Maxwell's equations

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Maxwell’s equations

Maxwell's equations

Equations linking electromagnetic field quantities have been introduced by James Clerk Maxwell in an elegant treatise first published in 1873.

We assume that a student reader is familiar with these equations. In what follows, we summarize Maxwell's equations in time and frequency domains.

It is customary to write Maxwell's equations in either local or in global form; we shall first consider their local form.

We also note that, unfortunately, it is customary to designate the local form as differential form and this generates some confusion with the general meaning that differential forms have.

Local Form of Maxwell's Equations

In three–dimensional vector notation, with vector **r** a indicating a position in space and t the time variable, Maxwell's equations are

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = -\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}\,, \qquad \text{Faraday's law} \tag{1a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}, t) = \frac{\partial \mathbf{D}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t)\,, \qquad \text{Ampère's law} \tag{1b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}, t) = \rho_e(\mathbf{r}, t)\,, \qquad \text{Gauss' law} \tag{1c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}, t) = 0\,, \qquad \text{Magnetic flux continuity} \tag{1d}$$

where bold face symbols denote vector quantities. The quantities are defined as

$\mathbf{E}(\mathbf{r}, t)$	electric field strength
$\mathbf{D}(\mathbf{r}, t)$	electric displacement
$\mathbf{B}(\mathbf{r}, t)$	magnetic flux density
$\mathbf{H}(\mathbf{r}, t)$	magnetic field strength
$\mathbf{J}(\mathbf{r}, t)$	electric current density
$\rho_e(\mathbf{r}, t)$	electric charge density

Equations dependence

Equations (1a)–(1d) are not independent since, for example, we may derive (1d) by taking the divergence of (1a).

Another fundamental relationship can be derived by introducing (1c) into the divergence of (1b)

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{\partial \rho_e(\mathbf{r}, t)}{\partial t} \tag{2}$$

which provides the conservation law for electric charge and current densities.

Actually, the set of three equations (1a), (1b) and (2) may be considered as the independent equations describing macroscopic electromagnetic fields, since the two Gauss equations (1c) and (1d) can be derived from this set.

Static case

Note that in the static case $\frac{\partial}{\partial t} = 0$ the electric and magnetic fields are not any more interdependent and the equations (1a) – (1d) become

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0\,, \tag{3a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r})\,, \tag{3b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r})\,, \tag{3c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0\,. \tag{3d}$$

Note that, if we assign the electric current density $\mathbf{J}(\mathbf{r})$ and the electric charge density $\rho_e(\mathbf{r})$, we have, from (1a) and (1b), two vector equations (i.e. six scalar equations) while we have four unknown vectors (i.e. twelve scalar quantities).

To complete the number of equations we have to account for the media properties expressed by the constitutive relations.

Integral Form of Maxwell's Equations

Maxwell's equations!integral form

The properties of an electromagnetic field may also be expressed globally by an equivalent system of integral relations through use of the two fundamental theorems of vector analysis: the divergence theorem and Stokes' theorem

Divergence or Gauss' Theorem

Let $\mathbf{U}(\mathbf{r})$ be any vector function of position, continuous together with its first derivative throughout a volume V bounded by a surface S. The divergence theorem states that

$$\oint_S \mathbf{U}(\mathbf{r}) \cdot \mathbf{n} \, dS = \int_V \nabla \cdot \mathbf{U}(\mathbf{r}) \, dV \,, \tag{4}$$

where $\mathbf{n}$ is the outward unit vector normal to S.

Gauss's theorem may also be used to *define* the divergence.

Stokes' Theorem

Stokes' theorem Let $\mathbf{U}(\mathbf{r})$ be any vector function of position, continuous together with its first derivatives throughout an arbitrary surface S bounded by a contour C, and assumed to be resolvable into a finite number of regular arcs.

Stokes' theorem (also called curl theorem) states that

$$\oint_C \mathbf{U}(\mathbf{r}) \cdot d\,\mathbf{l} = \int_S [\nabla \times \mathbf{U}(\mathbf{r})] \cdot \mathbf{n}\; dS\,, \tag{5}$$

where $d\,\mathbf{l}$ is an element of length along C, and $\mathbf{n}$ is a unit vector normal to the positive side of the element area dS as defined by the right–hand thumb rule.

