Electromagnetic field modeling through the use of Dirac matrices and geometric algebra

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Introduction

Traditional vector analysis relies on the approach proposed by Gibbs at the beginning of 1900 [1] and generally adopted in electromagnetic field engineering.

Later, several different approaches have been presented and elucidated.

As an example differential forms [2] and Geometric or Clifford Algebra, in the following briefly referred to as GA [3][4].

Generally GA has been employed in electromagnetic (EM) by considering the point of view of physicists and not of engineers [5][6].

As an example, in EM engineering wide use is made of time–harmonic analysis [8], while no publication deals with application of GA to the time–harmonic regime.

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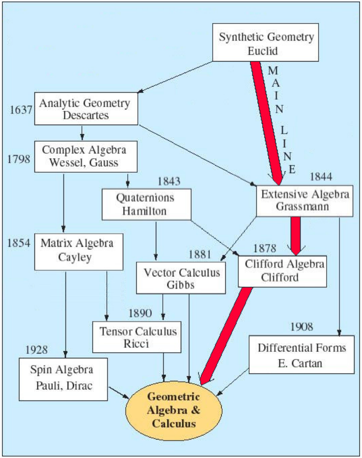

Figure 1: The descent of the various vector systems. The main path of development beginning from Euclid geometry down through Grassman and then to Clifford. Other parallel developments using complex numbers, quaternions, Gibbs' vectors, tensors, matrices and spinor algebra subsumed into the general formalism of Clifford geometric algebra with the inclusion of calculus.

Most of the engineering EM analysis are performed in a three–dimensional space (3D). In the 3D space a very interesting fact takes place: GA is equivalent to Pauli algebra.

Pauli matrices have been introduced by Wolfgang Pauli for the spin theory.

Very remarkably, by using Pauli matrices a vector can be represented as a 2x2 matrix. Thus it is possible to multiply vectors by multiplying the relative matrices.

It is also possible to make the inverse of a vector!

GA in 3D and Pauli matrices

In the 3D space, according to GA, we need to represent:

- a scalar;
- 3 vectors;
- 3 bivectors;
- pseudo–scalar

i.e. 8 numbers.

It is possible to introduce a multivector which is the sum of a scalar, a vector, a bivector and a pseudo–scalar.

Such multivector can be represented by a 2x2 matrix and can be constructed in terms of Pauli matrices.

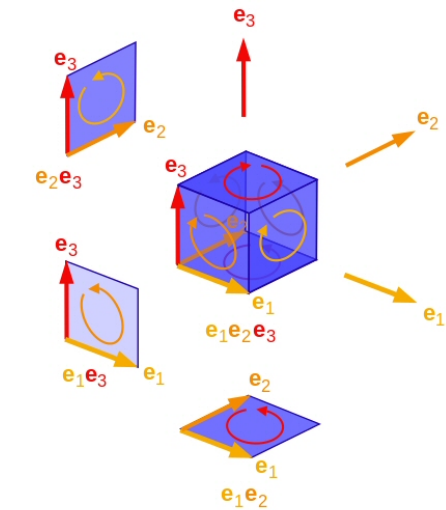

Figure 2: Identify the basis vectors as $e_1 = \sigma_1$, $e_2 = \sigma_2$, $e_3 = \sigma_3$.These are elements of Clifford's model for three–dimensional space. This consists of three unit vectors e_1, e_2 and e_3, three unit areas e_2e_3, e_3e_1 and e_1e_2 and a unit volume $i = e_1e_2e_3$. The pure scalars then defining points to form a complete algebraic description of three-dimensional physical space.

Pauli matrices and their properties

Wolfgang Pauli

Figure 3: Wolfgang Pauli

Wolfgang Ernst Pauli (25 April 1900 – 15 December 1958) was an Austrian-born Swiss and American theoretical physicist and one of the pioneers of quantum physics.

In 1945, after having been nominated by Albert Einstein, Pauli received the Nobel Prize in Physics for his "decisive contribution through his discovery of a new law of Nature, the exclusion principle or Pauli principle".

Pauli matrices

We first introduce the Pauli matrices and then discuss some of their properties and show how to operate with this new tool.

Note that engineers are quite well trained to operate with matrices and we will see that many standard vectors operations can be simplified and new important elements will be found.

The Pauli matrices are a set of three 2x2 complex matrices which are *Hermitian* and *unitary*.

The Pauli matrices have the following form

$$\begin{aligned} \sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_3 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \end{aligned} \tag{1}$$

Properties of the Pauli matrices

It is immediately noted that the trace of these matrices is always zero.

