S}(\mathbf{r},t)} = \mathrm{Re}\{\mathbf{T}(\mathbf{r})\}. \tag{4.284}$$

The real part of the complex Poynting's vector $\mathbf{T}$ denotes the power flowing through an unit area oriented perpendicular to $\mathbf{T}$. We write the time averages of the electric and magnetic energy densities $\overline{w}_e$ and $\overline{w}_m$ as

$$\overline{w}_e = \frac{\varepsilon}{2}\overline{\mathbf{E}(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t)} = \frac{\varepsilon'}{4} \mid \mathbf{E}(\mathbf{r}) \mid^2, \tag{4.285}$$

$$\overline{w}_m = \frac{\mu}{2}\overline{\mathbf{H}(\mathbf{r},t)\cdot\mathbf{H}(\mathbf{r},t)} = \frac{\mu'}{4} \mid \mathbf{H}(\mathbf{r}) \mid^2 . \tag{4.286}$$

We have to consider that the quantities ε' and μ' in the complex representation correspond to the quantities ε and μ in the time-dependent formulation. From equations (4.241), (4.281) and (**??**) we obtain the average electric power dissipation density

$$\overline{p}_{ve} = \frac{1}{2}\sigma \mid \mathbf{E}(\mathbf{r}) \mid^2 = \frac{1}{2}\omega\varepsilon'' \mid \mathbf{E}(\mathbf{r}) \mid^2 . \tag{4.287}$$

The introduction of the complex permittivity μ allows also to consider the magnetic losses. The average power dissipation density is given by

$$\overline{p}_v = \frac{w}{2}(\varepsilon'' \mid \mathbf{E}(\mathbf{r}) \mid^2 + \mu'' \mid \mathbf{H}(\mathbf{r}) \mid^2). \tag{4.288}$$

The complex power, which is added to the field due to the impressed current density $\mathbf{J}_0$ is given by

$$p_{s0} = -\frac{1}{2}\mathbf{E}\cdot\mathbf{J}_0^* \tag{4.289}$$

The real part of p_{s0} equals the time average $\overline{p}_{s0}$ according to equation (4.270).

$$\overline{p}_0 = \mathrm{Re}\{p_{s0}\}. \tag{4.290}$$

The proof is similar to the one of (4.284). After inserting of (4.281), (4.285), (4.286), (4.288) and (4.289) into (4.280), we can write down the complex Poynting's theorem in the following form

$$\mathrm{div}\,\mathbf{T} = -2j\omega(\overline{w}_m - \overline{w}_e) - \overline{p}_v + p_{s0}\,. \tag{4.291}$$

By integration over a volume V, we obtain the integral form of the complex Poynting's Theorem

$$\oint_{\partial V} \mathbf{T}\cdot\mathrm{d}A = \int_V p_{s0}\mathrm{d}V - 2j\omega\int_V (\overline{w}_m - \overline{w}_e)\mathrm{d}V - \int_V \overline{p}_v\mathrm{d}V\,. \tag{4.292}$$

We consider first the real part of (4.292).

$$\mathrm{Re}\left\{\oint_{\partial V}\mathbf{T}\cdot\mathrm{d}A\right\}=\mathrm{Re}\left\{\int_{V}p_{s0}\mathrm{d}V\right\}-\int_{V}\overline{p}_{v}\mathrm{d}V\,. \tag{4.293}$$

The left side of (4.293) equals the active power radiated from inside the volume V through the boundary ∂V. On the right side of this equation, the first term denotes the power added via the impressed current density $\mathbf{J}_0$; the second term describes the conductive losses, the dielectric losses and the magnetic losses inside the volume V.

The imaginary part of (4.292) is

$$\mathrm{Im}\left\{\oint_{\partial V}\mathbf{T}\cdot\mathrm{d}A\right\}=\mathrm{Im}\left\{\int_{V}p_{s0}\mathrm{d}V\right\}-2\omega\int_{V}(\overline{w}_{m}-\overline{w}_{e})\mathrm{d}V\,. \tag{4.294}$$

The first term on the right side gives the reactive power inserted into the volume V via the impressed current density $\mathbf{J}_0$. Let us first consider the case where the second term on the right side is vanishing. In this case we see that the left side of (4.294) denotes the power radiated from volume V. Since the volume V can be chosen arbitrarily, it follows that the imaginary part of the complex Poynting's vector $\mathbf{T}$ describes the reactive power radiated through an unit area normally oriented to the vector $\mathbf{T}$.

The second term on the right side of (4.294) contains the product of the double angle of frequency with the difference of the average stored magnetic and electric energies. This term yields no contribution, if the magnetic energy stored in the volume V equals the average electric energy stored in V. The magnetic energy as well as electric energy oscillate with an angular frequency 2ω. The energy is permanently converted between electric energy and magnetic energy. If the averages $\overline{w}_e$ and $\overline{w}_m$ are equal, electric and magnetic energies may be mutually converted completely. In this case the energy oscillates between electric and magnetic field inside the volume V. If the average electric and magnetic energies are not equal, energy as well oscillates between volume V and the space outside V. In this case there is a power flow between V and the outer region. For $\overline{w}_m > \overline{w}_e$ the reactive power flowing into volume V is positive, whereas for $\overline{w}_m < \overline{w}_e$ the reactive power flowing into V is negative.

IV.5 Field Theoretic Formulation of Tellegen's Theorem (*)

Tellegen's theorem states fundamental relations between voltages and currents in a network, and is of considerable versatility and generality in network theory [10–12]. A notable property of this theorem is that it is only based on Kirchhoff's current and voltage laws, i.e. on topological relationships, and that it is independent of the constitutive laws of the network. The same reasoning that leads from Kirchhoff's laws to Tellegen's theorem permits direct derivation of a field form of Tellegen's theorem from Maxwell's equations [12–14].

In order to derive Tellegen's theorem for partitioned electromagnetic structures, let us consider two cases based on the same partitioning but filled with different materials. The connection network is established via relating the tangential field components on both sides of the boundaries; since the connection network has zero volume, no field energy is stored therein. An important point for the following discussion is that the materials filling the subdomains may be completely different. Starting directly from Maxwell's equations we may derive for a closed volume V with boundary surface ∂V and normal unit vector n the following relation

$$\int_{\partial V} \mathbf{E}'(\rho,t') \times \mathbf{H}''(\rho,t'') \cdot n\, d\mathscr{A} = -\int_V \mathbf{E}'(\mathbf{r},t') \cdot \mathbf{J}''(\mathbf{r},t'') d\mathbf{r} - \int_V \mathbf{E}'(\mathbf{r},t') \cdot \frac{\partial \mathbf{D}''(\mathbf{r},t'')}{\partial t''}\, d\mathbf{r} - \int_V \mathbf{H}'(\mathbf{r},\mathbf{t}') \cdot \frac{\partial \mathbf{B}''(\mathbf{r},\mathbf{t}'')}{\partial t''}\, d\mathbf{r}. \quad (4.295)$$

The single and double primes relate to the case of a different choice of sources, different material parameters and also a different time reference. For volumes V of zero measure, we obtain the following equation

$$\int_{\partial V} \mathbf{E}'(\rho,t') \times \mathbf{H}''(\rho,t'') \cdot n d\mathscr{A} = 0. \quad (4.296)$$

The above equation may be considered as the field form of Tellegen's theorem. Since it applies to a volume of zero measure, it is independent of the domain equations.

IV.6 Uniqueness Theorem (*)

The uniqueness theorem indicates how a problem should be properly formulated in order to provide one and only one solution. Uniqueness of the solution is a consequence of the proper imposition of the boundary conditions, since overdetermination, i.e. too many boundary condi-

tions, may lead to no solution for a given problem, while a lack of boundary conditions may lead to multiple solutions.

For time–harmonic electromagnetic fields, the uniqueness theorem states that when the sources and the tangential components of the electric *or* magnetic field are specified over the *whole* boundary surface of a given region, then the solution within this region is unique. This is actually true only if the medium is slightly lossy; otherwise it is possible to have a multiplicity of solution as, for example, for a closed resonator.

The proof of the uniqueness theorem follows from considering two different solutions $\mathbf{E}_1$, $\mathbf{H}_1$ and $\mathbf{E}_2$, $\mathbf{H}_2$ in the volume V bounded by the surface S excited by the same system of sources. Let us define the difference fields $\delta\mathbf{E}$ and $\delta\mathbf{H}$ as

$$\delta\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2\,, \tag{4.297a}$$

$$\delta\mathbf{H} = \mathbf{H}_1 - \mathbf{H}_2\,. \tag{4.297b}$$

By linearity, and since the sources are the same, the difference fields satisfy the source–free Maxwell equations

$$\nabla \times \delta\mathbf{E} = -\,j\omega\mu\,\delta\mathbf{H}\,, \tag{4.298a}$$

$$\nabla \times \delta\mathbf{H} = j\omega\varepsilon\,\delta\mathbf{E}\,, \tag{4.298b}$$

where it has been assumed that the permittivity ε and the permeability μ are of the following form

$$\varepsilon = \varepsilon_r - j\varepsilon_i\,, \tag{4.299a}$$

$$\mu = \mu_r - j\mu_i\,, \tag{4.299b}$$

i.e. a small, positive, imaginary part is present. As noted in [4] the proof also holds when the imaginary parts are both negative. By scalar multiplication of (4.298a) by $\delta\mathbf{H}^*$ and of the complex conjugate of (4.298b) by $\delta\mathbf{E}$ we obtain

$$\nabla \cdot \left(\delta\mathbf{E} \times \delta\mathbf{H}^*\right) = j\omega\varepsilon^*|\delta\mathbf{E}|^2 - j\omega\mu|\delta\mathbf{H}|^2\,. \tag{4.300}$$

The complex conjugate of (4.300) also holds, giving

$$\nabla \cdot \left(\delta\mathbf{E}^* \times \delta\mathbf{H}\right) = -j\omega\varepsilon|\delta\mathbf{E}|^2 + j\omega\mu^*|\delta\mathbf{H}|^2 . \tag{4.301}$$

By adding (4.300) and (4.301), integrating over the volume V and applying the divergence theorem, we recover

$$\oint_S \left(\delta\mathbf{E} \times \delta\mathbf{H}^* + \delta\mathbf{E}^* \times \delta\mathbf{H}\right) \cdot dS = -2\omega \int_V \left(\varepsilon_i|\delta\mathbf{E}|^2 + \mu_i|\delta\mathbf{H}|^2\right) dV . \tag{4.302}$$

When the tangential components of $\mathbf{E}$ or $\mathbf{H}$ coincide on the boundary surface S, i.e. when either $\delta\mathbf{E}$ or $\delta\mathbf{H}$ are zero on S, we have

$$\oint_S \left(\delta\mathbf{E} \times \delta\mathbf{H}^* + \delta\mathbf{E}^* \times \delta\mathbf{H}\right) \cdot dS = 0 . \tag{4.303}$$

In this case, the right–hand side of (4.302) is zero only if $\delta\mathbf{E}$ and $\delta\mathbf{H}$ are identically zero in the region V. This proves the theorem.

As a last remark, observe that for lossless structures, when we look for modal spectra, we are seeking resonant solutions. In this case, the uniqueness theorem does not apply and an infinity of solutions is present.

IV.7 Equivalence Theorem (*)

There are several forms in which to state the *equivalence theorem* [4, 15] and, in view of its importance in the solution of electromagnetic field problems, it seems appropriate to examine relevant issues in detail.

Let us consider a volume V bounded by a surface S separating the internal region, labeled as region 1, from the external region, labeled as region 2. Our objective in applying the equivalence theorem is to maintain the field in region 2 even when modifying the field in region 1. By so doing, we obtain a modified field problem which, at least in region 2, and only in region 2, is equivalent to the original one.

We denote by $\mathbf{E}_1$, $\mathbf{H}_1$ and $\mathbf{E}_2$, $\mathbf{H}_2$ the original fields in regions 1 and 2, respectively, as shown in Figure 4.2. Now suppose that the field in region 1 is altered, thus changing the field in this region from $\mathbf{E}_1, \mathbf{H}_1$ to $\mathbf{E}_1', \mathbf{H}_1'$. In order to maintain the original field in region 2, we must insert

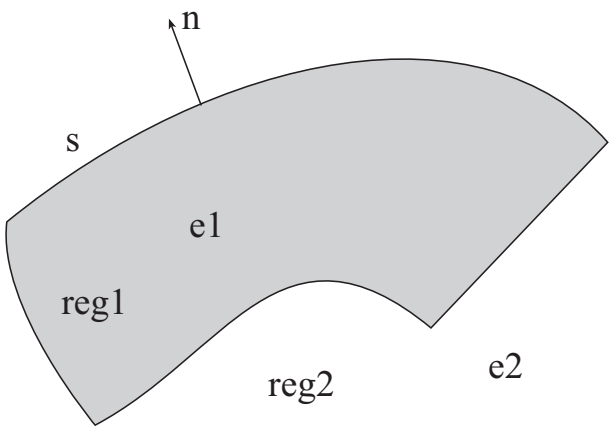

Fig. 4.2. Original field.

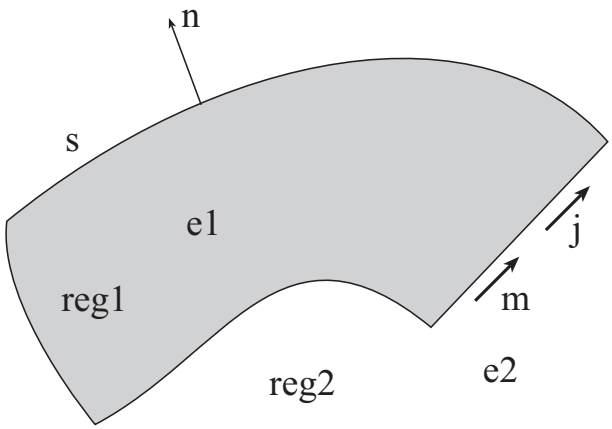

Fig. 4.3. The field in region 1 has been modified. By inserting the equivalent electric and magnetic currents on the surface S the field in region 2 is unchanged.

equivalent magnetic and electric currents, M_s and $\mathbf{J}_s$, respectively, on the surface S such that

$$\mathbf{J}_s = n \times \left(\mathbf{H}_2 - \mathbf{H}_1'\right) , \tag{4.304a}$$

$$M_s = -n \times \left(\mathbf{E}_2 - \mathbf{E}_1'\right) , \tag{4.304b}$$

as shown in Figure 4.3.

Love Equivalence Theorem

A particular case is to set the field in region 1 equal to zero. Thus we have the case shown in Figure 4.4 where the surface currents are now given by

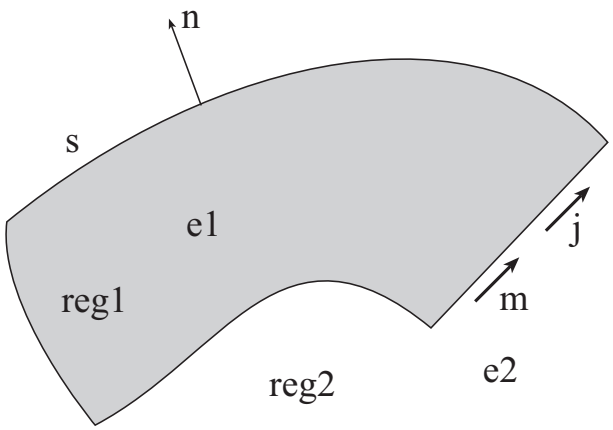

Fig. 4.4. The field in region 1 has been set to zero. Equivalent currents maintain the original field in region 2.

$$\mathbf{J}_s = n \times \mathbf{H}_2 \,, \tag{4.305a}$$

$$M_s = -\, n \times \mathbf{E}_2 \,, \tag{4.305b}$$

The *Love equivalence theorem* states that the field in region 2 produced by the given sources in region 1 is the same as that produced by a system of *virtual* sources on the surface S.

Perfect Electric Conductor

The Love theorem only specifies a zero field in region 1. This may be obtained by filling region 1 with a perfect electric conductor (PEC) as considered here, or by filling region 1 with a perfect magnetic conductor (PMC) as considered below. It is easy to see that electric currents $\mathbf{J}_s$ on the p.e.c. are short-circuited and therefore do not radiate any field. In fact, near the perfect conductor, the electric field is perpendicular to the surface *S*, while the magnetic field is parallel. The resulting Poynting vector $\mathbf{E} \times \mathbf{H}$ is thus parallel to the surface of the conductor and no power is radiated into space. A different proof of this fact may be obtained by using the Lorentz theorem. Thus, when region 1 is filled with a PEC, the resulting configuration is that shown in Figure 4.5. This form of equivalence theorem is used in practical applications where structures are bounded by metallic walls.

Perfect Magnetic Conductor

The other possibility of obtaining a null field in region 1, is to fill this region with a perfect magnetic conductor (PMC). In this case, since surface magnetic currents do not radiate, we

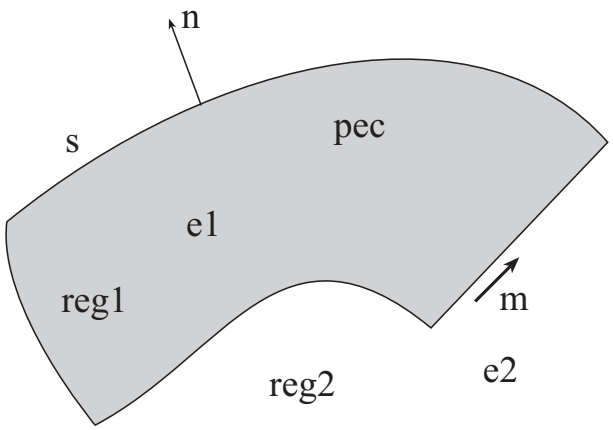

Fig. 4.5. Region 1 has been filled with a PEC. Only magnetic currents contribute since electric currents do not radiate.

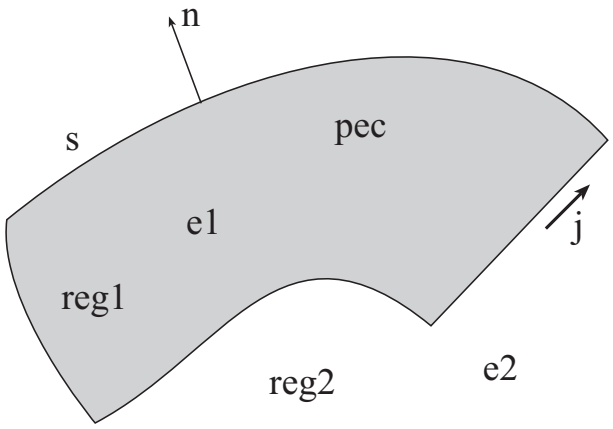

Fig. 4.6. Region 1 has been filled with a PMC Only electric currents contribute since magnetic currents do not radiate.

are left with the case of Figure 4.6. Note that as in the previously, when calculating the field produced by sources in region 2, we must take into account the presence of the PMC, since the Green's function to be considered must satisfy the appropriate boundary conditions on the surface S. On the contrary, when applying the Love theorem without filling region 1 with either a PMC or a PEC the Green's function to be considered is that of free–space.

The Circuit Theory Analog

The Circuit Theory analog of the equivalence theorem provides a simple and effective way to illustrate its utility [9]. Let region 2 be without sources, represented by the passive network in Figure 4.7(a), while region 1 is represented by the source–excited (active) network. We can set up an equivalent problem by

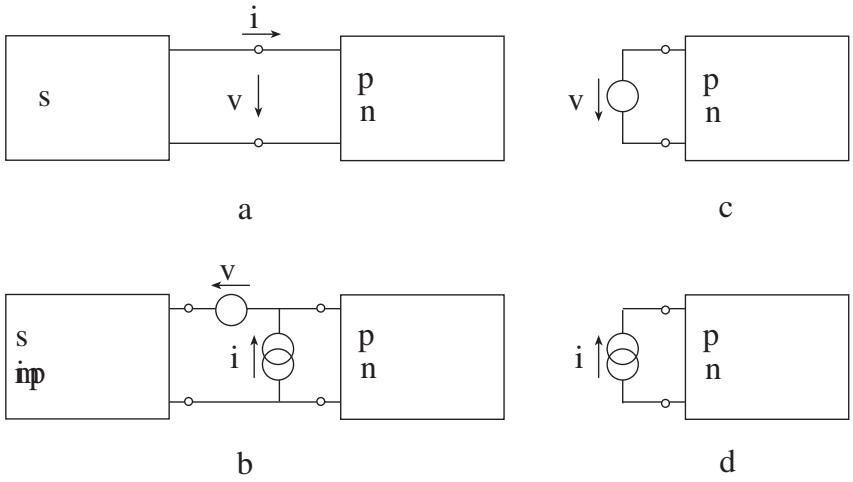

Fig. 4.7. Circuit analogue of the equivalence theorem: (a) original problem; (b) actual source deactivated, replaced by equivalent (virtual) sources; (c) source impedance replaced by a short circuit; (d) source impedance replaced by an open circuit.

- switching off the sources in the active network, leaving the source impedance connected;
- placing a shunt current generator I equal to the terminal current in the original problem;
- placing a series voltage generator V equal to the terminal voltages into the original problem.

This replaces the original sources in region 1, the active network, by the virtual sources at the interface as shown in Figure 4.7(b). From conventional circuit concepts, it is evident that there is no excitation of the source impedance from these equivalent sources whereas the excitation of the passive network is unchanged. This fact offers the possibility of replacing the source impedance by either a short or an open circuit. By considering a short circuit, we obtain the case of Figure 4.7(c), equivalent to considering a PEC when applying the Love theorem. When using an open circuit, we obtain the case of Figure 4.7(d), equivalent to considering a PMC in the Love theorem.

Duality

Returning to the Maxwell equations in (4.30a)–(4.30d) it is noted that performing the substitutions

$$\begin{array}{lll} \mathbf{E} \to \mathbf{H} & \varepsilon \to \mu & \mu \to \varepsilon \\ \mathbf{H} \to -\mathbf{E} & \mathbf{J} \to M & M \to -\mathbf{J}, \end{array} \tag{4.306}$$

equation (4.30a) becomes (4.30b), and vice versa. This is generally referred to as the "duality principle". However, the above substitutions imply a medium "dual" (or "adjoint") to free space, i.e. a medium with a permittivity of $4\pi \times 10^{-7}$ F/m and with permeability 8.854×10^{-12} H/m, which is undesirable.

A form of duality which is more suitable for antenna and radiation problems is established by the following equalities [4]

$$\begin{aligned} \mathbf{E} &\to \eta \mathbf{H}, & \mathbf{H} &\to -\frac{1}{\eta}\mathbf{E}, \\ \mathbf{J} &\to \frac{1}{\eta} M, & M &\to -\eta \mathbf{J}, \end{aligned} \tag{4.307}$$

with $\eta = \sqrt{\frac{\varepsilon}{\mu}}$ the free space impedance. With the substitutions in (4.307), equation (4.30a) becomes (4.30b) and vice versa, without need to replace free space with a different medium. Now, the form of duality in (4.307) does not apply to anisotropic or bianisotropic media, while (4.306) is more general.

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5

Electromagnetic Fields in Free space

I Introduction

In this chapter electromagnetic waves propagating in free–space when no sources are present are investigated.

II Scalar wave equation

The problem of finding the electromagnetic field in a source–free, homogeneous, region is solved once general solutions to the scalar Helmholtz equation are known. These solutions are generally obtained by the method called separation of variables. It has been shown by Eisenhart, [1], that the Helmholtz equation is separable in 11 three–dimensional orthogonal coordinate systems. In the next Sections we will use the separation of variables to solve the Helmholtz equation in the rectangular, circular cylindrical and spherical coordinate systems; the reader is referred to [2] for solutions of the wave equations in other coordinate systems. Emphasis is placed on the fact that when the domain of a variable is finite we have a discrete spectrum of eigenvalues and eigenfunctions; while for an unbounded (infinite) domain the discrete spectrum coalesces into a continuum. Another feature which deserves some attention is that, depending on the chosen coordinate system, we can represent the field in our domain in several different ways.

II.1 Plane wave expansion

The Helmholtz equation in a rectangular coordinate system is [3],

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + k^2 \psi = 0 \tag{5.1}$$

The method of separation of variables seeks solutions of (5.1) in the form

$$\psi(x,y,z) = X(x)Y(y)Z(z) \tag{5.2}$$

By introducing (5.2) into (5.1) and by dividing by ψ we get

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} + k^2 = 0 \tag{5.3}$$

In (5.3) each term depends on only one coordinate. The various terms depending on the x,y,z coordinates are linked only via k^2. Thus, by separating k^2 into three constants k_x, k_y, k_z, such that

$$k^2 = k_x^2 + k_y^2 + k_z^2 \tag{5.4}$$

we obtain the three equations

$$\begin{aligned} \frac{1}{X(x)}\frac{d^2X(x)}{dx^2} &= -k_x^2 \\ \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} &= -k_y^2 \\ \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} &= -k_z^2 \end{aligned} \tag{5.5}$$

Equation (5.4) is often called the *separation equation*. The solutions of (5.5) are the harmonic functions, i.e. functions like

$$h(k_x x) \sim \sin k_x x,\ \cos k_x x,\ e^{-jk_x x},\ e^{jk_x x} \tag{5.6}$$

Hence, the solutions of the Helmholtz equation are written in terms of the harmonic functions as

$$\psi(x,y,z) = h(k_x x)h(k_y y)h(k_z z) \tag{5.7}$$

and these solutions are often referred to as *elementary wave functions*.

It is noted that the vector $\mathbf{k} = (k_x, k_y, k_z)$ indicates the propagation direction and the type of field. In particular for $|\mathbf{k}| < k_0$ we have a field which is free to propagate in space; thus a radiative field; while for $|\mathbf{k}| > k_0$ the field is attenuated, at least in one direction, thus representing a localised, reactive field.

Finally, we observe that for spatial coordinates ranging over finite domains, such as in the case of closed waveguides and cavities, in order to satisfy the boundary conditions k_x, k_y, k_z may assume only discrete values; whenever the spatial domain becomes infinite, such as in the case of antennas, open waveguides, open resonators, etc., k_x, k_y, k_z vary over a continuous range.

II.2 Cylindrical wave expansion (*)

When boundaries coincide with cylindrical coordinate surfaces, it is expedient to solve the Helmholtz equation in cylindrical coordinates. Even when no finite boundaries are present, if we have an a priori knowledge that the field has cylindrical wavefronts it is obviously advantageous to represent the field in this coordinate system. It is therefore important to investigate solutions of the Helmholtz equation in cylindrical coordinates. The latter solutions can be obtained by separation of variables [3, p.198]. The scalar Helmholtz equation in circular cylindrical coordinates, henceforth referred to as cylindrical coordinates, is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0 \tag{5.8}$$

In order to solve (5.8) by the method of separation of variables we seek solutions in the form

$$\psi = R(\rho)\Phi(\phi)Z(z) \tag{5.9}$$

which, after introduction in (5.8) and division by ψ gives

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}+k^2=0 \tag{5.10}$$

In the above equation, the first term will be seen to depend only on ρ, the second only on ϕ, and the third only on z. By separating the free–space wavenumber k^2 into its z–component, k_z^2, and a transverse part k_ρ^2 as

$$k_z^2+k_\rho^2=k^2 \tag{5.11}$$

we get, for the z–dependent coordinate, an equation similar to that obtained for the rectangular coordinate system

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2 \tag{5.12}$$

After substitution of (5.12) into (5.10) we obtain

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \left(k^2 - k_z^2\right)\rho^2 = 0 \tag{5.13}$$

where, as before, the first term depends only on ρ, and the second only on ϕ. By imposing

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -n^2 \tag{5.14}$$

with n denoting a constant, (5.13) becomes

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) - n^2 + \left(k^2 - k_z^2\right)\rho^2 = 0 \tag{5.15}$$

which depends only on ρ. Thus, by using (5.9) we have separated (5.8) into the three following equations

$$\begin{aligned}
\rho\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \left[k_\rho^2\rho^2 - n^2\right]R = 0 \qquad & (a)\\
\frac{d^2\Phi}{d\phi^2} = -n^2\Phi \qquad & (b)\\
\frac{d^2Z}{dz^2} = -k_z^2 Z \qquad & (c)
\end{aligned} \tag{5.16}$$

Equation (5.16a) is the Bessel equation whose solutions are

$$B_n(k_\rho\rho) \sim J_n(k_\rho\rho), N_n(k_\rho\rho), H_n^{(1)}(k_\rho\rho), H_n^{(2)}(k_\rho\rho)$$

with

- $J_n(k_\rho\rho)$ Bessel function of the first kind
- $N_n(k_\rho\rho)$ Bessel function of the second kind
- $H_n^{(1)}(k_\rho\rho)$ Hankel function of the first kind

- $H_n^{(2)}(k_\rho \rho)$ Hankel function of the second kind

The equations (5.16b and c) are harmonic equations that give rise to solutions of the type already seen for rectangular coordinates. But now a fundamental difference is introduced. Let us consider, for example, the free–space case, where z varies over the infinite domain $-\infty < z < \infty$, while ϕ varies over the bounded domain $0 < \phi < 2\pi$. Hence, while k_z varies over a continuum of values, n is forced to assume only discrete values. In particular if we require that $\psi(\phi) = \psi(\phi + 2\pi)$ and if we take solutions of the type $\sin(n\phi), \cos(n\phi)$, n is forced to take only integer values. Note that also ρ has unlimited support $0 \leq \rho < \infty$, thus possessing a continuous spectrum of eigenvalues k_ρ. To summarise the free–space situation, we now have eigenfunctions along z that are of the same type for rectangular and cylindrical coordinates. In the transverse plane, for rectangular coordinate system we have two sets of eigenfunctions and eigenvalues (k_x, k_y) each varying over an unlimited support; while for cylindrical coordinates only the transverse wavenumber pertains to the continuous spectrum, since the angular variation is over a bounded domain. Note also that the transverse eigenfunctions, i.e. the Bessel functions, are now dependent on the angular solution. Hence, if we are considering a wave travelling along z, we can describe its modal spectrum in terms of either (k_x, k_y) or in terms of n, k_ρ.
Guided waves in cylindrical coordinates are often described by means of the TE and TM potentials. In particular, according to the formulas (4.236b) on page 355, by setting

$$\Pi_h = \mathbf{z}_0 \psi \tag{5.17}$$

we have the following expressions for TE fields

$$\begin{aligned} E_\rho &= -j\omega\mu \frac{1}{\rho}\frac{\partial \psi}{\partial \phi} \quad \mathrm{H}_\rho = \frac{\partial^2 \psi}{\partial \rho \partial z} \\ E_\phi &= j\omega\mu \frac{\partial \psi}{\partial \rho} \quad H_\phi = \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi \partial z} \\ E_z &= 0 \quad H_z = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\psi \end{aligned} \tag{5.18}$$

In a similar manner for TM fields we set

$$\Pi_e = \mathbf{z}_0 \psi \tag{5.19}$$

thus obtaining

$$E_\rho = \frac{\partial^2 \psi}{\partial \rho \partial z} \quad H_\rho = j\omega\varepsilon \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}$$
$$E_\phi = \frac{1}{\rho}\frac{\partial^2 \psi}{\partial \phi \partial z} \quad H_\phi = -j\omega\varepsilon \frac{\partial \psi}{\partial \rho}$$
$$E_z = \left(k^2 + \frac{\partial^2}{\partial z^2}\right)\psi \quad H_z = 0 \tag{5.20}$$

II.3 Spherical wave expansion (*)

The spherical coordinate system is represented in Figure **??** and is the natural coordinate system to adopt when dealing with spherical waves, or with boundaries conformal to such geometry. Solutions of the Helmholtz equation in the spherical coordinate system may be obtained by separation of variables by setting

$$\psi = R(r)\Theta(\theta)\Phi(\phi) \tag{5.21}$$

which is used in the Helmholtz equation expressed in spherical coordinates, i.e. in

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} + k^2\psi = 0 \tag{5.22}$$

By substituting (5.21) into (5.22), dividing by ψ, and multiplying by $r^2\sin^2\theta$, we obtain

$$\frac{\sin^2\theta}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + k^2r^2\sin^2\theta = 0 \tag{5.23}$$

from which the ϕ–dependence can be separated out by requiring the third term in the above equation to be equal to a constant $(-m^2)$ as

$$\frac{d^2\Phi}{d\phi^2} + m^2\Phi = 0 \tag{5.24}$$

After use of (5.24) in (5.23) and division by $\sin^2\theta$ we can separate (5.23) into

$$\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \frac{m^2}{\sin^2\theta} = -n(n+1) \tag{5.25}$$

and

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - n(n+1) + k^2r^2 = 0 \tag{5.26}$$

To summarise we now have the set of equations (5.24, 5.25 and 5.26) which represent the separated Helmholtz equation in spherical coordinates. The solutions of (5.24) are the well–known harmonic functions; they extend over a limited support, thus corresponding to discrete eigenvalues and eigenfunctions. The solutions (5.25) are called associated Legendre functions since (5.25) is related to Legendre's equation. Such solutions are referred to as

$$L_n^m(\cos\theta) \sim P_n^m(\cos\theta), Q_n^m(\cos\theta) \tag{5.27}$$

and they also represent a discrete eigensystem. The solutions of equations (5.26) are the spherical Bessel functions since (5.26) is closely related to the Bessel equation. The various different

Table 5.1. Types of spectra arising from the solution of the free–space Helmholtz equation in different coordinate systems.

Coordinate system	space domain	spectrum
rectangular	$-\infty < x < \infty$	continuous
	$-\infty < y < \infty$	continuous
	$-\infty < z < \infty$	continuous
cylindrical	$0 \le \rho < \infty$	continuous
	$0 \le \phi < 2\pi$	discrete
	$-\infty < z < \infty$	continuous
spherical	$0 \le r < \infty$	continuous
	$0 \le \phi < 2\pi$	discrete
	$0 \le \theta < \pi$	discrete

situations arising when describing the free–space with rectangular, cylindrical and spherical coordinate systems are summarised in Table 5.1.

II.4 Connections between different wave expansions (*)

The wave expansions introduced previously are not independent, as it is possible to represent a spherical or a cylindrical wave in terms of plane waves. These representations have particular interest in practical applications. The expansion of a spherical wave in terms of plane waves is

known as the *Weyl identity* [3, p.188] and may be written as

$$\frac{e^{-jkr}}{r} = -\frac{j}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-j(k_x x+k_y y)-jk_z|z|}}{k_z}dk_x dk_y \tag{5.28}$$

where in (5.28)

$$k_z = \sqrt{k^2-k_x^2-k_y^2}\;\; r = \sqrt{x^2+y^2+z^2} \tag{5.29}$$

Another useful relationship, called the *Sommerfeld identity* [4, p.312], relates a spherical wave to the cylindrical wave expansion and it states

$$\frac{e^{-jkr}}{r} = -\frac{j}{2}\int_{-\infty}^{\infty}\frac{k_\rho}{k_z}H_0^{(2)}(k_\rho\rho)e^{-jk_z|z|}dk_\rho \tag{5.30}$$

or, in terms of Bessel functions

$$\frac{e^{-jkr}}{r} = -j\int_{0}^{\infty}\frac{k_\rho}{k_z}J_0(k_\rho\rho)e^{-jk_z|z|}dk_\rho \tag{5.31}$$

where, in (5.30) and (5.31) we have set

$$k_z = \sqrt{k^2-k_\rho^2}\,. \tag{5.32}$$

II.5 Evaluation of the Sommerfeld–Weyl–type integrals (*)

The above identities (5.28, 5.30 and 5.31) are important not only in order to express a field in different coordinate systems, but also for quick evaluation of the Sommerfeld–Weyl–type integrals. In fact, in several applications such as microstrip antennas, frequency selective surfaces, etc., one needs to evaluate this type of integral in the radiation zone or the far–field region. As we will see later, solutions for this class of problem are often obtained in the spectral domain via Fourier or Hankel transforms. Naturally, when transforming back to the real space a Sommerfeld–Weyl–type integral is involved. Such integrals become difficult to evaluate by numerical integration routines, since when the observation point is far away from the source, the integrand becomes rapidly oscillatory: in this case the saddle–point method of integration has to be used. Nonetheless, as cleverly observed in [5], by using the above identities it is possible to

bypass most of the algebra involved in the steepest–descent calculations. As an example, consider a Weyl–type integral of the form commonly encountered in microwave integrated circuits, i.e.

