November 25, 2016

Geometric Algebra

Eric Chisolm

Abstract

This is an introduction to *geometric algebra*, an alternative to traditional vector algebra that expands on it in two ways:

1. In addition to scalars and vectors, it defines new objects representing subspaces of any dimension.
2. It defines a product that's strongly motivated by geometry and can be taken between any two objects. For example, the product of two vectors taken in a certain way represents their common plane.

This system was invented by William Clifford and is more commonly known as Clifford algebra. It's actually older than the vector algebra that we use today (due to Gibbs) and includes it as a subset. Over the years, various parts of Clifford algebra have been reinvented independently by many people who found they needed it, often not realizing that all those parts belonged in one system. This suggests that Clifford had the right idea, and that geometric algebra, not the reduced version we use today, deserves to be the standard "vector algebra." My goal in these notes is to describe geometric algebra from that standpoint and illustrate its usefulness. The notes are work in progress; I'll keep adding new topics as I learn them myself.

Contents

1. Introduction

1.1. Motivation

I'd say the best intuitive definition of a vector is "anything that can be represented by arrows that add head-to-tail." Such objects have *magnitude* (how long is the arrow) and *direction* (which way does it point). Real numbers have two analogous properties: a *magnitude* (absolute value) and a *sign* (plus or minus). Higher-dimensional objects in real vector spaces also have these properties: for example, a surface element is a plane with a *magnitude* (area) and an *orientation* (clockwise or counterclockwise). If we associate real scalars with zero-dimensional spaces, then we can say that scalars, vectors, planes, etc. have three features in common:

1. An *attitude*: exactly which subspace is represented.
2. A *weight*: an amount, or a length, area, volume, etc.
3. An *orientation*: positive or negative, forward or backward, clockwise or counterclockwise. No matter what the dimension of the space, there are always only two orientations.

If spaces of any dimension have these features, and we have algebraic objects representing the zero- and one-dimensional cases, then maybe we could make objects representing the other cases too. This is exactly what geometric algebra gives us; in fact, it goes farther by including all of these objects on equal footing in a single system, in which anything can be added to or multiplied by anything else. I'll illustrate by starting in three-dimensional Euclidean space.

My goal is to create a product of vectors, called the *geometric product*, which will allow me to build up objects that represent all the higher-dimensional subspaces. Given two vectors u and v, traditional vector algebra lets us perform two operations on them: the dot product (or inner product) and the cross product. The dot product is used to project one vector along another; the projection of v along u is

$$P_u(v) = \frac{u \cdot v}{|u|^2} u \tag{1}$$

where $u \cdot v$ is the inner product and $|u|^2 = u \cdot u$ is the square of the length of u. The cross product represents the oriented plane defined by u and v; it points along the normal to the plane and its direction indicates orientation. This has two limitations:

1. It works only in three dimensions, because only there does every plane have a unique normal.
2. Even where it works, it depends on an arbitrarily chosen convention: whether to use the right or left hand to convert orientations to directions. So the resulting vector does not simply represent the plane itself.

Because of this, I'll replace the cross product with a new object that represents the plane directly, and it will generalize beyond three dimensions as easily as vectors themselves do.

I begin with a formal product of vectors uv that obeys the usual rules for multiplication; for example, it's associative and distributive over addition. Given these rules I can write

$$uv = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu). \tag{2}$$

The first term is symmetric and bilinear, just like a generic inner product; therefore I set it equal to the Euclidean inner product, or

$$\frac{1}{2}(uv + vu) := u \cdot v. \tag{3}$$

I can immediately do something interesting with this: notice that $u^2 = u \cdot u = |u|^2$, so the square of any vector is just its squared length. Therefore, the vector

$$u^{-1} := \frac{u}{u^2} \tag{4}$$

is the multiplicative inverse of u, since obviously $uu^{-1} = u^2/u^2 = 1$. So in a certain sense we can divide by vectors. That's neat. By the way, the projection of v along u from Eq. (1) can now be written

$$P_u(v) = (v \cdot u)u^{-1}. \tag{5}$$

In non-Euclidean spaces, some vectors are null, so they aren't invertible. That means that this projection operator won't be defined. As it turns out, projection along noninvertible vectors doesn't make sense geometrically; I'll explain why in Section 7.1. Thus we come for the first time to a consistent theme in geometric algebra: algebraic properties of objects frequently have direct geometric meaning.

What about the second term in Eq. (2)? I call it the *outer product* or *wedge product* and represent it with the symbol $\wedge$, so now the geometric product can be decomposed as

$$uv = u \cdot v + u \wedge v. \tag{6}$$

To get some idea of what $u \wedge v$ is, I'll use the fact that it's antisymmetric in u and v, while $u \cdot v$ is symmetric, to modify Eq. (6) and get

$$vu = u \cdot v - u \wedge v. \tag{7}$$

Multiplying these equations together I find

$$uvvu = (u \cdot v)^2 - (u \wedge v)^2. \tag{8}$$

Now $vv = |v|^2$, and the same is true for u, while $u \cdot v = |u|\,|v| \cos\theta$, so

$$(u \wedge v)^2 = -|u|^2|v|^2 \sin^2\theta. \tag{9}$$

So whatever $u \wedge v$ is, its square has two properties:

1. It's a negative scalar. (Just like an imaginary number, without my having to introduce them separately. Hmm.)

2. Aside from the minus sign, it is the square of the magnitude of the cross product.

The first property means that $u \wedge v$ is neither scalar nor vector, while the second property makes it look like a good candidate for the plane spanned by the vectors. $u \wedge v$ will turn out to be something called a *simple bivector* or 2-*blade*, so 2-blades represent planes with an area and an orientation (interchange u and v and you change the sign of $u \wedge v$). There's no unique parallelogram associated with $u \wedge v$ because for any λ,

$$u \wedge (v + \lambda u) = u \wedge v. \tag{10}$$

So sliding the tip of one side along the direction of the other side changes the parallelogram but not the outer product. It is the plane (attitude), area (weight), and orientation that the outer product defines uniquely. With these definitions, the product of two vectors turns out to be the sum of two very different objects: a scalar and a bivector. For the moment think of such a sum as purely formal, like the sum of a real and an imaginary number.

Later I'll define the outer product of any number of vectors, and this product will be associative:

$$(u \wedge v) \wedge w = u \wedge (v \wedge w) = u \wedge v \wedge w. \tag{11}$$

This guy is called a *simple trivector* or 3-*blade*, and it represents the three-dimensional space spanned by its factors, again with a weight (volume) and orientation. We can also form 4-blades, 5-blades, and so on up to the dimension of whatever vector space we're in. Each of these represents a subspace with the three attributes of attitude, weight, and orientation. These r-blades and their sums, called *multivectors*, make up the entire geometric algebra. (Even scalars are included as 0-vectors.) The geometric product of vectors can be extended to the whole algebra; you can multiply any two objects together, which lets you do all sorts of useful things. Just multiplying vectors already lets us do a lot, as I'll show now.