This relationship may also be considered as an equation defining the *curl* or *circulation*.

Integral form of Maxwell's equations

By applying the curl theorem to (1a) and (1b), and the divergence theorem to (1c) and (1d), we get

$$\oint_C \mathbf{E}(\mathbf{r},t)\cdot d\,\mathbf{l} = -\int_S \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}\cdot \mathbf{n}\; dS\,, \tag{6a}$$

$$\oint_C \mathbf{H}(\mathbf{r},t)\cdot d\,\mathbf{l} = \int_S \frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t}\cdot \mathbf{n}\; dS + \int_S \mathbf{J}(\mathbf{r},t)\cdot \mathbf{n}\; dS\,, \tag{6b}$$

$$\int_V \nabla\cdot\mathbf{D}(\mathbf{r},t)dv = \oint_S \mathbf{D}(\mathbf{r},t)\cdot\mathbf{n}\; dS = \int_V \rho_e(\mathbf{r},t)\; dv\,, \tag{6c}$$

$$\int_V \nabla\cdot\mathbf{B}(\mathbf{r},t)dv = \oint_S \mathbf{B}(\mathbf{r},t)\cdot\mathbf{n}\; dS = 0\,. \tag{6d}$$

Maxwell's Equations in the Frequency Domain

Maxwell's Equations in the Frequency Domain

Electromagnetic fields operating at a particular frequency are known as time–harmonic steady–state or monochromatic fields.

By adopting the time dependence $e^{j\omega t}$ to denote a time–harmonic field with angular frequency ω, we write

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left\{\mathbf{E}(\mathbf{r})e^{j\omega t}\right\} , \tag{7}$$

where Re denotes the mathematical operator which selects the real part of a complex quantity.

The complex quantity is $\mathbf{E}(\mathbf{r})$ is called a *vector phasor*. In (7) we have used the same symbol to denote both the real quantity in the time domain, $\mathbf{E}(\mathbf{r}, t)$, and the complex quantity, $\mathbf{E}(\mathbf{r})$, in the frequency domain.

Example

By applying (7) to the field quantities appearing in (1a), (1b), (1c) and (1d) we obtain Maxwell's equations in the frequency domain.

As an example, let us consider (1a) for which we have

$$\mathrm{Re}\left\{[\nabla \times \mathbf{E}(\mathbf{r}) + j\omega \mathbf{B}(\mathbf{r})]\, e^{j\omega t}\right\} = 0\,. \tag{8}$$

Since this equation is valid for *all times t*, we may make use of the above lemma and state that the quantity inside the square bracket must be equal to zero.

Maxwell's equation in frequency domain

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega \mathbf{B}(\mathbf{r})\,, \tag{9a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega \mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r})\,, \tag{9b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r})\,, \tag{9c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0\,. \tag{9d}$$

Some identities for expressing Maxwell's Equations in GA

Let us consider a generic vector **a** expressed in terms of the Pauli representation $\tilde{a}$ as

$$\tilde{a} = \sigma_1\, a_x + \sigma_2\, a_y + \sigma_3\, a_z \tag{10}$$

where the elements σ_i constitutes the basis elements.

The ∇ operator, in cartesian coordinates, is given by:

$$\tilde{\nabla} = \sigma_1\, \partial_x + \sigma_2\, \partial_y + \sigma_3\, \partial_z\, . \tag{11}$$

Although we are referring to cartesian coordinates, the results derived next are valid in general.