The determinant of the Pauli matrices is always -1.

The following products hold:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I = \sigma_0 \,. \tag{2}$$

By multiplying e.g. σ_1 with σ_2 the result is $i\sigma_3$ and similarly for the other cases:

$$\begin{aligned} \sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\ \sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \\ \sigma_3\sigma_1 &= i\sigma_2 = -\sigma_1\sigma_3 \,. \end{aligned} \tag{3}$$

The above relations are very important. In fact, they show that, in the three-dimensional case, we can always replace the quantities $\sigma_i\sigma_j$ with the orthogonal vector $i\sigma_k$.

From the above properties it is seen that

$$(\sigma_1\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = -1 \tag{4}$$

i.e. $\sigma_1\sigma_2\sigma_3 = i$.

The three Pauli matrices, with the addition of the identity matrix σ_0, form a basis in the space of the 2x2 Hermitian matrices and a matrix A can be represented as:

$$A = a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3\,. \tag{5}$$

It is worthwhile to note that when the coefficients (a_0, a_1, a_2, a_3) are complex, also non Hermitian matrices can be described by the basis of $(\sigma_0, \sigma_1, \sigma_2, \sigma_3)$.

The Pauli vector

Let us introduce the Pauli vector which is a vector made by the three matrices $(\sigma_1, \sigma_2, \sigma_3)$.

$$\sigma = (\sigma_1\, \mathbf{x}_0, \sigma_2\, \mathbf{y}_0, \sigma_3\, \mathbf{z}_0)\ . \tag{6}$$

We can consider the vector $\mathbf{a} = (a_x\, \mathbf{x}_0 + a_y\, \mathbf{y}_0 + a_z\, \mathbf{z}_0)$ in the three-dimensional space and make the following product:

$$\begin{aligned}
\tilde{a} &= \sigma \cdot \mathbf{a} = \sigma_1 a_x + \sigma_2 a_y + \sigma_3 a_z && (7)\\
&= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} a_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a_z && (8)\\
&= \begin{pmatrix} a_z & a_x - i a_y \\ a_x + i a_y & -a_z \end{pmatrix} && (9)
\end{aligned}$$

The matrix $\tilde{a}$ is an equivalent description of the vector $\mathbf{a}$ in terms of a 2x2 matrix. Two different symbols have been used to denote the 2x2 matrix representation $\tilde{a}$ and the standard vector representation $\mathbf{a}$.

Let us try the product between two matrices $\tilde{a}, \tilde{b}$ and evaluate their product $\tilde{c} = \tilde{a}\tilde{b}$,

$$\begin{aligned}
\tilde{a} &= \begin{pmatrix} a_3 & a_1 - i\, a_2 \\ i\, a_2 + a_1 & -a_3 \end{pmatrix} \\
\tilde{b} &= \begin{pmatrix} b_3 & b_1 - i\, b_2 \\ i\, b_2 + b_1 & -b_3 \end{pmatrix} \\
\tilde{c} &= \begin{pmatrix} a_3\, b_3 + (a_2 + i\, a_1)\ b_2 + (a_1 - i\, a_2)\ b_1 & (i\, a_2 - a_1)\ b_3 - i\, a_3\, b_2 + a_3\, b_1 \\ (i\, a_2 + a_1)\ b_3 - i\, a_3\, b_2 - a_3\, b_1 & a_3\, b_3 + (a_2 - i\, a_1)\ b_2 + (i\, a_2 + a_1)\ b_1 \end{pmatrix}
\end{aligned}$$

- *the trace of $\tilde{c} = \tilde{a}\tilde{b}$ divided by 2 gives us the dot product*.

$$\mathbf{a} \cdot \mathbf{b} = (\tilde{c}_{11} + \tilde{c}_{22})/2 = a_1\, b_1 + a_2\, b_2 + a_3\, b_3 \tag{10}$$

It is straightforward to recognize that the component along x, y, z can be easily retrieved by the following operations

$$\begin{aligned}
c_x &= \frac{c_{21} + c_{12}}{2} \\
c_y &= -\frac{i\ (c_{21} - c_{12})}{2} \\
c_z &= \frac{c_{11} - c_{22}}{2}
\end{aligned} \tag{11}$$

By writing them explicitly we find:

$$\begin{aligned} c_x &= i\,(a_2\,b_3 - a_3\,b_2) \\ c_y &= -i\,(a_1\,b_3 - a_3\,b_1) \\ c_z &= i\,(a_1\,b_2 - a_2\,b_1) \end{aligned} \tag{12}$$