1.2. Simple applications

I'll start by solving two standard linear algebra problems. Let's suppose a plane is spanned by vectors a and b, and you have a known vector x in the plane that you want to expand in terms of a and b. Therefore you want scalars α and β such that

$$x = \alpha a + \beta b. \tag{12}$$

To solve this, take the outer product of both sides with a; since $a \wedge a = 0$, you get

$$a \wedge x = \beta a \wedge b. \tag{13}$$

It will turn out in Euclidean space that every nonzero vector, 2-blade, and so on is invertible, so this can be solved to get

$$\beta = (a \wedge x)(a \wedge b)^{-1}. \tag{14}$$

This makes sense geometrically: both $a \wedge x$ and $a \wedge b$ are bivectors in the same plane, so one should be a scalar multiple of the other. Since β is effectively a ratio of areas, I'm going to write instead

$$\beta = \frac{a \wedge x}{a \wedge b}. \tag{15}$$

The problem with this is that it could mean either $(a \wedge x)(a \wedge b)^{-1}$ or $(a \wedge b)^{-1}(a \wedge x)$; but in this case they're the same, so there's no harm. Taking the outer product of both sides with b similarly gets you $\alpha = (x \wedge b)/(a \wedge b)$, so now we know that

$$x = \left(\frac{x \wedge b}{a \wedge b}\right) a + \left(\frac{a \wedge x}{a \wedge b}\right) b. \tag{16}$$

This expression is called Cramer's Rule. Here I've derived it much more quickly than is done in regular vector algebra, it's expressed directly in terms of the vectors instead of in components, and the geometric meaning of the coefficients (ratios of areas in the plane) is immediately apparent. Also note that this expression is defined iff $a \wedge b \neq 0$, which is exactly the condition that a and b span the plane.

The generalization from planes to volumes is straightforward; if a, b, and c span the space then

$$x = \left(\frac{x \wedge b \wedge c}{a \wedge b \wedge c}\right) a + \left(\frac{a \wedge x \wedge c}{a \wedge b \wedge c}\right) b + \left(\frac{a \wedge b \wedge x}{a \wedge b \wedge c}\right) c \tag{17}$$

and so on for higher dimensions.

When you have a linear equation like this, taking the outer product with one of the terms, and thus removing that term, is often a handy trick. Here's another example. Suppose I have two lines that lie in a plane: The first passes through point p and points in direction a, while the second passes through point q and points in direction b. Assuming the lines aren't parallel, at what point x do they cross?

If the lines aren't parallel then a and b aren't parallel, so they span the plane. Therefore x is a linear combination of a and b as given by Eq. (16). That's nice but unhelpful, because this time x is unknown and we're trying to solve for it. But wait; x lies on the line through p pointing along a, or

$$x = p + \lambda a \tag{18}$$

for some λ. That means that $a \wedge x = a \wedge p$. And the fact that x lies on the line through q pointing along b tells me that $x \wedge b = q \wedge b$, so when I put all this in Eq. (16) I find that the intersection point x is

$$x = \left(\frac{q \wedge b}{a \wedge b}\right) a + \left(\frac{a \wedge p}{a \wedge b}\right) b, \tag{19}$$

expressing the unknown x in terms of the four known vectors defining the two lines.

The solutions to these last two exercises are expressed in an entirely intrinsic, coordinate-free way, which means that the results of this calculation can be used as inputs in any further calculations. Once you get to the end, of course, you can certainly use coordinates to perform the final computations. To do all this, though, you have to be comfortable with these new kinds of products and their inverses. I'm here to help with that.

Now for a little geometry. I'll start by looking at reflections, like the operation performed by a mirror. How do we perform a mirror reflection on a vector? Well, we often think of a reflection as happening in one of two complementary ways: either *through a plane* (components in the plane are left alone, the remaining component gets a minus sign) or *along an axis* (the component along the axis gets a minus sign and the other components are left alone). However, these ways of thinking are interchangeable only in three dimensions, because only there does any plane have a unique normal. I want a picture that works in any number of dimensions, and only the second one does that, because it works even in one dimension. So I'll use it from now on.

Let v be the vector we want to reflect and let n be a vector along the reflection axis. Then

$$\begin{aligned} v &= v(nn^{-1}) \\ &= (vn)n^{-1} \\ &= (v \cdot n)n^{-1} + (v \wedge n)n^{-1}. \end{aligned} \tag{20}$$

The first term looks like the right hand side of Eq. (5), so it represents the orthogonal projection of v along n. That means the other term is the component of v perpendicular to n, also called the *orthogonal rejection* of v from n. (I'll bet you've never heard that term before.) Now let v' be the reflected vector; its component along n has the opposite sign, while its perpendicular component is the same, so it is given by

$$v' = -(v \cdot n)n^{-1} + (v \wedge n)n^{-1}. \tag{21}$$

Using the symmetry and antisymmetry of the inner and outer products respectively, I can recast this as

$$\begin{aligned} v' &= -(n \cdot v)n^{-1} - (n \wedge v)n^{-1} \\ &= -(n \cdot v + n \wedge v)n^{-1} \\ &= -nvn^{-1}. \end{aligned} \tag{22}$$

This is a nifty little result; the equation for reflecting a vector along an axis is very tidy. Compare that to

$$v' = v - 2\frac{n \cdot v}{|n|^2} n, \tag{23}$$

which is the simplest one can do with traditional vector algebra.

The appearance of both n and n^{-1} in Eq. (22) guarantees that the result depends neither on the weight (length) nor the orientation of n, only its attitude (the axis it represents), as it should.

The next operation I'll describe is rotation. First note that the usual way one thinks of rotations, as being performed around an axis, works only in three dimensions. In general, it is better to think of a rotation of a vector as being performed *in a plane*; the component in the plane is rotated while the components perpendicular to the plane are left alone. Again, this picture works perfectly well in any number of dimensions.

Hamilton discovered a great way to perform rotations: To rotate through angle θ in a plane, perform two reflections in succession along any two axes in the plane, as long as (a) the angle between the axes is $\theta/2$ and (b) a rotation from the first axis to the second is in the same direction as the rotation to be performed.