External product of nabla with the vector a

The divergence is readily expressed as usual as:

$$\begin{aligned}\tilde{\nabla}\cdot\tilde{a} &= (\sigma_1\,\partial_x+\sigma_2\,\partial_y+\sigma_3\,\partial_z)\cdot(\sigma_1 a_x+\sigma_2\,a_y+\sigma_3\,a_z)\\ &= (\partial_x\,a_x+\partial_y\,a_y+\partial_z\,a_z)\,\sigma_0\,. \end{aligned}\tag{12}$$

The external product of nabla with the vector $\tilde{a}$ is (by using 1,2,3 instead of x,y,z)

$$\begin{aligned}\tilde{\nabla}\wedge\tilde{a} &= (\sigma_1\,\partial_1+\sigma_2\,\partial_2+\sigma_3\,\partial_3)\wedge(\sigma_1 a_1+\sigma_2\,a_2+\sigma_3\,a_3)\\ &= \sigma_1\sigma_2\,(\partial_1 a_2-\partial_2 a_1)+\sigma_1\sigma_3\,(\partial_1 a_3-\partial_3 a_1)+\sigma_2\sigma_3\,(\partial_2 a_3-\partial_3 a_2)\\ &= i\,[\sigma_1\,(\partial_2 a_3-\partial_3 a_2)-\sigma_2\,(\partial_1 a_3-\partial_3 a_1)+\sigma_3\,(\partial_1 a_2-\partial_2 a_1)]\\ &= i\nabla\times\mathbf{a}\,. \end{aligned}\tag{13}$$

Some identities for expressing Maxwell's Equations in GA: Bivectors

Let us consider a generic vector **B** expressed in terms of the Pauli representation $\tilde{B}$ as

$$\tilde{B} = \sigma_1 \, B_x + \sigma_2 \, B_y + \sigma_3 \, B_z \tag{14}$$

where the elements σ_i constitutes the basis elements.

The bivector $\hat{B} = i\sigma_0 \, \tilde{B}$ is expressed, using the identity $i\sigma_0 = \sigma_1 \, \sigma_2 \, \sigma_3$, as

$$\begin{aligned} \hat{B} &= i\sigma_0 \, (\sigma_1 \, B_x + \sigma_2 \, B_y + \sigma_3 \, B_z) \\ &= \sigma_2 \, \sigma_3 \, B_x + \sigma_3 \, \sigma_1 \, B_y + \sigma_1 \, \sigma_2 \, B_z \end{aligned} \tag{15}$$

The ∇ operator, in cartesian coordinates, is given by:

$$\nabla = \sigma_1 \, \partial_x + \sigma_2 \, \partial_y + \sigma_3 \, \partial_z \, . \tag{16}$$

Although we are referring to cartesian coordinates, the results derived next are valid in general.

External product of nabla with the bivector $\hat{B}$

The divergence is readily expressed as usual as:

$$\nabla \cdot \mathbf{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z \,. \tag{17}$$

The external product of nabla with the bivector $\hat{B}$ is

$$\begin{aligned} \nabla \wedge \hat{B} &= (\sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z) \wedge (\sigma_2 \sigma_3 B_x + \sigma_3 \sigma_1 B_y + \sigma_1 \sigma_2 B_z) \\ &= \sigma_1 \sigma_2 \sigma_3 (\partial_x B_x + \partial_y B_y + \partial_z B_z) \\ &= i \nabla \cdot \mathbf{B} \,. \end{aligned} \tag{18}$$

Note that the divergence of **B** is a scalar; when multiplied by i it becomes a pseudoscalar.

Also, when performing the external product of nabla with the bivector $\hat{B}$ a pseudoscalar is obtained.

Divergence of the bivector $\hat{B}$

The divergence of the bivector $\hat{B}$ is

$$\begin{aligned}
\nabla \cdot \hat{B} &= (\sigma_1\,\partial_x + \sigma_2\,\partial_y + \sigma_3\,\partial_z) \cdot (\sigma_2\,\sigma_3\,B_x + \sigma_3\,\sigma_1\,B_y + \sigma_1\,\sigma_2\,B_z) \\
&= -\sigma_3\,\partial_x\,B_y + \sigma_2\,\partial_x\,B_z + \sigma_3\,\partial_y\,B_x - \sigma_1\,\partial_y\,B_z - \sigma_2\,\partial_z\,B_x + \sigma_1\,\partial_z\,B_y \\
&= -\nabla \times \mathbf{B} = i\nabla \wedge \mathbf{B}\,. \qquad (19)
\end{aligned}$$

In summary we have derived the following important identities:

$$\nabla \wedge \hat{B} = i\nabla \cdot \mathbf{B} \qquad (20)$$

$$\nabla \cdot \hat{B} = -\nabla \times \mathbf{B} = i\,\nabla \wedge \mathbf{B} \qquad (21)$$

$$\nabla\,\hat{B} = i\,\nabla\,\mathbf{B} \qquad (22)$$

Geometric Algebra form of Maxwell's Equations

Geometric Algebra form of Maxwell's Equations

Faraday's law

By multiplying both sides of (1a) times i one obtains:

$$\nabla \wedge \mathbf{E} = -\partial_t \hat{B}\,. \tag{23}$$

It is noted that is an equation of grade 2, i.e. between bivectors.