It is now easy to recognize that, apart for the i factor, this is equal to the well known cross product. This new part is called *external product* and is denoted by $\mathbf{a} \wedge \mathbf{b}$. We have just obtained the important identity:

$$\mathbf{a} \wedge \mathbf{b} = i\,\mathbf{a} \times \mathbf{b}\,. \tag{13}$$

Naturally, to represent the wedge product in terms of the Pauli matrices it is sufficient to subtract the dot product from $\tilde{a}\tilde{b}$, thus obtaining

$$\mathbf{a} \wedge \mathbf{b} = \begin{pmatrix} i\,(a_1\,b_2 - a_2\,b_1) & i\,a_2\,b_3 - a_1\,b_3 - i\,a_3\,b_2 + a_3\,b_1 \\ i\,a_2\,b_3 + a_1\,b_3 - i\,a_3\,b_2 - a_3\,b_1 & -i\,(a_1\,b_2 - a_2\,b_1) \end{pmatrix}. \tag{14}$$

The quantity $\mathbf{a} \wedge \mathbf{b}$ is a *bivector* .

Once we have represented the vector **a** as a matrix $\tilde{a}$ it is possible to compute its determinant and its inverse.

It is immediately recognized that we have for the determinant

$$det(\tilde{a}) = -(a_1^2 + a_2^2 + a_3^2) \tag{15}$$

from which it is evident that, by taking the square root of the absolute value, we can recover the modulus of the vector.

In standard vector algebra the inverse of a vector is not defined. However we can perform the inverse of $\tilde{a}$ obtaining

$$\tilde{a}^{-1} = \frac{1}{{a_1}^2 + {a_2}^2 + {a_3}^2} \begin{pmatrix} a_3 & -i\,a_2 + a_1 \\ i\,a_2 + a_1 & -a_3 \end{pmatrix} \tag{16}$$

This is just the same vector divided by the square of its modulus!

Vector analysis with Pauli matrices

- Multiplication of two matrices corresponding to two vectors give us both the dot product and another new part, the external product. This multiplication corresponds to the geometric or Clifford product.

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \; = \tilde{a}\tilde{b}. \tag{17}$$

- dot product between two vectors is commutative:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = \frac{\tilde{a}\tilde{b} + \tilde{b}\tilde{a}}{2} \,, \tag{18}$$

- the external product between two vectors is anticommutative

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a} = \frac{\tilde{a}\tilde{b} - \tilde{b}\tilde{a}}{2} \,. \tag{19}$$

- the external product, in 3D, is related to the cross product as:

$$\mathbf{a} \wedge \mathbf{b} = i \, \mathbf{a} \times \mathbf{b} \,. \tag{20}$$

- the external product between two vectors introduces a new subject: *the bivector*.
- the bivector can be expressed either showing the two vector components or a single vector component but multiplied by i

$$\begin{aligned} \sigma_1\sigma_2 &= i\sigma_3 = -\sigma_2\sigma_1 \\ \sigma_2\sigma_3 &= i\sigma_1 = -\sigma_3\sigma_2 \\ \sigma_3\sigma_1 &= i\sigma_2 = -\sigma_1\sigma_3 \,. \end{aligned} \tag{21}$$

- A bivector $\hat{\mathbf{B}} = \mathbf{b} \wedge \mathbf{c}$ can be multiplied by a vector giving rise to a vector and a *trivector*:

$$\mathbf{a}\,\hat{\mathbf{B}} = \mathbf{a} \cdot \hat{\mathbf{B}} + \mathbf{a} \wedge \hat{\mathbf{B}} \tag{22}$$

- the internal product of a vector with a bivector is given by

$$\mathbf{a} \cdot \hat{\mathbf{B}} = \frac{1}{2}\left(\mathbf{a}\hat{\mathbf{B}} - \hat{\mathbf{B}}\mathbf{a}\right) = -\mathbf{a} \times \mathbf{b} \times \mathbf{c} = -\mathbf{a} \times \mathbf{B} \tag{23}$$

- the external product of a vector and a bivector is

$$\mathbf{a} \wedge \hat{\mathbf{B}} = \frac{1}{2}\left(\mathbf{a}\hat{\mathbf{B}} + \hat{\mathbf{B}}\mathbf{a}\right) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\ i = \mathbf{a} \cdot \mathbf{B}\, i \tag{24}$$

