# Complex numbers as an Introduction to Clifford Algebra

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# Complex numbers properties

# Argand plane

The fact that real and imaginary numbers map a plane seems to provide way to describe vectors in two dimensions.

In fact, given a vector $\mathbf{a} = a_r + i\, a_i$ we can plot it as illustrated in Fig. 1.

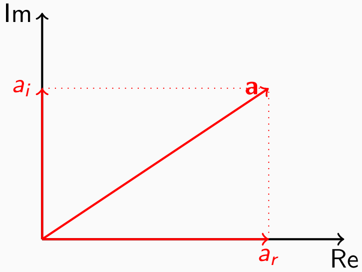


**Figure 1:** Representation of a vector via complex numbers. The vector $\mathbf{a} = a_r + i\, a_i$ is represented on the Argand plane.

# Dilation and Rotation

Multiplication of a complex number by a real number simply provide a *dilation*.

It is also very interesting that if we multiply the vector $\mathbf{a}$ by $i$ we obtain a rotation as

$$\mathbf{a}' = i\,\mathbf{a} = -a_i + i\,a_r \tag{1}$$

as reported in Fig. 2.

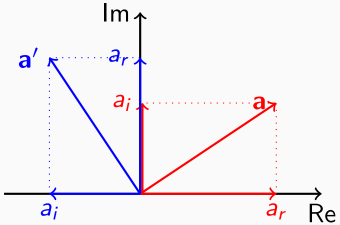


**Figure 2:** The vector $\mathbf{a} = a_r + i\,a_i$ when multiplied by $i$ is rotated by $\pi/2$ in the counterclockwise direction.

# The measure of a

It is rather intuitive that we want to have for the measure of **a** the following value:

$$a = \sqrt{a_r^2 + a_i^2}\,. \tag{2}$$

Since $a = a_r + i\, a_i$ we can try to evaluate its square as

$$a^2 = (a_r + ia_i)\,(a_r + ia_i) \tag{3}$$

which produces

$$\begin{aligned} a^2 &= (a_r + ia_i)\,(a_r + ia_i) \\ &= a_r^2 - a_i^2 + 2\, i\, a_r\, a_i\,. \end{aligned} \tag{4}$$

This is not the result we are looking for, i.e. a measure of the length of **a**.

# The complex conjugate remedy

Note that with vectors we are generally accustomed to perform either the dot or cross product. Here, instead, since we are dealing with complex quantities, we can perform their multiplication.

The *remedy* commonly used is to consider instead the product:

$$\begin{aligned} a\,a^* &= (a_r + ia_i)\,(a_r - ia_i) \\ &= a_r^2 + a_i^2\,, \end{aligned} \tag{5}$$

which correctly gives the square of the length of **a**.

# To summarize the concepts seen:

- complex numbers can have two different interpretations:
  - as a two–dimensional vector;
  - as an operator that apply a rotation and a scale factor to a vector.
- It can be observed that the product between two complex numbers

$$zw = |z|e^{i\phi_z}|w|e^{i\phi_w} = |z||w|e^{i(\phi_z+\phi_w)} \tag{6}$$

  represents:
  - the operator that results from the composition of $z$ and $w$ if $z$ and $w$ are operators;
  - the vector resulting from a rotation of $w$ by an angle $\phi_z$ and a scale factor $|z|$, if $z$ is an operator and $w$ is a vector;

- When $z$ and $w$ are two vectors, it is convenient to introduce the following product:

$$\begin{aligned} z^* w &= |z|e^{-i\phi_z}|w|e^{i\phi_w} = |z||w|e^{i(\phi_w - \phi_z)} = \\ &= |z||w|cos(\phi_w - \phi_z) + i|z||w|sin(\phi_w - \phi_z) \end{aligned} \tag{7}$$

  It can be noted that:

  - $Re[z^* w]$ is the scalar product between the vectors $z$ and $w$;
  - $Im[z^* w]$ is the area of the parallelogram defined by the two vectors with sign:
    - $+$ if $w$ if rotated counter clockwise with respect to $z$;
    - $-$ if it is rotated clockwise,

    and it will be related to the external product of the two vectors.
  - The result of the product can not be interpreted as a vector because the real part is a scalar product and the imaginary part is an area.

# Complex numbers Interpretations

- The fact that complex numbers can be interpreted in two different ways, it is necessary to define different products. This happens, for example, for the study of circuits in AC regime.
  In DC regime, the Ohm's law and the power are expressed as $v = Ri$ and $p = vi$.
  In AC regime, with the symbolic method:
  - The symbolic Ohm's law is written as $V = ZI$. In this expression $V$ and $I$ are phasors (i.e. vectors) but $Z$ is an operator that shows haw to modify the amplitude and phase of $I$ to obtain $V$.
  - The expression of the complex power is written as $N = VI^*$. In this case, $V$ and $I$ are phasors but $N$ is not a phasor.