Ampere's law

Let us now consider (1b) and make use of (21):

$$\nabla \cdot \hat{H} = -\partial_t \mathbf{D} - \mathbf{J}\,. \tag{24}$$

This is an equation of grade 1, i.e. between vectors.

Gauss' law This equation remains unchanged: in fact, by considering (1c):

$$\nabla \cdot \mathbf{D} = \rho_e \tag{25}$$

we have an equation of grade 0, i.e. a scalar equation.

Magnetic flux continuity

By considering (1d), multiplying by i and using (20):

$$\nabla \wedge \hat{B} = 0 \tag{26}$$

we have an equation of grade 3, i.e. a pseudoscalar equation.

GA equivalent of Maxwell's equations

In summary the following form are the GA equivalent of Maxwell's equations local form:

$$\nabla \wedge \mathbf{E} = -\partial_t \hat{B}\,, \qquad \text{grade 2} \tag{27a}$$

$$\nabla \cdot \hat{H} = -\partial_t \mathbf{D} - \mathbf{J}\,, \qquad \text{grade 1} \tag{27b}$$

$$\nabla \cdot \mathbf{D} = \rho_e\,, \qquad \text{grade 0} \tag{27c}$$

$$\nabla \wedge \hat{B} = 0\,, \qquad \text{grade 3} \tag{27d}$$

GA equivalent of Maxwell's equations with magnetic sources

Sometimes it is convenient to consider also magnetic sources obtaining the following local form of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon}\,, \qquad \text{grade } 0 \qquad (28a)$$

$$\nabla \cdot (i\eta\mathbf{H}) = -\frac{1}{c}\,\partial_t \mathbf{E} - \eta\,\mathbf{J}\,, \qquad \text{grade } 1 \qquad (28b)$$

$$\nabla \wedge \mathbf{E} = -\frac{1}{c}\,\partial_t\,(i\eta\mathbf{H}) - i\,\mathbf{M}, \qquad \text{grade } 2 \qquad (28c)$$

$$\nabla \wedge (i\eta\mathbf{H}) = -i\,c\,\rho_m\,, \qquad \text{grade } 3 \qquad (28d)$$

Geometric Algebra Global form of Maxwell's Equations

Let us refer to the equations (6a)–(6d). Instead of dealing with the term $\mathbf{n}\, dS$ we can now consider the bivector $d\,\hat{A}$ denoting the oriented surface area. In addition note that when a volume term dv is considered, it corresponds to a pseudoscalar as $dv = dx\, dy\, dz$.

$$\oint_C \mathbf{E} \cdot d\,\mathbf{l} = -\int_S \frac{\partial \hat{B}}{\partial t} \cdot d\,\hat{A}\,, \tag{29a}$$

$$\oint_C \hat{H} \wedge d\,\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \wedge d\,\hat{A} \tag{29b}$$

$$\int_V \nabla \cdot \mathbf{D}\, dv = \oint_S \mathbf{D} \wedge d\,\hat{A} = \int_V \rho_e\, dv\,, \tag{29c}$$

$$\int_V \nabla \cdot \mathbf{B}\, dv = \oint_S \hat{B} \cdot d\,\hat{A} = 0\,. \tag{29d}$$

It is noted that, in the above equations, the dot product between two bivectors give rise to a scalar, while the volume integral and the external product of a vector with a bivector produce a pseudoscalar.