- the dot product of a vector and a bivector is anticommutative and it is related to the cross product as:

$$\mathbf{a} \cdot \mathbf{b} \wedge \mathbf{c} = -\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{b})\mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\mathbf{b} = -\mathbf{b} \wedge \mathbf{c} \cdot \mathbf{a} \tag{25}$$

Space description

It is noted that elements three-dimensional space are described by eight numbers (i.e. a complex 2x2 matrix). In particular they are:

- one *scalar* (σ_0). Grade 0
- 3 basis *vectors* ($\sigma_1, \sigma_2, \sigma_3$) i.e. three directions. Grade 1
- 3 basis *bivectors* ($\sigma_1\sigma_2, \sigma_1\sigma_3, \sigma_2\sigma_3$). Grade 2
- one *pseudoscalar* ($\sigma_1\,\sigma_2\,\sigma_3$). Grade 3

All these elements are contained in a 2x2 matrix and, similarly to what we do for complex numbers, they can be written together in a *multivector M* as

$$M = a_0 + \mathbf{a} + \hat{\mathbf{B}} + \hat{t} \tag{26}$$

where a_0 is a scalar, $\mathbf{a}$ is a vector, $\hat{\mathbf{B}}$ is a bivector and $\hat{t}$ is a pseudoscalar.

Multivector

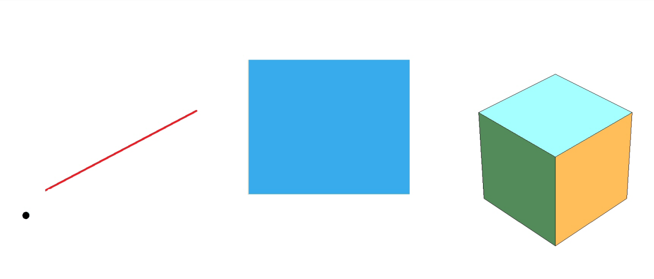

Figure 4: A multivector $M = a_0 + \mathbf{a} + \hat{\mathbf{B}} + \hat{t}$ representing a point, line, area and volume, that can be added, subtracted, multiplied or divided by other multivectors.

Pauli matrix representation of a multivvector

Let us see with more details the Pauli matrix representation of a multivector. The corresponding matrices of the multivector in (26) are given next:

$$
\begin{aligned}
\tilde{a_0} &= \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix} \\
\tilde{a} &= \begin{pmatrix} a_3 & a_1 - i\, a_2 \\ i\, a_2 + a_1 & -a_3 \end{pmatrix} \\
\tilde{B} &= \begin{pmatrix} i\, B_3 & B_2 + i\, B_1 \\ i\, B_1 - B_2 & -i\, B_3 \end{pmatrix} \\
\tilde{t} &= \begin{pmatrix} i\, t & 0 \\ 0 & i\, t \end{pmatrix} .
\end{aligned} \tag{27}
$$

The matrices in (27) can be summed together giving for the multivector M

$$\tilde{M} = \begin{pmatrix} i\, B_3 + i\, t + a_3 + a_0 & B_2 + i\, B_1 - i\, a_2 + a_1 \\ -B_2 + i\, B_1 + i\, a_2 + a_1 & -i\, B_3 + i\, t - a_3 + a_0 \end{pmatrix} . \tag{28}$$

Naturally, for a given Pauli matrix, it is straightforward to retrieve the elements of the different grades.

Let us assume that the matrix in (28) is given and we want to retrieve the various elements. It is convenient to extract the real and imaginary part of $\tilde{M}$ as

$$\begin{aligned}
\tilde{M}_r &= \mathrm{Re}\left\{\tilde{M}\right\} = \begin{pmatrix} a_3 + a_0 & B_2 + a_1 \\ a_1 - B_2 & a_0 - a_3 \end{pmatrix} \\
\tilde{M}_i &= \mathrm{Im}\left\{\tilde{M}\right\} = \begin{pmatrix} B_3 + t & B_1 - a_2 \\ B_1 + a_2 & t - B_3 \end{pmatrix} .
\end{aligned} \tag{29}$$