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(k_x,k_y)e^{-j(k_x x+k_y y)-jk_z|z|}dk_x\,dk_y \tag{5.33}$$

The integral (5.33) can be easily factorised in a form that resembles the Weyl identity; in fact we can write

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{g(k_x,k_y)k_z\right\}\left\{\frac{e^{-j(k_x x+k_y y)-jk_z|z|}}{k_z}\right\}dk_x dk_y \tag{5.34}$$

If x,y or z are large in the above integral, the integrand oscillates rapidly and contributions arise only from the neighbourhood of the point where the phase of the exponential function is stationary, i.e. when

$$\begin{aligned}\frac{\partial}{\partial k_x}(k_x x+k_y y+k_z|z|) &= 0\\ \frac{\partial}{\partial k_y}(k_x x+k_y y+k_z|z|) &= 0\end{aligned} \tag{5.35}$$

which may be solved giving

$$\begin{aligned}\frac{x}{k_x} &= \frac{z}{k_z} = \alpha\\ \frac{y}{k_y} &= \frac{z}{k_z} = \alpha\,.\end{aligned} \tag{5.36}$$

The parameter α has been introduced to denote the ratio between the spatial and spectral coordinates. Therefore we have

$$\begin{aligned}r^2 = x^2+y^2+z^2 &= \alpha^2\left(k_x^2+k_y^2+k_z^2\right)\\ \alpha &= \frac{r}{k}\,.\end{aligned} \tag{5.37}$$

The stationary phase point are given by

$$\begin{aligned}k_{x0} &= \frac{k}{r}x\\ k_{y0} &= \frac{k}{r}y\,.\end{aligned} \tag{5.38}$$

The physical interpretation of (5.38) is that only the plane wave whose **k**–vector points from the source to the observation point is important when evaluating the far field at the observation point. By replacing the slowly varying part of the integrand with its value at the points of stationary phase we obtain

$$I \approx \left\{ g(k_{x0}, k_{y0}) k_{z0} \right\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \frac{e^{-j(k_x x + k_y y) - jk_z|z|}}{k_z} \right\} dk_x dk_y \tag{5.39}$$

which after use of the Weyl identity provides the sought approximation

$$I \approx j2\pi g(k_{x0}, k_{y0}) k_{z0} \frac{e^{-jkr}}{r} \tag{5.40}$$

The location of the stationary phase point can be found by (5.38), which is easily remembered through its physical interpretation. In fact a source radiates plane waves in all directions, with wavenumbers given by $k_x^2 + k_y^2 + k_z^2 = k^2$. However only the wave which satisfies the condition

$$\frac{k_x}{x} = \frac{k_y}{y} = \frac{k_z}{z} = \frac{k}{r} \tag{5.41}$$

reaches the observation point; it is then apparent that the evaluation of (5.33) is achieved by simple inspection. A similar argument is also applicable to the Sommerfeld integral [5].

III The Sturm-Liouville Equation (*)

III.1 Source–Free Solutions: Eigenvalue Problem

Formulation

We consider the reduced one-dimensional differential equation of the generic form

$$L_\alpha(u) f_\alpha(u) = 0, \quad L_\alpha(u) \equiv \left[\frac{d}{du} \left(p(u) \frac{d}{du} \right) - q(u) + \lambda_\alpha w(u) \right], \tag{5.42}$$

where p, q and the weight function w are positive real functions of u, $f_\alpha(u)$ is the wave function, and λ_α is the separation parameter. Equation (5.42) is a homogeneous (source–free) *Sturm–Liouville* (SL) problem, and $L(u)$ is the Sturm-Liouville operator, defined, in general, for arbi-

trary λ, as

$$L(u,\lambda) \equiv \left[\frac{d}{du}\left(p(u)\frac{d}{du}\right) - q(u) + \lambda w(u)\right], \tag{5.43}$$

(5.42) is to be solved on the interval $u_1 \le u \le u_2$, subject to the linear homogeneous boundary conditions at the end points u_1 and u_2,

$$f_\alpha(u_{1,2}) + \gamma_{1,2}\left.\frac{df_\alpha}{du}\right|_{u_{1,2}} = 0, \tag{5.44}$$

where $\gamma_{1,2}$ are constants. A solution $f_\alpha(u)$ is called an *eigenfunction*, and the constant λ_α associated with $f_\alpha(u)$ is the corresponding *eigenvalue*. In general there will be a set of eigenfunction-eigenvalue pairs $\{(f_\alpha,\lambda_\alpha)\}$ which satisfy (5.42) and the boundary conditions in (5.44). Note that here and in the mathematical sections that follow, the *spectral* parameter λ in (5.43) plays a general role which, in the context of the wave equation, is equivalent to the squared wavenumber k^2. For the *eigenvalue* problem in (5.42), $\lambda \to \lambda_\alpha$ and $L(u,\lambda) \to L_\alpha(u)$.

Adjointness properties

Before proceeding further, we demonstrate that the SL operator $L(u)$ in (5.43) is *self-adjoint*; i.e., subject to the boundary conditions in (5.44), with f_α replaced by $F(u)$, $L(u)$ satisfies the property (suppressing the λ dependance)

$$\left\langle \hat{F}, LF \right\rangle \equiv \int_{u_1}^{u_2} \hat{F} LF \, du = \int_{u_1}^{u_2} F L\hat{F} \, du \equiv \left\langle F, L\hat{F} \right\rangle \tag{5.45}$$

(5.45) states that in the domain $u_1 \le u \le u_2$ of the operator $L(u)$ with the boundary conditions as in (5.44), the "inner product" $\langle \hat{F}, LF\rangle$ as defined on the left-hand side of (5.45) is equal to $\langle F, L\hat{F}\rangle$ on the right-hand side. Thus, the L-operation in the inner product is commutative. To prove (5.45), we construct

$$F L\hat{F} = F\left(\frac{d}{du}p\frac{d}{du} - q + \lambda w\right)\hat{F} \tag{5.46}$$

$$\hat{F} LF = \hat{F}\left(\frac{d}{du}p\frac{d}{du} - q + \lambda w\right)F, \tag{5.47}$$

whence

$$\begin{aligned} F L\hat{F} - \hat{F} L F &= F\frac{d}{du}p\frac{d\hat{F}}{du} - \hat{F}\frac{d}{du}p\frac{dF}{du} \\ &= \frac{d}{du}[p(F\hat{F}' - \hat{F}F')], \end{aligned} \tag{5.48}$$

Here and hereafter, $F(u)$ and $\hat{F}(u)$ are two different twice–differentiable functions of u with a prime denoting the derivative with respect to u. Integrating both sides of (5.48) between the limits u_1 and u_2 yields

$$\int_{u_1}^{u_2} (FL\hat{F} - \hat{F}LF)du = [p(F\hat{F}' - \hat{F}F')]_{u_1}^{u_2}, \tag{5.49}$$

The bracketed term on the right-hand side of (5.49),

$$W(\hat{F}, F) \equiv p(F\hat{F}' - \hat{F}F') = p\det\begin{vmatrix} F & F' \\ \hat{F} & \hat{F}' \end{vmatrix}, \tag{5.50}$$

is the λ–dependent *Wronskian* which plays an important role in the theory that follows (see (5.95) - (5.105)). Subject to the boundary conditions in (5.44), the Wronskian vanishes, thereby establishing (5.45). Vanishing of the Wronskian at the boundary is confirmed by noting that, in view of (5.44),

$$F_1' = -\frac{F_1}{\gamma_1}\ ,\ \hat{F}_1' = -\frac{\hat{F}_1}{\gamma_1}, \tag{5.51}$$

where $F_1 \equiv F(u_1)$, $\hat{F}_1 \equiv \hat{F}(u_1)$. The same holds for u_2.

Orthogonality, completeness relation, and eigenfunction expansions

In view of the adjointness property in (5.45), the eigenfunctions f_α satisfy an orthogonality property which can be derived as follows. (5.42) is written for an eigenfunction-eigenvalue pair $(f_\alpha, \lambda_\alpha)$ and for a different eigenfunction-eigenvalue pair (f_β, λ_β). Proceeding as in (5.46) to (5.48), the f_α–equation is multiplied by f_β^*, where the asterisk denotes the complex conjugate, and the complex conjugate of the f_β–equation is multiplied by f_α. The resulting equations are subtracted to obtain

$$\frac{d}{du}W(f_\alpha, f_\beta^*) + (\lambda_\alpha - \lambda_\beta^*)w f_\alpha f_\beta^* = 0. \tag{5.52}$$

(5.52) is now integrated with respect to u between u_1 and u_2 to give

$$(\lambda_\alpha - \lambda_\beta^*)\int_{u_1}^{u_2} w f_\alpha f_\beta^*\, du = 0, \tag{5.53}$$

since the endpoint contribution vanishes via (5.51). Therefore it follows that

$$\int_{u_1}^{u_2} f_\alpha f_\beta^* w\, du = 0, \quad \alpha \neq \beta \tag{5.54}$$

If $\lambda_\beta = \lambda_\alpha$, then from (5.53),

$$(\lambda_\alpha - \lambda_\alpha^*)\int_{u_1}^{u_2} |f_\alpha|^2 w\, du = 0. \tag{5.55}$$

Since w is positive and the trivial eigenfunction $f_\alpha = 0$ is not considered, the integral is nonzero. Thus,

$$\lambda_\alpha = \lambda_\alpha^*, \text{ i.e., } \textit{the eigenvalues are real} \tag{5.56}$$

Returning to (5.54), these considerations imply that the integral vanishes for $\lambda_\alpha \neq \lambda_\beta$. Since the integral in (5.54) represents, in the function space, the inner product of the functions f_α and f_β^*, (see the comments following (5.45)), vanishing of the integral implies that eigenfunctions corresponding to distinct eigenvalues are *orthogonal* with respect to the weighting function w. It is convenient to normalize the eigenfunctions (multiply by an appropriate constant) so that

$$\int_{u_1}^{u_2} |f_\alpha|^2 w\, du = 1. \tag{5.57}$$

This renders the set $\{f_\alpha\}$ orthonormal. (5.54) and (5.57) can then be written as the single expression

$$\int_{u_1}^{u_2} f_\alpha f_\beta^* w\, du = \delta_{\alpha\beta}, \tag{5.58}$$

with the Kronecker delta $\delta_{\alpha\beta}$ defined as $\delta_{\alpha\beta} = 0$ for $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ for $\alpha = \beta$.

Assuming that the eigenfunction set $\{f_\alpha(u)\}$ is complete, any "representable" function $F(u)$ can be expanded formally as

$$F(u) = \sum_\alpha A_\alpha f_\alpha(u); \tag{5.59}$$

here, "representable" implies that the expansion converges. Multiplying both sides of (5.59) by $w(u)f_\beta^*(u)$, integrating over the (u_1, u_2) interval, invoking the orthonormality condition given

by (5.58) and switching back to the index α, it follows that

$$A_\alpha = \int_{u_1}^{u_2} f_\alpha^* F w \, du. \tag{5.60}$$

Substitution of (5.60) into (5.59) gives, upon interchange of the orders of summation and integration,

$$F(u) = \int_{u_1}^{u_2} du' \, \{w(u') \sum_\alpha f_\alpha(u) f_\alpha^*(u')\} F(u'). \tag{5.61}$$

(5.61) implies that

$$w(u') \sum_\alpha f_\alpha(u) f_\alpha^*(u') = \delta(u-u'), \tag{5.62}$$

or

$$\frac{\delta(u-u')}{w(u')} = \sum_\alpha f_\alpha(u) f_\alpha^*(u'). \tag{5.63}$$

(5.63) expresses the *completeness statement* in compact symbolic form. The expansion of the weighted delta function in terms of the eigenfunctions implies that the set of eigenfunctions is complete, because any function $F(u)$ can be expressed by using the delta function property

$$F(u) = \int_{u_1}^{u_2} F(u') \delta(u-u') du' \tag{5.64}$$

Thus, to *apply* (5.63), the previous steps are *reversed* as follows. Each side of (5.63) is multiplied by $w(u')F(u')$ and integrated with respect to the variable u' from u_1 to u_2, giving

$$F(u) = \sum_\alpha f_\alpha(u) \int_{u_1}^{u_2} w(u') F(u') f_\alpha^*(u') du'. \tag{5.65}$$

(5.65) is of the form

$$F(u) = \sum_\alpha A_\alpha f_\alpha(u), \tag{5.66}$$

with the coefficients A_α given by

$$A_\alpha = \int_{u_1}^{u_2} w(u') F(u') f_\alpha^*(u') du'. \tag{5.67}$$

The implied orthonormality of the eigenfunctions is verified by setting $F(u)$ in (5.65) equal to the eigenfunction $f_\beta(u)$, giving

$$f_\beta(u) = \sum_\alpha f_\alpha(u) \int_{u_1}^{u_2} w(u') f_\beta(u') f_\alpha^*(u') du'. \tag{5.68}$$

To satisfy (5.68) one is led to (5.58).

Large $|\lambda|$ behavior of the source-free solutions

The source-free solutions $f(u)$ of the SL equation (see (5.42)) reduce to trigonometric functions for large values of λ, and when $w = p$. To demonstrate this behavior, we reduce the $L(u)$ operator to its normal form (without the first derivative d/du) by the transformation

$$f(u) = p^{-1/2} \hat{f}(u) \tag{5.69}$$

which changes $L(u)f(u) = 0$ to the normalized equation

$$\left[\frac{d^2}{du^2} + h(u) \right] \hat{f}(u) = 0 \tag{5.70}$$

where

$$h(u) = \frac{\lambda w}{p} - \frac{q}{p} - p^{-1/2} \frac{d^2}{du^2} p^{1/2} \tag{5.71}$$

For large λ, the $(\lambda w/p)$ term dominates, and when $w = p$, (5.70) reduces to

$$\left(\frac{d^2}{du^2} + \lambda \right) \hat{f}(u) \sim 0, \quad |\lambda| \gg 1, \quad w = p \tag{5.72}$$

Thus, the large-$|\lambda|$ solutions for $f(u)$ become

$$f(u) \sim p^{-1/2} \cdot (\sin \sqrt{\lambda} u, \cos \sqrt{\lambda} u, e^{\mp j\sqrt{\lambda} u}), \quad |\lambda| \gg 1, \quad w = p \tag{5.73}$$

For the eigenvalue problem, (5.73) applies with $f(u) \to f_\alpha(u)$, $\lambda \to \lambda_\alpha$, $\lambda_\alpha \gg 1$. For the Green's function problem in Sec. III.2, (5.73) applies to the synthesizing homogeneous solutions $\overleftarrow{f(u)}$ and $\overrightarrow{f(u)}$.

III.2 Source-Driven Solutions: Green's Function Problem

Properties of the Green's Function

The eigenvalue problem defined by (5.42) describes a one-dimensional physical system which is free or unforced. Problems in which forcing functions or sources exist are solved through the introduction of a Green's function. The one-dimensional Green's function $g(u,u';\lambda)$ satisfies the equation

$$L(u)g(u,u';\lambda) \equiv \left[\frac{d}{du}p(u)\frac{d}{du} - q(u) + \lambda w(u)\right] g(u,u';\lambda) = -\delta(u-u') \tag{5.74}$$

over the interval $u_1 \leq (u,u') \leq u_2$, with boundary conditions at $u = u_{1,2}$ of the form

$$g(u_{1,2}) + \gamma_{1,2}\left.\frac{dg}{du}\right|_{u_{1,2}} = 0. \tag{5.75}$$

The right-hand side of (5.74) represents a u-domain point source at location $u = u'$. Here, $L(u)$ is the general SL operator in (5.43), which is self-adjoint subject to the boundary conditions in (5.44). The parameter λ is now unrestricted and may range over the entire complex λ-plane, provided that $\lambda \neq \lambda_\alpha$. All eigenvalues $\lambda = \lambda_\alpha$ must be avoided because the source-free (5.74) has the eigensolutions $f_\alpha(u)$. Any eigensolution can be added to g and still satisfy (5.74) and (5.75), thereby rendering the resulting g nonunique.

Reciprocity

The Green's function $g(u,u';\lambda)$ is symmetric in its dependence on u and u'. This can be shown by referring to (5.49), with $F = g(u,u';\lambda)$ and $\hat{F} = g(u,u'';\lambda)$, where u' and u'' are source points in the interval $u_1 < (u',u'') < u_2$. Thus (omitting the λ-dependence),

$$\begin{aligned}\int_{u_1}^{u_2} &[g(u,u')L(u)g(u,u'') - g(u,u'')L(u)g(u,u')]du \\ &= \{p(u)[g(u,u')g'(u,u'') - g(u,u'')g'(u,u')]\}_{u_1}^{u_2}.\end{aligned} \tag{5.76}$$

Since $L(u)g(u,\overline{u}) = -\delta(u-\overline{u})$ and $g'(u_{1,2},\overline{u}) = -\gamma_{1,2}^{-1}g(u_{1,2},\overline{u})$ (see (5.74) and (5.75)), the endpoint contribution vanishes (self-adjointness property) and the integral is reduced via the delta functions, yielding the result

$$g(u'',u';\lambda) = g(u',u'';\lambda). \tag{5.77}$$

Thus, the self-adjoint SL Green's function $g(u,u';\lambda)$ is unchanged, i.e. *reciprocal* when u and u' are interchanged at any two locations in the interval (u_1,u_2).

Synthesis of the General Initial-Boundary Value Problem

The general SL initial-boundary value problem is of the form

$$L(u)F(u) = S(u), \quad u_1 \le u \le u_2, \tag{5.78}$$

subject to the initial-boundary condition

$$F(u_{1,2}) + \gamma_{1,2}F'(u_{1,2}) = \overline{S}(u_{1,2}), \tag{5.79}$$

where $S(u)$ are interior sources while $\overline{S}(u_{1,2})$ are sources impressed at the boundaries of the domain. The solution for $F(u)$ can be synthesized in terms of the Green's function $g(u,u';\lambda)$ defined in (5.74) together with (5.75). Returning to the adjointness relation in (5.48), let $\hat{F} = g(u,u';\lambda)$ and let $F(u)$ represent the solution of (5.78) and (5.79). Thus, omitting the λ-dependence,

$$\begin{aligned}\int_{u_1}^{u_2} &[F(u)L(u)g(u,u') - g(u,u')L(u)F(u)]du \\ &= p(u_2)[F(u_2)g'(u_2,u') - g(u_2,u')F'(u_2)] \\ &-p(u_1)[F(u_1)g'(u_1,u') - g(u_1,u')F'(u_1)].\end{aligned} \tag{5.80}$$

Inside the integral in (5.80), referring to (5.74) and (5.78), Lg and LF are replaced by $-\delta(u-u')$ and $S(u)$, respectively. On the right-hand side of (5.80), referring to (5.75) and (5.79), we use

$$g'(u_{1,2}) = -\frac{g(u_{1,2})}{\gamma_{1,2}}, \quad F'(u_{1,2}) = \frac{\overline{S}(u_{1,2}) - F(u_{1,2})}{\gamma_{1,2}}. \tag{5.81}$$

This reduces (5.80) to the expression

$$F(u') = -\int_{u_1}^{u_2} g(u,u')S(u)du + p(u_2)g(u_2,u')\overline{S}(u_2)/\gamma_2 - p(u_1)g(u_1,u')\overline{S}(u_1)/\gamma_1, \tag{5.82}$$

where u' is any point in the closed interval $u_1 \leq u' \leq u_2$.

Since u and u' in $g(u,u';\lambda)$ represent the field (observation) point and source point, respectively, it is customary to integrate the Green's function over the primed coordinates. The necessary interchange of u and u' can be implemented in view of the reciprocity property in (5.77) in the form (restoring the λ-dependence)

$$F(u,\lambda) = -\int_{u_1}^{u_2} g(u,u';\lambda)S(u')du' + p(u_2)g(u,u_2;\lambda)\overline{S}(u_2)/\gamma_2 - p(u_1)g(u,u_1;\lambda)\overline{S}(u_1)/\gamma_1. \tag{5.83}$$

Solution for the Green's Function

The Green's function $g(u,u';\lambda)$ can be evaluated directly. When $u \neq u'$, the Green's function satisfies the homogeneous equation obtained by setting the right-hand side of (5.74) equal to zero. Let $\overleftarrow{f}$ be a solution of the homogeneous equation which satisfies the boundary condition given by (5.75) at $u = u_1$, and let $\overrightarrow{f}$ be a solution of the homogeneous equation which satisfies the boundary condition given by (5.75) at $u = u_2$. The functions $\overleftarrow{f}$ and $\overrightarrow{f}$ can be constructed by superposition of any two linearly independent solutions $f^{(1)}$ and $f^{(2)}$ of the homogeneous (5.42) using the expressions

$$\overrightarrow{f}(u) = f^{(1)}(u) + \overrightarrow{\Gamma} f^{(2)}(u) \tag{5.84}$$

$$\overleftarrow{f}(u) = \overleftarrow{\Gamma} f^{(1)}(u) + f^{(2)}(u), \tag{5.85}$$

where

$$\overrightarrow{\Gamma} = -\frac{\left[f^{(1)}(u_2) + \gamma_2\left(\frac{df^{(1)}}{du}\right)_{u=u_2}\right]}{\left[f^{(2)}(u_2) + \gamma_2\left(\frac{df^{(2)}}{du}\right)_{u=u_2}\right]} \tag{5.86}$$

$$\overleftarrow{\Gamma} = -\frac{\left[f^{(2)}(u_1) + \gamma_1 \left(\frac{df^{(2)}}{du}\right)_{u=u_1}\right]}{\left[f^{(1)}(u_1) + \gamma_1 \left(\frac{df^{(1)}}{du}\right)_{u=u_1}\right]}. \tag{5.87}$$

To obtain the expression for $\overrightarrow{\Gamma}$, note from (5.75) that

$$\overrightarrow{f} = f^{(1)} + \overrightarrow{\Gamma} f^{(2)} = -\gamma_2 \frac{d\overrightarrow{f}}{du} = -\gamma_2 \left[\frac{df^{(1)}}{du} + \overrightarrow{\Gamma}\frac{df^{(2)}}{du}\right], \quad u = u_2. \tag{5.88}$$

The second equality follows from (5.44) applied to $\overrightarrow{f}$, whereas the third equality implements $d\overrightarrow{f}/du$ via (5.84). Solving the first and third equalities for $\overrightarrow{\Gamma}$ yields (5.86). A similar calculation gives the expression for $\overleftarrow{\Gamma}$ in (5.87).

Next, it is noted that g is continuous at $u = u'$ but has a discontinuous slope (first derivative) at $u = u'$, consistent with the recognition that the delta function singularity at $u = u'$ in (5.74) is generated by the highest derivative, (d^2g/du^2). Implementing continuity at u', with discontinuous slope, suggests the expression

$$g(u,u';\lambda) = \begin{cases} C\overleftarrow{f}(u)\overrightarrow{f}(u'), u < u' \\ C\overleftarrow{f}(u')\overrightarrow{f}(u), u > u' \end{cases} \tag{5.89}$$

which also satisfies both prescribed boundary conditions, as well as (5.74) for all $u \neq u'$. With the notation

$$u_> = \begin{cases} u, \ u > u' \\ u', u < u' \end{cases} \tag{5.90}$$

$$u_< = \begin{cases} u, \ u < u' \\ u', u > u', \end{cases} \tag{5.91}$$

(5.89) can be written as

$$g(u,u';\lambda) = C\overrightarrow{f}(u_>)\overleftarrow{f}(u_<). \tag{5.92}$$

To determine the constant C we integrate (5.74) over the interval $u' - \varepsilon < u < u' + \varepsilon$, $\varepsilon > 0$, and then allow $\varepsilon \to 0$. Since g is bounded at u' and q, w and p have no singularities at $u = u'$, the

contribution from the second and third terms in $L(u)$ vanishes in the limit. The result is

$$p\,\frac{dg}{du}\bigg|_{u'-\varepsilon}^{u'+\varepsilon} = -1, \tag{5.93}$$

which after using (5.92) gives

$$C = -\frac{1}{W(\overrightarrow{f},\overleftarrow{f})}, \tag{5.94}$$

with the Wronskian $W(\overrightarrow{f},\overleftarrow{f})$ defined as in (5.50),

$$W(\overrightarrow{f},\overleftarrow{f}) = p(u')\left[\overleftarrow{f}\,\frac{d\,\overrightarrow{f}}{du} - \overrightarrow{f}\,\frac{d\,\overleftarrow{f}}{du}\right]_{u=u'}. \tag{5.95}$$

Using (5.92) and (5.94), the Green's function $g(u,u',\lambda)$ can now be written as

$$g(u,u';\lambda) = -\frac{\overrightarrow{f}\,(u_>)\,\overleftarrow{f}\,(u_<)}{W(\overrightarrow{f},\overleftarrow{f})}. \tag{5.96}$$

The Wronskian $W(\overrightarrow{f},\overleftarrow{f})$ has the following properties (recall that the λ–dependence has been suppressed throughout):

1. W is a λ–dependent constant, independent of u'.
2. $W \neq 0$ if $\overrightarrow{f}$ and $\overleftarrow{f}$ are linearly independent functions over the interval $u_1 < u < u_2$.

To show that W is independent of u', the equation

$$\left[\frac{d}{du}p\frac{d}{du} - q + \lambda w\right]\overrightarrow{f} = 0 \tag{5.97}$$

is multiplied by $\overleftarrow{f}$, and the equation

$$\left[\frac{d}{du}p\frac{d}{du} - q + \lambda w\right]\overleftarrow{f} = 0 \tag{5.98}$$

is multiplied by $\overrightarrow{f}$. The resulting equations are subtracted to give

$$\overleftarrow{f}\,\frac{d}{du}p\frac{d\,\overrightarrow{f}}{du} - \overrightarrow{f}\,\frac{d}{du}p\frac{d\,\overleftarrow{f}}{du} = 0, \tag{5.99}$$

which is equivalent to

$$\frac{d}{du}\left[p\left(\overleftarrow{f}\frac{d\overrightarrow{f}}{du}-\overrightarrow{f}\frac{d\overleftarrow{f}}{du}\right)\right]=0, \tag{5.100}$$

or

$$\frac{d}{du}\left(W(\overrightarrow{f},\overleftarrow{f})\right)=0. \tag{5.101}$$

Equation (5.101) states that W is independent of u, i.e., W equals a λ–dependent constant. To show that W is nonzero if $\overrightarrow{f}$ and $\overleftarrow{f}$ are linearly independent, it will be shown conversely that $W=0$ implies linear dependence, i.e., that $\overrightarrow{f}$ is then a constant multiple of $\overleftarrow{f}$. If $W=0$, (5.95) gives

$$\overleftarrow{f}\frac{d\overrightarrow{f}}{du}=\overrightarrow{f}\frac{d\overleftarrow{f}}{du} \tag{5.102}$$

or

$$\frac{1}{\overleftarrow{f}}\frac{d\overleftarrow{f}}{du}=\frac{1}{\overrightarrow{f}}\frac{d\overleftarrow{f}}{du}. \tag{5.103}$$

Integration of (5.103) gives

$$\ln\frac{\overleftarrow{f}}{\overrightarrow{f}}=c=\text{const.} \tag{5.104}$$

or

$$\overleftarrow{f}=c'\,\overrightarrow{f} \tag{5.105}$$

which confirms that $W=0$ implies linear dependence of $\overleftarrow{f}$ and $\overrightarrow{f}$. (5.105) implies furthermore that $\overrightarrow{f}$ or $\overleftarrow{f}$ satisfy *both* boundary conditions at u_1 and u_2, in addition to satisfying the source-free (5.74); i.e., $\overrightarrow{f}$ or $\overleftarrow{f}$ are eigensolutions $f_\alpha(u)$ with forbidden eigenvalues $\lambda=\lambda_\alpha$. This is in accord with the result in (5.51). Evidently, the solution for g in (5.96) becomes invalid when $W=0$.

Large $|\lambda|$ Behavior of the Spectral Green's Function

In the investigation that follows, emphasis will be placed on the behavior of the Sturm–Liouville Green's function *throughout* the complex spectral $|\lambda|$–plane. For large values of λ, and when $w=p$, the synthesizing homogeneous solutions $\overrightarrow{f}$ and $\overleftarrow{f}$ in (5.96) reduce to trigonometric functions, as shown in Sec. III.1, (5.73). The formal solution for $g(u,u';\lambda)$ in (5.96) reduces

accordingly in the large-λ range. Consider the case where $g = 0$ at $u_1 = 0$ (no loss of generality) and at u_2. The synthesizing solutions of (5.72) are $\overleftarrow{f}\ (u_<) = \sin\left(\sqrt{\lambda}u_<\right)$, $\overrightarrow{f}\ (u_>) = \sin\left[\sqrt{\lambda}(u_2 - u_>)\right]$, whereas the u–independent Wronskian is given by $w = \sqrt{\lambda}\sin\left(\sqrt{\lambda}u_2\right)$. For $|\ \lambda\ | \gg 1$, $|\ \mathrm{Im}\lambda\ | \neq 0$, retaining only the dominant (growing) exponentials, one obtains

$$g(u,u';\lambda) \to \frac{e^{|\mathrm{Im}\ \sqrt{\lambda}|u_<} e^{|\mathrm{Im}\ \sqrt{\lambda}|(u_2-u_>)}}{\sqrt{\lambda}e^{|\mathrm{Im}\sqrt{\lambda}|u_2}} \to \frac{e^{|\mathrm{Im}\ \sqrt{\lambda}|(u_<-u_>)}}{\sqrt{\lambda}}$$
$$\frac{e^{-|\mathrm{Im}\ \sqrt{\lambda}|\ |u-u'|}}{\sqrt{\lambda}}, \qquad |\mathrm{Im}\ \sqrt{\lambda}| \neq 0. \tag{5.106}$$

which decays exponentially at infinity in the complex λ–plane, and therefore yields no contribution when integrated over a circular contour at $|\ \lambda\ | \to \infty$. This feature becomes relevant in the sections that follow (see Sec. **??**).

IV Examples of Sturm-Liouville Equations (*)

For a number of coordinate systems the Lapalce equation, the Helmholtz equation and the wave equation may be solved exactly by separation into ordinary differential equations of the Sturm-Liouville type. The solutions of these ordinary differential equations leads to special functions; here we summarize the case for the cartesian, circular cylindrical and spherical coordinate systems and some important formulae are introduced. For a detailed presentation of the mathematical background see for example [6,7]. A comprehensive presentation of the coordinate systems for which the partial differential equations mentioned above may be solved exactly and the methods of solution is given in [**?**]. Comprehensive collections of formulae and theorems for the special functions of mathematical physics are [**?**, 8].

IV.1 Harmonic Equation

Let us consider the equation

$$\frac{d^2\phi}{dx^2} + k_x^2\phi = 0 \tag{5.107}$$

with the boundary conditions at $x = 0, a$ requiring the vanishing of the function. The solution of the above eigenvalue problem is given by

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \tag{5.108}$$

The normalization constant in (6.48) has been chosen so that the eigenfunction set is normalized to unity; moreover the orthogonality is readily proved so that we have

$$\int_0^a \phi_m(x)\phi_n(x)dx = \delta_{mn} = \left\{ \begin{array}{l} 1 \; m = n \\ 0 \; m \neq n \end{array} \right\} \tag{5.109}$$

Delta function representation

In order to ascertain formally the completeness of the eigenfunction set we expand the delta function in terms of the above eigenfunctions as

$$\delta(x-x') = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

By multiplying both sides by $\phi_m(x)$ and by integrating over the interval $[0,a]$ we obtain

$$f_m = \int_0^a \delta(x-x')\phi_m(x)dx = \phi_m(x')$$

which inserted in the delta representation gives

$$\delta(x-x') = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(x') \tag{5.110}$$

i.e. a concise expression of completeness and orthonormality.

Green's function representation

So far we have considered a differential equation without sources. Now, let's consider the Green's function:

$$\frac{d^2}{dx^2}G(x,x') + k_x^2 G(x,x') = -\delta(x-x') \tag{5.111}$$

in which we have $G(x,x') = G(x-x')$.

We now expand the Green's function on the base of the eigenfunctions:

$$G(x,x') = \sum_{n=1}^{\infty} g_n \phi_n(x) \tag{5.112}$$

Moreover, we remember that

$$\frac{d^2}{dx^2}\phi_n(x) = -k_{x_n}^2 \phi_n(x) \tag{5.113}$$

Thus, we can write

$$\sum_{n=1}^{\infty} \left(k_x^2 - k_{x_n}^2\right) g_n \phi_n(x) = -\sum_{n=1}^{\infty} \phi_n(x)\phi_m(x') \tag{5.114}$$

Our aim is to find the g_n coefficients and to do this we can match the parts of the expression (5.114) that are different in both members:

$$\left(k_x^2 - k_{x_n}^2\right) g_n = \phi_m(x') \tag{5.115}$$

From this expression we obtain:

$$g_n = -\frac{\phi_m(x')}{k_x^2 - k_{x_n}^2} = \frac{\phi_m(x')}{k_{x_n}^2 - k_x^2} \tag{5.116}$$

In this way we can write the Green's function, for a finite domain, as follow:

$$G(x,x') = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_m(x')}{k_{x_n}^2 - k_x^2} \tag{5.117}$$

Green's function direct expansion

The Green's function is the response to the delta function and it has to be a continuous function. Let's consider the Green's function in the following way:

- G^+ = part of Green's function for $x' < x < a$;
- G^- = part of Green's function for $0 < x < x'$.