# Clifford algebra of order two

# Clifford algebra of order two

There is, however, a different way for proceeding. Let us introduce a *base* with two orthogonal elements $e_1$ and $e_2$ which satisfy the rule

$$e_1^2 = e_2^2 = 1\,. \tag{8}$$

In terms of the base elements $e_1, e_2$ the vector $\mathbf{a}$ can be written as

$$\mathbf{a} = a_r\, e_1 + a_i\, e_2 \tag{9}$$

and we can try to evaluate the following product:

$$\begin{aligned} \mathbf{a}\,\mathbf{a} &= (a_r\, e_1 + a_i\, e_2)\,(a_r\, e_1 + a_i\, e_2) \\ &= a_r^2 + a_i^2 + e_1 e_2\, a_r a_i + e_2 e_1\, a_r a_i\,. \end{aligned} \tag{10}$$

# Rules

In the multiplication the elements are generally assumed as commutative (i.e. $ab = ba$). But let us now assume that the basis elements are *anticommutative*, i.e. that our base verify the following rule:

$$e_1 e_2 = -e_2 e_1 \, . \tag{11}$$

If this is the case, then the product in (10) becomes the sought one, i.e.

$$\begin{aligned} \mathbf{a}\,\mathbf{a} &= (a_r\, e_1 + a_i\, e_2)\,(a_r\, e_1 + a_i\, e_2) \\ &= a_r^2 + a_i^2 \, . \end{aligned} \tag{12}$$

The two rules introduced in (8) and (11) are just what is sufficient to define the *Clifford algebra* of order 2.

# The space elements

Note that in this algebra we have four elements:

- the scalar
- the two vectors basis $e_1$ and $e_2$
- and the element $e_1 e_2$.

While we postpone to investigate what is the meaning of the element $e_1 e_2$ we just try to evaluate its square. We have:

$$e_1 e_2 \, e_1 e_2 = -e_1 e_2 \, e_2 e_1 \, = -1 \tag{13}$$

since both $e_2$ and $e_1$ square to 1. We have just found the important relationship that

$$e_1 e_2 = e_{12} = i \tag{14}$$

i.e. that the product of the base elements is equal to imaginary number $i$.

# Grade concept

Therefore the real part corresponds to the scalar;

the imaginary part corresponds to the product of the base elements $e_1 e_2$;

and we have two additional terms which are the base elements $e_1$ and $e_2$.

The scalar part is associated with grade zero, the base elements have grade one and the element $e_{12}$ has grade two.

It is apparent that by selecting only the even grades we recover the complex numbers from the Clifford algebra of order two.

# The multivector $\mathcal{M}$

In addition note that, similarly to the complex numbers, the Clifford algebra can sum together elements of different grades. We can therefore form the multivector $\mathcal{M}$ which contains the scalar, the two vectors components relative to $e_1, e_2$ and the component relative to $e_{12}$ as

$$\mathcal{M} = a_0 + a_1 e_1 + a_2 e_2 + a_{12} e_{12} \,. \tag{15}$$

In the case of the vector **a** we have therefore two different representations in terms of multivectors:

$$\begin{aligned} a' &= a_r + e_{12} a_i = a_r + i a_i && (16) \\ a'' &= a_r e_1 + a_i e_2 && (17) \end{aligned}$$

It is noted that graphically the two representations are very similar (see Fig. 3) but obviously refer to different multivectors.

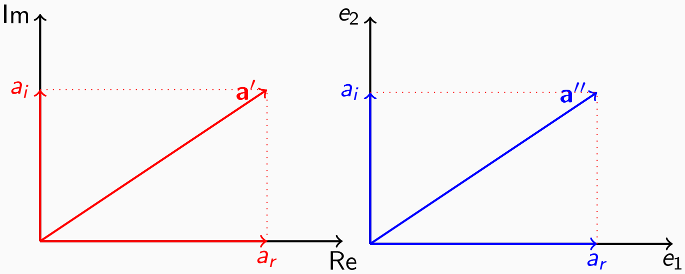


**Figure 3:** Representation of the multivectors $\mathbf{a}'$ and $\mathbf{a}''$. Although they look similar they represent different multivectors.

# The geometric product

# The geometric product

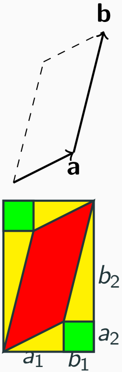


**Figure 4:** Let us consider two vectors $\mathbf{a} = (a_1, a_2)$ and $\mathbf{b} = (b_1, b_2)$ with e.g. the following amplitudes $\mathbf{a} = (2, 1)$ and $\mathbf{b} = (1, 4)$. The product $(a_1 + b_1)(a_2 + b_2)$ is the entire rectangle shown in the lower figure. In order to obtain the part in red we should subtract from the entire rectangle the green parts $(2a_2b_1)$ and the yellow parts $(a_1a_2 + b_1b_2)$.