By inspection, it is seen that we have the following identities:

$$\begin{aligned}
a_0 &= \frac{1}{2}\left(\tilde{M}_{r11} + \tilde{M}_{r22}\right) \\
a_1 &= \frac{1}{2}\left(\tilde{M}_{r12} + \tilde{M}_{r21}\right) \\
a_2 &= \frac{1}{2}\left(\tilde{M}_{i21} - \tilde{M}_{i12}\right) \\
a_3 &= \frac{1}{2}\left(\tilde{M}_{r11} - \tilde{M}_{r22}\right) \\
B_1 &= \frac{1}{2}\left(\tilde{M}_{i21} + \tilde{M}_{i12}\right) \\
B_2 &= \frac{1}{2}\left(\tilde{M}_{r12} - \tilde{M}_{r21}\right) \\
B_3 &= \frac{1}{2}\left(\tilde{M}_{i11} - \tilde{M}_{i22}\right) \\
t &= \frac{1}{2}\left(\tilde{M}_{i11} + \tilde{M}_{i22}\right) .
\end{aligned} \tag{30}$$

Nabla operator with Pauli matrices

By using Pauli matrices a field vector **F** may be written as

$$\tilde{F} \;=\; \begin{pmatrix} F_z & F_x - i\,F_y \\ F_x + i\,F_y & -F_z \end{pmatrix} \tag{31}$$

Similarly, the Pauli matrix representation of the ∇ operator takes the form

$$\tilde{\nabla} \;=\; \sigma \cdot \nabla = \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z = \begin{pmatrix} \partial_z & \partial_x - i\,\partial_y \\ \partial_x + i\,\partial_y & -\partial_z \end{pmatrix} . \tag{32}$$

From the fundamental identity (the geometric product) we have:

$$\nabla\,\mathbf{F} \;=\; \nabla \cdot \mathbf{F} + \nabla \wedge \mathbf{F} = \tilde{\nabla}\,\tilde{F}\,. \tag{33}$$

When evaluating $\nabla \mathbf{F}$ via Pauli matrices we simply need to perform the following matrix product:

$$
\begin{aligned}
\tilde{\nabla}\,\tilde{F} &= \begin{pmatrix} \partial_z & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & -\partial_z \end{pmatrix} \begin{pmatrix} F_z & F_x - i\,F_y \\ i\,F_y + F_x & -F_z \end{pmatrix} \\
&= \begin{pmatrix} \partial_z\,F_z + \partial_y\,F_y + \partial_x\,F_x & 0 \\ 0 & \partial_z\,F_z + \partial_y\,F_y + \partial_x\,F_x \end{pmatrix} + \\
&+ \begin{pmatrix} i\,(\partial_x\,F_y - \partial_y\,F_x) & i\,(\partial_y\,F_z) - \partial_x\,F_z - i\,(\partial_z\,F_y) + \partial_z\,F_x \\ i\,(\partial_y\,F_z) + \partial_x\,F_z - i\,(\partial_z\,F_y) - \partial_z\,F_x & -i\,(\partial_x\,F_y - \partial_y\,F_x) \end{pmatrix} . \qquad (34)
\end{aligned}
$$

In (34) we have separated the scalar part corresponding to $\nabla \cdot \mathbf{F}$ (diagonal matrix) from the external product $\nabla \wedge \mathbf{F}$.

In several instances it is necessary to form the second order expressions, e.g. $\nabla\nabla\,\mathbf{F}$. Computation of this quantity via Pauli matrices is embarrassing simple, since only matrix multiplication is required

$$
\begin{aligned}
\tilde{\nabla}\,\tilde{\nabla}\,\tilde{F} &= \begin{pmatrix} \partial_z & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & -\partial_z \end{pmatrix} \begin{pmatrix} \partial_z & \partial_x - i\,\partial_y \\ i\,\partial_y + \partial_x & -\partial_z \end{pmatrix} \begin{pmatrix} F_z & F_x - i\,F_y \\ i\,F_y + F_x & -F_z \end{pmatrix} \\
&= \begin{pmatrix} \Delta\,F_z & \Delta\,F_x - i\,\Delta\,F_y \\ i\,\Delta\,F_y + \Delta\,F_x & \Delta\,F_z \end{pmatrix} \qquad (35)
\end{aligned}
$$

where we have introduced the Laplacian Δ defined as

$$
\Delta = \partial_z^2 + \partial_y^2 + \partial_x^2 \qquad (36)
$$

Maxwell's Equations in compact form

Time–domain Maxwell's Equations in compact form

Time–domain Maxwell's equations are commonly expressed as:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{37}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{38}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{39}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{40}$$

It is convenient to express the above equations making use of the light velocity in the medium v and of the medium impedance η, recalling that:

$$v = \frac{1}{\sqrt{\mu\epsilon}} \tag{41}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{42}$$

$$\mu = \frac{\eta}{v} \tag{43}$$

$$\epsilon = \frac{1}{v\eta}\,. \tag{44}$$

With a few superficial changes we can make more evident the symmetries in Maxwell equations as,