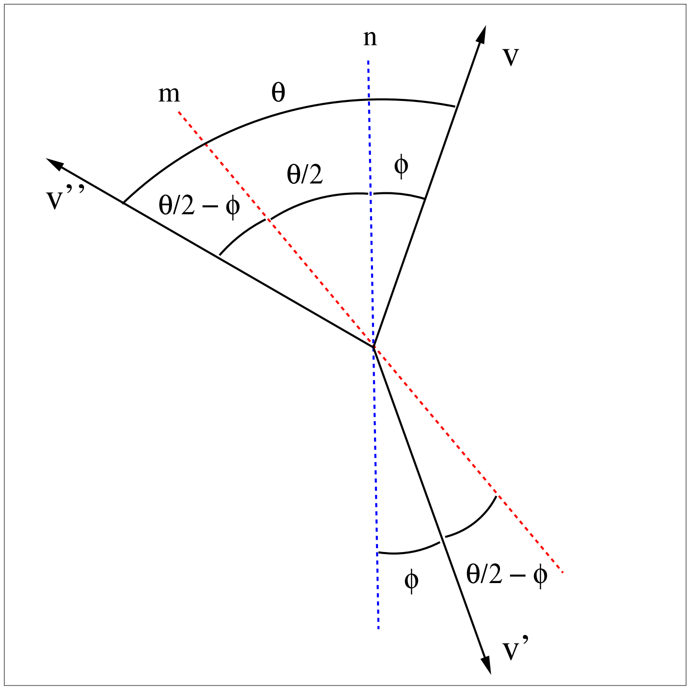

Figure 1: The vector v makes an angle ϕ with axis n. It is then reflected along n, producing vector v', then along axis m, producing vector v''. Notice that the angle between vectors v and v'' is θ, twice the angle between n and m, regardless of the value of ϕ.

This is shown in Figure 1. So if I want to rotate vector v, then I let m and n be vectors along axes satisfying the conditions, and the result of the rotation is

$$\begin{aligned} v' &= -m(-nvn^{-1})m^{-1} \\ &= (mn)v(mn)^{-1} \\ &= RvR^{-1} \end{aligned} \tag{24}$$

where $R = mn$. R is an object called a *rotor*. Typically a rotor is the product of unit vectors, in which case $m = m^{-1}$ and $n = n^{-1}$, which means $R^{-1} = nm$.

These two examples, reflections and rotations, introduce a second theme in geometric algebra: elements of the algebra represent both geometric objects (vectors, subspaces) and operations on those objects. There's no need to introduce any new elements (e.g. matrices) to represent operators.

I'll finish this section by looking at the rotor more closely.

$$\begin{aligned} R &= mn \\ &= m \cdot n + m \wedge n \\ &= m \cdot n - n \wedge m. \end{aligned} \tag{25}$$

I reversed the order of m and n because the sense of rotation of this rotor is from n to m (reflection n was applied first). Since m and n are now unit vectors and the angle between the two axes of reflection is $\theta/2$, $m \cdot n = \cos(\theta/2)$ and $(n \wedge m)^2 = -\sin^2(\theta/2)$. Therefore the bivector $B = (n \wedge m)/\sin(\theta/2)$ is a unit bivector: $B^2 = -1$. So now

$$\begin{aligned} R &= \cos(\theta/2) - \sin(\theta/2)B \\ &= \exp(-B\theta/2) \end{aligned} \tag{26}$$

where the exponential is defined by its power series; the scalar terms and the terms proportional to B can be grouped and summed separately. Now I have a rotation operator that explicitly displays the rotation's angle, plane (attitude of B), and direction (orientation of B), and all without coordinates.

Recall how little I started with: a product of vectors with the minimal algebraic properties to be useful, plus the extra bit that the symmetric part equals the inner product. From only that much, I've gotten a formula for rotating a vector that looks a lot like the formula for rotating a complex number, $z' = e^{i\theta}z$, except that it's double-sided and uses half of the rotation angle. The resemblance to complex numbers is no accident; as I will show later on, the complex numbers are contained in the geometric algebra of the real Euclidean plane. Therefore, all of complex algebra and analysis is subsumed into and generalized to arbitrary dimensions by geometric algebra. As for the half angles, in physics they normally show up in the quantum theory of half-integer spin particles, but this suggests that there's nothing particularly quantum about them; they arise simply because a rotation equals two reflections.

1.3. Where now?

When I first read about geometric algebra, examples like these immediately made me think it might have a lot to offer in terms of conceptual simplicity, unifying potential, and computational power. I was on the lookout for something like this because I had found standard mathematical physics dissatisfying in two main ways:

1. We use a hodgepodge of different techniques, particularly in more advanced work, each of which seems to find a home only in one or two specialized branches of theory. It seems like an unnecessary fragmentation of what should be a more unified subject.

2. As long as you stay in three dimensions and work only with vectors, everything is very concrete, geometrical, and easy to express in intrinsic form without coordinates. However, none of these desirable features seem to survive in more general situations (say, special relativity). Either you end up expressing everything in coordinates from the start, as in classical tensor analysis, or you use coordinate-free forms like those found in modern differential geometry, which I find hard to calculate with and which seem to leave their geometrical roots in favor of some abstract analytic form. (I put differential forms in this category.)

Despite signs of promise, however, I also have to admit I was taken aback by what looked like an enormous proliferation of new objects. After all, it seems like geometric algebra lets you multiply vectors all day long and keep getting new things, and I had very little sense of how they all related to each other. (I imagine this is why geometric algebra lost out to Gibbs' vector algebra in the first place.) I was also puzzled about how the rules for using these objects really worked. For example, if I had just read the previous two sections, I'd have questions like these.

1. In my first two examples in Section 1.2, I used the inverse of a 2-blade, $(a \wedge b)^{-1}$, and I mentioned that in Euclidean space every nonzero r-blade has an inverse. I've shown how to calculate the 2-blade itself: it's the antisymmetrized product. But how do you calculate the inverse?

2. In Eq. (22), I multiplied three vectors, nvn^{-1}, and the result was also a vector. However, you could tell that only by following the derivation. What's the product of three vectors in general? Is it always a vector? Is it something else? How can you tell?

3. Then I multiplied a bivector by a vector, $(v \wedge n)n^{-1}$. What's that? In this case the result was another vector, but again you had to follow the derivation to know that. In addition, I also said it was perpendicular to n. How do I check that? Presumably I should show that

$$\left[(v \wedge n)n^{-1}\right] \cdot n = 0, \tag{27}$$

but that looks scary.

To answer these and other questions for myself, I wrote these notes. I suspect I'm not the only one who reacted this way on seeing geometric algebra for the first time, so I hope the notes can help others understand geometric algebra and decide for themselves whether it's as good as advertised.

The structure of the notes reflects the best way I've found to explain geometric algebra to myself.