Now, we can define both G^+ and G^- so that they satisfy the homogeneous equation and the boundary conditions:

$$\begin{aligned} G^+ &= C^+ \sin\left[k_x(x-a)\right], x > x' \\ G^- &= C^- \sin\left(k_x x\right), x < x' \end{aligned} \tag{5.118}$$

In $x = x'$ it has to be $G^+ = G^-$:

$$C^+ \sin\left[k_x(x'-a)\right] = C^- \sin\left(k_x x'\right) \tag{5.119}$$

Now we chose C^+ and C^- in the following way, in order to satisfy the above expression:

$$\begin{aligned} C^+ &= \sin\left(k_x x'\right), x > x' \\ C^- &= \sin\left[k_x(x'-a)\right], x < x' \end{aligned} \tag{5.120}$$

So we can write:

$$\begin{aligned} G^+ &= \sin\left(k_x x'\right)\sin\left[k_x(x-a)\right], a \geq x \geq x' \\ G^- &= \sin\left[k_x(x'-a)\right]\sin\left(k_x x\right), 0 \leq x \leq x' \end{aligned} \tag{5.121}$$

Infinite domain

When the domain becomes infinite the set of discrete eigenvalues

$$k_n = \frac{n\pi}{a}$$

become a continuous range. By introducing

$$\Delta k_n = k_{n+1} - k_n = \frac{\pi}{a}$$

and rewriting (6.50) as

$$\delta(x-x') = \frac{2}{\pi}\sum_{k_n=\Delta k_n}^{\infty} \sin(k_n x)\sin\left(k_n x'\right)\Delta k_n$$

while performing the limit for $\Delta k_n \to 0$, we obtain the completeness relationship for the open region as

$$\delta(x-x') = \frac{2}{\pi}\int_0^{\infty} \sin(kx)\sin\left(kx'\right)dk \tag{5.122}$$

It is also apparent that the continuous eigenfunction is given by

$$\phi(x;k) = \sqrt{\frac{2}{\pi}}\sin(kx) \tag{5.123}$$

Note that the eigenfunctions appearing in (6.50) have infinite energy on the interval $[0,\infty)$; for this reason they are sometimes referred to as improper [9]. Note also that, while for finite domain the eigenfunctions have finite energy, in the infinite domain case only the integral of the energy is finite. The orthogonality between eigenfunctions may be obtained by noting that in (6.51) after replacing

$$x \to k; x' \to k'; k \to x$$

we obtain the orthogonality relationship

$$\delta(k-k') = \frac{2}{\pi}\int_0^\infty \sin(kx)\sin\left(k'x\right)dx \tag{5.124}$$

Table 5.2 compares some examples of eigenfunctions, completeness and orthogonality relationships for discrete and corresponding continuous cases. From Table 5.2, the following observations are in order:

Table 5.2. Summary of the completeness and orthogonality relationship for the discrete and continuous eigenvalue cases.

	discrete case	**continuous case**
eigenfunction	$\phi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$	$\phi(x;k) = \sqrt{\frac{2}{\pi}}\sin(kx)$
completeness (δ representation in the space domain) $\delta(x-x') =$	$\sum_{n=1}^{\infty}\phi_n(x)\phi_n(x')$	$\int_0^\infty \phi(x;k)\phi(x';k)dk$
orthogonality (δ representation in the spectral domain)	$\delta_{mn} = \left\{\begin{matrix} 1\ m=n \\ 0\ m\neq n \end{matrix}\right\} =$ $\int_0^a \phi_n(x)\phi_m(x)dx$	$\delta(k-k') =$ $\int_0^\infty \phi(x;k)\phi(x;k')dx$

1. in both cases, discrete and continuous, the eigenfunction set provides a basis which is both orthogonal and complete;

2. the eigenfunctions of the continuous spectrum satisfy the boundary condition at $x = 0$, but only finiteness is required at infinity;
3. from a physical point of view the continuous eigenfunction corresponds to a field that originates from a generator placed at infinity;
4. as such, the continuous eigenfunctions taken individually do not satisfy the radiation condition, i.e. that the field vanish at infinity; the only condition satisfied is that they remain finite at infinity.

However, on physical grounds it is quite obvious that the total field, which is given by a superposition of eigenfunctions each with its own amplitude, should vanish at infinity, thus satisfying the radiation condition. This is not in contrast to the above point 4 stating that each eigenfunction has a finite amplitude at infinity. It is only their sum, i.e. the wave packet formed by the superposition of all eigenfunctions, which vanishes at infinity.
When the interval is unbounded in both directions, e.g. by removing both plates, the delta representation over $(-\infty, \infty)$ is given by

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk(x-\xi)}dk \tag{5.125}$$

Consequently, the eigenfunctions, representing a plane wave spectrum, are:

$$\phi(x,k) = \frac{e^{-jkx}}{\sqrt{2\pi}} \tag{5.126}$$

Direct calculation of the Green's function

In several cases, the Green's function can be calculated almost immediately, at least for one dimensional cases, and without any previous knowledge of the eigensystem. The procedure for directly calculating the Green's function is here illustrated by an example [9, p.160]. Let us consider an operator:

$$L = -\frac{d^2}{dx^2} \tag{5.127}$$

with the boundary conditions

$$u(0) = 0 \;\; u(1) = 0 \tag{5.128}$$

In order to find the Green's function we must solve

$$\frac{d^2G(x,x')}{dx^2} = -\delta(x-x') \tag{5.129}$$

Let us now consider two solutions of the homogeneous equation corresponding to (5.129) such that they satisfy the boundary conditions for $x < x'$ and $x > x'$, respectively. One such solution can evidently be

$$\phi_0(x) = c_0 x \tag{5.130}$$

which satisfies the boundary condition at $x = 0$, and where c_0 is a constant still to be determined; another solution is

$$\phi_1(x) = c_1(1-x) \tag{5.131}$$

where c_1 is another suitable constant; (5.131) satisfies the prescribed condition for $x > x'$. However, the solution of (5.129) must be continuous at $x = x'$, which is obtained by setting

$$c_0 = (1-x') \;\; c_1 = x' \tag{5.132}$$

Therefore a solution of (5.129) of the form

$$G(x,x') = \begin{cases} (1-x')x \; x < x' \\ (1-x)x' \; x' < x \end{cases} \tag{5.133}$$

which satisfies both boundary conditions and continuity requirements. It is also apparent that the jump in the first derivative at $x = x'$ equals -1 as stated by (5.129).

For the same problem, the spectral expansion of the Green's function would have required first the determination of the eigenfunctions and eigenvalues, then the application of (**??**). Since in this simple case the eigenfunctions are

$$\sqrt{2}\sin(n\pi x) \tag{5.134}$$

while the eigenvalues are $n\pi$, the Green's function assumes the form

$$G(x,x') = \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin(n\pi x)\sin(n\pi x')}{n^2} \tag{5.135}$$

which looks quite different from that calculated with the direct approach (5.133).

IV.2 The Ordinary Bessel Functions

The separation of the Helmholtz or wave equation in circular cylindrical coordinates leads to *Bessel's differential equation*

$$z\frac{d}{dz}\left(z\frac{df}{dz}\right) + (z^2 - n^2)f = 0\,. \tag{5.136}$$

The variable z and the parameter n can be arbitrarily complex. However, in the following n will be assumed as real and integer or half-integer. The solutions of Bessel's differential equation

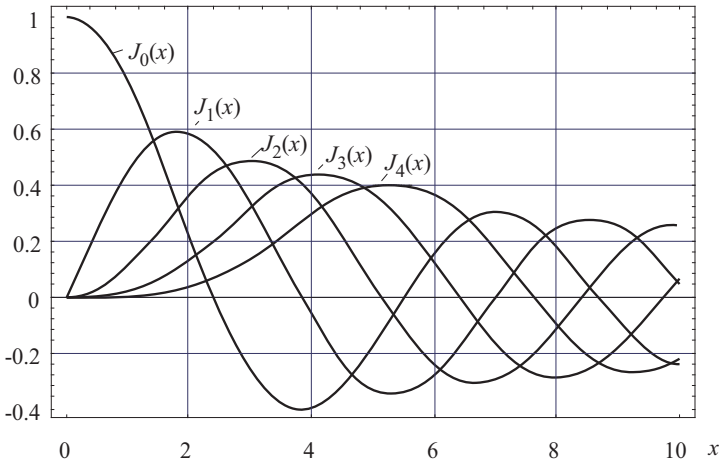

Fig. 5.1. The Bessel functions of the first kind of order 0,1,2,3 and 4.

are the *Bessel function* of the first kind $J_n(z)$, the *Neumann function* or Bessel function of the second kind $Y_n(z)$ and the *Hankel functions* of the first kind $H_n^{(1)}(z)$ and of the second kind $H_n^{(2)}(z)$. The index n denotes the order of the function. The Bessel functions of the first kind $J_n(z)$ are defined by

$$J_n(z) = \sum_{k=0}^{\infty}\frac{(-1)^k\left(z/2\right)^{2k+n}}{k!\,\Gamma(n+k+1)} \tag{5.137}$$

with the *factorial function* $n!$ and the *gamma function* $\Gamma(n)$ defined by

$$n! = n(n-1)\ldots 3\cdot 2\cdot 1 \quad \text{for integer } n \geq 1\,, \tag{5.138}$$

$$0! = 1\,, \tag{5.139}$$

$$\Gamma(n) = (n-1)! \qquad \text{for integer } n \geq 1\,, \tag{5.140}$$

$$\Gamma(n+\tfrac{1}{2}) = 2^{2n}\sqrt{\pi}\,\frac{n!}{(2n)!} \qquad \text{for integer } n \geq 0\,. \tag{5.141}$$

Figure 5.1 shows the Bessel functions of the first kind of order 0,1,2,3 and 4. The Neumann function $Y_n(z)$ is defined by

$$Y_n(z) = \frac{J_n(z)\cos(n\pi) - J_{-n}(z)}{\sin(n\pi)}\,. \tag{5.142}$$

Figure 5.2 shows the Neumann functions of order 0,1,2,3 and 4. The Hankel functions $H_n^{(1)}(z)$

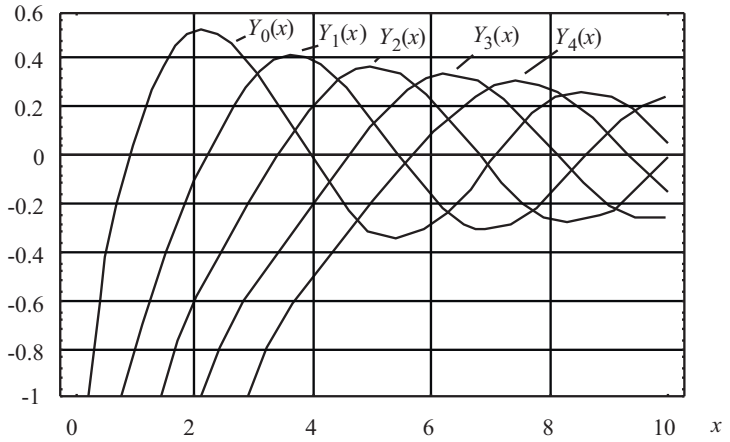

Fig. 5.2. The Neumann functions of order 0,1,2,3 and 4.

and $H_n^{(2)}(z)$ are defined by

$$H_n^{(1)}(z) = j\frac{J_{-n}(z) - J_n(z)e^{-jn\pi}}{\sin(n\pi)}\,, \tag{5.143}$$

$$H_n^{(2)}(z) = j\frac{J_n(z)e^{jn\pi} - J_{-n}(z)}{\sin(n\pi)}\,. \tag{5.144}$$

The Hankel functions functions are related to the Bessel functions of first and second kind via

$$H_n^{(1)}(z) = J_n(z) + jY_n(z)\,, \tag{5.145}$$

$$H_n^{(2)}(z) = J_n(z) - jY_n(z)\,. \tag{5.146}$$

Denoting by $f_n(z)$ any of the functions $J_n(z)$, $Y_n(z)$, $H_n^{(1)}(z)$, $H_n^{(2)}(z)$ the following recurrence relations are valid:

$$f_{n-1}(z) + f_{n+1}(z) = \frac{2n}{z} f_n(z) \,, \tag{5.147}$$

$$f_{n-1}(z) - f_{n+1}(z) = 2f_n'(z) \,, \tag{5.148}$$

The expression $f_n'(x)$ denotes the derivation of f_n with respect to x. The functions of positive and negative order are related via

$$f_{-\mathrm{n}}(z) = (-1)^n f_n(z) \,. \tag{5.149}$$

From (5.148) and (5.149) it follows

$$f_0'(z) = -f_1(z) \,. \tag{5.150}$$

The ordinary Bessel functions of the first kind are the Fourier series expansion coefficients of the *generating function*

$$\mathrm{e}^{jz\sin\varphi} = \sum_{n=-\infty}^{+\infty} J_n(z)\mathrm{e}^{jn\varphi} \,. \tag{5.151}$$

If ξ_{ni} and ξ_{nk} are the i^{th} and k^{th} zero of $J_n(x)$, i.e.

$$J_n(\xi_{ni}) = 0 \,, \quad J_n(\xi_{nk}) = 0 \,. \tag{5.152}$$

the following *orthogonality relation* is valid:

$$\int_0^a r J_n\left(\frac{\xi_{ni} r}{a}\right) J_n\left(\frac{\xi_{nk} r}{a}\right) dr = \begin{cases} 0 & \text{for } i \neq k \\ \frac{a^2}{2} J_n'^2(\xi_i) & \text{for } i = k \end{cases} \tag{5.153}$$

IV.3 The Modified Bessel Functions

The *modified Bessel functions* are the solutions of the *modified Bessel differential equation*

$$z\frac{d}{dz}\left(z\frac{df}{dz}\right) - (z^2 + n^2) f = 0 \,. \tag{5.154}$$

The modified Bessel differential equation is obtained by replacing z by jz in the Bessel differential equation (5.136). Solutions are the *modified Bessel function of the first kind* $I_n(x)$ and the *modified Bessel function of the second kind* $K_n(x)$. The index n denominates the order of the modified Bessel functions. The modified Bessel function of the first kind is defined by the following series expansion:

$$I_n(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+n}}{k!(n+k)!} . \tag{5.155}$$

Figure 5.3 shows the modified Bessel functions of the first kind. The modified Bessel function

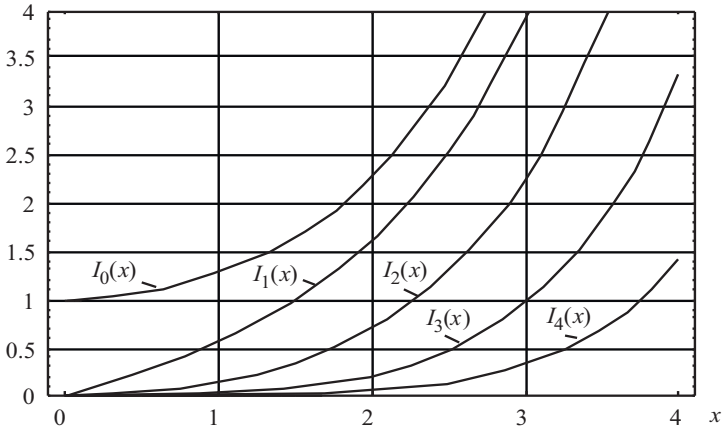

Fig. 5.3. The modified Bessel functions of the first kind of order 0,1,2,3 and 4.

of the second kind is related to the modified Bessel function of the first kind via

$$K_n(z) = \frac{\pi}{2} \frac{I_{-n}(z) - I_n(z)}{\sin(n\pi)} . \tag{5.156}$$

The modified Bessel functions are related to the ordinary Bessel functions via

$$I_n(x) = (-j)^n J_n(jx) , \tag{5.157}$$

$$K_n(x) = \frac{\pi}{2}(j)^{n+1}[J_n(jx) + jY_n(jx)] . \tag{5.158}$$

Figure 5.4 shows the modified Bessel functions of the second kind. For the modified Bessel functions the following recurrence relations are valid:

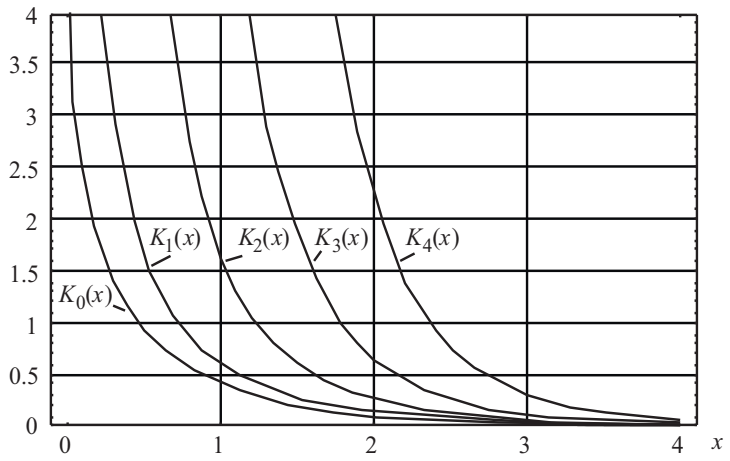

Fig. 5.4. The modified Bessel functions of the second kind of order 0,1,2,3 and 4.

$$I_{n-1}(z) - I_{n+1}(z) = \frac{2n}{z} I_n(z) \,, \tag{5.159}$$

$$I_{n-1}(z) + I_{n+1}(z) = 2I_n'(z) \,, \tag{5.160}$$

$$K_{n-1}(z) - K_{n+1}(z) = -\frac{2n}{z} K_n(z) \,, \tag{5.161}$$

$$K_{n-1}(z) + K_{n+1}(z) = -2K_n'(z) \,. \tag{5.162}$$

The modified Bessel functions of positive and negative order are related via

$$I_{-n}(z) = I_n(z) \,, \tag{5.163}$$

$$K_{-n}(z) = K_n(z) \,. \tag{5.164}$$

From (5.160), (5.163) and (5.162), (5.164) we obtain

$$I_0'(z) = I_1(z) \,, \tag{5.165}$$

$$K_0'(z) = -K_1(z) \,. \tag{5.166}$$

The modified Bessel functions of the first kind are the Fourier series expansion coefficients of the *generating function*

$$\mathrm{e}^{z\cos\varphi} = I_0(z) + 2\sum_{n=1}^{\infty} I_n(z)\cos(n\varphi) \,. \tag{5.167}$$

Integrals with Bessel Functions

We list some important itegrals involving Bessel functions. A large number of integrals involving Bessel functions is contained in [**?**, **?**, 8]. The functions $f_n(z)$, $g_n(z)$ may denote any of the functions $J_n(z)$, $Y_n(z)$, $I_n(z)$, $K_n(z)$.

$$\int x f_n(\alpha x) g_n(\beta x) dx = \frac{\beta x f_n(\alpha x) g_{n-1}(\beta x) - \alpha x f_{n-1}(\alpha x) g_n(\beta x)}{\alpha^2 - \beta^2} , \tag{5.168}$$

$$\int x f_n^2(\alpha x) dx = \frac{x^2}{2} \left[f_n^2(\alpha x) - f_{n-1}(\alpha x) f_{n+1}(\alpha x) \right] . \tag{5.169}$$

From (5.150) and (5.169) we obtain

$$\int x J_0^{2'}(\alpha x) dx = \frac{x^2}{2} \left[J_1^2(\alpha x) - J_0(\alpha x) J_2(\alpha x) \right] . \tag{5.170}$$

From (5.150) and (5.169) it follows

$$\int_0^{x_0} x J_1^2(\alpha x) dx = \int_0^{x_0} x J_0^{2'}(\alpha x) dx = \begin{cases} \frac{x_0^2}{2} J_1^2(\alpha x_0) \text{ for } J_0(\alpha x_0) = 0 \\ \frac{x_0^2}{2} J_0^2(\alpha x_0) \text{ for } J_0'(\alpha x_0) = 0 \\ \qquad\qquad \text{for } J_1(\alpha x_0) = 0 . \end{cases} \tag{5.171}$$

From (5.169) we obtain

$$\int_0^{x_0} x J_0^2(\alpha x) dx = \frac{x_0^2}{2} \left[J_0^2(\alpha x_0) + J_1^2(\alpha x_0) \right] . \tag{5.172}$$

IV.4 Spherical Bessel Functions

The *spherical Bessel functions* $j_n(x)$, $y_n(x)$ and the *spherical Hankel functions* $h_n^{(1)}(x)$, $h_n^{(2)}(x)$ are solutions of the differential equation

$$x^2 \frac{d^2 z_n(x)}{dx^2} + 2x \frac{dz_n(x)}{dr} + (x^2 - n^2) z_n(x) = 0 . \tag{5.173}$$

This differential equation is the the differential equation (**??**) normalized to $x = kr$ and describes the radial component of spherical wave functions. With the substitution

$$z_n(x) = \frac{1}{\sqrt{x}} f_{n+\frac{1}{2}} \tag{5.174}$$

we transform (5.173) into

$$x^2 \frac{d^2 f_{n+\frac{1}{2}}(x)}{dx^2} + r \frac{d f_{n+\frac{1}{2}}(x)}{dx} + \left[x^2 - \left(n + \frac{1}{2} \right)^2 \right] f_{n+\frac{1}{2}}(x) = 0 \,. \tag{5.175}$$

This is the Bessel's differential equation (5.136) for half odd integer order $n + \frac{1}{2}$. The spherical Bessel functions $j_n(x)$, $y_n(x)$ and the spherical Hankel functions $h_n^{(1)}(x)$, $h_n^{(2)}(x)$are related to the cylindrical Bessel and Hankel functions of order $n + 1/2$ by

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+\frac{1}{2}}(x) \,, \tag{5.176}$$

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+\frac{1}{2}}(x) \,, \tag{5.177}$$

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H^{(1)}_{n+\frac{1}{2}}(x) \,, \tag{5.178}$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H^{(2)}_{n+\frac{1}{2}}(x) \,. \tag{5.179}$$

The spherical Bessel functions of order 0 to 2 are given by

$$j_0(x) = \frac{\sin x}{x} \,, \tag{5.180}$$

$$j_1(x) = \frac{\sin x}{x^2} - \frac{\cos x}{x} \,, \tag{5.181}$$

$$j_2(x) = \left(\frac{3}{x^3} - \frac{1}{x} \right) \sin x - \frac{3}{x^2} \cos x \tag{5.182}$$

and

$$y_0(x) = -\frac{\cos x}{x} \,, \tag{5.183}$$

$$y_1(x) = -\frac{\sin x}{x} - \frac{\cos x}{x^2} \,, \tag{5.184}$$

$$y_2(x) = \left(-\frac{3}{x^3} + \frac{1}{x} \right) \cos x - \frac{3}{x^2} \sin x \,. \tag{5.185}$$

and the spherical Hankel functions of order 0 to 2 are given by

$$h_0^{(1)}(x) = h_0^{(2)*}(x) = -\frac{j}{x}e^{jx}\,, \tag{5.186}$$

$$h_1^{(1)}(x) = h_1^{(2)*}(x) = \left(-\frac{j}{x} - \frac{1}{x^2}\right)e^{jx}\,, \tag{5.187}$$

$$h_2^{(1)}(x) = h_2^{(2)*}(x) = \left(\frac{j}{x} - \frac{3}{x^2} - \frac{3j}{x^3}\right)e^{jx}\,. \tag{5.188}$$

IV.5 Legendre Polynomials

The separation of the Helmholtz or wave equation in spherical coordinates leads to the following differential equation for the function of the ϑ coordinate

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{\mathrm{f}}{d\vartheta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\vartheta}\right]f = 0\,. \tag{5.189}$$

With the substitution

$$x = \cos\vartheta \tag{5.190}$$

we can bring (5.189) in the form

$$(1-x^2)\frac{d^2f}{dx^2} + 2x\frac{df}{dx} + \left[n(n+1) - \frac{m^2}{1-x^2}\right]f = 0\,. \tag{5.191}$$

This is the the *Legendre differential equation*. Since The variable z and the parameters n and m can be arbitrarily complex. However, in the following m and n will be assumed as real and integer. Since in the spherical coordinate system we consider the interval $0 \le \vartheta \le \pi$ this yields $-1 \le x \le 1$. The solutions of the Legendre differential equation are the *associated Legendre function* of the first kind $P_n^m(z)$, and the second kind $Q_n^m(z)$. The subscript n denotes the degree of the Legendre function.

For $m = 0$ we obtain the *ordinary Legendre differential equation*

$$(1-x^2)\frac{d^2f}{dx^2} + 2x\frac{df}{dx} + n(n+1)f = 0\,. \tag{5.192}$$

The solutions of the ordinary Legendre differential equation are the *ordinary Legendre functions of the first kind* $P_n(x)$, and the second kind $Q_n(x)$. For integer n the ordinary Legendre functions of the first kind are polynomials of degree n and therefore ara also called *Legendre polynomials*

of degree n. The Ampèrecit expression for the Legendre polynomial is

$$P_n(x) = \sum_{m=0}^{N} (-1)^m \frac{(2n-2m)!x^{n-2m}}{2^n\, m!(n-m)!\,(n-2m)!}\,, \tag{5.193}$$

where $M = n/2$ for even n and $M = (n-1)/2$ for odd n. In terms of x and $\cos\vartheta$ the five

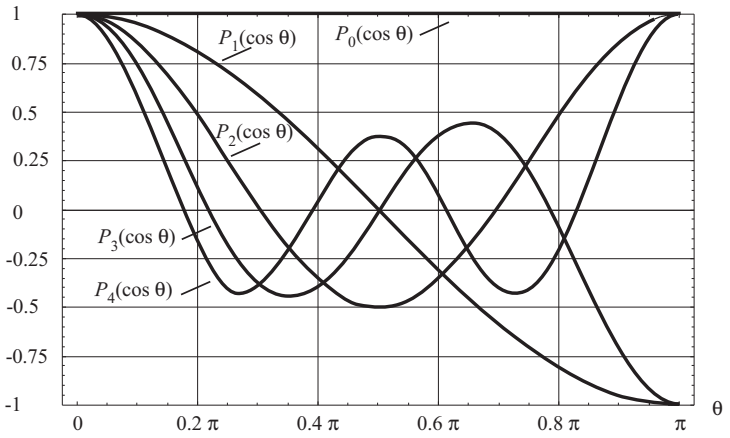

Fig. 5.5. The Legendre polynomials of the first kind of order 0,1,2,3 and 4.

Legendre polynomials of lowest order are

$$P_0(x) = 1\,, \tag{5.194}$$

$$P_1(x) = x = \cos\vartheta\,, \tag{5.195}$$

$$P_2(x) = \tfrac{1}{2}(3x^2-1) = \tfrac{1}{4}(1+3\cos 2\vartheta)\,, \tag{5.196}$$

$$P_3(x) = \tfrac{1}{2}(5x^3-3x) = \tfrac{1}{8}(3\cos\vartheta + 5\cos 3\vartheta)\,, \tag{5.197}$$

$$\begin{aligned} P_4(x) &= \tfrac{1}{8}(35x^4-30x^2+3) \\ &= \tfrac{1}{64}(9+20\cos 2\vartheta + 35\cos 4\vartheta)\,, \end{aligned} \tag{5.198}$$

Figure 5.5 shows the Legendre Polynomials of the first kind of order 0,1,2,3 and 4.

The Legendre functions of the second kind $Q_n(x)$ are infinite at $x = \pm 1$, or at $\vartheta = 0$ and $\vartheta = \pi$. The lowest order Legendre functions of the second kind are given by

$$Q_0(x) = \frac{1}{2}\log\frac{1+x}{1-x} = \log\cot\frac{\vartheta}{2}\,, \tag{5.199}$$

$$Q_1(x) = \frac{x}{2}\log\frac{1+x}{1-x} - 1 = \cos\vartheta\cot\log\frac{\vartheta}{2} - 1\,, \tag{5.200}$$

$$\begin{aligned} Q_2(x) &= \frac{3x^2-1}{4}\log\frac{1+x}{1-x} - \frac{3u}{2} \\ &= \frac{1}{4}(3\cos^2\vartheta - 1)\log\cot\frac{\vartheta}{2} - \frac{3}{2}\cos\vartheta\,. \end{aligned} \tag{5.201}$$

The n^{th} order Legendre function of the second kind is

$$Q_n(x) = Q_0(x)P_n(x) - \sum_{m=0}^{M}\frac{2n-4m+3}{(2m-1)(n-m+1)}P_{n-2m+1}(x)\,, \tag{5.202}$$

where $M = n/2$ for even n and $M = (n+1)/2$ for odd n. The Legendre functions with positive and negative argument are related via

$$P_n(-x) = (-1)^n P_n(x)\,, \tag{5.203}$$

$$Q_n(-x) = (-1)^{n+1} Q_n(x)\,. \tag{5.204}$$

Figure 5.6 shows the Legendre Polynomials of the first kind of order 0,1,2,3 and 4. The ass-

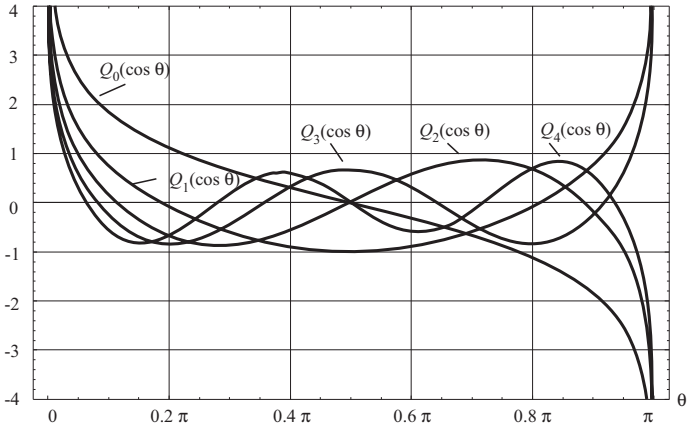

Fig. 5.6. The Legendre polynomials of the second kind of order 0,1,2,3 and 4.

ciated Legendre functions, i.e. the solutions of the associated Legendre differential equation (5.191) are

$$P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m} \,, \tag{5.205}$$

$$Q_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m Q_n(x)}{dx^m} \,. \tag{5.206}$$

Some lower order associated Legendre functions of the first kind are

$$P_1^1(x) = -(1-x^2)^{\frac{1}{2}} \,, \tag{5.207}$$

$$P_2^1(x) = -3(1-x^2)^{\frac{1}{2}} x \,, \tag{5.208}$$

$$P_2^2(x) = 3(1-x^2)^{\frac{1}{2}} \,, \tag{5.209}$$

$$P_3^1(x) = \frac{3}{2}(1-x^2)^{\frac{1}{2}}(1-5x^2) \,, \tag{5.210}$$

$$P_3^2(x) = 15(1-x^2)x \,, \tag{5.211}$$

$$P_3^3(x) = 1-5(1-x^2)^{\frac{3}{2}}(1-5x^2) \,. \tag{5.212}$$

Some lower order associated Legendre functions of the second kind are

$$Q_1^1(x) = -(1-x^2)^{\frac{1}{2}} \left(\frac{1}{2} \log \frac{1+x}{1-x} + \frac{x}{1-x^2} \right) \,, \tag{5.213}$$

$$Q_2^1(x) = -(1-x^2)^{\frac{1}{2}} \left(\frac{3}{2} \log \frac{1+x}{1-x} + \frac{3x^2-2}{1-x^2} \right) \,, \tag{5.214}$$

$$Q_2^2(x) = \ (1-x^2)^{\frac{1}{2}} \left(\frac{3}{2} \log \frac{1+x}{1-x} + \frac{5x-3x^2}{(1-x^2)^2} \right) \,. \tag{5.215}$$

The *orthogonality relations* for Legendre functions are

$$\int_{-1}^{1} P_{n_1}^m(x) P_{n_2}^m(x) = \frac{2}{2n_1+1} \frac{(n_1+m)!}{(n_1-m)!} \delta_{n_1,n_2} \,. \tag{5.216}$$

IV.6 Spherical Harmonics

Spherical harmonics are solutions of the partial differential equation

$$\frac{1}{\sin\vartheta} \frac{d}{d\vartheta} \left(\sin\vartheta \frac{df(\vartheta,\varphi)}{d\vartheta} \right) + \frac{1}{\sin^2\vartheta} \frac{d^2 f(\vartheta,\varphi)}{d\varphi^2} + n(n+1) f = 0 \,. \tag{5.217}$$

Setting

$$f(\vartheta,\varphi) = \Theta(\vartheta)\Phi(\varphi)\,. \tag{5.218}$$

with the integer separation parameter m with $|m| \leq n$ the partial differential equation (5.217) may be separated into (5.189) for $\Theta(\vartheta)$ and the second order differential equation for $\Phi(\varphi)$.

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\,\frac{df}{d\vartheta}\right) + \left[n(n+1) - \frac{m^2}{\sin^2\vartheta}\right] f = 0\,, \tag{5.219}$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2} + m^2\Phi(\varphi) = 0\,. \tag{5.220}$$

The normalized solution of (5.217) is the spherical harmonic

$$Y_n^m(\vartheta,\varphi) = \sqrt{\frac{2n+1}{2}\,\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\vartheta)\,e^{jm\varphi}\,. \tag{5.221}$$

Some lower order spherical harmonics are

$$Y_0^0(\vartheta,\varphi) = \frac{1}{4\pi}\,, \tag{5.222}$$

$$Y_1^1(\vartheta,\varphi) = -\sqrt{\frac{1}{8\pi}}\,\sin\vartheta\,e^{j\varphi}\,, \tag{5.223}$$

$$Y_1^0(\vartheta,\varphi) = \sqrt{\frac{1}{4\pi}}\,\cos\vartheta\,, \tag{5.224}$$

$$Y_1^{-1}(\vartheta,\varphi) = \sqrt{\frac{1}{8\pi}}\,\sin^2\vartheta\,e^{2j\varphi}\,, \tag{5.225}$$

$$Y_2^2(\vartheta,\varphi) = \sqrt{\frac{5}{96\pi}}\,3\sin^2\vartheta\,e^{2j\varphi}\,, \tag{5.226}$$

$$Y_2^1(\vartheta,\varphi) = -\sqrt{\frac{5}{24\pi}}\,3\sin\vartheta\cos\vartheta\,e^{j\varphi}\,, \tag{5.227}$$

$$Y_2^0(\vartheta,\varphi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\vartheta - \frac{1}{2}\right)\,, \tag{5.228}$$

$$Y_2^{-1}(\vartheta,\varphi) = \sqrt{\frac{5}{24\pi}}\,3\sin\vartheta\cos\vartheta\,e^{-j\varphi}\,, \tag{5.229}$$

$$Y_2^{-2}(\vartheta,\varphi) = \sqrt{\frac{5}{96\pi}}\,3\sin^2\vartheta\,e^{-2j\varphi}\,. \tag{5.230}$$

Figure 5.7 shows some of the lowest order spherical harmonics.