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial\, v\mathbf{B}}{\partial vt} &= 0 && (45) \\ \nabla \times v\mathbf{B} - \frac{\partial\, \mathbf{E}}{\partial vt} &= \eta \mathbf{J} && (46) \\ \nabla \cdot \mathbf{E} &= \frac{v\rho}{v\epsilon} = \eta v \rho && (47) \\ \nabla \cdot v\mathbf{B} &= 0 && (48) \end{aligned}$$

It is also noted that

$$v\,\mathbf{B} = \eta\,\mathbf{H} \qquad (49)$$

so that one can change the expression containing $v\,\mathbf{B}$ in $\eta\,\mathbf{H}$ or viceversa.

Equation (45) can be multiplied by i and (46) can be multiplied by i^2, obtaining

$$\begin{aligned} i\nabla \times \mathbf{E} + \frac{\partial\, i\,\eta \mathbf{H}}{\partial vt} &= 0 && (50)\\ i\nabla \times i\,\eta \mathbf{H} + \frac{\partial\, \mathbf{E}}{\partial vt} &= -\eta \mathbf{J} && (51)\\ \nabla \cdot \mathbf{E} &= \eta\, v \rho && (52)\\ \nabla \cdot i\,\eta \mathbf{H} &= 0 && (53) \end{aligned}$$

By using (33) we can write compactly (50)–(53) as

$$\begin{aligned} \nabla\, \mathbf{E} + \frac{\partial\, i\,\eta \mathbf{H}}{\partial vt} &= \eta\, v \rho && (54)\\ \nabla\, (i\,\eta \mathbf{H}) + \frac{\partial\, \mathbf{E}}{\partial vt} &= -\eta \mathbf{J}\,. && (55) \end{aligned}$$

A few observations are in order:

- In every place where t appears, we have arranged things so that vt appears, rather than t alone. The rationale is that vt has the same dimensions as x, y, and z.
- Similarly, the partner to $\mathbf{J}$ is not ρ but rather $v\,\rho$.
- Last but not least, the proper partner for $\mathbf{E}$ is not $\mathbf{H}$ but rather $\eta\,\mathbf{H}$. In every place where $\mathbf{H}$ appears, we have arranged things so the combination $\eta\,\mathbf{H}$ appears, rather than $\mathbf{H}$ alone. This is just an exercise in algebraic re–arrangement, and does not change the meaning of the equations. The rationale is that $\eta\,\mathbf{H}$ has the same dimensions as $\mathbf{E}$, and arranging things this way makes the equations more symmetric. It is also noted that since $\eta\,\mathbf{H}$ is always multiplied by i it is a bivector while $\mathbf{E}$ is a vector.

The field multivector: GA approach

The multivector $\mathcal{F}$, composed by a vector and a bivector part, is now introduced with the following definition:

$$\mathcal{F} = \mathbf{E} + i\,\eta\,\mathbf{H}\,. \tag{56}$$

By summing together (54) and (55), the well known results that allows to express the four Maxwell equation as a single one is recovered:

$$\left(\nabla + \frac{1}{v}\partial_t\right)\mathcal{F} = \eta\,\left(v\rho - \mathbf{J}\right)\,. \tag{57}$$

This expression, while being very synthetic, does not provide the same insight as the Dirac form, that we will introduce next.

The two equations (54,55) in the sourceless case, may be rewritten in terms of Pauli matrices as

$$\begin{aligned} \tilde{\nabla}\,\tilde{E} + \sigma_0 \frac{\partial\, i\,\eta \tilde{H}}{\partial vt} &= 0 && (58)\\ \tilde{\nabla}\left(i\,\eta \tilde{H}\right) + \sigma_0 \frac{\partial\,\tilde{E}}{\partial vt} &= 0\,. && (59) \end{aligned}$$

By using matrix notation we can write

$$\begin{pmatrix} \frac{1}{v}\partial_t\,\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\frac{1}{v}\partial_t\,\sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\,\eta \tilde{H} \end{pmatrix} = 0 \qquad (60)$$

where we have changed sign at (58). Equation (60) is ready to be cast in Dirac form, remembering that

$$\tilde{\nabla} = \sigma \cdot \nabla = \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z$$

Dirac Matrices

Paul Dirac

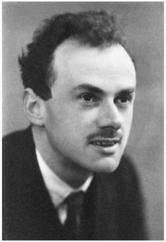

Figure 5: Paul Dirac

Paul Adrien Maurice Dirac (8 August 1902 – 20 October 1984) was an English theoretical physicist who made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics.