- In Section 2 I lay out a set of axioms. I find it helpful to present axioms first, so we can have all of our basic rules in place immediately, knowing we won't be surprised later by having to take anything else into account.
- With the axioms in hand, in Section 3 I answer the first question I asked myself when I saw all this: exactly what's in here? I describe both algebraically and geometrically what a generic multivector looks like, and I justify the claims I made at the end of Section 1.1 in terms of the axioms. By this point, a multivector should seem a lot more concrete than just "lots of vectors multiplied together."
- Having explained what a general multivector looks like, in Section 4 I explain what a general product of multivectors looks like. I also explain how to take the inner and outer products of any two multivectors, and I explain what they mean geometrically; this is a natural continuation of the geometrical discussion in Section 3. I claimed earlier that geometric algebra lets you take coordinate-free, intrinsic calculations much farther than standard methods; it does this because it has a large number of algebraic identities, which I'll start to derive here. These identities make vector algebra start to look a lot more like algebra with real numbers.
- A handful of additional operations are used all the time in calculations, and I collect them in Section 5. By describing them all together, I can show the relationships between them more easily.
- At this point even I think the reader needs a break, so I pause in Section 6 for an "application" by describing what our favorite vector spaces, two- and three-dimensional real Euclidean space, look like in these terms. I show how the complex numbers pop up all by themselves in the two-dimensional algebra, and in three dimensions I show how to convert back and forth from geometric algebra to the traditional language of cross products, triple products, and so on.
- With the full algebra at my disposal, in Section 7 I return with a vengeance to my initial examples: orthogonal projection, reflections, and rotations. Now I really get to show you why this isn't your grandpa's vector algebra. We can project vectors into subspaces, and even subspaces into other subspaces, far more easily than traditional methods ever made you think was possible. And wait till you see what rotations look like. Ever tried to rotate a plane? In geometric algebra, it's easy.
- Coordinates do have a role in geometric algebra, although it's vastly reduced, and I describe it in Section 8.
- Linear algebra looks very different when it's done not just on vector spaces but on geometric algebras; that's the subject of Section 9. I'll review the basics, but even familiar subjects like adjoints and skew symmetric operators take on a new flavor and significance in this system. And eigenvectors will be joined by eigenspaces of any dimension. I'll even show how to act with a linear operator on the whole vector space at once, and the eigenvalue of that operation will be our friend the determinant.
- Right now, the notes are very light on applications to physics; so far I have included only a brief discussion of classical angular momentum (which is no longer a vector, by the way) and the Kepler problem, which gets a pretty snazzy treatment. I'll add more applications soon.

All the important definitions and relations are listed together in Appendix A, and the topics I plan to include in future versions are listed in Appendix B.

1.4. References and comments

Although geometric algebra dates from the 19th century, it was recovered in the form described here only in the 20th century by David Hestenes [1,2,3], and it is slowly gaining popularity in various math and applied math communities. My primary sources are Hestenes' books; Doran and Lasenby [4], written for physicists; Dorst, Fontijne, and Mann [5], written for computer scientists; and the introductory linear algebra text by Macdonald [6], which includes geometric algebra alongside traditional linear algebra topics. You'll see their influence everywhere; for example, my axioms were inspired by [2], Section 1.1 and the second half of Section 1.2 come from [4], and the first half of Section 1.2 is lifted from [5]. I'll mention other areas where I'm particularly indebted my sources as I come to them. I follow [5] in defining two inner products, instead of Hestenes' one, but I continue to refer to them as inner products instead of "contractions" as Dorst *et al.* do. Finally, this approach to geometric algebra is far from the only one: Lounesto [7] describes this one and several others, and he gives a great overview of the history that has brought Clifford algebra to this point.

Given all the other introductions to geometric algebra out there, I hope this treatment is made distinctive by two elements. First, I have worked everything out in more detail than I've seen anywhere else, which I think is very helpful for getting one's initial bearings in the subject. Second, I don't believe this way of organizing the material is found in other sources either, and as I said in the previous section, this is the way I've found easiest to understand. I try to convey Hestenes' attitude toward Clifford algebra as not just another algebraic system but the natural extension of real numbers to include the geometric idea of direction, which I find very attractive.

I also prefer a more general treatment over a more specific one when the theory seems to be equally easy in either case. For example, all applications of geometric algebra I'm familiar with take the scalars to be $\mathbb{R}$, the real numbers, and an important part of Hestenes' view is that many of the other number systems used in mathematics are best understood not separately but as subsets of certain real Clifford algebras. (I dropped a hint about this regarding complex numbers in Section 1.1, to which I'll return in Section 6, where I'll handle the quaternions too.) However, I don't force the scalars to be $\mathbb{R}$ here, because the majority of results don't actually depend on what the scalars are. One thing that does change a bit, however, is the geometrical interpretation. For example, suppose the scalars are complex; how does the orientation of a vector change when you multiply by i? In fact, the two notions of weight and orientation make sense only for real vector spaces, and as a result they won't have a place in a general geometric algebra. They're still important for all those applications, however, so I'll make sure to explain them at the right time. And whenever the scalars have to be real for something to be true, I'll say so.

As part of my goal to work everything out in detail but keep the notes easy to follow, I've set the theorem proofs off from the rest of the text so they can be easily skipped. Nonetheless, I urge you to take a look at the shorter proofs; I tried to motivate them well and convey some useful insights. Even some of the long proofs consist of more than just turning the algebra crank. I like proofs that do more than show that something is true; they give a sense of why. I have tried to write those sorts of proofs here.

Because geometric algebra has found its way into most applied mathematics, albeit in a very fragmented way, everything I describe in these notes can be done using some other system: matrices, Grassmann algebras, complex numbers, and so on. The advantage that I see here is that one system, a natural extension of elementary vector algebra, can do all these things, and so far I've always found I can better understand what's going on when all the different results are related through a unified perspective.

2. Definitions and axioms

The purpose of this section is to define a geometric algebra completely and unambiguously. This is the rigorous version of the discussion from Section 1.1, and you'll see all of the basic ideas from that section reintroduced more precisely here.

A *geometric algebra* is a set $\mathcal{G}$ with two composition laws, addition and multiplication (also called the *geometric product*), that obey these axioms.

Axiom 1. *$\mathcal{G}$ is a ring with unit. The additive identity is called 0 and the multiplicative identity is called 1.*

Axiom 1 is the short way to say that (a) addition and multiplication in $\mathcal{G}$ are both associative, (b) both operations have identities, (c) every element has an additive inverse, (d) addition commutes, and (e) multiplication is left and right distributive over addition. So now I've said it the long way too.

A generic element of $\mathcal{G}$ is denoted by a capital Roman letter (A, B, etc.) and is called a *multivector*. Notice that a geometric algebra is one big system from the get-go: all multivectors, which will eventually include scalars, vectors, and much more, are part of the same set, and addition and multiplication are equally available to all. I'll continue to follow this philosophy as I introduce new operations by defining them for all multivectors. Also, this axiom formalizes the first requirement I made of the geometric product in Section 1.1; it gives addition and multiplication the minimal properties needed to be useful.