Maxwell's Equations in compact form

Time–domain Maxwell's Equations in compact form

Time–domain Maxwell's equations are commonly expressed as:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{30}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{31}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{32}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{33}$$

It is convenient to express the above equations making use of the light velocity in the medium v and of the medium impedance η, recalling that:

$$v = \frac{1}{\sqrt{\mu\epsilon}} \tag{34}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \tag{35}$$

$$\mu = \frac{\eta}{v} \tag{36}$$

$$\epsilon = \frac{1}{v\eta}\,. \tag{37}$$

With a few superficial changes we can make more evident the symmetries in Maxwell equations as,

$$\begin{aligned}
\nabla \times \mathbf{E} + \frac{\partial\, v\mathbf{B}}{\partial vt} &= 0 &(38)\\
\nabla \times v\mathbf{B} - \frac{\partial\, \mathbf{E}}{\partial vt} &= \eta \mathbf{J} &(39)\\
\nabla \cdot \mathbf{E} &= \frac{v\rho}{v\epsilon} = \eta v \rho &(40)\\
\nabla \cdot v\mathbf{B} &= 0 &(41)
\end{aligned}$$

It is also noted that

$$v\,\mathbf{B} = \eta\,\mathbf{H} \tag{42}$$

so that one can change the expression containing $v\,\mathbf{B}$ in $\eta\,\mathbf{H}$ or viceversa.

Equation (38) can be multiplied by i and (39) can be multiplied by i^2, obtaining

$$\begin{aligned} i\nabla \times \mathbf{E} + \frac{\partial\, i\,\eta\mathbf{H}}{\partial vt} &= 0 && (43) \\ i\nabla \times i\,\eta\mathbf{H} + \frac{\partial\,\mathbf{E}}{\partial vt} &= -\eta\mathbf{J} && (44) \\ \nabla \cdot \mathbf{E} &= \eta\, v\rho && (45) \\ \nabla \cdot i\,\eta\mathbf{H} &= 0 && (46) \end{aligned}$$

By using the Pauli identity we can write compactly (43)–(46) as

$$\begin{aligned} \nabla\,\mathbf{E} + \frac{\partial\, i\,\eta\mathbf{H}}{\partial vt} &= \eta\, v\rho && (47) \\ \nabla\,(i\,\eta\mathbf{H}) + \frac{\partial\,\mathbf{E}}{\partial vt} &= -\eta\mathbf{J}\,. && (48) \end{aligned}$$

A few observations are in order:

- In every place where t appears, we have arranged things so that vt appears, rather than t alone. The rationale is that vt has the same dimensions as x, y, and z.
- Similarly, the partner to $\mathbf{J}$ is not ρ but rather $v\,\rho$.
- Last but not least, the proper partner for $\mathbf{E}$ is not $\mathbf{H}$ but rather $\eta\,\mathbf{H}$. In every place where $\mathbf{H}$ appears, we have arranged things so the combination $\eta\,\mathbf{H}$ appears, rather than $\mathbf{H}$ alone. This is just an exercise in algebraic re–arrangement, and does not change the meaning of the equations. The rationale is that $\eta\,\mathbf{H}$ has the same dimensions as $\mathbf{E}$, and arranging things this way makes the equations more symmetric. It is also noted that since $\eta\,\mathbf{H}$ is always multiplied by i it is a bivector while $\mathbf{E}$ is a vector.

The field multivector: GA approach

The multivector $\mathcal{F}$, composed by a vector and a bivector part, is now introduced with the following definition:

$$\mathcal{F} = \mathbf{E} + i\,\eta\,\mathbf{H}\,. \tag{49}$$

By summing together (47) and (48), the well known results that allows to express the four Maxwell equation as a single one is recovered:

$$\left(\nabla + \frac{1}{v}\partial_t\right)\mathcal{F} = \eta\,\left(v\rho - \mathbf{J}\right)\,. \tag{50}$$

This expression, while being very synthetic, does not provide the same insight as the Dirac form, that we will introduce next.