Among other discoveries, he formulated the Dirac equation which describes the behaviour of fermions and predicted the existence of antimatter. Dirac shared the 1933 Nobel Prize in Physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory".[5]

A set of four anti–commuting matrices

As noted in Arfken, in 1927 P. A. M. Dirac was looking for a set of four anticommuting matrices.

The three Pauli matrices plus the unit matrix form a complete set, but this set presents only three anticommuting matrices. By extending the Pauli matrices to 4 by 4 matrices it is possible to find this set.

In 1928, building on 2 by 2 spin matrices which Dirac discovered independently of Wolfgang Pauli's work on non-relativistic spin systems, Dirac obtained the 4 by 4 matrices.

Abraham Pais quoted Dirac as saying "I believe I got these (matrices) independently of Pauli and possibly Pauli got these independently of me".

The sixteen forms, which form a complete basis for representing four by four matrices, are reported in Table 1.

Table 1: Dirac Matrices as reported in Arfken, mathematical methods for physicists, pag. 213, third edition, academic press. In addition the notation with the g_{ij} has been introduced for compactness.

	σ_0	σ_1	σ_2	σ_3
ρ_0	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ $g_{00}, 1, \alpha_0$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$ g_{01}, σ_1	$\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&-i\\0&0&i&0\end{pmatrix}$ g_{02}, σ_2	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$ g_{03}, σ_3
ρ_1	$\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$ $g_{10}, \rho_1, -\gamma_5$	$\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$ g_{11}, α_1	$\begin{pmatrix}0&0&0&-i\\0&0&i&0\\0&-i&0&0\\i&0&0&0\end{pmatrix}$ g_{12}, α_2	$\begin{pmatrix}0&0&1&0\\0&0&0&-1\\1&0&0&0\\0&-1&0&0\end{pmatrix}$ g_{13}, α_3
ρ_2	$\begin{pmatrix}0&0&-i&0\\0&0&0&-i\\i&0&0&0\\0&i&0&0\end{pmatrix}$ $g_{20}, \gamma_0, \rho_2, \alpha_5$	$\begin{pmatrix}0&0&0&-i\\0&0&-i&0\\0&i&0&0\\i&0&0&0\end{pmatrix}$ g_{21}, γ_1	$\begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}$ g_{22}, γ_2	$\begin{pmatrix}0&0&-i&0\\0&0&0&i\\i&0&0&0\\0&-i&0&0\end{pmatrix}$ g_{23}, γ_3
ρ_3	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$ $g_{30}, \delta_0, \rho_3, \alpha_4, \gamma_4, \beta$	$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&-1\\0&0&-1&0\end{pmatrix}$ g_{31}, δ_1	$\begin{pmatrix}0&-i&0&0\\i&0&0&0\\0&0&0&i\\0&0&-i&0\end{pmatrix}$ g_{32}, δ_2	$\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&1\end{pmatrix}$ g_{33}, δ_3

Table 2: Dirac matrices in terms of Pauli matrices.

	σ_0	σ_1	σ_2	σ_3
ρ_0	$\begin{pmatrix} \sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}$ $g_{00}, 1, \alpha_0$	$\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}$ g_{01}, σ_1	$\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}$ g_{02}, σ_2	$\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ g_{03}, σ_3
ρ_1	$\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$ $g_{10}, \rho_1, -\gamma_5$	$\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$ g_{11}, α_1	$\begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$ g_{12}, α_2	$\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$ g_{13}, α_3
ρ_2	$i\begin{pmatrix} 0 & -\sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$ $g_{20}, \gamma_0, \rho_2, \alpha_5$	$i\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}$ g_{21}, γ_1	$i\begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix}$ g_{22}, γ_2	$i\begin{pmatrix} 0 & -\sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$ g_{23}, γ_3
ρ_3	$\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$ $g_{30}, \delta_0, \rho_3, \alpha_4, \gamma_4, \beta$	$\begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}$ g_{31}, δ_1	$\begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix}$ g_{32}, δ_2	$\begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix}$ g_{33}, δ_3

The Dirac Gamma matrices are defined, with index $i = 1, 2, 3$ as:

$$\begin{aligned} \gamma^0 = \gamma^4 &= \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \\ \gamma^i &= \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \\ \gamma^5 &= -\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \end{aligned} \tag{61}$$

We note that γ^0 squares to plus one, all the γ^i with $i = 1, 2, 3$ square to minus one. In addition they anti-commute. As a consequence the set of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ form a Clifford basis $Cl(1, -3)$.