Axiom 2. *$\mathcal{G}$ contains a field $\mathcal{G}_0$ of characteristic zero which includes 0 and 1.*

A member of $\mathcal{G}_0$ is called a *0-vector*, a *homogeneous* multivector of *grade* 0, or a *scalar*. Scalars are denoted by lower case Greek letters (λ, μ, etc.). Being a field means that $\mathcal{G}_0$ is closed under addition and multiplication, it contains all inverses of its elements, and it obeys all the rules that $\mathcal{G}$ obeys from Axiom 1 plus the additional rules that (a) everything except 0 has a multiplicative inverse and (b) multiplication commutes. The rational numbers, real numbers, and complex numbers are all fields. The property of having characteristic zero saves me from getting in trouble in the following way. Since $\mathcal{G}_0$ doesn't have to be $\mathbb{R}$, the integers aren't actually the usual integers, but sums of terms all equaling 1, the multiplicative identity of $\mathcal{G}_0$. (So, for example, by 2 I literally mean $1+1$.) If I don't specify any further properties of $\mathcal{G}_0$, then I haven't ruled out $1+1=0$, which would be bad when I try to divide by 2. (Which I'll be doing frequently; see Eq. (2).) Having characteristic zero means that no finite sum of terms all equaling 1 will ever add up to 0, so I can divide by integers to my heart's content.

Axiom 3. *$\mathcal{G}$ contains a subset $\mathcal{G}_1$ closed under addition, and $\lambda \in \mathcal{G}_0, v \in \mathcal{G}_1$ implies $\lambda v = v\lambda \in \mathcal{G}_1$.*

A member of $\mathcal{G}_1$ is called a *1-vector*, a *homogeneous* multivector of *grade* 1, or just a *vector*. Vectors are denoted by lower case Roman letters (a, b, u, v, etc.). The axioms imply that $\mathcal{G}_1$ obeys all the rules of a vector space with $\mathcal{G}_0$ as scalars, justifying their names. However, all is not the same as what you're used to. In standard vector algebra, the scalars and vectors are usually separate sets. For example, consider the vector space $\mathbb{R}^3$ with the real numbers as scalars; the zero scalar is the number 0, but the zero vector is the ordered triple $(0,0,0)$. In geometric algebra this is not the case, and here's why.

1. 0 is a scalar by Axiom 2.
2. $0v = 0$ for any vector v by Axiom 1.
3. A scalar times a vector is a vector by Axiom 3.
4. Therefore, 0 is also a vector.

It will turn out that 0 is a whole lot of other things too.

So far the axioms have told us how to add scalars, add vectors, multiply scalars, and multiply a scalar and a vector. Multiplying vectors is next.

Axiom 4. *The square of every vector is a scalar.*

As it was in Section 1.1, this is the most important axiom of the bunch. Here's the first consequence: for any vectors u and v,

$$\frac{1}{2}(uv+vu) = \frac{1}{2}\left[(u+v)^2 - u^2 - v^2\right] \tag{28}$$

(you can easily prove this by expanding out the right hand side), and the right side is a scalar thanks to Axiom 4, so it follows that the symmetrized product of any two vectors is a scalar. In fact, this is not merely implied by Axiom 4; it's equivalent to it. (Assume the statement is true. Since the square of a vector is its symmetrized product with itself, Axiom 4 follows.) The symmetrized product of two vectors defined above is called their *inner product* and is denoted either $u \rfloor v$ or $u \lfloor v$. It is symmetric and linear in both terms, thus obeying the usual rules for an inner product on a real vector space (but not a complex vector space). Vectors u and v are said to be *orthogonal* if $u \rfloor v = 0$, u is a *unit* vector if $u^2 = \pm 1$, and u is *null* if $u^2 = 0$. Notice that vectors are orthogonal iff they anticommute. This turns out to be handy. Recall my earlier comment that if v is non-null, then v is invertible and $v^{-1} = v/v^2$.

I have two inner products, $\rfloor$ and $\lfloor$, instead of just the usual $\cdot$, for reasons that won't be clear until Section 4.3. However, the two products are equal when both factors are vectors, so I can continue to use the standard terminology of inner products as I please. The next axiom is an example.

Axiom 4 by itself is a little too general; for instance, it would allow the product of any two vectors to be zero. That seems pointless. To prevent that, I'll add another axiom.

Axiom 5. *The inner product is nondegenerate.*

This means that the only vector orthogonal to all vectors, including itself, is 0. This axiom is true in every application I can imagine, and I use it to prove some useful results in Section 5.5. However, it is possible to replace it with a weaker axiom that accomplishes most of the same things; I discuss that in Section 5.5 too. So if you ever find yourself reading other treatments of Clifford algebras, watch out to see whether they use this axiom or put something else in its place.

Now I'll name other elements of $\mathcal{G}$. Let $r > 1$; then an *r-blade* or *simple r-vector* is a product of r orthogonal (thus anticommuting) vectors. A finite sum of r-blades is called an *r-vector* or *homogeneous* multivector of *grade* r. (I'll bet you didn't see that coming.) 2-vectors are also called *bivectors*, 3-vectors *trivectors*. The set of r-vectors is called $\mathcal{G}_r$. Notice that this definition of simple r-vectors uses the geometric product of orthogonal vectors, not the outer product of arbitrary vectors as I did in Section 1.1. The definitions are equivalent, as I'll show later.

Products of vectors play an important role, so they get their own name. An *r-versor* is a product of r vectors. So far we've seen two types of versor: blades (where the vectors in the product are orthogonal) and rotors, introduced in Section 1.2. A rotor was defined to be a product of two invertible vectors, so a rotor is an invertible biversor. Later, a rotor will be any invertible even versor.

From these definitions and Axiom 3 it follows that that if $A \in \mathcal{G}_r$ and λ is a scalar,

$$\lambda A = A\lambda \in \mathcal{G}_r. \tag{29}$$

So multiplication by a scalar doesn't change the grade of an r-vector. This in turn implies that (a) each $\mathcal{G}_r$ is a vector space with $\mathcal{G}_0$ as scalars and (b) $0 \in \mathcal{G}_r$ for every r. So all the results I gave right after Axiom 3 generalize fully.

Now we know that $\mathcal{G}$ contains all the $\mathcal{G}_r$, and we know a few things about how different $\mathcal{G}_r$ are related. For example, suppose u and v are orthogonal and consider the 2-blade uv. It anticommutes with both u and v, which means that it can't have a scalar part, because that part would have commuted with all vectors. In fact, for this same reason no even blade can have a scalar part; and no odd blade can either, as long as there's another vector orthogonal to all the factors in the blade. You can continue on this line and deduce a few more results, but it's not clear to me that you can use only the axioms so far to show that all the $\mathcal{G}_r$ are completely independent of each other. So I add one final axiom for cleaning up.