# Product of two vectors

Let us perform the product of two vectors according to the rules of Clifford algebra. We have a vector $\mathbf{a} = (a_1, a_2)$ and a vector $\mathbf{b} = (b_1, b_2)$ which we may write as:

$$\begin{aligned} \mathbf{a} &= a_1\, e_1 + a_2\, e_2 \\ \mathbf{b} &= b_1\, e_1 + b_2\, e_2 \,. \end{aligned} \tag{18}$$

By performing the multiplication we get

$$\begin{aligned} \mathbf{a}\,\mathbf{b} &= (a_1\, e_1 + a_2\, e_2)\,(b_1\, e_1 + b_2\, e_2) \\ &= a_1 b_1 + a_2 b_2 + e_{12}\,(a_1 b_2 - a_2 b_1) \\ &= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b} \end{aligned} \tag{19}$$

# Geometric product

with the dot and wedge products corresponding, respectively, to

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= a_1 b_1 + a_2 b_2 \\ \mathbf{a} \wedge \mathbf{b} &= e_{12} \left( a_1 b_2 - a_2 b_1 \right) \end{aligned} \tag{20}$$

for this two dimensional case. While the dot product has the ordinary meaning, let us see the geometric meaning of the wedge product. With reference to Fig. 4 it is easy to see that the wedge product of **a** and **b** corresponds to the part in red.

# External product

In fact, the product $(a_1 + b_1)(a_2 + b_2)$ is the entire rectangle shown in the lower part of Fig. 4.

In order to obtain the part in red we should subtract from the entire rectangle the green parts $(2a_2b_1)$ and the yellow parts $(a_1a_2 + b_1b_2)$.

In this way we obtain:

$$\begin{aligned}(a_1 + b_1)(a_2 + b_2) - 2a_2b_1 - a_1a_2 - b_1b_2 &= \\ a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 - 2a_2b_1 - a_1a_2 - b_1b_2 &= \\ a_1b_2 - a_2b_1\,. & \end{aligned} \tag{21}$$

# Geometric product

The product introduced in (19) is the *geometric product* and is a new entity.

Similarly to the case of complex number in the geometric product we sum together a scalar quantity with a surface (like real and imaginary parts).

In addition, it is immediately recognized that, contrary to the case of complex number, the product of two vectors gives something different (i.e. the scalar $(\mathbf{a} \cdot \mathbf{b})$ and a bivector $(\mathbf{a} \wedge \mathbf{b})$).

It is useful to consider also the following product:

$$\begin{aligned} \mathbf{b}\,\mathbf{a} &= (b_1\, e_1 + b_2\, e_2)\,(a_1\, e_1 + a_2\, e_2) \\ &= a_1 b_1 + a_2 b_2 - e_{12}\,(a_1 b_2 - a_2 b_1) \\ &= \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \wedge \mathbf{a} = \mathbf{b} \cdot \mathbf{a} - \mathbf{a} \wedge \mathbf{b}\,. \end{aligned} \tag{22}$$

# Alternative definitions of the dot and wedge product

By comparing (19) and (22) it is seen that when changing the order of multiplication the part in the wedge product change sign. Equivalently, by summing and subtracting (19) and (22) we obtain alternative definitions of the dot and wedge product as

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2}(\mathbf{a}\,\mathbf{b} + \mathbf{b}\,\mathbf{a}) \tag{23}$$

$$\mathbf{a} \wedge \mathbf{b} = \frac{1}{2}(\mathbf{a}\,\mathbf{b} - \mathbf{b}\,\mathbf{a})\,. \tag{24}$$

The above relations are valid not only in two dimensions but apply in general.

Let us also note that the wedge product of a vector with itself is null, i.e.

$$\mathbf{a} \wedge \mathbf{a} = a_1 a_2 - a_2 a_1 = 0\,. \tag{25}$$

# Inverse of a vector

The geometric product of vectors is *invertible* for all vectors with non-zero square $a^2 \neq 0$

$$\mathbf{a}^{-1} := \mathbf{a}/a^2, \quad \mathbf{a}\mathbf{a}^{-1} = \mathbf{a}\mathbf{a}/a^2 = 1,$$
$$\mathbf{a}^{-1}\mathbf{a} = \frac{\mathbf{a}}{a^2}\mathbf{a} = a^2/a^2 = 1. \tag{26}$$

The inverse vector $\mathbf{a}/a^2$ is a rescaled version (reflected at the unit circle) of the vector $\mathbf{a}$. This invertibility leads to significant simplifications and ease in computations.