The *orthogonality relation* for Spherical harmonics is

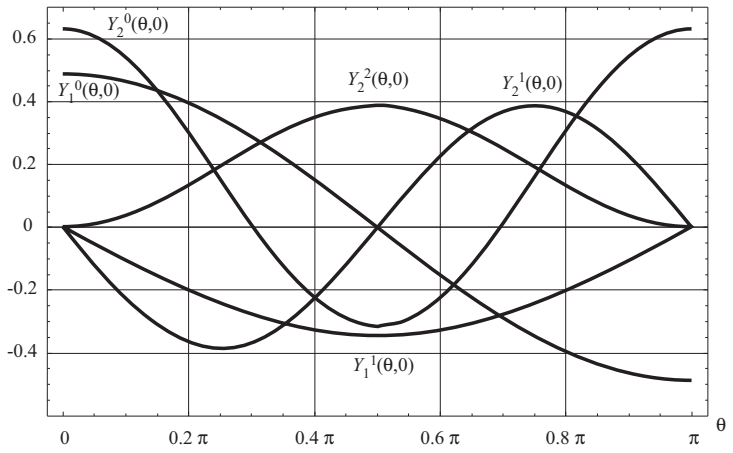

Fig. 5.7. The spherical harmonics Y_1^0, Y_1^1, Y_2^0, Y_2^1, Y_2^2.

$$\int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} Y_{n_1}^{m_1*}(\vartheta,\varphi)\,Y_{n_2}^{m_2}(\vartheta,\varphi)\,d\vartheta\,d\varphi = \delta_{n_1,n_2}\delta_{m_1,m_2}\,. \tag{5.231}$$

The spherical harmonics constitute a *complete system of functions* within the interval $0 \leq \vartheta\pi$, $0 \leq \varphi 2\pi$. Any function $f(\vartheta,\varphi)$ with sufficient continuity properties may be expanded in this interval in a series

$$f(\vartheta,\varphi) = \sum_{m,n} a_{m,n} Y_m(\vartheta,\varphi)\,. \tag{5.232}$$

The expansion coefficients a_{mn} are

$$a_{m,n} = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi} Y_n^{m*}(\vartheta,\varphi) f(\vartheta,\varphi)\,d\vartheta\,d\varphi\,. \tag{5.233}$$

V Plane waves

Let start from Maxwell's equations without sources:

$$\partial\!\!\!/ \bar{F} = 0 \tag{5.234}$$

that is

$$\not{\partial}\bar{F} = \begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} E_z \\ iE_y + E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x - H_y\right) \end{pmatrix} \tag{5.235}$$

Multiplying equation (5.235) by $\not{\partial}$ we obtain:

$$\not{\partial}^2\bar{F} = \left(\partial_1^2 + \partial_2^2 + \partial_3^2 - \partial_0^2\right)\bar{F} = 0 \tag{5.236}$$

which is the wave equation without sources. For plane waves propagation it is convenient to use the cartesian reference system, so that:

$$\begin{aligned} \partial_1 &= \partial_x \\ \partial_2 &= \partial_y \\ \partial_3 &= \partial_z \\ \partial_0 &= \tfrac{1}{v}\partial_t \end{aligned} \tag{5.237}$$

then the wave equation becomes:

$$\left(\partial_x^2 + \partial_y^2 + \partial_z^2 - \frac{1}{v^2}\partial_t^2\right)\bar{F} = 0 \tag{5.238}$$

The expression (5.238) is composed of six equations, one for each field component. For example, the equation for the component E_z is:

$$\partial_x^2 E_z + \partial_y^2 E_z + \partial_z^2 E_z - \frac{1}{v^2}\partial_t^2 E_z = 0 \tag{5.239}$$

or

$$\nabla^2 E_z - \frac{1}{v^2}\partial_t^2 E_z = 0 \tag{5.240}$$

which is the wave equation for the component E_z. The other components satisfy the same equation and the wave equation can be written is the compact form:

$$\begin{aligned} \nabla^2 \bar{E} - \tfrac{1}{v^2}\partial_t^2 \bar{E} = 0 \\ \nabla^2 \bar{H} - \tfrac{1}{v^2}\partial_t^2 \bar{H} = 0 \end{aligned} \tag{5.241}$$

Writing the same equations in frequency domain, we have:

$$\begin{aligned} \bar{E}(x,y,z,t) = \mathrm{Re}\{\mathbf{E}(x,y,z)e^{j\omega t}\} \\ \bar{H}(x,y,z,t) = \mathrm{Re}\{\mathbf{H}(x,y,z)e^{j\omega t}\} \end{aligned}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the phasors of the electric and magnetic fields. With time dependence of the form $e^{j\omega t}$:

$$\partial_t = j\omega$$

and expressing the speed as

$$v = \frac{1}{\sqrt{\varepsilon\mu}}$$

the wave equations can be rewritten in the following form:

$$\begin{aligned} \nabla^2\mathbf{E} + \omega^2\varepsilon\mu\mathbf{E} = \nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \\ \nabla^2\mathbf{H} + \omega^2\varepsilon\mu\mathbf{H} = \nabla^2\mathbf{H} + k^2\mathbf{H} = 0 \end{aligned} \tag{5.242}$$

which are Helmoltz' equations. The quantity

$$k = \omega\sqrt{\varepsilon\mu} \qquad [m^{-1}]$$

is defined as *wave number*. It is worth noting that:

$$\frac{1}{v}\partial_t = \sqrt{\varepsilon\mu}\, j\omega = jk \tag{5.243}$$

Equations (5.242) are the wave equations for the electromagnetic field in the free space. They are expressed in a general form and can be greatly simplified in the next section to express the propagation of a uniform plane wave.

V.1 Uniform plane waves

Considering the equation (5.242), the electric field expressed in cartesian coordinates is:

$$\mathbf{E} = \hat{x}\mathbf{E}_x(x,y,z) + \hat{y}\mathbf{E}_y(x,y,z) + \hat{z}\mathbf{E}_z(x,y,z) \tag{5.244}$$

so that the equation can be split into three equations, one for each component, that is:

$$\nabla^2\mathbf{E}_x + k^2\mathbf{E}_x = 0 \tag{5.245}$$

and similar equations for the other components. Now a *uniform plane wave* is characterized by the fact that the elecromagnetic field is constant (uniform) at all points across an infinite plane. It is convenient to map the coordinate system in such a way that the plane is the xy one. In this case the electromagnetic field is constant with respect the coordinates x and y or, in other words, the first derivatives of the field with respect to x and y are zero, hence $\partial_x = 0$ and $\partial_y = 0$. The equations (5.242) become:

$$\begin{aligned} \tfrac{d^2\mathbf{E}}{dz^2} + k^2\mathbf{E} = 0 \\ \tfrac{d^2\mathbf{H}}{dz^2} + k^2\mathbf{H} = 0 \end{aligned} \tag{5.246}$$

The solution is a wave propagating along the z direction. We can also demonstrate that the longitudinal components $\mathbf{E}_z$ and $\mathbf{H}_z$ are zero. In fact, taking the first equation of (5.235) we have:

$$jk\mathbf{E}_z + \partial_z\eta\, i\mathbf{H}_z + \eta(i\partial_x\mathbf{H}_x - \partial_x\mathbf{H}_y) - \eta(-\partial_y H_x + i\partial_y\mathbf{H}_y) = 0 \tag{5.247}$$

considering only the real part:

$$jk\mathbf{E}_z - \eta\,\partial_x\mathbf{H}_y + \eta\,\partial_y\mathbf{H}_x = 0 \tag{5.248}$$

since $\partial_x = 0$ and $\partial_y = 0$, the field component $\mathbf{E}_z$ is also zero. A similar demostration applied to $\mathbf{H}_z$.

The conclusion is that the electromagnetic field of a unifrom plane wave is transverse to the direction of propagation of the wave. This is an example of TEM (Transverse ElectroMagnetic) wave. The remaining wave equations are:

$$\begin{aligned} \partial_z^2 \mathbf{E}_x + k^2 \mathbf{E}_x = 0 \\ \partial_z^2 \mathbf{E}_y + k^2 \mathbf{E}_y = 0 \end{aligned} \tag{5.249}$$

and

$$\begin{aligned} \partial_z^2 \mathbf{H}_x + k^2 \mathbf{H}_x = 0 \\ \partial_z^2 \mathbf{H}_y + k^2 \mathbf{H}_y = 0 \end{aligned} \tag{5.250}$$

The solutions are of the type:

$$\begin{aligned} \mathbf{E}_x(z) &= E_{x0}^+ e^{-jkz} + E_{x0}^- e^{jkz} \\ \mathbf{E}_y(z) &= E_{y0}^+ e^{-jkz} + E_{y0}^- e^{jkz} \\ \mathbf{H}_x(z) &= H_{x0}^+ e^{-jkz} + H_{x0}^- e^{jkz} \\ \mathbf{H}_y(z) &= H_{y0}^+ e^{-jkz} + H_{y0}^- e^{jkz} \end{aligned} \tag{5.251}$$

where the first term identifies a progressive wwave (a wave propagating along positive z) and the second term identifies a regressive wave (propagating along negative z). At this point we search if there is a relation between the amplitudes of electric and magnetic fields. In order to find this relation we can used the fourth equation of (5.235):

$$\partial_z (i\mathbf{E}_y + \mathbf{E}_x) - jk\eta\,(i\mathbf{H}_x - \mathbf{H}_y) = 0 \tag{5.252}$$

If we take the real part the relation is:

$$\partial_z \mathbf{E}_x + jk\eta \mathbf{H}_y = 0 \tag{5.253}$$

hence:

$$\mathbf{H}_y = -\frac{1}{-jk\eta}\partial_z \mathbf{E}_x = \frac{j}{\eta k}\left[-jkE_{x0}^{+}e^{-jkz} + -jkE_{x0}^{-}e^{-jkz}\right] = \frac{E_{x0}^{+}}{\eta}e^{-jkz} - \frac{E_{x0}^{-}}{\eta}e^{jkz} \tag{5.254}$$

This is an expression very similar to the one found for transmission lines. The amplitude of the waves are realted to each other through η which is called *wave impedance* and is measured in Ohms. The free space wave impedance is usually expressed as $\eta_0 = \sqrt{\mu_0/\varepsilon_0} \approx 120\pi \approx 377$ Ohm. This expression says that for a uniform plane wave propagating along z direction, the components E_x and H_y are tied toghether through the impedance η. If the medium is lossless, electric and magnetic fields are in phase for a progressive wave and in opposition of phase for a regressive wave. This has to do with the power flow as we will see in the following.

By taking the imaginary part, with respect to i, of (5.252) we obtain:

$$\mathbf{H}_x = -\frac{E_{y0}^{+}}{\eta}e^{-jkz} + \frac{E_{y0}^{-}}{\eta}e^{jkz} \tag{5.255}$$

The two relations can be written in a compact form as:

$$\mathbf{H} = \frac{1}{\eta}\hat{k} \times \mathbf{E} \tag{5.256}$$

or

$$\mathbf{E} = \eta \mathbf{H} \times \hat{k} \tag{5.257}$$

where $\hat{k}$ is the versor identifying the propagation of the wave, that is $\hat{k} = \hat{z}$ for the progressive wave and $\hat{k} = -\hat{z}$ for the regressive wave.

V.2 Polarization

We have seen that, for a plane wave propagating along the z-axis, there are in general both E_x and E_y. However, their amplitude and phase can be different. If we imagine the electric field as a vector with starting point at the origin, the tip of this vector describe a trajectory depending on amplitudes and phases. First of all we can define the *polarization* of a wave as the locus described by the tip of the $\mathbf{E}$ vector, in the plane orthogonal to the direction of propagation, at a given point in space as a function of time.

The different amplitudes and phases of the two electric field components gives different cases. The most general case is that the polarization is *elliptical* so that the tip describes an ellipsis in the plane orthogonal to the direction of prapagation. Under certain conditions, which will be discussed in the following, the polarization can be linear or circular .

Let we start with the electric field of a wave propagating along z:

$$\mathbf{E}(z) = \hat{x}\mathbf{E}_x(z) + \hat{y}\mathbf{E}_y(z) \tag{5.258}$$

with

$$\begin{aligned}\mathbf{E}_x(z) &= E_{x0}^{+} e^{-jkz} \\ \mathbf{E}_y(z) &= E_{y0}^{+} e^{-jkz}\end{aligned}$$

the polarization depends on the relative amplitude and phase between the two component and not on the absolute values. For this reason we can take the phase of $\mathbf{E}_x$ as reference or, in other words, we can assign the phase of the x component as zero and use the phase difference of the y component, which is denoted by δ:

$$\begin{aligned}E_{x0}^{+} &= |E_{x0}| \\ E_{y0}^{+} &= |E_{y0}| e^{j\delta}\end{aligned}$$

then, we can write:

$$\mathbf{E}(z) = (\hat{x}|E_{x0}| + \hat{y}|E_{y0}|e^{j\delta})e^{-jkz} \tag{5.259}$$

or, in time domain:

$$\begin{aligned}E(z,t) &= \mathrm{Re}\left(\mathbf{E}(z)e^{i\omega t}\right) \\ &= \hat{x}|E_{x0}|\cos(\omega t - kz) + \hat{y}|E_{y0}|\cos(\omega t - kz + \delta)\end{aligned} \tag{5.260}$$

The amplitude is then equal to:

$$|E(z,t)| = \sqrt{|E_{x0}|^2\cos^2(\omega t - kz) + |E_{y0}|^2\cos^2(\omega t - kz + \delta)} \tag{5.261}$$

We can also define the angle of inclination as the angle of the vector **E** while the wave is propagating:

$$\Psi(z,t) = \text{atan}\left(\frac{E_y(z,t)}{E_x(z,t)}\right) \tag{5.262}$$

Linear polarization

The wave is linearly polarized if the tip of the electric field moves along a line. This happens when the relative phase between the filed components δ is equal to 0 or $\pm\pi$.
In phasor form:

$$\mathbf{E}(z) = (\hat{x}|E_{x0}| \pm \hat{y}|E_{y0}|e^{j\delta})e^{-jkz} \tag{5.263}$$

in time domain format:

$$E(z,t) = (\hat{x}|E_{x0}| \pm \hat{y}|E_{y0}|)\cos(\omega t - kz) \tag{5.264}$$

Then the amplitude is:

$$|E(z,t)| = \sqrt{|E_{x0}|^2 + |E_{y0}|^2}\cos(\omega t - kz) \tag{5.265}$$

and the inclination angle is:

$$\Psi(z,t) = \text{atan}\left(\frac{\pm|E_{y0}|}{|E_{x0}|}\right) \tag{5.266}$$

It is worth noting that the amplitude depends either on z or t while the inclination is not dependent on neither z nor t.

Circular polarization

The wave is circular polarized if the tip of the **E** vector moves along a circumference. This happens when the amplitudes of the two component of the field are equal and the phase δ is $\pm$ 90. In this first case we have a Left Handed Circula Polarization (LHCP). In the second case we have a Right Hand Circular Polarization (RHCP).
In the LHCP the amplitude and phase are:

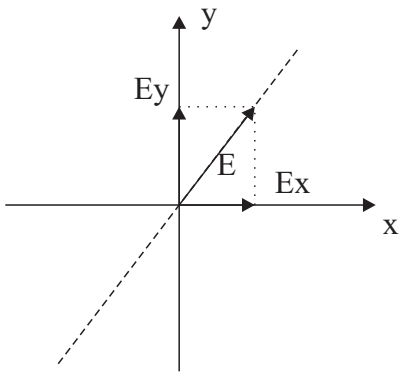

Fig. 5.8. Linear polarized electric field

$$|E_{x0}| = |E_{y0}| = a$$

and

$$\delta = \frac{\pi}{2}$$

so that in phasor form the total field is:

$$\mathbf{E}(z) = (\hat{x}a + \hat{y}ae^{j\pi/2})e^{-jkz} = (\hat{x}a + j\hat{y}a)e^{-jkz} \tag{5.267}$$

or in time domain:

$$E(z,t) = \hat{x}\,a\cos(\omega t - kz) + \hat{y}\,a\cos\big(\omega t - kz + \pi/2\big) = \hat{x}\,a\cos(\omega t - kz) - \hat{y}\,a\sin(\omega t - kz) \tag{5.268}$$

The total amplitude is:

$$|E(z,t)| = \sqrt{a^2\cos^2(\omega t - kz) + a^2\sin^2(\omega t - kz)} = a \tag{5.269}$$

and the inclination of the vector is:

$$\Psi(z,t) = \operatorname{atan}\left(\frac{-a\sin(\omega t - kz)}{a\cos(\omega t - kz)}\right) = -(\omega t - kz) \tag{5.270}$$

From the last two equations we can see that the amplitude is constant while the inclination depends both on z and time. If we observe the electric field vector at a given z (for example

$z = 0$) the inclination angle decreases as t increases. This mean that if look at an inbound wave we can see the vector rotating in a clockwise direction. on the otneh hand, is we identify with our thumb the direction of propagation, the field rotate in the same way we close the left hand fingers. This is why it is called LHCP.

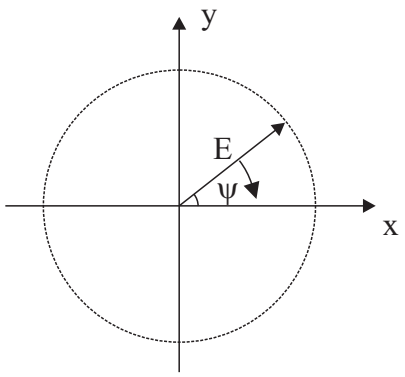

Fig. 5.9. Left Hand Circular Polarization

On the contrary, considering a RHCP wave, we have:

$$|E_{x0}| = |E_{y0}| = a$$

and

$$\delta = -\frac{\pi}{2}$$

so that in phasor form the total field is:

$$\mathbf{E}(z) = (\hat{x}a + \hat{y}a\,e^{-j\pi/2})e^{-jkz} = (\hat{x}a - j\hat{y}a)e^{-jkz} \tag{5.271}$$

or in time domain:

$$E(z,t) = \hat{x}a\cos(\omega t - kz) + \hat{y}a\cos\big(\omega t - kz - \pi/2\big) = \hat{x}a\cos(\omega t - kz) + \hat{y}a\sin(\omega t - kz) \tag{5.272}$$

The total amplitude is:

$$|E(z,t)| = \sqrt{a^2\cos^2(\omega t - kz) + a^2\sin^2(\omega t - kz)} = a \tag{5.273}$$

and the inclination of the vector is:

$$\Psi(z,t) = \text{atan}\left(\frac{a\sin(\omega t - kz)}{a\cos(\omega t - kz)}\right) = (\omega t - kz) \tag{5.274}$$

From the last two equations we can see that the amplitude is constant while the inclination depends both on z and time. If we observe the electric field vector at a given z (for example $z = 0$) the inclination angle increases as t increases. This mean that if look at an inbound wave we can see the vector rotating in a counter-clockwise direction. Similar to the first case, if we identify with our thumb the direction of propagation, the field rotate in the same way we close the right hand fingers.

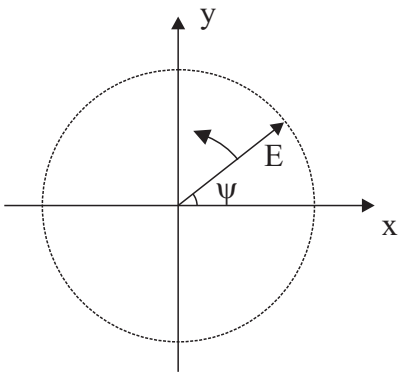

Fig. 5.10. Right Hand Circular Polarization

V.3 Power density

The Poynting's vector is defined as:

$$\bar{S} = \bar{E} \times \bar{H} \qquad \left[\frac{W}{m^2}\right] \tag{5.275}$$

The vector $\bar{S}$ is directed aong the propagation direction of the wave and it indicates the power density of the wave. If we integrates this expression over a finite surface A, the result is:

$$P = \int_A \bar{S} \cdot \hat{n} dS \tag{5.276}$$

where $\hat{n}$ is the verso normal to the surface A. It is important not to confuse $\hat{n}$ with $\hat{k}$, the latter indicating the direction of propagation of the wave, since they could be not parallel. In phasor form the vector is:

$$\mathbf{S}_{av} = \frac{1}{2}\mathrm{Re}\left(\mathbf{E}\times\mathbf{H}^{*}\right) \tag{5.277}$$

It is worth noting that, considering an ideal planar wave, the poynting's vector should be integrated over and infinite surface. This means that the total power carried by the wave is also infinite. Such condition is not possible and in fact an ideal planar wave does not exist in reality. However, the concept of planar EM wave is a good approximation of a spherical wave at a sufficient distance from the source. As it is showed in Fig. 5.11, if we have an observation point at great distance from the source and if we consider only a limited surface, the wave is almost planar over that surface and the derived equations describe very well the behavior of the electromagnetic field in that area.

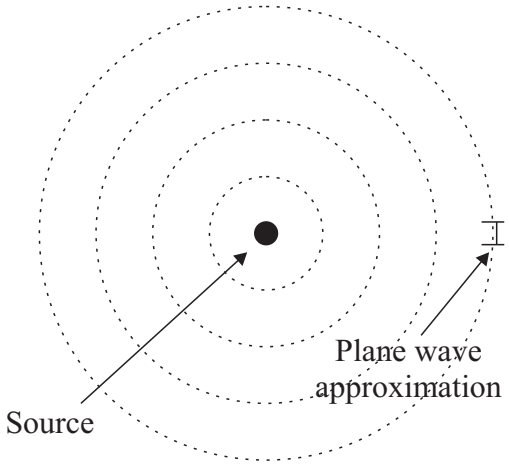

Fig. 5.11. Plane wave approximation

V.4 Wave reflection and transmission at normal incidence

In this section we will see the behaviour of an uniform plane wave orthogonally incident to a boundary surface between two media. The two media are fully described by their permettivity ε and permeability μ. Fig. 5.12 depicts the situation with the wave travelling along z direction and the incident electromagnetic field ($\mathbf{E}_i$ and $\mathbf{H}_i$). The two media are identified by their wave impedances η_1 and η_2.

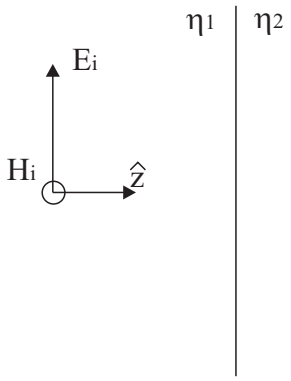

Fig. 5.12. Plane wave normal incidence to a boundary surface

The incident field in phasor form is:

$$\mathbf{E}_i = \hat{x}E_x^i e^{-jk_1 z} + \hat{y}E_y^i e^{-jk_1 z} \tag{5.278}$$

$$\begin{aligned}\mathbf{H}_i &= \tfrac{1}{\eta_1}\hat{k}_1 \times \mathbf{E}_i = \tfrac{1}{\eta_1}\hat{z} \times \left(\hat{x}E_x^i e^{-jk_1 z} + \hat{y}E_y^i e^{-jk_1 z}\right) \\ &= \hat{y}\tfrac{E_x^i}{\eta_1}e^{-jk_1 z} - \hat{x}\tfrac{E_y^i}{\eta_1}e^{-jk_1 z}\end{aligned} \tag{5.279}$$

When this field hit the surfeca between the two media, in general some of the power is transmitted to the second medium and some is reflected to the first medium. In the same way as the transmission lines, a transmitted wave and a reflected wave is generated (Fig. 5.13)

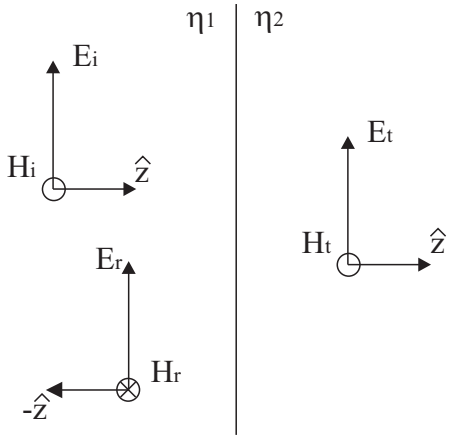

Fig. 5.13. Incident, reflected and transmitted wave in the normal incidence case

Let us consider only the $\hat{x}$ component of the electric field in order to simplify the formulation. Similar results can be obtained also for the $\hat{y}$ component. The total electromagnetic field in the medium 1 is:

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \hat{x}E_x^i e^{-jk_1 z} + \hat{x}E_x^r e^{jk_1 z} \tag{5.280}$$

$$\mathbf{H}_1 = \hat{y}\frac{E_x^i}{\eta_1} e^{-jk_1 z} - \hat{y}\frac{E_x^r}{\eta_1} e^{jk_1 z} \tag{5.281}$$

In the medium 2 the transmitted field is:

$$\mathbf{E}_2 = \mathbf{E}_t = \hat{x}E_x^t e^{-jk_2 z} \tag{5.282}$$

$$\mathbf{H}_2 = \mathbf{H}_t = \hat{y}\frac{E_x^t}{\eta_2} e^{-jk_2 z} \tag{5.283}$$

In $z = 0$ the boundary conditions must be satisfied. This means that the tangential components of both electric and magnetic fields must be the same in the two media. By imposing the conditions at $z = 0$ we obtain:

$$\begin{array}{c} E_x^i + E_x^r = E_x^t \\ \frac{E_x^i}{\eta_1} - \frac{E_x^r}{\eta_1} = \frac{E_x^t}{\eta_2} \end{array} \tag{5.284}$$

By solving the system of equations we can compute the reflection coefficient Γ and the transmissione coefficient T that are defined as:

$$\Gamma = \frac{E_x^r}{E_x^i} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} \tag{5.285}$$

and

$$T = \frac{E_x^t}{E_x^i} = \frac{2\eta_2}{\eta_2 + \eta_1} = 1 + \Gamma \tag{5.286}$$

By looking at the equations (5.285) and (5.286) we can see that they are very similar to the equations (3.44) and (3.47) found for transmission lines. In fact, the problem of a plane wave incident to a boundary surface between two media can be solved as a transmission line problem, substituting the following quantities:

$$\begin{aligned}\mathbf{E}_x &\leftrightarrow V\\ \mathbf{H}_y &\leftrightarrow I\\ \eta &\leftrightarrow Z_0\end{aligned} \tag{5.287}$$

or, considering the other components:

$$\begin{aligned}\mathbf{E}_y &\leftrightarrow V\\ -\mathbf{H}_x &\leftrightarrow I\\ \eta &\leftrightarrow Z_0\end{aligned} \tag{5.288}$$

Since in the first medium there are two waves (incident and reflected), in general we have a stationary wave and a Standing Wave Ratio can be also found.

$$\mathbf{E}_1 = \hat{x}\left(E_x^i e^{-jk_1 z} + \Gamma E_x^i e^{jk_1 z}\right) \tag{5.289}$$

$$\mathbf{H}_1 = \hat{y}\frac{1}{\eta_1}\left(E_x^i e^{-jk_1 z} - \Gamma E_x^i e^{jk_1 z}\right) \tag{5.290}$$

The Poynting's vector is:

$$\begin{aligned}\mathbf{S}_{av1}(z) &= \tfrac{1}{2}\,\mathrm{Re}\left(\mathbf{E}_1 \times \mathbf{H}_1^*\right)\\ &= \tfrac{1}{2}\,\mathrm{Re}\left(\hat{x}E_x^i\left(e^{-jk_1 z} + \Gamma e^{jk_1 z}\right) \times \hat{y}\tfrac{E_x^{i*}}{\eta_1}\left(e^{jk_1 z} - \Gamma^* e^{-jk_1 z}\right)\right)\\ &= \tfrac{1}{2}\hat{z}\tfrac{|E_x^i|^2}{\eta_1}\,\mathrm{Re}\left(1 - \Gamma^* e^{-2jkz} + \Gamma e^{2jkz} - |\Gamma|^2\right)\\ &= \tfrac{1}{2}\hat{z}\tfrac{|E_x^i|^2}{\eta_1}\left(1 - |\Gamma|^2\right)\end{aligned} \tag{5.291}$$

whereas in the medium 2 it is:

$$\mathbf{S}_{av2}(z) = \frac{1}{2}\hat{z}\frac{|E_x^t|^2}{\eta_2} \tag{5.292}$$

It is worth noting that, for lossless media, both quantities are independent of the coordinate z. Moreover, since there are no losses, the power flowingin the first medium is the same as the power flowing in the second:

$$\mathbf{S}_{av1} = \mathbf{S}_{av2} \tag{5.293}$$

so that:

$$\frac{1-|\Gamma|^2}{\eta_1} = \frac{|T|^2}{\eta_2} \tag{5.294}$$

V.5 Oblique incidence

If the plane wave incidence is not perpendicular to the boundary surface the problem becomes slightly more complicated. Having a plane wave with oblique incidence to a separation surface we can rearrange the coordinate system in such a way to fall in a combination of the two cases depicted in Fig. 5.14.

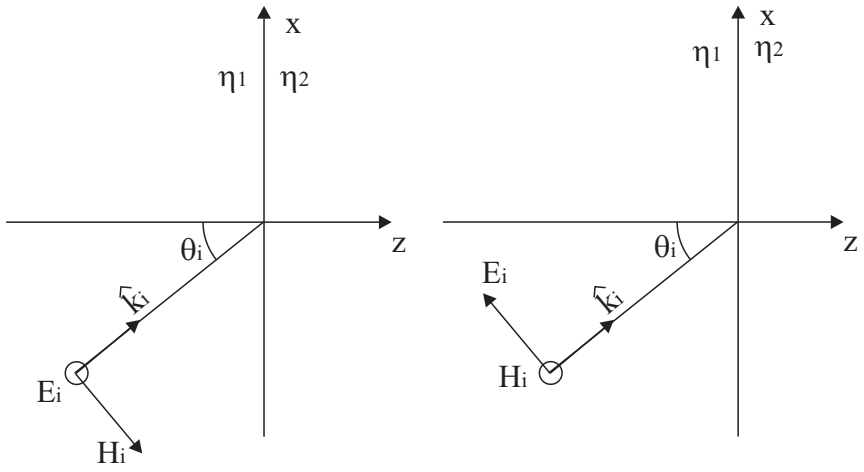

Fig. 5.14. Oblique incidence of a plane wave. (left) Perpendicular polarization. (right) Parallel polarization

The two cases differ from each other with the orientation of the fields. Let us define the *plane of polarization* as the plane containing the $\hat{k}$ vector the the normal to the boundary plane between the two media. It is easy to understand the this plane is the plane of the sheet. In this way, in Fig. 5.14 (left) the electric field is perpendicular to the polarization plane and this case is called *perpendicular polarization*. In Fig. 5.14 (right) the electric field is parallel to the plane and it is called *parallel polarization*. It is worth noting that an oblique incidence problem can be fully solved by dividing the problem into the two cases.

Parallel polarization

The incident electric field in parallel to the plane of polarization (i.e. the xz plane). The angle of incidence with respect to the normal to the boundary plane is θ_i. When an incident plane wave hit the plane of separation two other waves are generated. One is a reflected wave on the first medium and the other is a transmitted (or refracted) wave to the second medium. The inclination of the reflected wave is, in general, θ_r and the inclination of the trasmitted wave is θ_t. The problem is fully outlined in Fig. 5.15.

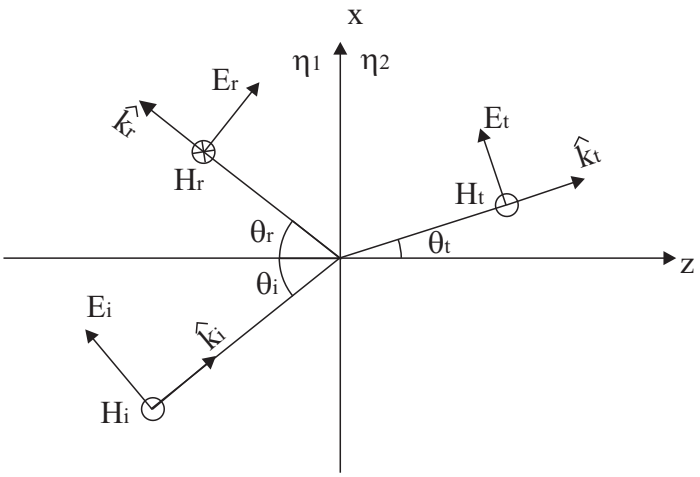

Fig. 5.15. Oblique incidence of a plane wave. Parallel polarization

The incident field in phasor form is:

$$\mathbf{E}_{\parallel}^{i} = E_{\parallel 0}^{i} \hat{k}_i e^{-jk_1 x_i} \tag{5.295}$$

It should be noted that the versor $\hat{k}_i$ is a comination of versors $\hat{x}$ and $\hat{z}$, as well as the coordinate x_i is a combination of the coordinates x and z. Using basic trigonometry, they can be found as:

$$\hat{k}_i = \hat{x}\cos\theta_i - \hat{z}\sin\theta_i$$

and

$$x_i = x\sin\theta_i + z\cos\theta_i$$

The incident field is then:

$$\begin{aligned}\mathbf{E}_{\parallel}^{i} &= E_{\parallel 0}^{i}(\hat{x}\cos\theta_i - \hat{z}\sin\theta_i)e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \\ \mathbf{H}_{\parallel}^{i} &= \frac{1}{\eta_1}\hat{k}_i \times \mathbf{E}_i = \hat{y}\frac{1}{\eta_1}E_{\parallel 0}^{i}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)}\end{aligned} \tag{5.296}$$

With similar considerations, the reflected wave is:

$$\begin{aligned}\mathbf{E}_{\parallel}^{r} &= E_{\parallel 0}^{r}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \\ \mathbf{H}_{\parallel}^{r} &= -\hat{y}\frac{1}{\eta_1}E_{\parallel 0}^{r}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)}\end{aligned} \tag{5.297}$$

and the trasmitted wave is:

$$\begin{aligned}\mathbf{E}_{\parallel}^{t} &= E_{\parallel 0}^{t}(\hat{x}\cos\theta_t - \hat{z}\sin\theta_t)e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \\ \mathbf{H}_{\parallel}^{t} &= \hat{y}\frac{1}{\eta_2}E_{\parallel 0}^{t}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)}\end{aligned} \tag{5.298}$$

The parallel components (i.e. the $\hat{x}$-components) of the electric and magnetic field must be continuous in $z = 0$ for every x. This means that:

$$\begin{aligned}&E_{\parallel 0}^{i}\cos\theta_i e^{-jk_1 x\sin\theta_i} + E_{\parallel 0}^{r}\cos\theta_r e^{-jk_1 x\sin\theta_r} = E_{\parallel 0}^{t}\cos\theta_t e^{-jk_2 x\sin\theta_t} \\ &\frac{E_{\parallel 0}^{i}}{\eta_1}e^{-jk_1 x\sin\theta_i} - \frac{E_{\parallel 0}^{r}}{\eta_1}e^{-jk_1 x\sin\theta_r} = \frac{E_{\parallel 0}^{t}}{\eta_2}e^{-jk_2 x\sin\theta_t}\end{aligned} \tag{5.299}$$

At this point we can define a reflection and a transmission coefficient:

$$\begin{aligned}\Gamma_{\parallel} &= \frac{E_{\parallel 0}^{r}}{E_{\parallel 0}^{i}} \\ T_{\parallel} &= \frac{E_{\parallel 0}^{t}}{E_{\parallel 0}^{i}}\end{aligned} \tag{5.300}$$

and substitute these quantities in (5.299) and simplify the incident electric field $E_{\parallel 0}^{i}$:

$$\begin{aligned}&\cos\theta_i e^{-jk_1 x\sin\theta_i} + \Gamma_{\parallel}\cos\theta_r e^{-jk_1 x\sin\theta_r} = T_{\parallel}\cos\theta_t e^{-jk_2 x\sin\theta_t} \\ &\frac{1}{\eta_1}e^{-jk_1 x\sin\theta_i} - \frac{\Gamma_{\parallel}}{\eta_1}e^{-jk_1 x\sin\theta_r} = \frac{T_{\parallel}}{\eta_2}e^{-jk_2 x\sin\theta_t}\end{aligned} \tag{5.301}$$

The conditions in (5.301) are satisfyed for every x, then the phase terms must be equal, that is:

$$k_1 x\sin\theta_i = k_1 x\sin\theta_r = k_2 x\sin\theta_t \tag{5.302}$$

From this equation we can derive two different conditions:

$$\begin{aligned} \sin\theta_i &= \sin\theta_r \\ k_1 \sin\theta_i &= k_2 \sin\theta_t \end{aligned} \tag{5.303}$$

The expression in (5.303) are the Snell's laws for oblique indicende with parallel polarization. They can be further simplified so that:

$$\begin{aligned} \theta_i &= \theta_r \\ \sqrt{\varepsilon_1 \mu_1}\sin\theta_i &= \sqrt{\varepsilon_2 \mu_2}\sin\theta_t \end{aligned} \tag{5.304}$$

The second expression in (5.304) can be further simplified if the media are not magnetic (i.e. $\mu_1 = \mu_2 = \mu_0$) and using the refraction index $n = \sqrt{\varepsilon_r}$:

$$n_1 \sin\theta_i = n_2 \sin\theta_t \tag{5.305}$$

which is the well known Snell's law for geometric optics.