Axiom 6. *If $\mathcal{G}_0 = \mathcal{G}_1$, then $\mathcal{G} = \mathcal{G}_0$. Otherwise, $\mathcal{G}$ is the direct sum of all the $\mathcal{G}_r$.*

The first part of the axiom covers a special case: a field by itself, without any vectors, can be a geometric algebra. When there are vectors around, the axiom says that every $A \in \mathcal{G}$ may be expressed one and only one way as $A = \sum_r A_r$ where $A_r \in \mathcal{G}_r$ and all but finitely many A_r vanish. Therefore, every $A \neq 0$ is either an r-vector for only one r or is of mixed grade.

For each r, let the grade operator $\langle\,\rangle_r : \mathcal{G} \to \mathcal{G}_r$ project each $A \in \mathcal{G}$ onto its unique grade-r component. Then

(a) A is an r-vector iff $A = \langle A \rangle_r$.

(b) $\langle A + B \rangle_r = \langle A \rangle_r + \langle B \rangle_r$.

(c) $\langle \lambda A \rangle_r = \langle A\lambda \rangle_r = \lambda \langle A \rangle_r$.

(d) $\langle \langle A \rangle_r \rangle_s = \langle A \rangle_r \, \delta_{rs}$. (Thus the $\langle\,\rangle_r$ are independent projection operators.)

(e) $\sum_r \langle A \rangle_r = A$ for any $A \in \mathcal{G}$. (Thus the $\langle\,\rangle_r$ are a complete set of projection operators.)

It will turn out to be convenient to define $\langle\,\rangle_r$ even when r is negative, so let me add one final property:

(f) $\langle A \rangle_r = 0$ if $r < 0$ for all $A \in \mathcal{G}$.

Because we take the scalar part of multivectors so often, I will let $\langle\,\rangle$ mean $\langle\,\rangle_0$.

The notation A_r will usually mean that A_r is an r-vector. The exception is vectors: a_1, a_2, etc., are all vectors in a single enumerated set (not objects of increasing grades). Sometimes A_r will represent the grade-r component of multivector A, which is more properly denoted $\langle A \rangle_r$, but that notation is cumbersome so sometimes I drop it. You can always tell from the context. A blade is indicated by boldface; for example,

$\boldsymbol{A_r}$ is an r-blade. The exceptions are scalars (0-blades) and vectors (1-blades). I want a special notation for blades because they have geometric meaning while general r-vectors don't, as I'll show in Section 3.

Axiom 6 tells us that the relation $\lambda A = A\lambda$, which I proved above for any homogeneous multivector A, is true for any $A \in \mathcal{G}$, homogeneous or not. Another consequence of Axiom 6 is that $\mathcal{G}$ is the direct sum of subspaces $\mathcal{G}_+$ and $\mathcal{G}_-$ consisting of the even-grade and odd-grade multivectors respectively. Since many identities will contain factors of $(-1)^r$, they will only depend on whether the multivectors are even or odd. Also, I'll show in Section 4 that the product of even multivectors is also even; this means that the even subspace is actually a subalgebra, which will turn out to be important. For these reasons it's good to extend some of my notation to cover even and odd cases; the notations A_+ and A_- will mean that these objects have only even-grade or odd-grade terms, respectively, and for any multivector A, $\langle A\rangle_+$ (resp. $\langle A\rangle_-$) is the even-grade (resp. odd-grade) part of A.

By the way, I haven't actually proved that anything satisfying these axioms exists. That's done in [8].

3. The contents of a geometric algebra

According to Axiom 6, a geometric algebra consists of r-blades and their sums. However, the axioms and my comments at the end of Section 1.1 give us two different pictures of what r-blades are. According to the axioms, an r-blade is a product of r orthogonal vectors; according to the end of Section 1.1, an r-blade is an outer product of r arbitrary vectors. I also said that blades represent subspaces, with weights and orientations when the scalars are real. I'll spend this section relating these two pictures; first I'll show the two definitions of r-blades are equivalent, and then I'll justify the geometric interpretation. Then we'll have a good intuitive feel for what a geometric algebra really is: sums of subspaces, with orientations and weights if the algebra is real. To do this, I'll be using some concepts that I haven't fully explained yet. Everything left hanging here will be fixed up in the next few sections.

First I want to show that outer products of vectors really are r-blades in the axiomatic sense of Section 2. To do this, I define the outer product of vectors $\{a_i\}_{i=1,\ldots,r}$ to be their fully antisymmetrized product, or

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r := \frac{1}{r!}\sum_\sigma (\operatorname{sgn}\sigma)\, a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(r)} \tag{30}$$

where σ is a permutation of 1 through r, sgn σ is the sign of the permutation (1 for even and -1 for odd), and the sum is over all $r!$ possible permutations. If $r = 2$ this reduces to the outer product of two vectors defined previously. Here's the result I need.

Theorem 1. *The outer product of r vectors is an r-blade, and every r-blade is the outer product of r vectors.*

A corollary is that the outer product of two vectors is a 2-blade, as I said in Section 1.1. This means that I could have used this as the definition of an r-blade in the axioms, but the definition I did use is more convenient in many situations. Now, however, I'll use either definition as I need to.

Proof. To begin, I'll show that if the a_i all anticommute then the outer product reduces to the geometric product, so the result is an r-blade. Let $\{e_i\}_{i=1,\ldots,r}$ anticommute with one another, and consider their outer product

$$e_1 \wedge e_2 \wedge \cdots \wedge e_r = \frac{1}{r!}\sum_\sigma (\operatorname{sgn}\sigma)\, e_{\sigma(1)}e_{\sigma(2)}\cdots e_{\sigma(r)}. \tag{31}$$

In each term, the e_i can be reordered so they're in ascending numerical order, and each interchange of two e_i introduces a minus sign. The end result is a factor of the form $\operatorname{sgn}\sigma$, which cancels the $\operatorname{sgn}\sigma$ that's already there. The result is

$$\begin{aligned} e_1 \wedge e_2 \wedge \cdots \wedge e_r &= \frac{1}{r!}\sum_\sigma e_1e_2\cdots e_r \\ &= e_1e_2\cdots e_r \end{aligned} \tag{32}$$

since there are $r!$ permutations to sum over and all $r!$ terms are the same. So when the vectors all anticommute, the wedges can be retained or dropped as desired. The $r = 2$ version of this result,

$$e_1 \wedge e_2 = e_1\, e_2, \tag{33}$$

was already obvious from Eq. (6) since $e_1 \cdot e_2 = 0$, or $e_1 \rfloor e_2 = 0$ as I would say it now.