Maxwell's equations in Dirac form

It is convenient to introduce the following notation:

$$
\begin{aligned}
x_0 &= vt \\
x_1 &= x \\
x_2 &= y \\
x_3 &= z
\end{aligned}
\tag{62}
$$

which is valid for the cartesian coordinate system. Similarly, we use for the derivatives the symbol

$$
\partial_i = \frac{\partial}{\partial x_i} \,. \tag{63}
$$

By changing the sign of eq. (54) it is possible to rewrite the Maxwell equations in a Dirac like notation in terms of the gamma matrices as

$$
\sum_{i=0}^{3} \partial_i \gamma^i \begin{pmatrix} E_z \\ i\, E_y + E_x \\ \eta\, i\, H_z \\ \eta\, (i\, H_x - H_y) \end{pmatrix} = -\eta \begin{pmatrix} J_z \\ i\, J_y + J_x \\ v\, \rho_e \\ 0 \end{pmatrix} \tag{64}
$$

By introducing the Feynman slash notation

$$\partial\!\!\!/ = \sum_{i=0}^{3} \partial_i \gamma^i \tag{65}$$

and the shorthand notation for the quadrivectors

$$\begin{aligned}
\bar{F} &= \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_0 - b_0 \\ a_1 - b_1 \end{pmatrix} \\
\bar{J} &= \begin{pmatrix} J_z \\ i\,J_y + J_x \\ v\,\rho_e \\ 0 \end{pmatrix}
\end{aligned} \tag{66}$$

we simply have

$$\partial\!\!\!/ \bar{F} = -\eta \bar{J} \tag{67}$$

which presents a form similar to the Dirac equation.

Explicit representation of $\not{\partial}\bar{F}$

The representation of $\not{\partial}\bar{F}$ in matrix terms is:

$$\not{\partial}\bar{F} = \begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\,\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix}$$

It is possible to note that the four equations are coupled: we need to solve 4 coupled equations.

It is possible to separate the problem into two systems of only two coupled equations by using the Weyl decomposition.

Weyl decomposition

Weyl decomposition

In this section the sourceless case, which applies eg. to propagation problems, is considered. The four eqs. (67) are coupled, but it is possible to separate them in two independent sets of two equations. To this end it is convenient to introduce the two matrices

$$\begin{aligned} A^- &= \frac{1}{\sqrt{2}}\left(\gamma^4 - \gamma^5\right) \\ A^+ &= \frac{1}{\sqrt{2}}\left(\gamma^4 + \gamma^5\right) . \end{aligned} \tag{68}$$

These matrices square to the identity matrix thus being equal to their inverse. It is also noted that

$$\begin{pmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{pmatrix} = \frac{1}{\sqrt{2}} A^- \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_0 - b_0 \\ a_1 - b_1 \end{pmatrix} . \tag{69}$$

It is therefore possible to transform eq. (67) as

$$A^+ \not{\partial} A^- \; A^- \bar{F} = 0 \tag{70}$$

or, explicitly, in cartesian coordinates

$$A^+ \not{\partial} A^- = \begin{pmatrix} \partial_3 + \partial_0 & \partial_1 - i\,\partial_2 & 0 & 0 \\ i\,\partial_2 + \partial_1 & \partial_0 - \partial_3 & 0 & 0 \\ 0 & 0 & \partial_3 - \partial_0 & \partial_1 - i\,\partial_2 \\ 0 & 0 & i\,\partial_2 + \partial_1 & -\partial_3 - \partial_0 \end{pmatrix}$$

i.e. the problem is decomposed into two sub-problems

$$
\begin{aligned}
\left(\tilde{\nabla} + \sigma_0 \partial_0\right) \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} &= 0 \\
\left(\tilde{\nabla} - \sigma_0 \partial_0\right) \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} &= 0 \, .
\end{aligned} \tag{71}
$$

which are the generalization of two transmission lines problems for incident $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ and reflected $\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$ waves.

Conclusions

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Pauli matrices allow to represent vectors in the form of a 2x2 matrix.

By using this representation and Dirac gamma matrices, we can write Maxwell's equations in a compact form, which is similar to the Dirac equation.

Moreover, in the sourceless case, the four equations can be decoupled into two sets of two equations.