The two equations (47,48) in the sourceless case, may be rewritten in terms of Pauli matrices as

$$\begin{aligned} \tilde{\nabla}\,\tilde{E} + \sigma_0 \frac{\partial\, i\,\eta \tilde{H}}{\partial vt} &= 0 && (51)\\ \tilde{\nabla}\left(i\,\eta \tilde{H}\right) + \sigma_0 \frac{\partial\, \tilde{E}}{\partial vt} &= 0\,. && (52) \end{aligned}$$

By using matrix notation we can write

$$\begin{pmatrix} \frac{1}{v}\partial_t\,\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\frac{1}{v}\partial_t\,\sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\,\eta \tilde{H} \end{pmatrix} = 0 \qquad (53)$$

where we have changed sign at (51). Equation (53) is ready to be cast in Dirac form, remembering that

$$\tilde{\nabla} = \sigma \cdot \nabla = \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z$$

Maxwell's equations in Dirac form

Maxwell's equations

It is convenient to introduce the following notation:

$$\begin{aligned} x_0 &= vt \\ x_1 &= x \\ x_2 &= y \\ x_3 &= z \end{aligned} \tag{54}$$

which is valid for the cartesian coordinate system. Similarly, we use for the derivatives the symbol

$$\partial_i = \frac{\partial}{\partial x_i} . \tag{55}$$

By changing the sign of eq. (47) it is possible to rewrite the Maxwell equations in a Dirac like notation in terms of the gamma matrices as

$$\sum_{i=0}^{3} \partial_i \gamma^i \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\,i\,H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = -\eta \begin{pmatrix} J_z \\ i\,J_y + J_x \\ v\,\rho_e \\ 0 \end{pmatrix} \tag{56}$$

Maxwell's equations

We have already seen eq. (53), here repeated for convenience for the sourceless case and expressed in the present notation:

$$\begin{pmatrix} \partial_0\,\sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -\partial_0\,\sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{E} \\ i\,\eta\tilde{H} \end{pmatrix} \quad = \quad 0 \tag{57}$$

with $\tilde{\nabla}$ being

$$\tilde{\nabla} = \sigma \cdot \nabla = \sigma_1\partial_1 + \sigma_2\partial_2 + \sigma_3\partial_3\,.$$

Therefore, if we write explicitly eq. (57) we have

$$\begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\,\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = 0 \tag{58}$$

In (58) we have considered the sourceless case and we have used just the first column of the Pauli matrices representing the fields $\tilde{E}$ and $i\,\eta\tilde{H}$. By using the Dirac gamma matrices we have

$$\left(\gamma^0\partial_0 + \gamma^1\partial_1 + \gamma^2\partial_2 + \gamma^3\partial_3\right) \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = 0 \tag{59}$$

Maxwell–Dirac

By taking into account also of the sources it is possible to rewrite the Maxwell equations in a Dirac like notation in terms of the gamma matrices as

$$\sum_{i=0}^{3} \partial_i \gamma^i \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = -\eta \begin{pmatrix} J_z \\ i\,J_y + J_x \\ v\,\rho_e \\ 0 \end{pmatrix} \tag{60}$$

By introducing the Feynman slash notation

$$\not{\partial} = \sum_{i=0}^{3} \partial_i \gamma^i \tag{61}$$

Maxwell–Dirac

and the shorthand notation for the quadrivectors

$$
\begin{aligned}
\bar{F} &= \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} \\
\bar{J} &= \begin{pmatrix} J_z \\ i\,J_y + J_x \\ v\,\rho_e \\ 0 \end{pmatrix}
\end{aligned}
\tag{62}
$$

we simply have

$$
\partial\!\!\!/ \bar{F} = -\eta \bar{J} \tag{63}
$$

which presents a form similar to the Dirac equation for null mass.

The quadrivector $\bar{F}$

The quadrivector $\bar{F}$ can also be written in terms of waves

$$\bar{F} \quad = \quad \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_0 - b_0 \\ a_1 - b_1 \end{pmatrix}$$

Explicit representation of $\partial\!\!\!/\bar{F}$

The representation of $\partial\!\!\!/\bar{F}$ in matrix terms is:

$$\partial\!\!\!/\bar{F} = \begin{pmatrix} \partial_0 & 0 & \partial_3 & \partial_1 - i\partial_2 \\ 0 & \partial_0 & \partial_1 + i\,\partial_2 & -\partial_3 \\ -\partial_3 & -\partial_1 + i\partial_2 & -\partial_0 & 0 \\ -\partial_1 - i\partial_2 & \partial_3 & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix}$$

It is possible to note that the four equations are coupled: we need to solve 4 coupled equations.