Turning to the general case, I can show that this is an r-blade by examining the matrix M with entries $M_{ij} = a_i \rfloor a_j$. This is a real symmetric matrix, so it can be diagonalized by an orthogonal transformation, meaning that there exists an orthogonal matrix R and a set of vectors $\{e_i\}_{i=1,\ldots,r}$ such that

$$a_i = \sum_j R_{ij} e_j \quad \text{and} \quad e_i \rfloor e_j = e_i^2\, \delta_{ij}, \tag{34}$$

so the e_i anticommute with each other. In that case

$$\begin{aligned} a_1 \wedge a_2 \wedge \cdots \wedge a_r &= \sum_{i,j,\ldots,m} R_{1i} e_i \wedge R_{2j} e_j \wedge \cdots \wedge R_{rm} e_m \\ &= \det(R)\, e_1 \wedge e_2 \wedge \cdots \wedge e_r. \end{aligned} \tag{35}$$

Now $\det(R)$=± 1, and if it equals -1 I interchange e_1 and e_2 and relabel them, with the result

$$\begin{aligned} a_1 \wedge a_2 \wedge \cdots \wedge a_r &= e_1 \wedge e_2 \wedge \cdots \wedge e_r \\ &= e_1 e_2 \cdots e_r \end{aligned} \tag{36}$$

where the final line relies on the result from the previous paragraph. So the outer product of r vectors can be re-expressed as the product of r anticommuting vectors, making it an r-blade. Further, every r-blade is such an outer product (since for anticommuting vectors the wedges can be added or dropped at will), so an object is an r-blade iff it's the outer product of r vectors. □

Since every multivector is a unique sum of r-vectors by Axiom 6, and every r-vector is a sum of r-blades, I can now say that a multivector is a sum of a scalar, a vector, and a bunch of outer products. Now let's take the geometric point of view. I know what scalars and vectors are geometrically, but what are the outer products? To answer that, I need to look at when they vanish. For example, $a \wedge a = 0$ for any a by antisymmetry. The more general case is given by this theorem.

Theorem 2. *The simple r-vector $a_1 \wedge a_2 \wedge \cdots \wedge a_r = 0$ iff the vectors $\{a_i\}_{i=1,\ldots,r}$ are linearly dependent.*

Proof. The outer product is clearly antisymmetric under interchange of any pair of factors, so it vanishes if any factor repeats. It is also linear in each of its arguments, so if one factor is a linear combination of the others, the outer product vanishes. So if the vectors are dependent, their product vanishes. The other half of the proof, that the product of independent vectors doesn't vanish, is given in Theorem 25, which we don't have the tools to prove yet, so I'll defer it until later. □

So an r-blade $\boldsymbol{A_r}$ is nonzero exactly when its factors span an r-dimensional subspace. Thus I associate $\boldsymbol{A_r}$ with that subspace (attitude).

To solidify the connection between subspaces and r-blades, here's a really cool result. It uses $a \wedge \boldsymbol{A_r}$, which I haven't defined yet, but for now let's just say that it equals the outer product of a and the factors of $\boldsymbol{A_r}$.

Theorem 3. *If $\boldsymbol{A_r}$ is a nonzero r-blade with $r \geq 1$, then vector a lies in the span of the factors of $\boldsymbol{A_r}$ iff $a \wedge \boldsymbol{A_r} = 0$.*

Proof. $a \wedge \boldsymbol{A_r} = 0$ iff a and the factors of $\boldsymbol{A_r}$ are linearly dependent. Now the factors of $\boldsymbol{A_r}$ are themselves independent because $\boldsymbol{A_r}$ is nonzero, so $a \wedge \boldsymbol{A_r}$ vanishes iff a is a linear combination of the factors of $\boldsymbol{A_r}$. □

Therefore $\boldsymbol{A_r}$ does indeed define a subspace: the set of all vectors a such that $a \wedge \boldsymbol{A_r} = 0$.

The proof of this theorem actually shows a bit more. If $a \wedge \boldsymbol{A_r} \neq 0$, then it's an $r+1$-blade, and it represents the direct sum of $\boldsymbol{A_r}$ and the one-dimensional subspace defined by a. I'll use this fact later when I show how to interpret outer products geometrically.

Theorem 3 implies another useful fact.

Theorem 4. *Two nonzero r-blades $\boldsymbol{A_r}$ and $\boldsymbol{B_r}$ define the same subspace iff each is a nonzero multiple of the other.*

Proof. If $\boldsymbol{A_r} = \lambda \boldsymbol{B_r}$ for some nonzero λ, then clearly $a \wedge \boldsymbol{A_r} = 0$ iff $a \wedge \boldsymbol{B_r} = 0$, so they represent the same subspace. Conversely, suppose $\boldsymbol{A_r}$ and $\boldsymbol{B_r}$ represent the same subspace; then $\boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$ and $\boldsymbol{B_r} = b_1 \wedge b_2 \wedge \cdots \wedge b_r$ for some linearly independent sets of vectors $\{a_i\}_{i=1,\ldots,r}$ and $\{b_j\}_{j=1,\ldots,r}$, and each of the b_j is a linear combination of the a_i. Substituting those linear combinations into the expression for $\boldsymbol{B_r}$, removing the terms where any a_i appears twice, and reordering the factors in each term, I find that $\boldsymbol{B_r}$ equals $\boldsymbol{A_r}$ multiplied by some scalar. This scalar can't be zero because $\boldsymbol{B_r}$ is nonzero, so that completes the proof. □

Frankly, something would be wrong if this weren't true. In turn, Theorem 4 gives me another result I'll use a lot.

Theorem 5. *If $\boldsymbol{A_r}$ represents a proper subspace of $\boldsymbol{A_s}$, then $\boldsymbol{A_r}$ can be factored out of $\boldsymbol{A_s}$; that is, there exists a blade $\boldsymbol{A_{s-r}}$ such that $\boldsymbol{A_s} = \boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$.*

Proof. Let $\boldsymbol{A_r} = a_1 \wedge \cdots \wedge a_r$; then $\{a_j\}_{j=1,\ldots,r}$ is a linearly independent set lying in $\boldsymbol{A_s}$, so it can be extended to a basis $\{a_j\}_{j=1,\ldots,s}$ of $\boldsymbol{A_s}$. That means $a_1 \wedge \cdots \wedge a_s$ defines the same subspace as $\boldsymbol{A_s}$, so it differs from $\boldsymbol{A_s}$ by a scalar factor; absorb that factor in the newly-added vectors and we have $\boldsymbol{A_s} = \boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$, where $\boldsymbol{A_{s-r}}$ is the outer product of the newly-added vectors. □

I'll show later that $A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r$ for any r- and s-vectors, so $\boldsymbol{A_r}$ can be factored out of $\boldsymbol{A_s}$ from either side and the other blade doesn't have to change by more than a sign.