From (5.301) and substituting the relations (5.304) we obtain:

$$\begin{aligned} \cos\theta_i + \Gamma_\parallel \cos\theta_r &= T_\parallel \cos\theta_t \\ \frac{1}{\eta_1} - \frac{\Gamma_\parallel}{\eta_1} &= \frac{T_\parallel}{\eta_2} \end{aligned} \tag{5.306}$$

that can be used to find the reflection and transmission coefficients:

$$\Gamma_\parallel = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i} \tag{5.307}$$

that is called the Fresnel reflection coefficient, and

$$T_\parallel = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i} \tag{5.308}$$

that is defined as the Fresnel transmission coefficient.

It is worth noting that, from equation (5.305), we can increase the incident angle θ_i in such a way that θ_t becoms equal to $\pi/2$. This happens when:

$$n_1 \sin\theta_{ic} = n_2$$

or

$$\theta_{ic} = \arcsin\frac{n_2}{n_1} \tag{5.309}$$

where θ_{ic} is called *critical angle*. This angle exists only if $n_2 < n_1$. A tipical example of the critical angle is shown in Fig. 5.16. In Fig. 5.16 (left) $n_1 < n_2$ so that the angle $\theta_t < \theta_i$. In this case the critical angle does not exist. This is the case, for example, when a incident wave is travelling on air and hit a glass. In Fig. 5.16 (right), on the contraty, $n_2 < n_1$ so that the angle $\theta_t > \theta_i$. In this case, for example when an incident wave travelling inside a glass hit a boundary surface with air, the critical angle does exist (dashed lines in Fig. 5.16 (right)).

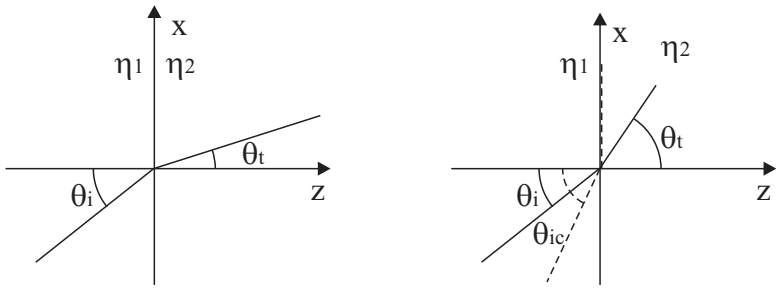

Fig. 5.16. Critical angle

Perpendicular polarization

The incident electric field is perpendicular to the plane of polarization (i.e. the xz plane). The angle of incidence with respect to the normal to the boundary plane is θ_i. As in the parrallel polarization case, when the incident wave hit the plane of separation two other waves are generated. One is a reflected wave on the first medium and the other is a transmitted (or refracted) wave to the second medium. The inclination of the reflected wave is, in general, θ_r and the inclination of the trasmitted wave is θ_t. The problem is fully outlined in Fig. 5.17.
The incident field in phasor form is:

$$\begin{aligned} \mathbf{E}^i_\perp &= E^i_{\perp 0}\hat{y}e^{-jk_1x_i} \\ \mathbf{H}^i_\perp &= \tfrac{1}{\eta_1}\hat{k}_i \times \mathbf{E}^i_\perp = \tfrac{1}{\eta_1}\hat{k}_i \times \hat{y}E^i_{\perp 0}e^{-jk_1x_i} \end{aligned} \tag{5.310}$$

where the versor $\hat{k}_i$ is a comination of versors $\hat{x}$ and $\hat{z}$ and the coordinate x_i is a combination of the coordinates x and z. Using basic trigonometry, they can be found as:

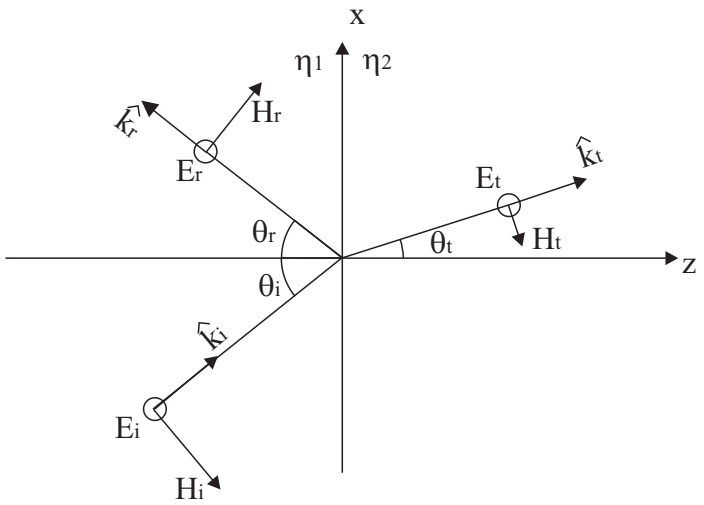

Fig. 5.17. Oblique incidence of a plane wave. Perpendicular polarization

$$\hat{k}_i = \hat{x}\cos\theta_i - \hat{z}\sin\theta_i$$

and

$$x_i = x\sin\theta_i + z\cos\theta_i$$

The incident field is then:

$$\begin{aligned} \mathbf{E}^i_\perp &= \hat{y}E^i_{\perp 0}e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \\ \mathbf{H}^i_\perp &= \frac{E^i_{\perp 0}}{\eta_1}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)e^{-jk_1(x\sin\theta_i + z\cos\theta_i)} \end{aligned} \tag{5.311}$$

With similar considerations, the reflected wave is:

$$\begin{aligned} \mathbf{E}^r_\perp &= \hat{y}E^r_{\perp 0}e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \\ \mathbf{H}^r_\perp &= \frac{E^r_{\perp 0}}{\eta_1}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)e^{-jk_1(x\sin\theta_r - z\cos\theta_r)} \end{aligned} \tag{5.312}$$

and the trasmitted wave is:

$$\begin{aligned} \mathbf{E}^t_\perp &= \hat{y}E^t_{\perp 0}e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \\ \mathbf{H}^t_\perp &= \frac{E^t_{\perp 0}}{\eta_2}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)e^{-jk_2(x\sin\theta_t + z\cos\theta_t)} \end{aligned} \tag{5.313}$$

The parallel components to the boundary surface (i.e. the $\hat{x}$-components) of the electric and magnetic field must be continuous in $z = 0$ for every x. This means that:

$$\begin{aligned} &E^i_{\perp 0}e^{-jk_1 x\sin\theta_i} + E^r_{\perp 0}e^{-jk_1 x\sin\theta_r} = E^t_{\perp 0}e^{-jk_2 x\sin\theta_t} \\ &\frac{E^i_{\perp 0}}{\eta_1}(-\cos\theta_i)e^{-jk_1 x\sin\theta_i} + \frac{E^r_{\perp 0}}{\eta_1}(\cos\theta_r)e^{-jk_1 x\sin\theta_r} = \frac{E^t_{\perp 0}}{\eta_2}(-\cos\theta_t)e^{-jk_2 x\sin\theta_t} \end{aligned} \tag{5.314}$$

At this point we can define a reflection and a transmission coefficient:

$$\begin{aligned} \Gamma_\perp &= \frac{E^r_{\perp 0}}{E^i_{\perp 0}} \\ T_\perp &= \frac{E^t_{\perp 0}}{E^i_{\perp 0}} \end{aligned} \tag{5.315}$$

and substitute these quantities in (5.314) and simplify the incident electric field $E^i_{\perp 0}$:

$$\begin{aligned} &e^{-jk_1 x\sin\theta_i} + \Gamma_\perp e^{-jk_1 x\sin\theta_r} = T_\perp e^{-jk_2 x\sin\theta_t} \\ &\frac{1}{\eta_1}(-\cos\theta_i)e^{-jk_1 x\sin\theta_i} + \Gamma_\perp \frac{1}{\eta_1}(\cos\theta_r)e^{-jk_1 x\sin\theta_r} = T_\perp \frac{1}{\eta_2}(-\cos\theta_t)e^{-jk_2 x\sin\theta_t} \end{aligned} \tag{5.316}$$

The conditions in (5.316) must be satisfyed for every x, then the phase terms must be equal, that is:

$$k_1 x\sin\theta_i = k_1 x\sin\theta_r = k_2 x\sin\theta_t \tag{5.317}$$

From this equation we can derive two different conditions:

$$\begin{aligned} \sin\theta_i &= \sin\theta_r \\ k_1\sin\theta_i &= k_2\sin\theta_t \end{aligned} \tag{5.318}$$

The expression in (5.318) are the Snell's laws for oblique indicende with perpendicular polarization. They can be further simplified so that:

$$\begin{aligned} \theta_i &= \theta_r \\ \sqrt{\varepsilon_1\mu_1}\sin\theta_i &= \sqrt{\varepsilon_2\mu_2}\sin\theta_t \end{aligned} \tag{5.319}$$

For not magnetic media (i.e. $\mu_1 = \mu_2 = \mu_0$), using the refraction index $n = \sqrt{\varepsilon_r}$:

$$n_1\sin\theta_i = n_2\sin\theta_t \tag{5.320}$$

which is the well known Snell's law for geometric optics. It is worth noting that these equations are exactly the same as the ones derived for the parallel polarization.

From (5.316) and substituting the relations (5.319) we obtain:

$$\begin{aligned} 1+\Gamma_\perp &= T_\perp \\ -\tfrac{\cos\theta_i}{\eta_1}+\Gamma_\perp\tfrac{\cos\theta_r}{\eta_1} &= -T_\perp\tfrac{\cos\theta_t}{\eta_2} \end{aligned} \tag{5.321}$$

that can be used to find the reflection and transmission coefficients:

$$\Gamma_\perp = \frac{\eta_2\cos\theta_i - \eta_1\cos\theta_t}{\eta_2\cos\theta_i - \eta_1\cos\theta_t} \tag{5.322}$$

that is called the Fresnel reflection coefficient, and

$$T_\perp = \frac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_i - \eta_1\cos\theta_t} \tag{5.323}$$

that is defined as the Fresnel transmission coefficient.

As in the parallel polarization case, there exists a critical angle at which the field is completely reflected:

$$\theta_{ic} = \arcsin\frac{n_2}{n_1} \tag{5.324}$$

As in the previous case, this angle exists only when $n_2 < n_1$.

Brewster angle

Looking at the Fresnel reflection coefficient (5.307) we can ask if exist an angle such that this coefficient vanishes. The equation to be solved is:

$$\Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t - \eta_1\cos\theta_i} = 0 \tag{5.325}$$

This happens when the numerator of (5.325) is equal to zero:

$$\eta_2\cos\theta_t = \eta_1\cos\theta_i$$

in order to simplify the problem, we can consider non magnetic media ($\mu_1 = \mu_2 = \mu_0$) so that:

$$\frac{1}{\sqrt{\varepsilon_2}}\cos\theta_t = \frac{1}{\sqrt{\varepsilon_1}}\cos\theta_i \tag{5.326}$$

by squaring and substituting $\cos^2$ with $1-\sin^2$ we obtain:

$$\frac{1}{\varepsilon_2}(1-\sin^2\theta_t) = \frac{1}{\varepsilon_1}(1-\sin^2\theta_i) \tag{5.327}$$

but, from equation (5.305) we have that:

$$\sin^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i \tag{5.328}$$

substituting equation (5.328) into equation (5.326) we obtain an equation with only the angle θ_i. By solving this equation we obtain:

$$\theta_B = \theta_i = \arcsin\sqrt{\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}} \tag{5.329}$$

θ_B is defined as Brewster angle and is the angle at which the reflection coefficient vanishes. It is worth noting that this angle exists for any media since the argument of the arcsin is always less than the unity.

On the contray, the Brewster angle for perpendicuar polarization does not exist. In fact, imposing:

$$\Gamma_\perp = 0$$

yield the condition:

$$\eta_2\cos\theta_i = \eta_1\cos\theta_t$$

squaring this expression and substituting $\cos^2$ with $1-\sin^2$ we find:

$$\frac{1}{\varepsilon_2}(1-\sin^2\theta_i) = \frac{1}{\varepsilon_1}(1-\sin^2\theta_t) \tag{5.330}$$

Using the Snell's law:

$$\varepsilon_1\sin^2\theta_i = \varepsilon_2\sin^2\theta_t \tag{5.331}$$

and substituting θ_t in (5.330), we find that:

$$\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\sin^2\theta_i = \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\sin^2\theta_i$$

which gives:

$$\sin^2\theta_i = -1 \tag{5.332}$$

that does not admit a solution for real angles.

If we draw a graph with the Fresnel reflections coefficients for parallel and perpendicular polarizations as a function of the angle of incidence θ_i (Fig. 5.18). The example is computed considering $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 3$. We can see that $\Gamma_{\parallel}$ vanishes at the Brewster angle (60° in this particular case) whereas $\Gamma_{\perp}$ is always non zero.

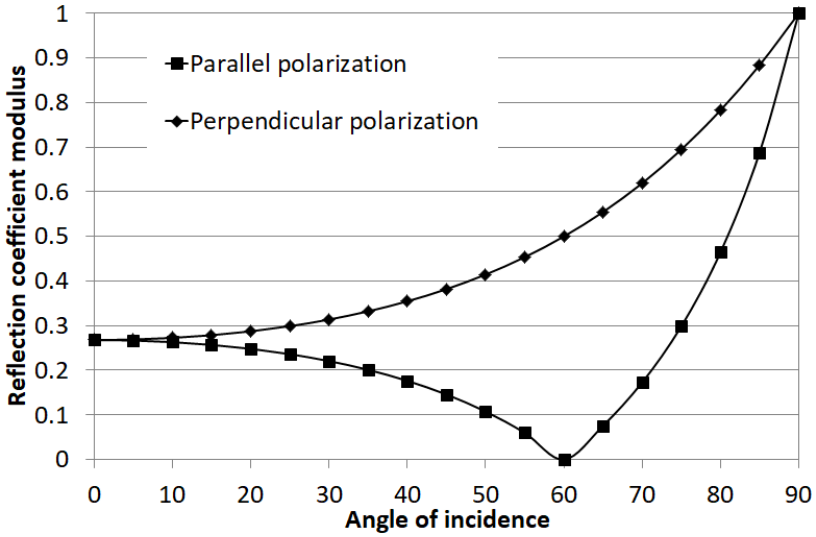

Fig. 5.18. Brewster angle. The graph is computed considering $\varepsilon_{r1} = 1$ and $\varepsilon_{r2} = 3$

From Fig. 5.18 we can see that for $\theta_i = 0$ the two coefficients are equal, as it is the case of a normal incidence of a wave to a boundary surface. At the right extreme the amplitude of both reflection coefficient go to the unity. The angle at the right extreme can be θ_{ic}, if the critical angle exists, or $\pi/2$.

It is worth noting that if two polarizations incides to a surface with an incident angle equal to θ_B, only one (i.e. the perpendicular polarization) is totally transmitted to the second medium. This means that the reflected wave if composed of onle one polarization. For this reason the Brewster angle is also called *polarizing angle*.

V.6 Plane waves in lossy media

When a plane wave is propagating through a lossy media, there are an induced current due to the presence of the electric field. Such current is proportional to the electric field as the relation between them is the Ohm's law:

$$\mathbf{J} = \sigma \mathbf{E}$$

Then, the electromagnetic field is the solution of the Maxwell equations with sources:

$$\not{\partial}\mathbf{F} = -\eta \mathbf{J}$$

By appling again the $\not{\partial}$ operator, we obtain:

$$\not{\partial}^2\mathbf{F} = -\not{\partial}\eta \mathbf{J}$$

that can be rewritten using ():

$$-(\partial_x^2+\partial_y^2+\partial_z^2+k^2)\mathbf{F} = -\eta \begin{pmatrix} jk & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & jk & \partial_x + i\partial_y & -\partial_z \\ -\partial_z & -\partial_x + i\partial_y & -jk & 0 \\ -\partial_x - i\partial_y & \partial_z & 0 & -jk \end{pmatrix} \begin{pmatrix} J_z \\ iJ_y + J_x \\ 0 \\ 0 \end{pmatrix} \tag{5.333}$$

Taking the first equation and applying the Ohm's law:

$$-(\partial_x^2+\partial_y^2+\partial_z^2+k^2)E_z = -\eta(jk)\sigma E_z \tag{5.334}$$

that can be rewritten as:

$$\begin{aligned} &(\partial_x^2+\partial_y^2+\partial_z^2)E_z + \omega^2\mu\left(\varepsilon - j\tfrac{\sigma}{\omega}\right)E_z = 0 \\ &(\partial_x^2+\partial_y^2+\partial_z^2)E_z + \omega^2\mu\varepsilon_c E_z = 0 \end{aligned} \tag{5.335}$$

The quantity

$$\varepsilon_c = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon'' \tag{5.336}$$

is a complex dielectric constant, that takes into account for the losses of the medium. Same equations can be obtained for the other components (E_x and E_y), so that wave equation can be rewritten as:

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\mathbf{E} - \gamma^2\mathbf{E} = 0 \tag{5.337}$$

by defining

$$\gamma^2 = -\omega^2\mu\varepsilon_c = -\omega^2\mu\left(\varepsilon' - j\varepsilon''\right) \tag{5.338}$$

where $\gamma = \alpha + j\beta$ is the complex propagation constant, α is the attenuation constant and β is the phase constant. α and β can be computed by considering that:

$$\gamma^2 = (\alpha + j\beta)^2 = \alpha^2 - \beta^2 + j2\alpha\beta \tag{5.339}$$

by putting together equations (5.338) and (5.339) we obtain the following system of equations:

$$\begin{aligned} \alpha^2 - \beta^2 &= -\omega^2\varepsilon'\mu \\ 2\alpha\beta &= \omega^2\varepsilon''\mu \end{aligned} \tag{5.340}$$

The solutions are:

$$\begin{aligned} \alpha &= \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right]\right\}^{1/2} \qquad [Np/m] \\ \beta &= \omega\left\{\frac{\mu\varepsilon'}{2}\left[\sqrt{1+\left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right]\right\}^{1/2} \qquad [rad/m] \end{aligned} \tag{5.341}$$

Looking at equations (5.341), we can see that if $\varepsilon'' = 0$ then $\alpha = 0$ and $\beta = \omega\sqrt{\mu\varepsilon'}$, that is the lossless case.

Considering a plane wave propagating along z-direction (i.e. $\partial_x = \partial_y = 0$), the wave equation for the x-component is:

$$\partial_z^2 E_x(z) - \gamma^2 E_x(z) = 0 \tag{5.342}$$

the solution is of the type:

$$E_x(z) = E_{x0}^{+} e^{-\alpha z} e^{-j\beta z} + E_{x0}^{-} e^{\alpha z} e^{j\beta z} \tag{5.343}$$

The corresponding magnetic field can be computed with the formula:

$$\mathbf{H} = \frac{1}{\eta_c} \hat{k} \times \mathbf{E} \tag{5.344}$$

where

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}}$$

is the complex wave impedance. For example, if the electric field has only the x-component, the only component of the magnetic field is directed along y:

$$H_y(z) = \frac{E_{x0}^{+}}{\eta_c} e^{-\alpha z} e^{-j\beta z} - \frac{E_{x0}^{-}}{\eta_c} e^{\alpha z} e^{j\beta z}$$

It is worth noting that the electric field and the magnetic field are now out of phase and the phase shift depends on η_c.

Let us now consider only the propagating wave and only the x-component of the electric field. The amplitude of the field is the following:

$$|E_x(z)| = |E_{x0}^{+} e^{-\alpha z} e^{-j\beta z}| = |E_{x0}^{+}| e^{-\alpha z}$$

we can see that the field decreases with exponential law. In particular after a distance $\delta_s = 1/\alpha$ the field is rughly the 37% of the incident one. After a distance of $5\delta_s$ the field is almost zero. The distance δ_s is called *skin depth*.

The ratio between the imaginary and the real parts of the complex dielectric constant (5.336) is defined as:

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} \tag{5.345}$$

From this ratio we can divide the materials into three classes:

- If $\tan\delta \ll 1$ the medium is a low loss dielectric
- If $\tan\delta \gg 1$ the medium is a good conductor
- In between the medium is a semiconductor

Low loss dielectric material

Usually for a low loss dielectric material $10^{-4} < \tan\delta < 10^{-2}$. In this range, for example, we have many substrate suited for printed circuit working at very high frequencies (> 1 GHz). In this case we can approximate the propagation constant:

$$\gamma = j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{1/2}$$

since $\varepsilon''/\varepsilon' \ll 1$ we can approximate the term under the square root so that:

$$\gamma \approx j\omega\sqrt{\mu\varepsilon'}\left(1 - j\frac{\varepsilon''}{2\varepsilon'}\right)$$

then the attenuation and phase constants are:

$$\begin{aligned} \alpha &\approx \tfrac{\omega\varepsilon''}{2}\sqrt{\tfrac{\mu}{\varepsilon'}} \qquad [Np/m] \\ \beta &\approx \omega\sqrt{\mu\varepsilon'} \qquad [rad/m] \end{aligned} \tag{5.346}$$

Looking at the equations (5.346), we can see that the phase constant is the same as in the lossless case, with the addition of am attenuation due to α. Even the approximated wave impedance is equal to the lossless case ($\eta_c \approx \sqrt{\mu/\varepsilon'}$).

Good conductor

In the case of a good conductor $\varepsilon''/\varepsilon' \gg 1$. The we can approximate:

$$\gamma^2 = -\omega^2\mu\varepsilon'\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)$$

with:

$$\gamma^2 \approx -\omega^2\mu\varepsilon'\left(-j\frac{\varepsilon''}{\varepsilon'}\right) = j\omega^2\mu\varepsilon'' = j\omega\mu\sigma$$

where $\varepsilon'' = \sigma/\omega$. Now:

$$\gamma \approx \sqrt{j\omega\mu\sigma} = \frac{1+j}{\sqrt{2}}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\sigma\mu}$$

So that the attenuation and propagation constants are:

$$\begin{aligned} \alpha &\approx \sqrt{\pi f \sigma \mu} \\ \beta &\approx \sqrt{\pi f \sigma \mu} \end{aligned} \tag{5.347}$$

and the wave impedance is:

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon' \left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)}}$$

that can be approximated as:

$$\eta_c \approx \sqrt{\frac{\mu}{-j\varepsilon''}} = \frac{1+j}{\sqrt{2}}\sqrt{\frac{\mu}{\varepsilon''}} = (1+j)\frac{\pi f \mu}{\sigma}$$

V.7 Current flow in a good conductor

Let us consider a plane wave incident to a good conductor. The boundary surface between the free space and the conductor is the *xy* plane and the wave is propagation along z direction (i.e. the incidence is orthogonal to the boundary surface) (Fig. xx)

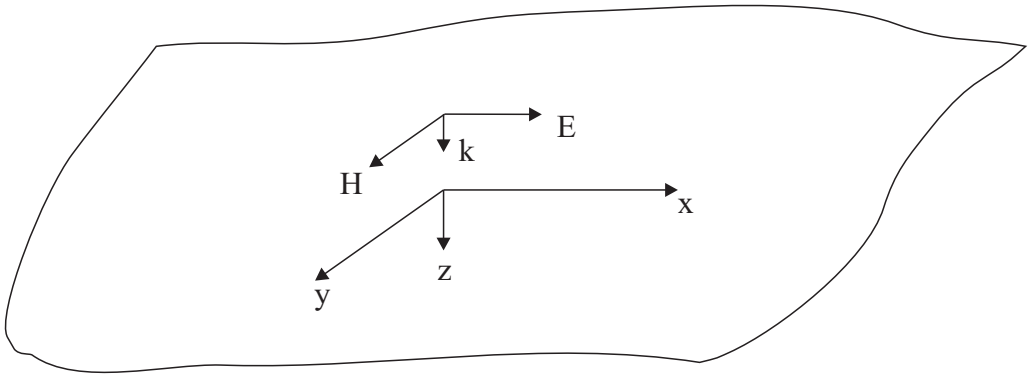

Fig. 5.19. Plane wave incident to a good gonductor

Moreover, the boundary surface is at $z = 0$ and the plane wave is linearly polarized with only the E_x component. At $Z < 0$ we have:

$$\begin{aligned} \mathbf{E}(z) &= \hat{x} E_0 e^{-j\beta z} + \hat{x}\Gamma E_0 e^{j\beta z} \\ \mathbf{H}(z) &= \hat{y}\frac{E_0}{\eta} e^{-j\beta z} - \hat{y}\Gamma\frac{E_0}{\eta} e^{j\beta z} \end{aligned} \tag{5.348}$$

At $Z > 0$ we have:

$$\begin{aligned}\mathbf{E}(z) &= \hat{x}E_0 e^{-\alpha z}e^{-j\beta z} \\ \mathbf{H}(z) &= \hat{y}\frac{E_0}{\eta_c}e^{-\alpha z}e^{-j\beta z}\end{aligned} \tag{5.349}$$

Considering the Ohm's law, the current density in the conductor is proportional to the electric field:

$$\mathbf{J}(z) = \hat{x}\mathbf{J}_x(z) \tag{5.350}$$

with

$$\mathbf{J}_x(z) = \sigma E_0 e^{-\alpha z}e^{-j\beta z} = J_0 e^{-\alpha z}e^{-j\beta z} \tag{5.351}$$

where α and β can be computed using equations (5.347).

The skin depth is

$$\delta_s = \frac{1}{\alpha} \tag{5.352}$$

so that

$$\mathbf{J}_x(z) = J_0 e^{-(1+j)\frac{z}{\delta_s}} \tag{5.353}$$

The total current flowing in the conductor is:

$$\mathbf{I} = w\int_0^\infty \mathbf{J}_x(z)dz = w\int_0^\infty J_0 e^{-(1+j)\frac{z}{\delta_s}} = \frac{J_0 w\delta_s}{1+j} \tag{5.354}$$

Since the currenct decays with exponential law, after a depth equal to $5\delta_s$ the current density is very small and can be neglected. In this way, we can compute the current with a finite upper extreme of the integral. The voltage drop along a distance l is:

$$\mathbf{V} = E_0 l = \frac{J_0}{\sigma}l \tag{5.355}$$

The ratio between $\mathbf{V}$ and $\mathbf{I}$ is:

$$Z = \frac{\mathbf{V}}{\mathbf{I}} = \frac{\frac{J_0}{\sigma} l}{\frac{J_0 w \delta_s}{1+j}} = \frac{1+j}{\sigma \delta_s} \frac{l}{w} = Z_s \frac{l}{w} \tag{5.356}$$

the quantity Z_s is defined as *surface impedance* and it is:

$$Z_s = \frac{1+j}{\sigma \delta_s} \qquad [\Omega] \tag{5.357}$$

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6

Guided–Wave Electromagnetic Fields

I Introduction

In this chapter electromagnetic waves propagating in waveguiding structures studied. We investigate the eigensolution, i.e. the solutions that can arise also when no sources are present in the considered region of space. We will start with the simple example of the parallel plate waveguide in order to introduce the most relevant phenomenology with a minimum of analytical burden. Practical transmission lines as e.g. rectangular, circular and coaxial waveguides are studied. Examples of more complex structures are then considered.

II Introduction to waveguides

Electromagnetic energy and signals can be transferred either in free–space or by guided waves. In this chapter we deal with guided wave propagation. The classification of waveguides can be done on several, different, basis. As an example we can distinguish between

- closed or open structures
- separable or non separables waveguides
- dielectric waveguides, or multi–conductors waveguides, or single–conductor waveguides
- power handling capabilities
- manufacturing characteristics etc.

In general, a waveguide is a structure with longitudinal symmetry capable to support electromagnetic waves. The following hypothesis are made.

- the medium is homogeneous and without ferrites therefore $\mathbf{B} = \mu_0 \mathbf{H}$;

- the medium is isotropic and homogeneous, so that $\mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon \mathbf{E}$;
- $\mathbf{J} = 0$
- $\rho = 0$
- sinusoidal time–dependence of the type $e^{j\omega t}$ from which $\frac{d}{dt} = j\omega$

In order to study the various types of electromagnetic energy propagation it is advantageous to see how the various components of the electromagnetic field are related between each other.
We have already seen that in the sourceless case

$$\underline{\partial}^2 \bar{F} = \left(\partial_x^2 + \partial_y^2 + \partial_z^2 + k^2\right) \bar{F} = 0 \tag{6.1}$$

By separating also the real and imaginary parts we note that each component of the electromagnetic field, say Ψ, must satisfy the Helmholtz equation

$$\left(\nabla^2 + k^2\right) \Psi = 0 \tag{6.2}$$

in the frequency domain. Thus in the waveguide this equation should be satisfied for each field component.
In section II.3 of chapter 4 we have seen that eq. (4.111), which is here repeated for convenience in the sourceless case, is

$$
\begin{aligned}
\not{\partial}\bar{F} &= \begin{pmatrix} jk & 0 & \partial_z & \partial_x - i\partial_y \\ 0 & jk & \partial_x + i\,\partial_y & -\partial_z \\ -\partial_z & -\partial_x + i\partial_y & -jk & 0 \\ -\partial_x - i\partial_y 2 & \partial_z & 0 & -jk \end{pmatrix} \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i H_z \\ \eta\,\left(i H_x - H_y\right) \end{pmatrix} \\
&= \begin{pmatrix} jkE_z - \eta\,\left(\frac{\partial}{\partial x}H_y - \frac{\partial}{\partial y}H_x\right) \\ jkE_x - \eta\,\left(\frac{\partial}{\partial y}H_z + \frac{\partial}{\partial z}H_y\right) \\ -\nabla\cdot\mathbf{E} \\ jk\,\eta\,H_y - \frac{\partial}{\partial x}E_z + \frac{\partial}{\partial z}E_x \end{pmatrix} \\
&\quad + i \begin{pmatrix} \eta\,\nabla\cdot\mathbf{H} \\ jkE_y + \eta\,\left(\frac{\partial}{\partial x}H_z - \frac{\partial}{\partial z}H_x\right) \\ -jk\,\eta\,H_z - \frac{\partial}{\partial x}E_y + \frac{\partial}{\partial y}E_x \\ -jk\,\eta\,H_x - \frac{\partial}{\partial y}E_z + \frac{\partial}{\partial z}E_y \end{pmatrix} \\
&= 0
\end{aligned}
\tag{6.3}
$$

It is instructing to analyze the cases for the field–dependence on x, y, z, (3D–case), on x, z (2D–case) and just on z (1D–case). These field dependencies can be derived directly from the above equations or by using the code

```
wxm/Maxwell_Dirac_v03.wxm
```

II.1 Sourceless 3D–case

In this case all the field components are linked to each other, as apparent from (6.4). Since this is the most complex case we will study it as the last one.

II.2 2D problem: parallel plate waveguide

Let us assume that the structure does not change with y coordinate, as e.g. in a parallel plate waveguide infinite in the y direction. Accordingly, we have $\partial_y = 0$. A possible example is the parallel plate waveguide, with the structure composed by two infinite planes of perfect electric conductor (p.e.c.) placed at $x = 0$ and at $x = a$.

As a result for the $\underline{\not{\partial}}\bar{F}$ operator we have

$$
\begin{aligned}
\underline{\partial}\bar{F} &= \begin{pmatrix} jk & 0 & \partial_z & \partial_x \\ 0 & jk & \partial_x & -\partial_z \\ -\partial_z & -\partial_x & -jk & 0 \\ -\partial_x & \partial_z & 0 & -jk \end{pmatrix} \begin{pmatrix} E_z \\ iE_y + E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x - H_y\right) \end{pmatrix} \\
&= \begin{pmatrix} jkE_z - \eta\,\left(\frac{\partial}{\partial x} H_y\right) \\ jkE_x - \eta\,\left(\frac{\partial}{\partial z} H_y\right) \\ -\nabla\cdot\mathbf{E} \\ jk\eta H_y - \frac{\partial}{\partial x} E_z + \frac{\partial}{\partial z} E_x \end{pmatrix} \\
&\quad + i \begin{pmatrix} \eta\,\nabla\cdot\mathbf{H} \\ jkE_y + \eta\,\left(\frac{\partial}{\partial x} H_z - \frac{\partial}{\partial z} H_x\right) \\ -jk\eta H_z - \frac{\partial}{\partial x} E_y \\ -jk\eta H_x + \frac{\partial}{\partial z} E_y \end{pmatrix} \\
&= 0
\end{aligned} \tag{6.4}
$$

It is possible to observe that the six equations are not anymore coupled. Rather, there are two blocks composed by three equations that are coupled. In particular, from the imaginary part we have a block involving $E_y, H_x, H_z (TE)$

$$jkE_y + \eta\,\left(\frac{\partial}{\partial x} H_z - \frac{\partial}{\partial z} H_x\right) = 0 \tag{6.5}$$

$$-jk\eta H_z - \frac{\partial}{\partial x} E_y = 0 \tag{6.6}$$

$$-jk\eta H_x + \frac{\partial}{\partial z} E_y = 0 \tag{6.7}$$

while from the real part we have a block that only contains $E_x, H_y, E_z (TM)$

$$jkE_z - \eta\,\left(\frac{\partial}{\partial x} H_y\right) = 0 \tag{6.8}$$

$$jkE_x - \eta\,\left(\frac{\partial}{\partial z} H_y\right) = 0 \tag{6.9}$$

$$jk\eta H_y - \frac{\partial}{\partial x} E_z + \frac{\partial}{\partial z} E_x = 0 \tag{6.10}$$

II.3 1D problem: parallel plate waveguide and transmission lines

Let us assume that the fields does not change with x, y coordinate. We obtain for real and imaginary part of Maxwell's equations

$$\begin{aligned}
\underline{\partial}\bar{F} &= \begin{pmatrix} jk & 0 & \partial_z & 0 \\ 0 & jk & 0 & -\partial_z \\ -\partial_z & 0 & -jk & 0 \\ 0 & \partial_z & 0 & -jk \end{pmatrix} \begin{pmatrix} E_z \\ iE_y + E_x \\ \eta\, iH_z \\ \eta\,\left(iH_x - H_y\right) \end{pmatrix} \\
&= \begin{pmatrix} jkE_z \\ jkE_x + \eta\,\left(\frac{\partial}{\partial z} H_y\right) \\ -\frac{\partial}{\partial z} E_z \\ jk\eta H_y + \frac{\partial}{\partial z} E_x \end{pmatrix} \\
&\quad + i \begin{pmatrix} \eta\,\frac{\partial}{\partial z} H_z \\ jkE_y - \eta\,\left(\frac{\partial}{\partial z} H_x\right) \\ -jk\eta H_z \\ -jk\eta H_x + \frac{\partial}{\partial z} E_y \end{pmatrix} \\
&= 0\,.
\end{aligned} \tag{6.11}$$

It is apparent that the z components of the fields, and their derivatives, are null. The real part provides the following equations

$$\begin{aligned}
jkE_x &= -\eta\,\left(\frac{\partial}{\partial z} H_y\right) \\
jk\eta H_y &= -\frac{\partial}{\partial z} E_x
\end{aligned} \tag{6.12}$$

and in a parallel plate waveguide composed by two infinite metallic planes placed at $x = 0$ and at $x = a$ they represent the TEM mode.