It is possible to separate the problem into two systems of only two coupled equations by using the Weyl decomposition.

Weyl decomposition

Weyl decomposition

Let us consider the case without sources, which applies e.g. to propagation problems.

The four eqs. (60) are coupled, but it is possible to separate them in two independent sets of two equations.

To this end it is convenient to introduce the two matrices

$$\begin{aligned} A^- &= \frac{1}{\sqrt{2}}\left(\gamma^4 - \gamma^5\right) \\ A^+ &= \frac{1}{\sqrt{2}}\left(\gamma^4 + \gamma^5\right) . \end{aligned} \tag{64}$$

These matrices square to the identity matrix thus being equal to their inverse.

$$A^- A^- = A^+ A^+ = I_4 \tag{65}$$

Weyl decomposition in matrix form

It is also noted that

$$\begin{pmatrix} a_0 \\ a_1 \\ b_0 \\ b_1 \end{pmatrix} = \frac{1}{\sqrt{2}} A^- \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \\ a_0 - b_0 \\ a_1 - b_1 \end{pmatrix} . \tag{66}$$

We can therefore transform eq. (63) as

$$A^+ \partial\!\!\!/ A^- \; A^- \bar{F} = 0 \tag{67}$$

Weyl decomposition in matrix form

or, explicitly, in cartesian coordinates, we have

$$A^+ \not\partial A^- = \begin{pmatrix} \partial_3 + \partial_0 & \partial_1 - i\,\partial_2 & 0 & 0 \\ i\,\partial_2 + \partial_1 & \partial_0 - \partial_3 & 0 & 0 \\ 0 & 0 & \partial_3 - \partial_0 & \partial_1 - i\,\partial_2 \\ 0 & 0 & i\,\partial_2 + \partial_1 & -\partial_3 - \partial_0 \end{pmatrix}$$

i.e. the problem is decomposed in two systems as following:

$$\begin{aligned} \left(\tilde{\nabla} + \sigma_0 \partial_0\right) \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} &= \tilde{\partial}^+ \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0 \\ \left(\tilde{\nabla} - \sigma_0 \partial_0\right) \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} &= \tilde{\partial}^- \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0\,. \end{aligned} \qquad (68)$$

An alternative approach

At the same result we can arrive directly starting from (51), (52) and by introducing the column vectors u, w defined as

$$
\begin{aligned}
u &= \begin{pmatrix} E_z \\ i\,E_y + E_x \end{pmatrix} = \begin{pmatrix} a_0 + b_0 \\ a_1 + b_1 \end{pmatrix} \\
w &= \begin{pmatrix} \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = \begin{pmatrix} a_0 - b_0 \\ a_1 - b_1 \end{pmatrix}
\end{aligned}
\tag{69}
$$

we have

$$
\begin{aligned}
\tilde{\nabla} u + \sigma_0\, \partial_0\, w &= 0 \\
\tilde{\nabla} w + \sigma_0\, \partial_0\, u &= 0\,.
\end{aligned}
\tag{70}
$$

Wave propagation equations

By summing and subtracting the above eqs. we have:

$$\begin{aligned}\tilde{\nabla}\,(u+w) + \sigma_0\,\partial_0\,(u+w) &= 0\\ \tilde{\nabla}\,(u-w) + \sigma_0\,\partial_0\,(u-w) &= 0\end{aligned} \tag{71}$$

and since

$$\begin{aligned}u+w &= 2\,a = 2\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}\\ u-w &= 2\,b = 2\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}\end{aligned} \tag{72}$$

we have obtained eqs. (73) or

$$\begin{aligned}\left(\tilde{\nabla} + \sigma_0\partial_0\right)\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} &= \tilde{\partial}^{+}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = 0\\ \left(\tilde{\nabla} - \sigma_0\partial_0\right)\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} &= \tilde{\partial}^{-}\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = 0\,.\end{aligned} \tag{73}$$