III General procedure for finding the modes: PPW example

III.1 TM case

In the TM case we have, in the longitudinal direction, only the E_z field. It has already been noted that, the field potential A_z should satisfy the same boundary conditions as E_z. Let us therefore find the E_z field. We have seen that each component (denoted by Ψ) of the electromagnetic field should satisfy the equation

$$\nabla^2\Psi + k^2\Psi = 0 \tag{6.13}$$

By applying the technique of separation of variables, assuming a solution of the type $e^{-jk_z z}$ for the z dependence, and since there is no variation along the y coordinate, we have only $k_x, k_t = k_x = \sqrt{k^2 - k_z^2}$ and the resulting equation is e.g.:

$$\frac{d^2E_z(x)}{dx^2} + k_x^2 E_z(x) = 0 \tag{6.14}$$

The solution of the above equation is well known and can be expressed as a combination of exponential functions or sine and cosine in the form

$$E_z(x) = A\cos(k_x x) + B\sin(k_x x) \tag{6.15}$$

with the constants A, B and the values of k_x still to be determined. Naturally the resulting field should satisfy the boundary conditions, that for the electric field $E_z(x)$, on the metallic surfaces, are:

- $E_z(x=0) = 0$,
- $E_z(x=a) = 0$.

In order to satisfy the above condition at $x = 0$ we have $A = 0$. Accordingly, the solution takes the form

$$E_z(x) = B\sin(k_x x)\,. \tag{6.16}$$

For this function the condition at $x = 0$ is automatically satisfied, while the condition at $x = a$ requires:

$$k_x a = n\pi \tag{6.17}$$

or

$$k_x = \frac{n\pi}{a} = k_t \tag{6.18}$$

The above values are the eigenvalues of the field equation (6.14).

As a consequence we can select the following potential for finding the modes.

$$A_z = \sin\left(\frac{n\pi x}{a}\right) e^{-jk_z z} \tag{6.19}$$

which corresponds to the quadrivector

$$\bar{A} = \begin{pmatrix} v\sin\left(\frac{n\pi x}{a}\right) e^{-jzk_z} \\ 0 \\ \phi \\ 0 \end{pmatrix}. \tag{6.20}$$

By taking into account that the scalar function ϕ should satisfy the Lorenz gauge (4.151), here repeated for convenience,

$$\phi = \frac{jv}{k}\nabla\cdot\mathbf{A} \tag{6.21}$$

the quadrivector potential becomes

$$\bar{A} = \begin{pmatrix} v\sin\left(\frac{n\pi x}{a}\right) e^{-jk_z z} \\ 0 \\ \frac{1}{k}v\sin\left(\frac{n\pi x}{a}\right) e^{-jk_z z} k_z \\ 0 \end{pmatrix}. \tag{6.22}$$

Application of the equation

$$\bar{F} = -\partial\!\!\!/\bar{A} \tag{6.23}$$

provides the sought result for the field

$$
\begin{aligned}
E_x &= -\frac{n\pi}{a}\frac{k_z v}{k}\cos\left(\frac{n\pi x}{a}\right)e^{-jk_z z} \\
E_z &= -jv\frac{k^2-k_z^2}{k}\sin\left(\frac{n\pi x}{a}\right)e^{-jk_z z} \\
H_y &= -\frac{v}{\eta}\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)e^{-jk_z z}
\end{aligned}
\tag{6.24}
$$

It is instructive to consider separately the two cases when considering the expression for $k_z = \sqrt{k^2 - k_x^2}$.

- When $k > k_x$ we have $k_z > 0$ and propagation of the type $e^{-jk_z z}$ in the positive z direction; this condition is called *above cut–off*.
- when $k < k_x$ $k_z = -j\gamma$ with γ a real number and we have $e^{-jk_z z} = e^{-\gamma z}$ i.e. when the wave propagates in the positive z direction it is exponentially attenuated. Naturally when the wave propagates in the negative z direction we have $k_z = j\gamma$ and therefore attenuation in the negative z direction. This condition is called *below cut–off*.

When increasing the frequency a mode that was below cut-off may start to propagate. If we further increase the frequency we can have several modes above cut-off. This regime is called *multimodal*. From a communication viewpoint only the single mode regime is usable. In fact, if we have different modes they propagate with different phase velocity and therefore changing the relative phases. Typically, practical devices are used in order to avoid the propagation of unwanted modes (generated e.g. by discontinuities).
The procedure for the TM case are covered in the following code

```
wxm/Potentials_v02_TM.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

/* Define Dirac Gammas */
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

g0 :  kronecker_product(%sigma[3],%sigma[0])$
g1 :  kronecker_product(%i*%sigma[2],%sigma[1])$
g2 :  kronecker_product(%i*%sigma[2],%sigma[2])$
g3 :  kronecker_product(%i*%sigma[2],%sigma[3])$
g4 :  g0$
g5 :  -kronecker_product(%sigma[1],%sigma[0])$

/*
The block Dslash perform the d-slash operation
in rectangular coordinates on a quadrivector
*/
/********** Block start **********************/
Dslash(Ap) :=
block([],
DAp :  factor(j*k*g0.Ap+diff(g1.Ap,x)+diff(g2.Ap,y)+diff(g3.Ap,z))
)$
/********** Block end **********************/

print("----------------------------------------")$
print("The vector E and H in general depend on x,y,z,")$

/* 3D case */
depends([Ex,Ey,Ez],[x,y,z]);
depends([Hx,Hy,Hz],[x,y,z]);
```

```
depends([Ax,Ay,Az],[x,y,z]);
depends([%Phi],[x,y,z]);

/* For recovering the Hertzian potential
depends(Psie,[x,y,z]);
Ax : 0$
Ay : 0$
Az : j * k/(v)*Psie$
*/

/* For potential in a rectangular waveguide */
snx : sin(n*%pi/a*x)$
smy : sin(m*%pi/b*y)$
propz : %e^(-j * k[z]*z)$
Ax:0 $
Ay:0$
Az : snx * smy * propz;


/* Potential in a parallel plate waveguide
snx : sin(n*%pi/a*x)$
propz : %e^(-j * k[z]*z)$
Ax:0 $
Ay:0$
Az : snx  * propz;
*/

print("----------------------------------------")$
print("compose the Field Quadrivector")$
A:  matrix([v* Az],[v* Ax+%i * Ay*v],[%Phi],[0]);

print("compose the Field Quadrivector")$
F: matrix([Ez],[Ex+%i * Ey],[%i * %eta *Hz],[%i*%eta*(Hx + %i*Hy)]);

/*
print("compose the Source Quadrivector")$
J: matrix([Jz],[Jx+%i * Jy],[%rho*v],[0]);
```

```
*/

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the potential gives")$
Apot : rectform(A)$
Dslash(Apot)$
DslApot : rectform(DAp);
/*
print("further application gives")$
Dslash(DslApot)$
DAp2 : DAp$
DAp2 :subst(-1,j^2,DAp2);
print("Potentials equations")$
DAp2 - %eta * J;
*/
print("setting to zero the scalar part gives Lorenz condition")$
re3 :realpart(DslApot[3,1]);
solphi : solve(re3,%Phi);
%Phi : rhs(solphi[1])$
print("compose again the Potential Quadrivector satisfying Lorenz gauge")$
A: matrix([v* Az],[v* Ax+%i * Ay*v],[%Phi],[0]);

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the Potential gives")$
Ap : rectform(A)$
Dslash(Ap)$
DAp1 : - rectform(DAp)$
DAp1 :subst(-1,j^2,DAp1);
/* DAp1 :subst(-1/j,j,DAp1);*/

Ex : factor(realpart(DAp1[2][1]))$
Ey : factor(imagpart(DAp1[2][1]))$
Ez : factor(DAp1[1][1])$
Hy : -1/%eta *ratsimp(realpart(DAp1[4][1]))$
Hx : 1/%eta *ratsimp(imagpart(DAp1[4][1]))$
Hz : 1/%eta * ratsimp(imagpart(DAp1[3][1]))$
```

```
print("Which provides the field equations")$
print('Ex," = ",Ex)$
print('Ey," = ",Ey)$
print('Ez," = ",Ez)$
print('Hx," = ",Hx)$
print('Hy," = ",Hy)$
print('Hz," = ",Hz)$

/*
Ey/Hx;
Ex/Hy;
*/
print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

III.2 TE case

For the TE we can notice that the potential C_z is proportional to H_z. As a consequence we can select

$$C_z = \cos(k_x x)\, e^{-jk_z z} \tag{6.25}$$

where, as before,

$$k_x = \frac{n\pi}{a} = k_t \tag{6.26}$$

is obtained from the boundary conditions. In the TE case the potential quadrivector is

$$\bar{C} = \begin{pmatrix} -i\eta\,\psi \\ 0 \\ -i\eta\, v \cos\left(\frac{n\pi x}{a}\right) e^{-jk_z z} \\ 0 \end{pmatrix}. \tag{6.27}$$

By taking into account that the scalar function ψ should satisfy the Lorenz gauge (4.173), here repeated for convenience,

$$\psi = \frac{jv}{k}\nabla\cdot\mathbf{C} \tag{6.28}$$

the quadrivector potential satisfying Lorenz gauge becomes

$$\bar{C} = \begin{pmatrix} -i\,\eta\,v\,\frac{k_z}{k}\cos\left(\frac{n\pi x}{a}\right)e^{-jk_z z} \\ 0 \\ -i\,\eta\,v\cos\left(\frac{n\pi x}{a}\right)e^{-jk_z z} \\ 0 \end{pmatrix}. \tag{6.29}$$

Application of the equation

$$\bar{F} = -\partial\!\!\!/\bar{C} \tag{6.30}$$

provides the sought result for the field

$$\begin{aligned} E_y &= -\frac{n\pi}{a}\,\eta\,v\sin\left(\frac{n\pi x}{a}\right)e^{-jk_z z} \\ H_x &= \frac{v k_z}{k}\frac{n\pi}{a}\sin\left(\frac{n\pi x}{a}\right)e^{-jk_z z} \\ H_z &= -j\,v\,\frac{k^2-k_z^2}{k}\cos\left(\frac{n\pi x}{a}\right)e^{-jk_z z}. \end{aligned} \tag{6.31}$$

The ratio between the modulus of E_y and H_x provides the TE modal impedance Z^{TE}

$$Z^{TE} = \eta\,\frac{k}{k_z} \tag{6.32}$$

It is convenient to refer to the code

`wxm/Potentials_v02_TE.wxm`

here reported in the following page.

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

/* Define Dirac Gammas */
%sigma[0] : matrix([1,0],[0,1])$
%sigma[1] : matrix([0,1],[1,0])$
%sigma[2] : matrix([0,-%i],[%i,0])$
%sigma[3] : matrix([1,0],[0,-1])$

g0 :  kronecker_product (%sigma[3],%sigma[0])$
g1 :  factor(kronecker_product (%i*%sigma[2],%sigma[1]))$
g2 :  kronecker_product (%i*%sigma[2],%sigma[2])$
g3 :  kronecker_product (%i*%sigma[2],%sigma[3])$
g4 :  g0$
g5 :  - kronecker_product (%sigma[1],%sigma[0])$



/*
The block Dslash perform the d-slash operation
in rectangular coordinates on a quadrivector
*/
/********** Block start **********************/
Dslash(Ap) :=
block([],
DAp :  factor(j*k*g0.Ap+diff(g1.Ap,x)+diff(g2.Ap,y)+diff(g3.Ap,z))
)$
/********** Block end **********************/

print("----------------------------------------")$
print("The vector E and H in general depend on x,y,z,")$

/* 3D case */
depends([Ex,Ey,Ez],[x,y,z]);
depends([Hx,Hy,Hz],[x,y,z]);
```

```
depends([Cx,Cy,Cz],[x,y,z]);
depends([%psi],[x,y,z]);

/* For recovering the Hertzian potential
depends(Psih,[x,y,z]);
Cx : 0$
Cy : 0$
Cz : j * k/v*Psih$
*/

/* For potential in a rectangular waveguide */
cnx : cos(n*%pi/a*x)$
cmy : cos(m*%pi/b*y)$
propz : %e^(-j * k[z]*z)$
Cx:0 $
Cy:0$
Cz : cnx * cmy * propz;

/* Potential in a parallel plate waveguide
cnx : cos(n*%pi/a*x)$
propz : %e^(-j * k[z]*z)$
Cx:0 $
Cy:0$
Cz : cnx  * propz;
 */
print("----------------------------------------")$
print("compose the Potential Quadrivector")$
C:  -%i * %eta * matrix([%psi],[0],[v* Cz],[v* Cx+%i * Cy*v]);

print("compose the Field Quadrivector")$
F: matrix([Ez],[Ex+%i * Ey],[%i * %eta *Hz],[%i*%eta*(Hx + %i*Hy)]);
/*
print("compose the Source Quadrivector")$
Jm: matrix([%rho[m]*v],[0],[Mz],[Mx+%i * My]);
*/
print("----------------- D-slah ----------------------------------------")$
```

```
print("Application of the d-slash operator on the potential gives")$
Cpot : rectform(C)$
Dslash(Cpot)$
DslCpot : rectform(DAp);
/*
print("further application gives")$
Dslash(DslCpot)$
DCp2 : DAp$
DCp2 :subst(-1,j^2,DCp2);
print("Potentials equations")$
DCp2 +%i * Jm;
*/
print("setting to zero the imaginary part gives Lorenz condition")$
ie1 :imagpart(DslCpot[1,1]);
solpsi : solve(ie1,%psi);
%psi : rhs(solpsi[1])$
print("compose again the Potential Quadrivector satisfying Lorenz gauge")$
C: -%i * %eta * matrix([%psi],[0],[v* Cz],[v* Cx+%i * Cy*v]);

print("----------------- D-slah ----------------------------------------")$
print("Application of the d-slash operator on the Potential gives")$
Cp : rectform(C)$
Dslash(Cp)$
DCp1 : - rectform(DAp)$
DCp1 :subst(-1,j^2,DCp1);
/* DCp1 :subst(-1/j,j,DCp1);*/

Ex : ratsimp(realpart(DCp1[2][1]))$
Ey : ratsimp(imagpart(DCp1[2][1]))$
Ez : ratsimp(DCp1[1][1])$
Hy : -1/%eta *ratsimp(realpart(DCp1[4][1]))$
Hx : 1/%eta *ratsimp(imagpart(DCp1[4][1]))$
Hz : 1/%eta * ratsimp(imagpart(DCp1[3][1]))$

Hx:ratsimp(Hx)$
Hy:ratsimp(Hy)$
Hz:factor(Hz)$
```

```
print("Which provides the field equations")$
print('Ex," = ",Ex)$
print('Ey," = ",Ey)$
print('Ez," = ",Ez)$
print('Hx," = ",Hx)$
print('Hy," = ",Hy)$
print('Hz," = ",Hz)$
/* */

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

IV Rectangular waveguide modes

IV.1 TM case

The procedure for finding the modes in a rectangular waveguide is identical to the PPW case (and, in fact they share the same listing). The only difference being in the form of E_z component. In the *TM* case we have, in the longitudinal direction, only the E_z field. It has already been noted that, the field potential A_z should satisfy the same boundary conditions as E_z.
As a consequence we can select the following potential for finding the modes.

$$A_z = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \tag{6.33}$$

which corresponds to the quadrivector

$$\bar{A} = \begin{pmatrix} v\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\ 0 \\ \phi \\ 0 \end{pmatrix}. \tag{6.34}$$

By taking into account that the scalar function ϕ should satisfy the Lorenz gauge (4.151), here repeated for convenience,

$$\phi = \frac{jv}{k}\nabla\cdot\mathbf{A} \tag{6.35}$$

the quadrivector potential becomes

$$\bar{A} = \begin{pmatrix} v\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\ 0 \\ \frac{1}{k}v\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z}k_z \\ 0 \end{pmatrix}. \tag{6.36}$$

Application of the equation

$$\bar{F} = -\partial\!\!\!/\bar{A} \tag{6.37}$$

provides the sought result for the field

$$\begin{aligned}
E_x &= -\frac{n\pi}{a}\frac{k_z v}{k}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\
E_y &= -\frac{m\pi}{b}\frac{k_z v}{k}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\
E_z &= -jv\frac{k^2-k_z^2}{k}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\
H_x &= \frac{v}{\eta}\frac{m\pi}{b}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\
H_y &= -\frac{v}{\eta}\frac{n\pi}{a}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)e^{-jk_z z} \\
H_z &= 0
\end{aligned} \tag{6.38}$$

IV.2 TE case

For the TE we can notice that the potential C_z is proportional to H_z. As a consequence we can select

$$C_z = \cos(k_x x)\cos\left(k_y y\right)e^{-jk_z z} \tag{6.39}$$

where, as before,

$$k_x = \frac{n\pi}{a}$$
$$k_y = \frac{m\pi}{b} \tag{6.40}$$

are obtained from the boundary conditions. In the TE case the potential quadrivector is

$$\bar{C} = \begin{pmatrix} -i\eta\,\psi \\ 0 \\ -i\,\eta\,v\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\,e^{-jk_z z} \\ 0 \end{pmatrix}. \tag{6.41}$$

By taking into account that the scalar function ψ should satisfy the Lorenz gauge (4.173), here repeated for convenience,

$$\psi = \frac{jv}{k}\nabla\cdot\mathbf{C} \tag{6.42}$$

the quadrivector potential satisfying Lorenz gauge becomes

$$\bar{C} = \begin{pmatrix} -i\,\eta\,v\,\frac{k_z}{k}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\,e^{-jk_z z} \\ 0 \\ -i\,\eta\,v\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\,e^{-jk_z z} \\ 0 \end{pmatrix}. \tag{6.43}$$

Application of the equation

$$\bar{F} = -\partial\!\!\!/\,\bar{C} \tag{6.44}$$

provides the sought result for the field

$$
\begin{aligned}
E_x &= \frac{m\pi}{b}\eta\, v \cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) e^{-jk_z z} \\
E_y &= -\frac{n\pi}{a}\eta\, v \sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) e^{-jk_z z} \\
E_z &= 0 \\
H_x &= \frac{v k_z}{k}\frac{n\pi}{a}\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) e^{-jk_z z} \\
H_y &= \frac{v k_z}{k}\frac{m\pi}{b}\cos\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) e^{-jk_z z} \\
H_z &= -jv\frac{k^2-k_z^2}{k}\cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right) e^{-jk_z z}
\end{aligned}
\tag{6.45}
$$

The ratio between the modulus of E_y and H_x provides the TE modal impedance Z^{TE}

$$
Z^{TE} = \eta\,\frac{k}{k_z}. \tag{6.46}
$$

V Spectral representations for closed and open regions (*)

We have seen in the previous Chapter several examples of open waveguides and of their applications. It has been noted that a peculiar characteristic of open waveguides is to possess, apart from a few bound modes, a continuous spectrum. While, generally, the reader is quite familiar with the discrete spectrum of closed waveguides, as studied from basic e.m. courses the notion of a continuous spectrum is relatively uncommon. In order to introduce the latter in a simple way, we refer in the following to the case of a parallel plate waveguide where we move one of the plates to infinity. By considering this example the properties of completeness and orthogonality for the continuous spectrum are introduced, also showing the analogy between discrete and continuous cases.

Let us consider a parallel plate region such as that shown in Figure **??** with a TE mode present. In this region $E_y = \phi$ satisfies the equation

$$
\frac{d^2\phi}{dx^2} + k_x^2\phi = 0 \tag{6.47}
$$

with the boundary conditions at $x = 0, a$ (see Figure **??**), requiring the vanishing of the tangential component of the electric field on the metallic planes, thus $\phi = 0$. The solution of the above

eigenvalue problem is given by

$$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \tag{6.48}$$

The normalisation constant in (6.48) has been chosen so that the mode set is normalised to unity; moreover the orthogonality is readily proved so that we have

$$\int_0^a \phi_m(x)\phi_n(x)dx = \delta_{mn} = \left\{\begin{array}{l} 1\ m = n \\ 0\ m \neq n \end{array}\right\} \tag{6.49}$$

In order to ascertain formally the completeness of the eigenfunction set we expand the delta function in terms of the above eigenfunctions as

$$\delta(x-x') = \sum_{n=1}^{\infty} f_n \phi_n(x)$$

By multiplying both sides by $\phi_m(x)$ and by integrating over the interval $[0,a]$ we obtain

$$f_m = \int_0^a \delta(x-x')\phi_m(x)dx = \phi_m(x')$$

which inserted in the delta representation gives

$$\delta(x-x') = \sum_{n=1}^{\infty} \phi_n(x)\phi_n(x') \tag{6.50}$$

i.e. a concise expression of completeness and orthonormality.

When the plate separation becomes infinite, the parallel plate guide turns into a half–space region over a ground plane and the set of discrete eigenvalues

$$k_n = \frac{n\pi}{a}$$

become a continuous range. By introducing

$$\Delta k_n = k_{n+1} - k_n = \frac{\pi}{a}$$

and rewriting (6.50) as

$$\delta(x-x') = \frac{2}{\pi} \sum_{k_n=\Delta k_n}^{\infty} \sin(k_n x) \sin\left(k_n x'\right) \Delta k_n$$

while performing the limit for $\Delta k_n \to 0$, we obtain the completeness relationship for the open region as

$$\delta(x-x') = \frac{2}{\pi} \int_0^{\infty} \sin(kx) \sin\left(kx'\right) dk \tag{6.51}$$

It is also apparent that the continuous eigenfunction is given by

$$\phi(x;k) = \sqrt{\frac{2}{\pi}} \sin(kx) \tag{6.52}$$

Note that the eigenfunctions appearing in (6.50) have infinite energy on the interval $[0,\infty)$; for this reason they are sometimes referred to as improper. Note also that, while in the parallel plate case the eigenfunctions have finite energy, in the half space only the integral of the energy is finite. The orthogonality between eigenfunctions may be obtained by noting that in (6.51) after replacing

$$x \to k; x' \to k'; k \to x$$

we obtain the orthogonality relationship

$$\delta(k-k') = \frac{2}{\pi} \int_0^{\infty} \sin(kx) \sin\left(k'x\right) dx \tag{6.53}$$

Table 6.1 compares some examples of eigenfunctions, completeness and orthogonality relationships for discrete and corresponding continuous cases. From Table 6.1, the following observations are in order:

1. in both cases, discrete and continuous, the eigenfunction set provides a basis which is both orthogonal and complete;
2. the eigenfunctions of the continuous spectrum satisfy the boundary condition at $x = 0$, but only finiteness is required at infinity;

Table 6.1. Summary of the completeness and orthogonality relationship for the discrete and continuous eigenvalue cases.

	discrete case	**continuous case**
eigenfunction	$\phi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$	$\phi(x;k) = \sqrt{\frac{2}{\pi}} \sin(kx)$
completeness (δ representation in the space domain) $\delta(x-x') =$	$\sum_{n=1}^{\infty} \phi_n(x)\phi_n(x')$	$\int_0^{\infty} \phi(x;k)\phi(x';k)dk$
orthogonality (δ representation in the spectral domain)	$\delta_{mn} = \begin{Bmatrix} 1 \; m=n \\ 0 \; m \neq n \end{Bmatrix} =$ $\int_0^a \phi_n(x)\phi_m(x)dx$	$\delta(k-k') =$ $\int_0^{\infty} \phi(x;k)\phi(x;k')dx$

3. from a physical point of view the continuous eigenfunction corresponds to a field that originates from a generator placed at infinity;
4. as such, the continuous eigenfunctions taken individually do not satisfy the radiation condition, i.e. that the field vanish at infinity; the only condition satisfied is that they remain finite at infinity.

However, on physical grounds it is quite obvious that the total field, which is given by a superposition of eigenfunctions each with its own amplitude, should vanish at infinity, thus satisfying the radiation condition. This is not in contrast to the above point 4 stating that each eigenfunction has a finite amplitude at infinity. It is only their sum, i.e. the wave packet formed by the superposition of all eigenfunctions, which vanishes at infinity. As we will see in the following discussion, it is exactly this point as elucidated by [**?**] which allows us to determine the spectrum of open waveguides.

For reference purposes, it is advantageous to tabulate the spectral representation for some cases which occur frequently in applications [1]. In the first column of Table 6.2 are specified the boundary conditions, while in the second and third columns are given the eigenfunction expansion and the spectral representation for the operator

$$L = -\frac{d^2}{dx^2}$$

over $[0,a]$.

Boundary conditions	Eigenfunction	$\delta(x-\xi)$
$u(0)=u(a)=0$	$\sqrt{\frac{2}{a}}\sin(\frac{n\pi}{a}x)$	$\frac{2}{a}\sum_{n=1}^{\infty}\sin(\frac{n\pi}{a}x)\sin(\frac{n\pi}{a}\xi)$
$u'(0)=u'(a)=0$	$\sqrt{\frac{1}{a}}\ n=0$ $\sqrt{\frac{2}{a}}\cos(\frac{n\pi}{a}x)\ n\geq 1$	$\frac{1}{a}+\frac{2}{a}\sum_{n=1}^{\infty}\cos(\frac{n\pi}{a}x)\cos(\frac{n\pi}{a}\xi)$
$u(0)=u'(a)=0$	$\sqrt{\frac{2}{a}}\sin(\frac{[n+1/2]\pi}{a}x)$	$\frac{2}{a}\sum_{n=1}^{\infty}\sin(\frac{[n+1/2]\pi}{a}x)\sin(\frac{[n+1/2]\pi}{a}\xi)$
$u(0)=u(a),$ $u'(0)=u'(a)$		$\frac{1}{a}+\frac{2}{a}\sum_{n=1}^{\infty}\cos(\frac{2n\pi}{a}x)\cos(\frac{2n\pi}{a}\xi)+$ $\frac{2}{a}\sum_{n=1}^{\infty}\sin(\frac{2n\pi}{a}x)\sin(\frac{2n\pi}{a}\xi)+$

Table 6.2. Spectral representation over the finite domain $[0,a]$.

Boundary conditions	Eigenfunction	$\delta(x-\xi)$
$u(0)=0$	$\sqrt{\frac{2}{\pi}}\sin(kx)$	$\frac{2}{\pi}\int_0^{\infty}\sin(kx)\sin(k\xi)dk$
$u'(0)=0$	$\sqrt{\frac{2}{\pi}}\cos(kx)$	$\frac{2}{\pi}\int_0^{\infty}\cos(kx)\cos(k\xi)dk$
$u'(0)=\alpha u(0)$		$-2\alpha\, e^{\alpha(x+\xi)}+$ $\frac{2}{\pi}\int_0^{\infty}\left[\cos(kx)+\frac{a}{k}\sin(kx)\right]\left[\cos(k\xi)+\frac{a}{k}\sin(k\xi)\right]\frac{k^2}{k^2+\alpha^2}dk$ The first term is missing if $Re(\alpha)>0$

Table 6.3. Spectral representation on the infinite domain $[0,\infty)$.

When the domain of the operator becomes infinite the above eigenfunction expansion is modified as in Table 6.3 in the interval $[0,\infty)$. Note that in the lower right–hand box of Table 6.3, apart from the usual superposition integral, another term also appears. The latter corresponds to a surface wave, which will be investigated in greater detail in Chapter 3, referring to the case of planar dielectric waveguides. For the moment it is enough to note that, since for 2D TE modes we have

$$\frac{dE_y}{dx}=-j\omega\mu H_z \tag{6.54}$$

then identifying E_y with u and using (6.54) yields

$$\frac{E_y}{H_z}=-\frac{j\omega\mu}{\alpha} \tag{6.55}$$

where the latter quantity may be interpreted as the impedance of a wave travelling in the x-direction. By varying α between 0 and infinity we recover the two cases of a magnetic and metallic plate, respectively. Hence, when the plate has a finite, negative impedance, in addition to the continuous spectrum of modes, also a surface wave exists. The reason why we have a surface wave contribution only for positive values of $Re(\alpha)$ will become clear in Chapter 3, where we discuss surface waves as obtained from transverse resonance.

Finally, we observe that when the interval is unbounded in both directions, e.g. by removing both plates, the delta representation over $(-\infty,\infty)$ is given by

$$\delta(x-\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-jk(x-\xi)}dk \tag{6.56}$$

Consequently, the eigenfunctions, representing a plane wave spectrum, are:

$$\phi(x,k) = \frac{e^{-jkx}}{\sqrt{2\pi}}\,. \tag{6.57}$$

V.1 LSE, LSM potentials (*)

Let us now consider a region vertically stratified in a given direction, say y, as shown in Figure 6.1. As noted before, the field can be represented in terms of two scalars when no sources are present. The choice of these two potentials is somewhat arbitrary, as we can choose, for instance, an electric potential in the x–direction or a magnetic potential in the z–direction. However, it appears that the field obtained in terms of just one potential (for example the field obtained only from the z–directed magnetic potential) cannot satisfy all the boundary conditions at the dielectric interfaces. In general, it is necessary to consider both scalar potentials together in order to generate a field satisfying the boundary conditions at the various dielectric interfaces.

Fig. 6.1. Geometry of a multi–layer planar dielectric structure.

A favourable situation arises if we take potentials directed along the stratification (y–direction in Figure 6.1). In this case, fields generated by each individual potential fulfill the boundary conditions independently and are therefore separate solutions of Maxwell's equations. The possibility of using each potential separately greatly simplifies calculations, so that the above constitutes

the common choice when dealing with a layered region. A time–harmonic dependence is assumed and suppressed. Moreover, since we are interested in modes propagating along z, a space dependence of the type $e^{-j\beta z}$ is assumed. With the above choices, with reference to Figure 6.1, the field may be represented in terms of the following potentials

$$\begin{aligned} \boldsymbol{\Pi}_e &= \hat{\mathbf{y}}\,\psi_e(x,y)e^{-j\beta z} \\ \boldsymbol{\Pi}_h &= \hat{\mathbf{y}}\,\psi_h(x,y)e^{-j\beta z} \end{aligned} \tag{6.58}$$

which satisfy the scalar wave equation

$$\nabla_t^2\psi_{e,h}(x,y) + \left(\varepsilon_r k_0^2 - \beta^2\right)\psi_{e,h}(x,y) = 0 \tag{6.59}$$

By introducing (6.58) into (4.236b), we get in cartesian coordinates the following expression for the electric and magnetic fields

$$\begin{aligned} E_x &= \partial_x\partial_y\psi_e + \omega\mu_0\beta\,\psi_h \\ E_y &= \partial_y^2\psi_e + \varepsilon_r k_0^2\psi_e \\ E_z &= -j\beta\,\partial_y\psi_e - j\omega\mu_0\partial_x\psi_h \\ H_x &= -\omega\varepsilon\beta\,\psi_e + \partial_x\partial_y\psi_h \\ H_y &= \partial_y^2\psi_h + \varepsilon_r k_0^2\psi_h \\ H_z &= j\omega\varepsilon\partial_x\psi_e - j\beta\,\partial_y\psi_h \end{aligned} \tag{6.60}$$

In (6.60) and in the following the z–dependence $e^{-j\beta z}$ is understood but not shown for brevity. If we retain just the magnetic Hertzian potential ψ_h in (6.60), we obtain the following field

$$
\begin{aligned}
E_x &= \omega\mu_0\beta\,\psi_h \\
E_y &= 0 \\
E_z &= -j\omega\mu_0\partial_x\psi_h \\
H_x &= \partial_x\partial_y\psi_h \\
H_y &= \partial_y^2\psi_h + \varepsilon_r k_0^2\psi_h \\
H_z &= -j\beta\,\partial_y\psi_h
\end{aligned}
\tag{6.61}
$$

In (6.61) the electric field has no component normal to the interface and lies therefore on a plane parallel to the latter. This solution is referred to as longitudinal section electric (LSE). Actually, there are infinite longitudinal sections; since this mode lies on a constant y–plane, or equivalently it has no component in the y–direction, it is completely identified by saying that it is of the LSE^y type. Such a field can also be referred to as a TE^y, since the electric field is transverse with respect to the y–direction. The admittance looking in the positive y–direction is given by the ratio of a pair of fields transverse to y, that is

$$
Y_0^{LSE} = -\frac{H_z}{E_x} = \frac{j\beta\,\partial_y\psi_h}{\omega\mu_0\beta\,\psi_h} = -\frac{1}{j\omega\mu_0}\frac{\partial_y\psi_h}{\psi_h}
\tag{6.62}
$$

and, if the hypothesis of separability holds, that is, if

$$
\partial_y\psi_h = -jk_y\psi_h
\tag{6.63}
$$

we get for TE^y modes

$$
Y_0^{LSE} = \frac{k_y}{\omega\mu_0}
\tag{6.64}
$$

It is also interesting to note from (6.61) that, for slow changes in the x–direction, the terms containing the x–derivative are smaller than the other terms, thus giving

$$
\begin{aligned}
|E_z| &<< |E_x| \\
|H_x| &<< |E_x|
\end{aligned}
\tag{6.65}
$$

We are left with a field with three components, namely E_x, H_y, H_z, that corresponds to a two–dimensional TE field.

In a similar manner, an LSM mode is obtained from (6.60) by setting

$$\psi_h = 0 \tag{6.66}$$

which, upon use in (6.60), gives

$$\begin{aligned} E_x &= \partial_x \partial_y \psi_e \\ E_y &= \partial_y^2 \psi_e + \varepsilon_r k_0^2 \psi_e \\ E_z &= -j\beta \partial_y \psi_e \\ H_x &= -\omega \varepsilon \beta \psi_e \\ H_y &= 0 \\ H_z &= j\omega \varepsilon \partial_x \psi_e \end{aligned} \tag{6.67}$$

In this case we have for the impedance in the positive y–direction

$$Z_0^{LSM} = \frac{E_z}{H_x} = \frac{-j\beta \partial_y \psi_e}{-\omega \varepsilon \beta \psi_h} = -\frac{1}{j\omega \varepsilon} \frac{\partial_y \psi_h}{\psi_h} \tag{6.68}$$

Also for the LSM case, if the hypothesis of separability holds, we have

$$\partial_y \psi_e = -jk_y \psi_e \tag{6.69}$$

so that we obtain,

$$Z_0^{LSM} = \frac{k_y}{\omega \varepsilon} \tag{6.70}$$

Similarly to the LSE case, when a slow variation in the x–direction is present, we recover a two-dimensional LSM field.

References

[1] B. Friedman, *Principles and Techniques of Applied Mathematics*. New York: John Wiley & Sons Inc., 1956.

7

Sources in Free space

I Introduction

The simplest example of antenna is the Hertz dipole. Its study will be considered after having introduced the radiation condition and the delta representation in three dimensions.

I.1 Radiation Condition

For an unbounded region it is necessary to specify the field behavior on a surface at infinity. By assuming that all sources are contained in a finite region, only *outgoing waves* can be present at large distances from the sources. In other words, the field behavior at large distances from the sources must meet the physical requirement that energy travel away from the source region. This requirement is the Sommerfeld "*radiation condition*" and constitutes a boundary condition on the surface at infinity. It assumes different expressions when dealing with 2D– or 3D–regions.

3D region.

Let A denote any field component transverse to the radial distance r. The transverse field of a spherically diverging wave in a homogeneous isotropic medium decays as $1/r$ at large distances r from the source region; locally the spherical wave behaves like a plane wave traveling in the *outward* r direction. As such (for an implied $e^{j\omega t}$ time dependence) each field component transverse to r must behave like $\exp(-jkr)/r$, where $k = \omega/c$ is the free–space wavenumber and c is the speed of light in vacuum. This requirement may be phrased mathematically as

$$\lim_{r\to\infty} r\left(\frac{\partial A}{\partial r} + jkA\right) = 0\,. \tag{7.1}$$

Observe that the above boundary condition is not self–adjoint in the Hermitian sense. The adjoint boundary condition would be

$$\lim_{r\to\infty} r\left(\frac{\partial A}{\partial r} - jkA\right) = 0\,. \tag{7.2}$$

corresponding to waves impinging from infinity.

2D region.

Let ρ denote the radial variable in the transverse plane, perpendicular to the direction of uniformity. The transverse to ρ field component A in a cylindrically diverging wave in a homogeneous isotropic medium decays as $1/\sqrt{\rho}$ at large distances ρ from the source region; locally A behaves like a plane wave travelling in the outward ρ direction. As such, each field component transverse to ρ must behave like $\exp(-jk\rho)/\sqrt{\rho}$ This requirement may be phrased mathematically as

$$\lim_{\rho\to\infty} \sqrt{\rho}\left(\frac{\partial A}{\partial \rho} + jkA\right) = 0\,. \tag{7.3}$$

The above equations apply to non–dissipative media. When the media are slightly lossy one may use the simpler requirement that all fields excited by sources in a finite region should vanish at infinity (i.e. k has a small *negative* imaginary part).

II Three–dimensional delta–functions (*)

The three-dimensional δ–function is defined by the sifting property

$$\langle \delta_0, \phi\rangle = \phi(0) = \iiint \delta(\mathbf{r})\,\phi(\mathbf{r})dV\,; \tag{7.4}$$

here and in the future, the omission of the integration limits means that the integral is extended over all space. In Cartesian coordinates, the volume element is $dx\,dy\,dz$, and $\delta(\mathbf{r})$ can be written explicitly as

$$\delta(\mathbf{r}) = \delta(x)\,\delta(y)\,\delta(z)\,. \tag{7.5}$$

In a more general coordinate system, the form of dV determines that of $\delta(\mathbf{r})$. Let (u,v,w) be a set of curvilinear coordinates. The volume element at a regular point is $J\,du\,dv\,dw$, where J denotes the Jacobian of the transformation from the (x,y,z) coordinates into the (u,v,w) coordinates. More explicitly:

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The three–dimensional delta function can be expressed in terms of one–dimensional functions by the relationship

$$\delta\,(u-u_0,\,v-v_0,\,w-w_0) = \frac{\delta(u-u_0)\,\delta(v-v_0)\,\delta(w-w_0)}{J(x_0,y_0,z_0)} \tag{7.6}$$

The singular points of the coordinate system are those at which the Jacobian vanishes. At such points, the transformation from (x,y,z) into (u,v,w) is no longer of the one–to–one type, and some of the (u,v,w) coordinates become ignorable, i.e. they need not be known to find the corresponding (x,y,z). Let J_k be the integral of J over the ignorable coordinates. Then δ is the product of the δ's relative to the non–ignorable coordinates, divided by J_k.

II.1 Cylindrical coordinates

In cylindrical coordinates, for example, J is equal to ρ, and

$$\begin{aligned} \delta(\mathbf{r}-\mathbf{r}_0) &= \delta(\rho-\rho_0,\phi-\phi_0,z-z_0) \\ &= \frac{1}{\rho}\,\delta(\rho-\rho_0)\,\delta(\phi-\phi_0)\,\delta(z-z_0)\,. \end{aligned} \tag{7.7}$$

Points on the z axis are singular, and ϕ is ignorable there. We therefore write:

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{\delta(\rho)\delta(z-z_0)}{\int_0^{2\pi}\rho\,d\phi} = \frac{1}{2\pi\rho}\delta(\rho)\delta(z-z_0)\,. \tag{7.8}$$

This representation is valid with the convention

$$\int_0^\infty \delta(r)\,dr = 1\,. \tag{7.9}$$

II.2 Spherical coordinates

Express the rectangular coordinates in terms of the spherical ones as:

$$\mathrm{x}(r,\theta,\phi) = r\sin(\theta)\cos(\phi) \tag{7.10}$$

$$\mathrm{y}(r,\theta,\phi) = r\sin(\theta)\sin(\phi) \tag{7.11}$$

$$\mathrm{z}(r,\theta,\phi) = r\cos(\theta) \tag{7.12}$$

and compute the Jacobian of the transformation as

$$J_s = \begin{vmatrix} \cos(\phi)\sin(\theta) & \cos(\phi)\, r\cos(\theta) & -\sin(\phi)\, r\sin(\theta) \\ \sin(\phi)\sin(\theta) & \sin(\phi)\, r\cos(\theta) & \cos(\phi)\, r\sin(\theta) \\ \cos(\theta) & -r\sin(\theta) & 0 \end{vmatrix} \tag{7.13}$$

which gives for the Jacobian in spherical coordinates

$$J_s = r^2\sin(\theta)\,. \tag{7.14}$$

Therefore we have

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0)}{r^2\sin\theta}\,. \tag{7.15}$$

On the polar axis, where θ is 0 or π, the azimuth ϕ is ignorable, and we have

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{\delta(r-r_0)\delta(\theta-\theta_0)}{2\pi r^2\sin\theta}\,. \tag{7.16}$$

At the origin, both θ and ϕ are ignorable, and

$$\delta(\mathbf{r}-\mathbf{r}_0) = \frac{\delta(r)}{4\pi r^2}\,. \tag{7.17}$$

wxm/spherical&polar_delta.wxm

Listing for delta function representations in spherical and cylindrical coordinate systems

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */
```

```
/* [wxMaxima: input   start ] */
kill(all)$

print("The delta function placed in the origin is rather simple
to be expressed in cartesian coordinates as:")$
dc : delta(x)*delta(y)*delta(z);

print("In a more general coordinate system, the form of dV
determines that of delta(r). Let (u,v,w) be a set of curvilinear coordinates.")$

print("The volume element at a regular point is J du dv dw,
where J denotes the Jacobian of the transformation from
the (x, y, z) coordinates into the (u, v, w) coordinates.")$

print("The three-dimensional delta function can be expressed
in terms of one- dimensional functions by the relationship")$
delta(u-u[0],v-v[0],w-w[0]);
print(" = ")$
delta(u-u[0])*delta(v-v[0])*delta(w-w[0])/J(x[0], y[0], z[0]);

print("Example for spherical coordinates")$

print("Express the rectangular coordinates in terms of the spherical ones")$
x(r,theta,phi):=r*sin(theta)*cos(phi);
y(r,theta,phi):=r*sin(theta)*sin(phi);
z(r,theta,phi):=r*cos(theta);

print("now compute the jacobian of the transformation
 (see eq 1.24 of Van Bladel)")$
Jac : jacobian([x(r,theta,phi),y(r,theta,phi),z(r,theta,phi)],[r,theta,phi]);
DJ : determinant(Jac);
DJ : trigsimp(DJ);

print("In general the delta expression in spherical coordinates is")$
delta(r-r[0])*delta(theta-theta[0])*delta(phi-phi[0])/ (r^2 * sin(theta));
```

```
print("The singular points of the coordinate system
are those at which the jacobian vanishes.")$
print("At such points, the transformation from (x,y,z) into (u,v,w)
 is no longer of the one-to-one type,")$
print("and some of the (u,v,w) coordinates become ignorable,
i.e. they need not be known to find the corresponding (x,y,z)")$

print("On the polar axis (where theta[0] is zero or pi),
the azimuth phi is ignorable, and we need to integrate the Jacobian over phi")$
print(" The integral of the Jacobian over phi [0,2pi] gives ",2*%pi)$
print("on the polar axis the delta function is thus expressed as")$
delta(r-r[0])*delta(theta-theta[0])/ (2 * %pi * r^2 * sin(theta));



print("At the origin both ", theta," And ",phi,"are ignorable.")$
print("Therefore we need to integrate also sin(theta) over theta from 0 to
pi, which gives")$
integrate(sin(theta),theta,0,%pi);
print("Accordingly, the delta function at the origin in spherical coordinates is")$
delta(r)/ (4 * %pi * r^2 );

print("Now we'll try for the polar (circular cylindrical) coordinates")$
print("Express the rectangular coordinates in terms of the polar ones")$
x(rho,phi,z):=rho*cos(phi);
y(rho,phi,z):=rho*sin(phi);
z(rho,phi,z):=z;

print("now compute the jacobian of the transformation (see eq 1.24 of Van Bladel)")$
Jac : jacobian([x(rho,phi,z),y(rho,phi,z),z(rho,phi,z)],[rho,phi,z]);
DJ : determinant(Jac);
DJ : trigsimp(DJ);

print("In cylindrical coordinates, for example, J is equal to rho, and")$
1/rho[0]*delta(rho-rho[0])*delta(phi-phi[0])*delta(z-z[0]);
print("Points on the z axis are singular, and phi is ignorable there.
We therefore write")$
```

```
1/(2 * %pi * rho)*delta(rho)*delta(z-z[0]);
print("Bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

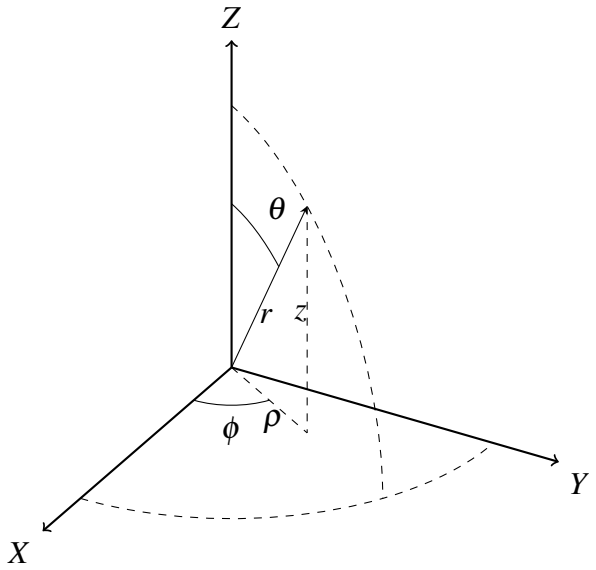

Fig. 7.1. The short current filament and the spherical coordinate system.

III Hertzian dipole radiation

III.1 Frequency–domain Potentials

It is has been shown in (4.192) that the field F can be recovered from a scalar potential ϕ and a vector potential $\mathbf{A}$, using the following expression:

$$F = (\nabla - jk)(v\mathbf{A} - \phi)\,. \tag{7.18}$$

The potentials satisfy the scalar and vector Helmholtz equations:

$$\left(\nabla^2 + k^2\right)\phi = -\frac{\rho}{\varepsilon} \tag{7.19}$$

$$\left(\nabla^2 + k^2\right)\mathbf{A} = -\mu\mathbf{J}\,. \tag{7.20}$$

The fields can be conveniently recovered as:

$$\phi = j\frac{v}{k}\nabla\cdot\mathbf{A} \tag{7.21}$$

$$i\eta\mathbf{H} = v\nabla\wedge\mathbf{A} \tag{7.22}$$

$$\mathbf{E} = \frac{j}{k}\nabla(i\eta\,\mathbf{H}) \tag{7.23}$$

III.2 Radiation from a short current filament

Figure 7.1 illustrates a short, thin filament of current located at the origin and oriented along the z axis. The vector potential corresponding to this source is A_z and is a solution of the Helmholtz equation,

$$\left(\nabla^2 + k^2\right) A_z = -\mu J_z \tag{7.24}$$

where $J_z = I/dS$ and S is the cross–sectional area of the current filament of length dl. The volume $dV = dS\,dl$ occupied by the current is of infinitesimal size so the source can be considered as located at a point. There is a spherical symmetry in the source distribution and therefore we have that the potential only depends on r as

$$A_z = A_z(r) \tag{7.25}$$

but do not depend on the polar angle θ and do not depend from the azimuth angle ϕ. Away from the source, i.e. for r not equal to zero, the potential should satisfy the following equation (see (2.101))

$$\frac{1}{r^2}\partial_r\left(r^2\partial_r A_z\right) + k^2 A_z = 0 \tag{7.26}$$

where we have expressed the laplacian in spherical coordinates and we have dropped the derivatives with respect to θ and ϕ. Equation (7.26) is conveniently solved by setting

$$A_z = \frac{\psi}{r} \tag{7.27}$$

and noting that

$$\partial_r A_z = \frac{1}{r}\partial_r \psi - \frac{\psi}{r^2}\,. \tag{7.28}$$

By substitution into (7.26) becomes

$$\frac{d^2\,\psi}{dr^2} + k^2\,\psi = 0 \tag{7.29}$$

This is a simple harmonic–motion equation with solutions $C_1 e^{-jkr}$ and $C_2 e^{+jkr}$ corresponding, respectively, to an outward wave and an inward propagating spherical wave. The inward propagating is neglected and our solution is therefore

$$A_z = C_1 \frac{e^{-jkr}}{r} \tag{7.30}$$

Determination of the constant C_1.

In order to relate the constant C_1 to the source strength, we integrate both sides of (7.24) over a small spherical volume of radius r_0. We note that $\nabla^2 A_z = \nabla \cdot \nabla A_z$ and we make use of the divergence theorem to write

$$\begin{aligned} \int_V \nabla^2 A_z \, dV &= \int_V \nabla \cdot \nabla A_z \, dV \\ &= \oint_S \nabla A_z \cdot \mathbf{a}_r \, r_0^2 \sin\theta \, d\theta \, d\phi \\ &= -k^2 \int_V A_z \, dV - \mu \int_V J_z \, dV \, . \end{aligned} \tag{7.31}$$

Now $dV = r^2 \sin\theta \, d\theta \, d\phi \, dr$ and A_z varies as $1/r$; consequently, if we choose r_0 vanishingly small the volume integral of A_z, which is proportional to r_0^2, vanishes. The volume integral of the current J_z gives $J_z \, dS \, dl = I \, dl$, which is the total source strength.
In addition we note that

$$\nabla A_z \cdot \mathbf{a}_r = \partial_r A_z = -(1 + jkr) C_1 \frac{e^{-jkr}}{r^2} \tag{7.32}$$

so

$$\lim_{r_0 \to 0} \int_0^{2\pi} \int_0^{\pi} -(1 + jkr) \, C_1 \, e^{-jkr_0} \sin\theta \, d\theta \, d\phi = -4\pi C_1 = -\mu I \, dl \tag{7.33}$$

or

$$C_1 = \frac{\mu I \, dl}{4\pi} \tag{7.34}$$

IV Green's function (*)

Instead of solving the field problem for each different excitation, it is advantageous to investigate the solution for an elementary source, the Dirac delta function $\delta(\mathbf{r} - \mathbf{r}')$. As an example, referring to the case of A_z and J_z, we can introduce $G_z^A(\mathbf{r} - \mathbf{r}')$ such that

$$\left(\nabla^2 + k^2\right) G_z^A(\mathbf{r} - \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{7.35}$$

is verified.

If, instead of considering (7.24), here reported in the following,

$$\left(\nabla^2 + k^2\right) A_z = -\mu J_z \tag{7.36}$$

we can consider the following equation with the three–dimensional delta function placed at the origin

$$\left(\nabla^2 + k^2\right) G(r) = -\delta(\mathbf{r})\,. \tag{7.37}$$

In this case, the solution will be of the type

$$G(r) = C_1' \frac{e^{-jkr}}{r} \tag{7.38}$$

and the integration over the volume V of the delta functions, using (7.17) and (7.9), will give

$$-\iiint \delta(\mathbf{r}) = -C_1 \int_0^{2\pi} \int_0^{\pi} \int_0^{r_0} \frac{\delta(r)}{r^2} r^2 dr \sin\theta\, d\theta\, d\phi = -1\,. \tag{7.39}$$

or

$$C_1' = \frac{1}{4\pi} \tag{7.40}$$

Hence equation (7.37) has the solution

$$G(r) = \frac{e^{-jkr}}{4\pi r}\,. \tag{7.41}$$

When the delta is located at the position $\mathbf{r}'$ we have

$$\begin{aligned} \left(\nabla^2 + k^2\right) G(\left|\mathbf{r} - \mathbf{r}'\right|) &= -\delta(\mathbf{r} - \mathbf{r}') \\ G(\left|\mathbf{r} - \mathbf{r}'\right|) &= \frac{e^{-jk\left|\mathbf{r}-\mathbf{r}'\right|}}{4\pi\left|\mathbf{r} - \mathbf{r}'\right|}\,. \end{aligned} \tag{7.42}$$

It is convenient to define

$$R = \left|\mathbf{r} - \mathbf{r}'\right| \tag{7.43}$$

where $\mathbf{r}$ is the coordinate of the observation point and $\mathbf{r}'$ is the coordinate of the source point. It is apparent that when a distribution of currents, directed along z is present, the potential A_z may be obtained as

$$A_z = \mu \iiint_V J_z(\mathbf{r}')\, G(R)\, d\mathbf{r}' . \tag{7.44}$$

A corresponding solution of the same type exists when the current are directed along x as e.g.

$$\left(\nabla^2 + k^2\right) A_x = -\mu J_x \tag{7.45}$$

has the solution

$$A_x = \mu \iiint_V J_x(\mathbf{r}')\, G(R)\, d\mathbf{r}' . \tag{7.46}$$

and similarly for the y coordinate. By summing together these three solution allows us to write the solution for the vector wave equation as

$$\mathbf{A} = \mu \iiint_V \mathbf{J}(\mathbf{r}')\, G(R)\, d\mathbf{r}' . \tag{7.47}$$

wxm/Helmholtz_v1.wxm

Listing for scalar Helmholtz equation in free-space

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$

print("the package vect allows to perform vector operations")$
load(vect);

/* By using this block you can use directly j instead of %i as unit imaginary part !!!! */
block([simp:false],
?putprop(%i,?j,'?texword),
?aliaslist: ?list(?cons(%i,j)),
alias(j,%i))$
```

```
assume(k[0]>0)$
assume(sin(%theta)>0)$

depends([Az,%psi],r);

print("scalefactors spherical")$
scalefactors([[r*sin(%theta)*cos(phi),r*sin(%theta)*sin(phi),r*cos(%theta)],r,%theta,phi]);

/* depends(Az,[r,%theta,phi]); */
depends(Az,[r]);
print(" express...")$
l_spher : express(laplacian(Az));
print(" evaluate...")$
l_spher : ev(l_spher,diff);
print(" ratexpand...")$
lap : ratexpand(l_spher);
print("Now that we have the laplacian form the Helmholtz equation")$
eq220 : lap + k[0]^2 * Az;
print("and try to find out if we can get a solution")$
ode2(eq220,Az,r);

print("Then try the other approach...")$
print("================================")$
print("Using the substitution")$
Az : %psi/r;

print("First find d/dr ")$
diffAz : diff(Az,r);
print("Then substitute into our equation")$
tt : 1/r^2 * diff(r^2 * diffAz,r) + k[0]^2 * Az;
print("and put the solution in a more suitable form")$
eqr : expand(tt);

print("now try to find the solution")$
sol : rhs(ode2(eqr,%psi,r));

print("or in exponential form")$
```

```
Azr: exponentialize(sol);

print("by choosing the constants (%k2=%i*%k1) we can keep only the outward wave")$
Azr : expand(Azr)$
at(Azr, %k2=%i * %k1);

Az : C[1] * %e^(- %i * k[0] *r)/r;

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

The potential is therefore given by

$$\mathbf{A} = \frac{\mu I\, dl}{4\pi} \frac{e^{-jkr}}{r} \mathbf{a}_z \,. \tag{7.48}$$

Note that

- $I\,dl$ is the so–called dipole moment and is given by the current I times the dipole length dl
- $k = \omega\sqrt{\varepsilon\mu}$ is the wavenumber
- the potential $\mathbf{A}$ is directed as the currents (directed along z)
- the surface with the same phase is a sphere

IV.1 Field evaluation using Pauli matrices

We have seen that, by using Pauli matrices, the representation of a vector is independent of the coordinate system. In particular we have for the potential:

$$\tilde{A} = \begin{pmatrix} \frac{e^{-jkr}}{r} & 0 \\ 0 & -\frac{e^{-jkr}}{r} \end{pmatrix} \tag{7.49}$$

A procedure for finding the fields given the potential is based on the following two steps:

- compute the Pauli matrix of the magnetic field as

$$i\eta\tilde{H} = v\tilde{\nabla} \wedge \tilde{A} = v\left(\tilde{\nabla}\tilde{A} - \tilde{\nabla}\cdot\tilde{A}\right) \tag{7.50}$$

- compute the electric field from the magnetic field as

$$\tilde{E} = \frac{j}{k}\tilde{\nabla}\left(i\eta\tilde{H}\right) \tag{7.51}$$

The quantity $\tilde{\nabla} \wedge \tilde{A}$ is readily evaluated as

$$\tilde{\nabla} \wedge \tilde{A} = \begin{pmatrix} 0 & \frac{\sin(\theta)\,(jkr+1)\,e^{-jkr-i\phi}}{r^2} \\ -\frac{\sin(\theta)\,(jkr+1)\,e^{i\phi-jkr}}{r^2} & 0 \end{pmatrix} \tag{7.52}$$

which, after moltiplication times v and C_1 and grade extraction in spherical coordinates, provides the following expression for the magnetic field

$$\begin{aligned} H_r &= 0 \\ H_\theta &= 0 \\ H_\phi &= \frac{\sin(\theta)\, I dl\, (jkr+1)\, e^{-jkr}}{4\pi r^2}. \end{aligned} \tag{7.53}$$

By using (7.51) the Electric field is recovered as

$$\begin{aligned} E_r &= \frac{\eta \cos(\theta)\, I dl\, (kr-j)\, e^{-jkr}}{2\pi k r^3} \\ E_\theta &= \frac{\eta \sin(\theta)\, I dl \left(jk^2 r^2 + kr - j\right) e^{-jkr}}{4\pi k r^3} \\ E_\phi &= 0. \end{aligned} \tag{7.54}$$

Note that the following code also provides the grade extraction.

wxm/3Dipole_sph_3.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02)$
functions;
/* 3Dipole_sph_2.wxm */

print("introducing the dependencies")$
depends([a11,a12,a21,a22],[r,%theta,phi]);
Ap : matrix([a11,a12],[a21,a22]);

print("------------------------------------------------------")$
print("vector A in spherical coordinates")$
print("------------------------------------------------------")$

print("It is assumed that the dipole is oriented along z")$
```

```
print("also the potential will be oriented along z")$
Az : %e^(-j * k *r)/r;

print("Remeber that the Pauli matrix of a vector is independent of the coordinates")$
print("Accordingly, we write it in rectangular coordinates,")$
print("but it can be used for computation also in spherical coordinates")$

print("The constant c1 is also defined")$
c1 : %mu * I*dl/(4*%pi);
Ap : matrix([Az,0],[0,-Az]);

print("------------------------------------------------------------")$
print("Solution procedure")$
print("------------------------------------------------------------")$
print("The solution procedure is organized in 2 steps")$
print("------------------------------------------------------------")$
print("------------------------------------------------------------")$
print("1)  The magnetic field is obtained from i %eta H = v nabla wedge A ")$
print("------------------------------------------------------------")$
NA  : Nablasph(Ap);
print("The divergence is")$
divA : 1/2 *(NA[1,1]+NA[2,2]);
print("Nabla wedge A is")$
NWA : NA - divA*%sigma[0];

M : NWA$
print("Extracting the Grades")$
Gradesph(M)$
print("It is seen that i %eta H is only in the phi direction and is ")$
vBap : v * Bap$
Hr : factor(1/%mu * Bar)*c1$
Ht : factor(1/%mu * Bat)*c1$
Hphi : factor(1/%mu * Bap)*c1$
print('Hr," = ",Hr)$
print('Ht," = ",Ht)$
print('Hphi," = ",Hphi)$
```

```
print("------------------------------------------------------------")$
print("2)  The electric field is recoverd from j/k * nabla(i %eta H) ")$
print("------------------------------------------------------------")$
print("The Pauli matrix representing H is ")$
HP : %i * vBap * %sigma[phi];
EFA : j/k * Nablasph(HP)$
EFA :subst(-1,j^2,EFA );
M : EFA$
print("Extracting the Grades")$
Gradesph(M)$
print("The electric field is equal to")$
Er : ar$
Et : at$
Ep : ap$
Er : subst(-1,j^2,Er)$
Et : subst(-1,j^2,Et)$
Er : factor(Er)*c1$
Er : subst(%eta/v, %mu,Er)$
Et : factor(Et)*c1$
Et : subst(%eta/v, %mu,Et)$
print('Er," = ",Er)$
print('Et," = ",Et)$
print('Ep," = ",Ep)$

print("end")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

Dipole field using multivectors

However, by using the multivector concept, the field can be retrieved in a more direct way. In fact, we can start from the potential expression as a Pauli matrix

$$\tilde{A} = \begin{pmatrix} \frac{\mu I dl e^{-jkr}}{4\pi r} & 0 \\ 0 & -\frac{\mu I dl e^{-jkr}}{4\pi r} \end{pmatrix} \tag{7.55}$$

and evaluate its nabla obtaining

$$\tilde{\nabla}\tilde{A} = \frac{\mu I dl}{4\pi}\frac{e^{-jkr}}{r}(jkr+1)\begin{pmatrix} \cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{i\phi} & \cos\theta \end{pmatrix}. \tag{7.56}$$

From the latter expression the divergence is readily evaluated and therefore also the scalar potential

$$\psi = \frac{jv}{k}\nabla\cdot\mathbf{A} = \frac{\eta\cos(\theta)\, I dl\, j\,(jkr+1)\, e^{-jkr}}{4\pi k r^2}. \tag{7.57}$$

It is now possible to form the multivector potential as

$$\tilde{P} = v\tilde{A} - \tilde{\psi} = \begin{pmatrix} \frac{\eta\cos(\theta) I dl\, j(jkr+1)e^{-jkr}}{4\pi k r^2} + \frac{\eta I dl e^{-jkr}}{4\pi r} & 0 \\ 0 & \frac{\eta\cos(\theta) I dl\, j(jkr+1)e^{-jkr}}{4\pi k r^2} - \frac{\eta I dl e^{-jkr}}{4\pi r} \end{pmatrix} \tag{7.58}$$

and apply the operator $\left(\tilde{\nabla} - jk\sigma_0\right)$ on the multivector potential $\tilde{P}$

$$\tilde{F} = \left(\tilde{\nabla} - jk\sigma_0\right)\tilde{P}, \tag{7.59}$$

thus recovering the field in (7.53), (7.69) simply from grade extraction.

IV.2 Dipole field using Dirac matrices

We start by composing the potential quadrivector as

$$\bar{P} = \begin{pmatrix} \frac{\eta I dl e^{-jkr}}{4\pi r} \\ 0 \\ \psi \\ 0 \end{pmatrix} \tag{7.60}$$

where we consider the component A_z and the scalar potential here denoted with ψ. The result of the $\not{\partial}$ operator on the potential $\bar{P}$ is

$$\not{\partial}\bar{P} = \begin{pmatrix} \frac{e^{-jkr}\left(4\pi\left(\frac{d}{dr}\psi\right)\cos(\theta) r e^{jkr} + \eta I dl\, jk\right)}{4\pi r} \\ \left(\frac{d}{dr}\psi\right)\sin(\theta)\, e^{i\phi} \\ -\frac{e^{-jkr}\left(4\pi\psi\, jkr^2 e^{jkr} - \eta\cos(\theta) I dl\, jkr - \eta\cos(\theta) I dl\right)}{4\pi r^2} \\ \frac{\eta\sin(\theta) I dl\, e^{i\phi}(jkr+1) e^{-jkr}}{4\pi r^2} \end{pmatrix} \tag{7.61}$$

Thus, we have for the ψ scalar part that the following condition has to be satisfied (zeroing of H_z):

$$\psi = \frac{\eta I dl}{4\pi}\frac{\cos(\theta)\,(kr - j)\, e^{-jkr}}{kr^2}. \tag{7.62}$$

By inserting the latter expression of ψ inside the potential we have the new potential quadrivector satisfying Lorenz condition

$$\bar{P} = \begin{pmatrix} \frac{\eta I dl e^{-jkr}}{4\pi r} \\ 0 \\ \frac{\eta\cos(\theta) I dl (jkr+1) e^{-jkr}}{4\pi jkr^2} \\ 0 \end{pmatrix}. \tag{7.63}$$

It is now possible to perform $\not{\partial}\bar{P}$

$$\not{\partial}\bar{P} = \begin{pmatrix} -\frac{\eta I dl\left(-\cos(\theta)^2 k^2 r^2 + k^2 r^2 - \sin(\theta)^2 jkr + 2\cos(\theta)^2 jkr - \sin(\theta)^2 + 2\cos(\theta)^2\right) e^{-jkr}}{4\pi jkr^3} \\ -\frac{\eta\cos(\theta)\sin(\theta) I dl\, e^{i\phi}\left(-k^2 r^2 + 3jkr + 3\right) e^{-jkr}}{4\pi jkr^3} \\ 0 \\ \frac{\eta\sin(\theta) I dl\, e^{i\phi}(jkr+1) e^{-jkr}}{4\pi r^2} \end{pmatrix} \tag{7.64}$$

Note that an output similar to the code has been reported in order to facilitate the analysis. In order to recover the field the Pauli matrices corresponding to the vector part and to the bivector part have been created. From the grade evaluation in spherical coordinates we recover the field expressions:

$$
\begin{aligned}
E_r &= \frac{\eta \cos(\theta)\, Idl\, (kr-j)\, e^{-jkr}}{2\pi k r^3} \\
E_\theta &= \frac{\eta \sin(\theta)\, Idl\, \left(jk^2r^2+kr-j\right) e^{-jkr}}{4\pi k r^3} \\
E_\phi &= 0 \\
H_r &= 0 \\
H_\theta &= 0 \\
H_\phi &= \frac{\sin(\theta)\, Idl\, (jkr+1)\, e^{-jkr}}{4\pi r^2}\,.
\end{aligned}
\tag{7.65}
$$

The code that realize both the multivector analysis with Pauli matrices and the approach with Dirac matrices is

```
wxm/3Dipole_dslash_v03.wxm

/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);

load(Pauli_v02);
load(Dirac_v01);
functions$

print("------------------------------------------------------")$
print("Computation using Pauli matrices")$
print("------------------------------------------------------")$
print("vector A in spherical coordinates as Pauli matrix")$
print("------------------------------------------------------")$
Az : %e^(-j*k*r)/r$
```

```
print("speed of light defined as")$
v: %eta/%mu;
print("Potential in the form")$
Az : Az *I *dl /(4*%pi)*%mu;
print("Pauli matrix of the potential")$
Ap : matrix([Az,0],[0,-Az]);
print("nabla A")$
Nablasph(Ap)$
M:NAp;
Gradesph(M)$

print("div A")$
divA : (NAp[1,1]+NAp[2,2])/2;
print("Lorenz Potential")$
pot : j*v/k*divA;

print("from Pauli matrices")$
print("Construct the potential multivector v*Ap - j*v/k*divA")$
App : v*Ap - pot*%sigma[0];
print("apply the operator (nabla -j*k* %sigma[0]) and obtain the field")$
Nablasph(App)$
NApp : NAp$
print("Field")$
Field : factor(NApp -j*k*App)$
Field : subst(-1,j^2,Field)$
Field : subst(-j,j^3,Field);

M: Field$
Gradesph(M)$

Er : ar$
Et : at$
Ep : ap$
print("The field is the following")$
print('Er," = ",Er)$
print('Et," = ",Et)$
print('Ep," = ",Ep)$
```

```
Hr : 1/%eta*Bar$
Ht : 1/%eta*Bat$
Hp : 1/%eta*Bap$
print("The field is the following")$
print('Hr," = ",Hr)$
print('Ht," = ",Ht)$
print('Hp," = ",Hp)$

print("------------------------------------------------------------")$
print("Computation using Dirac matrices")$
print("------------------------------------------------------------")$

depends(%psi,r);

print("compose the Potential Quadrivector")$
A :  matrix([v* Az],[0],[%psi],[0])$
A :  ratsimp(A)$
A :  factor(subst(-1,j^2,A));
Ap:  A$
DAp : Dslashsph(Ap)$
DAp : subst(-1,j^2,DAp);

print("Find the potential %psi which makes Hz = 0 ")$
eqpsi :DAp[3][1]$
spsol : solve(eqpsi,%psi);
%psi : rhs(spsol[1])$
print("and create the new quadrivector")$
A : matrix([v* Az],[0],[%psi],[0])$
A : ratsimp(A)$
A : factor(subst(-1,j^2,A));
Ap: A$
print("Perform the d-slash operator on the new potential")$
DAp : Dslashsph(Ap)$
DAp : subst(-1,j^2,DAp);

print("From the 4vector compose the Pauli matrices")$
```

```
print("vector part")$
PPz : realpart(DAp[1][1])$
 xpiy : DAp[2][1]$
 xmiy : conjugate(xpiy)$
PP : matrix([PPz ,xmiy ],[xpiy,-PPz]);

print("Bivector part")$
PPz: imagpart(DAp[3][1])/%eta$
/*  xpiy :  DAp[4][1]/(%i * %eta)$ */
xpiy : - %i*DAp[4][1]$
 xmiy : conjugate(xpiy)$
PPBi :  matrix([PPz ,xmiy ],[xpiy,-PPz]);

M : PP+ %i *  PPBi$
Gradesph(M);

print("Final result")$
ar : -j^2*ar$
at : -j^2*at$

Er : -ar$
Et : -at$
Ep : -ap$
print("The field is the following")$
print('Er," = ",Er)$
print('Et," = ",Et)$
print('Ep," = ",Ep)$

Hr : -Bar/%eta$
Ht : -Bat/%eta$
Hp : -Bap/%eta$
print("The field is the following")$
print('Hr," = ",Hr)$
print('Ht," = ",Ht)$
print('Hp," = ",Hp)$

print("end")$
```

```
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

IV.3 Conventional approach

The first step is to transform the potential **A** from cartesian to spherical coordinates. It suffices to take the **A** vector expressed in cartesian coordinates and to multiply it by the appropriate transformation matrix to obtain:

$$\begin{pmatrix} \frac{\mu I dl e^{-jkr}\cos(\theta)}{4\pi r} \\ -\frac{\mu \sin(\theta) I dl e^{-jkr}}{4\pi r} \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\phi)\sin(\theta) & \sin(\phi)\sin(\theta) & \cos(\theta) \\ \cos(\phi)\cos(\theta) & \sin(\phi)\cos(\theta) & -\sin(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \frac{\mu I dl e^{-jkr}}{4\pi r} \end{pmatrix}. \tag{7.66}$$

The next step is to evaluate

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \tag{7.67}$$

in spherical coordinates. By doing this evaluation, either according to standard expressions or by using the listing reported in the following, we get

$$\begin{aligned} H_r &= 0 \\ H_\theta &= 0 \\ H_\phi &= \frac{\sin(\theta)\, I dl\, (jkr+1)\, e^{-jkr}}{4\pi r^2}. \end{aligned} \tag{7.68}$$

The electric field is conveniently evaluated as the curl of the magnetic fields appropriately scaled. By performing this evaluation one obtains:

$$\begin{aligned} E_r &= -j\frac{\eta}{k} \frac{\cos(\theta)\, I dl\, (jkr+1)\, e^{-jkr}}{2\pi r^3} \\ E_\theta &= -j\frac{\eta}{k} \frac{\sin(\theta)\, I dl \left(-k^2 r^2 + jkr + 1\right) e^{-jkr}}{4\pi r^3} \\ E_\phi &= 0. \end{aligned} \tag{7.69}$$

The above computations are easily performed by using computer algebra. The reader is invited to perform the computations by hand as an exercise. It is important to consider the following cases:

- The static (or quasi–static) case (i.e. when $k = 0$);
- The high frequency case (i.e. when $k = \omega\sqrt{\mu\varepsilon}$);
- the far–field radiation terms (large r or $r \gg \lambda$);

- the near–field radiation terms (small r or $\lambda \gg r$).

Far field approximation

In the far field approximation $kr \gg 1$ we recover the following expressions for the radiated fields

$$E_\theta = \frac{j\eta \sin(\theta)\, I dl\, k e^{-jkr}}{4\pi r}$$
$$H_\phi = \frac{j \sin(\theta)\, I dl\, k e^{-jkr}}{4\pi r}. \tag{7.70}$$

It is possible to observe the following facts, which are general properties of radiated fields and not only of the Hertzian dipole

- The ratio between electric and magnetic fields is given by

$$\frac{E_\theta}{H_\phi} = \eta = \sqrt{\frac{\mu}{\varepsilon}}. \tag{7.71}$$

- only transverse components are present (orthogonal to $\mathbf{r}_0$) and related between them as

$$\mathbf{E} = -\eta\, \mathbf{r}_0 \times \mathbf{H}$$
$$\mathbf{H} = \frac{1}{\eta}\, \mathbf{r}_0 \times \mathbf{E} \tag{7.72}$$

  i.e. the radial direction (propagation direction), $\mathbf{E}$ and $\mathbf{H}$

- for $r = cost$ both fields are dependent on $\sin\theta$ (i.e. the fields cannot be radiated isotropically)

Another important quantity to consider is the Poynting vector given by:

$$\mathbf{P} = \frac{1}{2}\mathbf{E} \times \mathbf{H}^* = \frac{\eta \sin(\theta)^2 I I^* dl^2 k^2}{32\pi^2 r^2}\, \mathbf{r}_0. \tag{7.73}$$

It is often necessary to compute the total radiated power P_{tot} which is given by

$$P_{tot} = \iint_S \mathbf{P} \cdot \mathbf{n}\, dS = \int_0^\pi \int_0^{2\pi} \frac{\eta \sin(\theta)^2 I I^* dl^2 k^2}{32\pi^2 r^2}\, r^2 \sin\theta\, d\theta\, d\phi \tag{7.74}$$

that, after evaluation, as shown in the next listings provides

$$P_{tot} = \frac{\eta I I^* dl^2 k^2}{12\pi}. \tag{7.75}$$

Most of the previous results and some other useful computations are available in the code reported next.

wxm/Hertz_dipole_v05.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all);
assume(r>0);
assume(%rho>0);

/* Hertz_dipole_v05.wxm */

/* To use directly j instead of %i as unit imaginary part */
block([simp:false],
?putprop(%i,?j,'?texword),
?aliaslist: ?list(?cons(%i,j)),
alias(j,%i))$

print("COMPUTATION OF THE FIELD PRODUCED BY AN HERTZIAN DIPOLE")$
print("")$
/* print("Start from eq. 2.26 of Collin book")$ */
print("Start from the potential expressed in cartesian coordinates")$
print("Load the package vect which allows to perform vector operations")$
load(vect)$

print("=======================================================")$

/*  Vector function definition*/
print("Consider a vector potential A of the form")$
print("in cartesian coordinates and transform it in spherical coordinates")$

%psi : %e^(-j * k * r)/r$
```

```
const : (%mu * I * dl)/(4 * %pi)$

Ax : 0$
Ay : 0$
Az : const * %psi$

print("vector A")$
[Ax, Ay, Az]$

Arect: matrix([Ax],[Ay],[Az]);

print("Transform A from cartesian to spherical using the matrix")$
r2s : matrix(
[sin(%theta)* cos(%phi), sin(%theta)*sin(%phi), cos(theta)],
[cos(%theta)*cos(%phi), cos(%theta)*sin(%phi),-sin(%theta)],
[-sin(%phi),cos(%phi),0]);

Asp : r2s . Arect$
Asp : factor(trigsimp(Asp));

print("=======================================================")$



cost : %mu[0]*I*dl $

print("Consider spherical coordinates")$

print("radial component of the vector magnetic potential A")$
A[r] : cost * %e^(-j * k * r)/(4 * %pi *r) * cos(%theta);

print("theta component of the vector magnetic potential A")$
A[%theta] : - cost * %e^(-j * k * r)/(4 * %pi *r) * sin(%theta);

print("phi component of the vector magnetic potential A")$
A[%phi] : 0;
```

```
print("In spherical coordinates evaluate curl(A)")$
print("scalefactors spherical")$
print("Curl")$

scalefactors([[r*sin(%theta)*cos(%phi),
        r*sin(%theta)*sin(%phi),r*cos(%theta)],r,%theta,%phi]);

print("The magnetic field H is obtained from 2.27 as")$
d : curl ([A[r], A[%theta], A[%phi]])$
express (d)$
H : 1/%mu[0] * ev (%, diff)$
H : factor(H);
print("Only the phi component of the magnetic field is present")$

Hphi : H[3]$
/* Hphi : ratsimp(Hphi); */
Hphi : factor(Hphi);

print("============================================================")$
print("Electric field computed as the curl of magnetic field")$
E :  curl([0,0,Hphi])$
express (E)$
EfromH :  ev (%, diff)$
EfromH : factor(EfromH);

print("Electric field:")$
print("note that: 1/ (j * %omega * %epsilon) = -j * %eta/k")$
EfromH : - j * %eta / k * EfromH ;

print("There is a discordancy of sign with Collin in Er ??? ")$

Er : EfromH[1]$
print("The radial component of the Electric field is Er")$
Er : trigsimp(Er)$
Er : factor(Er);

/*
```

```
limit(Er,r,inf);
limit(Er,k,0);
*/

Et : EfromH[2]$
print("The theta component of the Electric field is Et")$
Et : trigsimp(Et)$
Et : factor(Et);

Ep : EfromH[3]$
print("The phi component of the Electric field is Ep")$
Ep : trigsimp(Ep);

print("=========================================")$
print("radiated Electric field : theta component ")$
Etr : ((j*%eta*sin(%theta)*dl*k^2*r^2)*%e^(-j*k*r)*I)/(4*%pi*k*r^3);

print("=========================================")$
print("radiated Magnetic field : phi component ")$
Hphir : ((j*sin(%theta)*dl*k*r)*%e^(-j*k*r)*I)/(4*%pi*r^2);

print("=========================================")$
print("Compute power: start from Etheta x Hphi: radiated Poynting vector")$
Poynt[r] : 1/2 * Etr * conjugate(Hphir);

print("Now try to evaluate the total radiated power")$
print("The infinitesimal solid angle is r^2*sin(theta) d%theta d%phi")$
print("perform the integral in %phi")$

P1 : integrate(Poynt[r],x,0,2*%pi);

print("and now perform the integral in %theta (cfr. eq. 2.35)")$
P2 : integrate(P1*r^2*sin(%theta),%theta,0,%pi);

print("Directivity (cfr. 2.36)")$
Dir : Poynt[r] / P2 * 4 * %pi *r^2;
```

```
print("Radiation resistance (cfr. 2.39)")$
Ra : P2 * 2 / I^2;



/*
depends(t,[r,%theta,%phi]);

l_spher : express(laplacian(t));
l_spher : ev(l_spher,diff);
ratexpand(l_spher);
*/

print("Bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

IV.4 Electric field Green's function via the Weyl identity (*)

We have seen in (5.28) a relationship between the spherical wave and the expansion in terms of plane waves, referred to as Weyl's identity, which is here reported for ease of reading:

$$\frac{e^{-jkr}}{r} = -\frac{j}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-j(k_x x+k_y y)-jk_z|z|}}{k_z}\,dk_x\,dk_y \tag{7.76}$$

Let us define

$$\phi = \frac{1}{2\pi}e^{-jzk_z-jyk_y-jxk_x} \tag{7.77}$$

Using the identity in (7.76) the potential vector $\mathbf{A}$ of the Hertzian dipole source, in terms of Pauli matrices, may therefore be written as

$$\tilde{A} = -j\,\mu\,c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\phi}{k_z}\,\sigma_3\,dk_x\,dk_y \tag{7.78}$$

where, for simplicity, we have considered the case $z>0$. The operation $\nabla\mathbf{A}$ is readily performed, and in terms of Pauli matrices we have:

$$\tilde{\nabla}\,\tilde{A} = -\mu\,c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\left(\sigma_0 + i\,\frac{k_y}{k_z}\sigma_1 - i\,\frac{k_x}{k_z}\sigma_2\right)dk_x\,dk_y \tag{7.79}$$

from which we can see the trace, which corresponds to the divergence, given by the term

$$\nabla\cdot\mathbf{A} = -\mu\,c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\,dk_x\,dk_y\,. \tag{7.80}$$

Therefore, the magnetic field vector is given by (we have divided (7.79) by $i\,\mu$)

$$\tilde{H} = -c_1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\left(\frac{k_y}{k_z}\sigma_1 - \frac{k_x}{k_z}\sigma_2\right)dk_x\,dk_y\,. \tag{7.81}$$

The electric field is obtained by applying the nabla operator $\tilde{\nabla}$ to the magnetic field (which, since the divergence of the magnetic field is zero) gives

$$\mathbf{E} = \frac{\eta}{jk}\nabla\times\mathbf{H}, \tag{7.82}$$

or explicitly:

$$\mathbf{E} = c_1 \frac{\eta}{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left(k_x \sigma_1 + k_y \sigma_2 - \frac{k_x^2 + k_y^2}{k_z} \sigma_3 \right) dk_x dk_y \tag{7.83}$$

This is the electric field generated by a current source placed in the origin and directed along z. By setting $Idl = 1$ and by post multiplying with σ_3 we obtain the Green's function that relates a current source along z to the electric field components as:

$$\mathscr{G}_z^E = \frac{1}{4\pi} \frac{\eta}{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi \left(i k_y \sigma_1 - i k_x \sigma_2 - \frac{k_x^2 + k_y^2}{k_z} \sigma_0 \right) dk_x dk_y \,. \tag{7.84}$$

Two observations are in order:

- if the current source is in a direction different from z we can easily recover the relative results by simple matrix multiplication. As an example, if the current is in the x direction, one can make use of $-\sigma_3\sigma_2 = \sigma_2\sigma_3 = i\sigma_1$ and recover the relative results.
- when we need to compute the magnetic field field produced by a magnetic current, by use of duality, we can employ the above expressions.

Listing for Green's function evaluation in cartesian coordinates with Pauli matrices using Weyl's identity

wxm/Dipole_weyl_v03.wxm

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$
load(Pauli_v02);

/* code name : Dipole_weyl_v03 */
/*
write the potential as a Pauli matrix,
using the Weyl identity
perform the nabla operation and obtain the H components
and from H  obtain the E field
and the Green's function
```

```
*/

depends([a11,a12,a21,a22],[x,y,z]);
Ap : matrix([a11,a12],[a21,a22]);

/* Vector function definition*/
print("Consider a Vector potential of the form")$

r : sqrt(x^2+y^2+z^2)$
%psi : %e^(-j * k * r)/r$
const : (%mu * I * dl)/(4 * %pi)$

print("The double integral is omitted for the moment")$
%psi : -j/(2*%pi) * %e^(-j*(k[x]*x+k[y]*y) -j*k[z]*z)/k[z];

Ax : 0$
Ay : 0$
Az : const * %psi$

[Ax, Ay, Az];

print("------------------------------------------------------------")$
print("vector A as Pauli matrix")$
Ap : Ax * %sigma[1] + Ay * %sigma[2] + Az * %sigma[3];

print("------------------------------------------------------------")$
print("Nabla A is")$

NAp : Nablarect(Ap)$

NAp : subst(-1,j^2,NAp );

print("The trace corresponds to the divergence and is")$
traceNAp : factor(ratsimp(1/2 * (NAp[1,1] + NAp[2,2])))$
traceNApxyz : traceNAp;

divA : %sigma[0] * traceNApxyz$
```

```
print("------------------------------------------------------------")$
print("Nabla ^ A is")$
NestA : ratsimp(NAp - divA)$
NestA : factor(NestA);
print("The magnetic field is")$
Hp : gfactor(NestA/%mu/%i);

print("Electric field")$
Ep : ratsimp(1/j * %eta / k *Nablarect(Hp)/%i);
fac : (%eta * I * dl) / (k * 8 * %pi^2)$
face : %e^(-j*z*k[z]-j*y*k[y]-j*x*k[x])$
print("By sorting out ",fac*face)$
Epsimp : ratsimp(Ep / fac /face);

print("Multiplication with %sigma[3] gives the Green function")$
ratsimp(Epsimp . %sigma[3]);

print("Bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

V Algebraic derivation of dyadic Green's functions

In section IV.4 we have refferred to the Weyl identity (7.76) here reported for convenience:

$$\frac{e^{-jkr}}{r} = -\frac{j}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-j(k_x x+k_y y)-jk_z|z|}}{k_z}\,dk_x\,dk_y \tag{7.85}$$

together with the definition of the function Φ

$$\Phi = \frac{1}{2\pi}e^{-jzk_z-jyk_y-jxk_x}\,. \tag{7.86}$$

This identity, in conjunction with the Dirac matrices and the field, sources and potentials quadrivectors, allows to derive in an algebraic manner the dyadic Green's functions. For convenience we repeat here Table 4.2 which summarizes the relevant equations.

Table 7.1. Table summarizing spinors definitions. The first column refers to sources, $\bar{S}$, the second to the field, $\bar{F}$, the third to the potential $\bar{P}$. The last two row reports the relevant equations and definitions.

$\bar{S}$	$\bar{F}$	$\bar{P}$
$\begin{pmatrix} \eta J_z - i\nu\rho_m \\ \eta(J_x+iJ_y) \\ -iM_z+\eta\nu\rho_e \\ -i(M_x+iM_y) \end{pmatrix}$	$\begin{pmatrix} E_z \\ iE_y+E_x \\ \eta\, iH_z \\ \eta\,(iH_x-H_y) \end{pmatrix}$	$\begin{pmatrix} \nu A_z - i\eta\psi \\ \nu(A_x+iA_y) \\ -i\eta\nu C_z+\phi \\ -i\eta\nu(C_x+iC_y) \end{pmatrix}$
$\not{\partial}\bar{F} = -\bar{S}$	$\not{\partial}\bar{P} = -\bar{F}$	$-\not{\partial}^2\bar{P} = \not{\partial}\bar{F} = -\bar{S}$
Lorenz	$\phi = \frac{j\nu}{k}\nabla\cdot\mathbf{A}$	$\psi = \frac{j\nu}{k}\nabla\cdot\mathbf{C}$

In order to introduce the methodology we first refer to a unitary current and a unitary moment dipole (i.e. $Idl = 1$) directed along the z–direction. As a consequence, the source vector $\bar{S}$ is given by

$$\bar{S} = \eta\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7.87}$$

Such a current generates a potential A_z which is given by

$$\begin{aligned} A_z &= \frac{\mu}{4\pi}\frac{e^{-jkr}}{r} \\ &= -j\frac{\mu}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Phi}{k_z}\,dk_x dk_y \\ &= -j\frac{\eta}{v}\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\Phi}{k_z}\,dk_x dk_y \end{aligned} \tag{7.88}$$

In the potential $\bar{P}$ in addition to the A_z component also the ϕ scalar potential satisfying Lorentz condition should be present. In the present case the ϕ scalar potential is thus given by

$$\phi = -j\frac{\eta}{k}\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Phi\,dk_x dk_y \tag{7.89}$$

Thus the quadrivector of the potential $\bar{P}$ is given by

$$\bar{P} = -j\frac{\eta}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix}\frac{1}{k_z}\\0\\\frac{1}{k}\\0\end{pmatrix}\Phi\,dk_x dk_y\,. \tag{7.90}$$

We can therefore relate the current source to the potential via the following matrix

$$T_{ps} = -j\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix}\frac{1}{k_z}\,0\,0\,0\\0\;0\;0\;0\\\frac{1}{k}\;0\,0\,0\\0\;0\;0\;0\end{pmatrix}\Phi\,dk_x dk_y\,. \tag{7.91}$$

In this way the potential $\bar{P}$ is obtained from the sources as

$$\bar{P} = T_{ps}\,\bar{S}. \tag{7.92}$$

As reported in Table 7.1, the field are obtained from the potentials by the $\underline{\partial}$ operator in the frequency domain. It is noted that in this case the $\underline{\partial}\bar{P}$ is the following:

$$
\begin{aligned}
\underline{\not\partial}\bar{P} &= \left(jk\gamma^0 + \gamma^1\partial_x + \gamma^2\partial_y + \gamma^3\partial_z\right)\bar{P} \\
&= \left(jk\gamma^0 - jk_x\gamma^1 - jk_y\gamma^2 - jk_z\gamma^3\right)\bar{P} \\
&= j\not{k}\bar{P} \\
&= j\begin{pmatrix} k & 0 & -k_z & ik_y - k_x \\ 0 & k & -ik_y - k_x & k_z \\ k_z & k_x - ik_y & -k & 0 \\ ik_y + k_x & -k_z & 0 & -k \end{pmatrix}\bar{P} \\
&= \frac{\eta}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} k & 0 & -k_z & ik_y - k_x \\ 0 & k & -ik_y - k_x & k_z \\ k_z & k_x - ik_y & -k & 0 \\ ik_y + k_x & -k_z & 0 & -k \end{pmatrix}\begin{pmatrix} \frac{1}{k_z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\Phi\, dk_x dk_y
\end{aligned}
\tag{7.93}
$$

where the matrix $\not{k} = k\gamma^0 - k_x\gamma^1 - k_y\gamma^2 - k_z\gamma^3$ has been introduced. By multiplying the first and second matrices that appear in 7.93 we obtain the dyadic Green's function for a current excitation in the z direction. In other words we have

$$
\begin{aligned}
\bar{F} &= -\underline{\not\partial}\bar{P} \\
&= -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} \frac{(k-k_z)(k_z+k)}{kk_z} & 0 & 0 & 0 \\ -\frac{\left(ik_y+k_x\right)}{k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\left(ik_y+k_x\right)}{k_z} & 0 & 0 & 0 \end{pmatrix}\Phi\,\bar{S}\, dk_x dk_y \\
&= G\bar{S}
\end{aligned}
\tag{7.94}
$$

with the dyadic Green's function G given by

$$
G = -\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} \frac{(k-k_z)(k_z+k)}{kk_z} & 0 & 0 & 0 \\ -\frac{\left(ik_y+k_x\right)}{k} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\left(ik_y+k_x\right)}{k_z} & 0 & 0 & 0 \end{pmatrix}\Phi\, dk_x dk_y\,. \tag{7.95}
$$

Note that the procedure can be repeated for all the sources providing the complete dyadic Green's function expressed as a four by four matrix.

V.1 Complete Dyadic Green's function

Let us start from a general source distribution which presents both electric and magnetic currents as

$$\bar{S} = \begin{pmatrix} \eta J_z \\ i\eta J_y + \eta J_x \\ -iM_z \\ M_y - iM_x \end{pmatrix}. \tag{7.96}$$

After integration over a region of volume containing the sources we get the following quadrivector for the sources expressed in terms of dipole moments

$$\bar{S}_V = \begin{pmatrix} \eta D_z \\ i\eta D_y + \eta D_x \\ -iN_z \\ N_y - iN_x \end{pmatrix}. \tag{7.97}$$

The quantity $D_x = I_x dl_x$ is the electric moment dipole in the x–direction and similarly for the other coordinates. The quantity $N_x = M_x dl_x$ is the magnetic moment dipole in the x– direction with an analogous meaning for the other components.
Linearity provides the following expression for the potentials in the general case:

$$\bar{P} = -\frac{j}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{pmatrix} -i\mathbf{N}\cdot\frac{\mathbf{k}}{k} + \eta D_z \\ i\eta D_y + \eta D_x \\ \eta\,\mathbf{D}\cdot\frac{\mathbf{k}}{k} - iN_z \\ N_y - iN_x \end{pmatrix} \frac{\Phi}{k_z}\, dk_x\, dk_y\,. \tag{7.98}$$

As before the potential $\bar{P}$ can be obtained from the sources as

$$\bar{P} = T_{ps}\,\bar{S}_V \tag{7.99}$$

with

$$T_{ps} = -j\frac{1}{4\pi k_z}\begin{pmatrix} 1 & 0 & \frac{k_z}{k} & -\frac{i\left(N_y k_y + N_x k_x\right)}{k\left(N_y - iN_x\right)} \\ 0 & 1 & 0 & 0 \\ \frac{k_z}{k} & \frac{D_y k_y + D_x k_x}{k\left(iD_y + D_x\right)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{7.100}$$

and the general dyadic Green's function G is recovered as

$$G = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} j\not{k}\, T_{ps}\, \Phi\, dk_x\, dk_y \tag{7.101}$$

or explicitly as

$$G = \frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\begin{pmatrix} k & 0 & -k_z & ik_y - k_x \\ 0 & k & -ik_y - k_x & k_z \\ k_z & k_x - ik_y & -k & 0 \\ ik_y + k_x & -k_z & 0 & -k \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{k_z}{k} & -\frac{i\left(N_y k_y + N_x k_x\right)}{k\left(N_y - iN_x\right)} \\ 0 & 1 & 0 & 0 \\ \frac{k_z}{k} & \frac{D_y k_y + D_x k_x}{k\left(iD_y + D_x\right)} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\frac{\Phi}{k_z}\, dk_x\, dk_y\,.$$

The product of the two matrices $\not{k}\, T_{ps}$ may be expressed as

$$\not{k}\, T_{ps} = \begin{pmatrix} -\frac{(k_z - k)(k_z + k)}{k k_z} & -\frac{D_y k_y + D_x k_x}{k\left(iD_y + D_x\right)} & 0 & -\frac{k_x N_y - N_x k_y}{\left(N_y - iN_x\right)k_z} \\ -\frac{ik_y + k_x}{k} & g_{22} & -\frac{ik_y + k_x}{k_z} & 1 \\ 0 & -\frac{iD_x k_y - ik_x D_y}{\left(iD_y + D_x\right)k_z} & \frac{(k_z - k)(k_z + k)}{k k_z} & -\frac{i\left(N_y k_y + N_x k_x\right)}{k\left(N_y - iN_x\right)} \\ \frac{ik_y + k_x}{k_z} & -1 & \frac{ik_y + k_x}{k} & g_{44} \end{pmatrix} \tag{7.102}$$

with the terms g_{22} and g_{44} given by

$$g_{22} = \left[\frac{k}{k_z} - \frac{\left(D_x k_x + D_y k_y\right)^2}{k k_z\left(D_x^2 + D_y^2\right)}\right] - \frac{i\left(D_x k_y - k_x D_y\right)\left(D_y k_y + D_x k_x\right)}{k\left(D_y^2 + D_x^2\right)k_z}$$

$$g_{44} = \left[-\frac{k}{k_z} + \frac{\left(N_x k_x + N_y k_y\right)^2}{k k_z\left(N_x^2 + N_y^2\right)}\right] + \frac{i\left(N_x k_y - k_x N_y\right)\left(N_y k_y + N_x k_x\right)}{k\left(N_y^2 + N_x^2\right)k_z}\,.$$

VI Radiation from a small current loop (*)

In Fig. 7.2 a photograph of a loop antenna is reported. The loop antenna is the equivalent of a magnetic dipole, presenting a field which is dual to the electric dipole.

Fig. 7.2. Figure of a loop antenna. Place the z axis orthogonal to the loop plane.

Fig. 7.3. Loop antenna geometry.

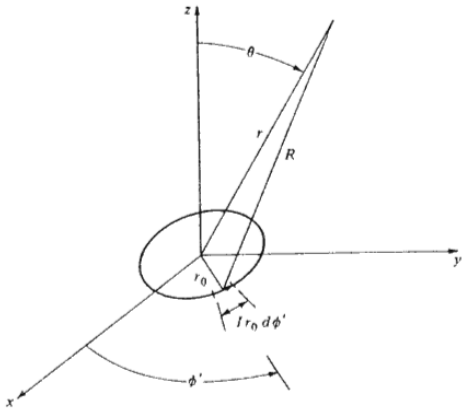

Figure 7.2 shows a small current loop of radius r_0 area πr_0^2 and with a current I. The axis of the loop antenna is oriented in the z direction. For $\lambda \gg r_0$ the loop may be treated as a point source. A small loop of current is called a *magnetic dipole* and its magnetic dipole moment equals the product of the area with the current, i.e.

$$\mathbf{M} = \pi r_0^2 I \mathbf{a}_z \tag{7.103}$$

The basic idea for the analysis: is the foolowing. It is conceivable that we have many infinitesimal Hertzian dipoles oriented along $\mathbf{a}_\phi$. With reference to Fig. 7.3, consider the current filament at ϕ', which has the vector orientation

$$\mathbf{a}_\phi = -\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi' . \tag{7.104}$$

For the Hertzian dipole along z we have:

$$\mathbf{A} = \mu_0 \frac{I\,dl}{4\pi} \frac{e^{-jkr}}{r} \mathbf{a}_z \tag{7.105}$$

for each segment $Ir_0\,d\phi'$ we have to sum the various contributions. It is convenient to define:

$$R = \left[\left(x - r_0 \cos\phi'\right)^2 + \left(y - r_0 \sin\phi'\right)^2 + z^2 \right]^{\frac{1}{2}} \tag{7.106}$$

so that we have to sum the contributions

$$\mu_0 \frac{I r_0}{4\pi} \frac{e^{-jkR}}{R} \left(-\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi'\right) d\phi' \tag{7.107}$$

which leads to the following integral to be evaluated:

$$\mathbf{A} = \mu_0 \frac{I\,dl}{4\pi} \int_0^{2\pi} \frac{e^{-jkR}}{R} \left(-\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi'\right) d\phi' . \tag{7.108}$$

This integral is difficult to be solved analytically unless we make certain approximation in the expression for R. Note that we have R under the integral. Since we are primarily interested in the far–zone radiation we can assume that

$$r \gg r_0 \tag{7.109}$$

Two different types of approximation are used: one concerning the amplitude and one concerning the phase.

- for the amplitude we set

$$\frac{1}{R} \approx \frac{1}{r} \tag{7.110}$$

- for the phase we need a more precise approximation.

First express R in spherical coordinates as

$$R = \left[r^2 + r_0^2 - 2 r r_0 \sin\theta \left(\cos\phi \cos\phi' + \sin\phi \sin\phi'\right) \right]^{\frac{1}{2}} . \tag{7.111}$$

Then we can factor r^2 and we can drop the term r_0^2/r^2. We are therefore left with a term of type $(1+u)^{1/2}$ and we can use the binomial expansion for $1 \gg |u|$ and obtain

$$\sqrt{1+u} \approx 1 + \frac{u}{2} + \ldots \tag{7.112}$$

which gives

$$R \approx r - r_0 \sin\theta \left(\cos\phi \cos\phi' + \sin\phi \sin\phi'\right) . \tag{7.113}$$

In the exponential function e^{-jkR} we will have the term involving kr_0 when we substitute our approximate expression for R. But $1 \gg kr_0$, so we can use the approximation

$$e^u \approx 1 + u + \ldots \tag{7.114}$$

for $1 \gg |u|$ to obtain the following simplified form

$$e^{-jkR} = e^{-jkr} \left[1 + jkr_0 \sin\theta \left(\cos\phi \cos\phi' + \sin\phi \sin\phi'\right)\right] . \tag{7.115}$$

By using these approximations the integral in (7.108) becomes

$$\mathbf{A} = \mu_0 \frac{I r_0}{4\pi} \frac{e^{-jkr}}{r} \int_0^{2\pi} \left(-\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi'\right) \left[1 + jk_0 r_0 \sin\theta \left(\cos\phi \cos\phi' + \sin\phi \sin\phi'\right)\right] d\phi' \tag{7.116}$$

and note that the only terms that do not integrate to zero are the $\cos^2\phi'$ and $\sin^2\phi'$ as

$$\int_0^{2\pi} \cos^2\phi' \, d\phi' = \int_0^{2\pi} \sin^2\phi' \, d\phi' = \pi \tag{7.117}$$

and

$$\int_0^{2\pi} \cos\phi' \sin\phi' \, d\phi' = 0 \tag{7.118}$$

The final expression for the vector potential becomes

$$\mathbf{A} = jk_0\mu_0 \frac{I\pi r_0^2}{4\pi} \frac{e^{-jkr}}{r} \sin\theta \, \mathbf{a}_\phi \tag{7.119}$$

where use has been made of the relation $\mathbf{a}_\phi = -\mathbf{a}_x \sin\phi' + \mathbf{a}_y \cos\phi'$. From the potential we can find the magnetic field:

$$\begin{aligned}\mathbf{H} &= \frac{1}{\mu_0}\nabla\times\mathbf{A} = -\frac{1}{\mu_0 r}\frac{\partial}{\partial r}\left(rA_\phi\right)\mathbf{a}_\theta \\ &= -Mk_0^2\sin\theta\,\frac{e^{-jkr}}{4\pi r}\mathbf{a}_\theta\end{aligned} \tag{7.120}$$

In the far field, we have for the electric component

$$\begin{aligned}\mathbf{E} &= -Z_0\mathbf{a}_r\times\mathbf{H} = \\ &= -Z_0 Mk_0^2\sin\theta\,\frac{e^{-jkr}}{4\pi r}\mathbf{a}_\phi\end{aligned} \tag{7.121}$$

These expressions show that the role of electric and magnetic fields for magnetic dipole radiation have been interchanged w.r.t. electric dipole.

The total radiated power is given by

$$\begin{aligned}P_r &= \frac{1}{2}\mathrm{Re}\int_0^{2\pi}\int_0^{\pi}E_\phi H_\theta^* r^2\sin\theta d\theta\, d\phi \\ &= \frac{M^2 Z_0 k_0^4}{16\pi^2}\int_0^{2\pi}\int_0^{\pi}\sin^3\theta d\theta\, d\phi \\ &= \frac{M^2 Z_0 k_0^4}{12\pi}\end{aligned} \tag{7.122}$$

The radiation pattern and directivity have not changed.

The radiation resistance is found by considering

$$\frac{1}{2}|I|^2 R_a = P_r \tag{7.123}$$

which gives

$$R_a = 320\pi^6\left(\frac{r_0}{\lambda_0}\right)^4 \tag{7.124}$$

This is typically very low but, fortunately, if N turns of wire are used the radiation resistance is increased by a factor N^2. Small loop antennas are often used as receiving antennas for portable radios.

Listing for loop antenna analysis

```
wxm/loop_v03.wxm
```

```
/* [wxMaxima batch file version 1] [ DO NOT EDIT BY HAND! ]*/
/* [ Created with wxMaxima version 11.08.0 ] */

/* [wxMaxima: input   start ] */
kill(all)$

assume(r>0);     /* Arghh */
assume(%rho>0); /* Arghh */

print("the package vect allows to perform vector operations")$
load(vect);



/* Loop antenna analysis */

print("R is given by")$
R : sqrt((x-r[0]*cos(pp))^2 + (y - r[0] * sin(pp))^2 + z^2);

print("define the variable")$
const : %mu[0] * I *r[0] /(4 *%pi * 'R) * %e^(-%i * k * 'R);
const : %mu[0] * I *r[0] /(4 *%pi * R) * %e^(-%i * k * R);

/* eq. before 2.41 */
print("equation 2.41 of Collin")$

print("dA")$
dA[x] : - const * sin(pp);
dA[y] :  const * cos(pp);
dA[z] : 0$

print("try to integrate")$
A[x] : integrate(dA[x], pp,0, 2 * %pi);

print("Direct integration of 2.41 failed! Proceed with some approximations...")$

print("rewrite R as")$
/*
```

```
x : r * sin(%theta) * cos(p);
y : r * sin(%theta) * sin(p);
z : r * cos(%theta)
*/
R : subst (r * sin(%theta) * cos(p), x,R);
R : subst (r * sin(%theta) * sin(p), y,R);
R : subst (r * cos(%theta) , z,R);
R : trigsimp(R);
print("passages...")$
print("we now drop r[0]...")$
R :subst(0,r[0]^2,R);



print("expansion of sqrt(1 + u)")$
taylor(sqrt(1+u), u, 0, 2);

print("u is equal to")$
ur : ((-2*r[0]*sin(%theta)*sin(p)*sin(pp)-2*r[0]*sin(%theta)*cos(p)*cos(pp))*r)/r^2;

print("Therefore R is given by")$
R : r*(1-ur/2);

print("expansion of %e^u")$
taylor(%e^u, u, 0, 2);

uexp : %i * k * (r[0]*sin(%theta)*sin(p)*sin(pp)+r[0]*sin(%theta)*cos(p)*cos(pp));
expterm : %e^(-%i * k * r) * (1 + uexp);
Ac : %mu[0] * I *r[0] /(4 *%pi * r) ;

print("Ax component ")$
Axint : integrate(- expterm *sin(pp),pp,0,2*%pi);
Axint : trigsimp(Axint);

print("Ay component ")$
Ayint : integrate( expterm *cos(pp),pp,0,2*%pi);
Ayint : trigsimp(Ayint);
```

```
print("Now express the Ax, Ay, Az")$
A[x] : Axint * Ac;
A[y] : Ayint * Ac;
A[z] : 0;

print("By noting that -ax * sin(p) + ay * cos(p) = ap ")$
print("we now switch to SPHERICAL COORDINATES")$


scalefactors([[r*sin(%theta)*cos(%phi),r*sin(%theta)*sin(%phi),r*cos(%theta)],r,%theta,%phi]);

As[r] :0;
As[%theta] :0;
As[%phi] : 1/2 * (-A[x]/sin(p)+A[y]/cos(p));
/* in the above eq. the factor 1/2 is not clear ...there should be a mistake somewhere..*/


print("Curl")$
d : curl ([As[r], As[%theta], As[%phi]]);
express (d);
Hp : ev (%, diff)$
Hp : ratsimp(Hp/%mu[0]);
print("note that the radial component of H decreases as 1/r^2")$
print("thus remains only the phi component")$

print("By introducing the dipole moment M = ",%pi * r[0]^2 *I," we have for Hphi")$
Hphi : Hp[2]$
Hphi : subst(M/%pi,r[0]^2,Hphi)$
Hphi : subst(1,I,Hphi);

print("the far-field electric field may be obtained by multiplying with Z[0]")$
Eth : -Z[0] * Hphi;
print("in the above expression we have used ar x aphi = - ",a[%theta])$

print("Poynting vector")$
poynt : - 1/2 * Eth * conjugate(Hphi);
```

```
print("Total radiated power")$
intpoynt: 2 * %pi * integrate(poynt * sin(%theta) * r^2,%theta,0,%pi);

print("Try to find the radiation resistance")$
eqr : 1/2 * Ra * I^2 - intpoynt;
Rsol : solve(eqr,Ra);
print("select the solution")$
Ra : rhs(Rsol[1]);

print("now perform the following substitutions")$
print("substitute M = ", %pi * r[0]^2 *I)$
Ra : subst(%pi * r[0]^2 *I,M,Ra);

print("substitute  Z0 = ", 120 * %pi )$
Ra : subst(120*%pi,Z[0],Ra) ;

print("substitute  k = ", 2 * %pi / %lambda)$
Ra : subst(2*%pi/%lambda,k,Ra) ;

print("Bye")$
/* [wxMaxima: input   end   ] */

/* Maxima can't load/batch files which end with a comment! */
"Created with wxMaxima"$
```

References