While I'm here, let me also give a necessary and sufficient condition for a vector to be orthogonal to a subspace. This theorem uses $a \rfloor \boldsymbol{A_r}$, the left inner product of a vector and an r-blade, which once again I haven't defined yet. For now, think of it as taking the inner product of a with each factor of $\boldsymbol{A_r}$ separately, as in Eq. (37) below. (Now that I'm taking the inner product of objects of different grades, it matters which of the two products I use; notice that the "floor" of the product points toward the vector.)

Theorem 6. *If $\boldsymbol{A_r}$ is a nonzero r-blade with $r \geq 1$, a is orthogonal to the subspace $\boldsymbol{A_r}$ iff $a \rfloor \boldsymbol{A_r} = 0$.*

Proof. To show this, let $\boldsymbol{A_r} = a_1 \wedge a_2 \wedge \cdots \wedge a_r$ and let's look at

$$\begin{aligned} a \rfloor \boldsymbol{A_r} &= a \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_r) \\ &= \sum_{j=1}^{r} (-1)^{j-1} (a \rfloor a_j)\, a_1 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_r, \end{aligned} \tag{37}$$

where I used Eq. (77) in the second line. (I'll derive it later. The check over a_j means it's not included in the outer product.) If a is orthogonal to $\boldsymbol{A_r}$ then it's orthogonal to all the a_j and $a \rfloor \boldsymbol{A_r} = 0$. If instead a is orthogonal to all the a_j but one, then $a \rfloor \boldsymbol{A_r}$ is the product of a nonzero scalar and the outer product of the remaining a_j, which is nonzero because they're linearly independent. So $a \rfloor \boldsymbol{A_r} \neq 0$. The remaining case is a nonorthogonal to multiple a_j, in which case let $\{a_j\}_{j=1,\ldots,s}$ where $1 < s \leq r$ be the vectors for which $a \rfloor a_j \neq 0$, and for $j = 2, \ldots, s$ let b_j be defined by

$$b_j = a_j - \left(\frac{a \rfloor a_j}{a \rfloor a_1}\right) a_1. \tag{38}$$

None of the b_j equal 0 because the a_j are linearly independent, and $a \rfloor b_j = 0$; better yet, because each of the b_j is just a_j with a multiple of a_1 added to it, the outer product is unchanged by replacing the a_j with the b_j:

$$\begin{aligned}\boldsymbol{A_r} &= a_1 \wedge a_2 \wedge \cdots \wedge a_r \\ &= a_1 \wedge b_2 \wedge \cdots \wedge b_s \wedge a_{s+1} \wedge \cdots \wedge a_r.\end{aligned} \tag{39}$$

Now I'm back to the previous case where only one vector in $\boldsymbol{A_r}$ is nonorthogonal to a, so I get the same result as before. Therefore if a is not orthogonal to $\boldsymbol{A_r}$ then $a \rfloor \boldsymbol{A_r} \neq 0$. □

The *orthogonal complement* of a subspace is the set of all vectors orthogonal to every vector in the subspace. Theorem 6 says that the orthogonal complement of $\boldsymbol{A_r}$ is the set of all vectors a satisfying $a \rfloor \boldsymbol{A_r} = 0$.

Just as with Theorem 3, the proof of Theorem 6 actually shows a bit more. If $a \rfloor \boldsymbol{A_r} \neq 0$, then it's an $r-1$-blade, and it represents the subspace of $\boldsymbol{A_r}$ that is orthogonal to a. I'll use this fact later when I interpret inner products geometrically.

So not only do we know that r-blade $\boldsymbol{A_r}$ represents an r-dimensional subspace, we have an easy way to tell whether vector a is in that subspace ($a \wedge \boldsymbol{A_r} = 0$) or orthogonal to it ($a \rfloor \boldsymbol{A_r} = 0$). Theorems 3 and 6 are also our first examples of a general fact: *algebraic relations between multivectors reflect geometric relations between subspaces.* We'll see more advanced examples later.

Now let's suppose the scalars are real, in which case blades are also supposed to have orientation and weight. To give $\boldsymbol{A_r}$ an orientation, I note that it's the product of r vectors in a given order. That order defines an orientation: follow the vectors in their given order, and then follow their negatives in the same order until you're back where you started. For example, the orientation of the 2-blade $a \wedge b$ is found by moving along a, then b, then $-a$, then $-b$ back to the beginning. A little experimentation shows that interchanging any two vectors reverses the orientation, and it also changes the sign of the blade. Therefore there are two orientations, and changing orientations is associated with changing the sign of the blade or equivalently interchanging vectors.

Now this definition had nothing to do with what the scalars are; the problem with non-real algebras arises when you try to decide what scalar multiplication does to the orientation. In real algebras it's easy: every nonzero scalar is positive or negative, which either leaves the orientation alone or reverses it. Other sets of scalars are not well-ordered like this, so we can't say unambiguously what they do to the orientation of blades; this is why I define orientation only for real algebras.

All that remains now is to define a blade's weight, which should include the notions of length, area, and volume, but generalize them to arbitrary dimensions. That's most easily done in Section 5.4 on scalar products, so I will defer it until then. (I'll also explain why I define it only in the real case.) Taking that for granted at the moment, I can now conclude that a general multivector is a sum of terms that represent different subspaces of all dimensions from 0 on up, with weights and orientations if the scalars are real. This of course is what I was after in the very beginning.

4. The inner, outer, and geometric products

Before we go on, I'll repeat what we know about the product of two vectors using all my new notation from Section 2.

$$uv = u \rfloor v + u \wedge v \tag{40}$$

where the two terms are the symmetric and antisymmetric parts of the product. We also know that the first term is a scalar and the second is a bivector. That means the inner and outer products can also be written

$$\begin{aligned}u \rfloor v &= \langle uv \rangle \\ u \wedge v &= \langle uv \rangle_2 .\end{aligned} \tag{41}$$

My next job is to extend this knowledge to the inner, outer, and geometric product of any two multivectors at all. Once that's done, I will also geometrically interpret the inner and outer products. Along the way I'll build up a set of tools and identities I'll use later to do calculations.

4.1. The inner, outer, and geometric products of a vector with anything

As a steppingstone, I first define the inner and outer products of a vector with any multivector. Sometimes I will want to explicitly indicate that a particular vector is absent from a product; I do this by including the vector anyway with a check ˇ over it. For example, if $\{a_i\}_{i=1,\dots,r}$ is a collection of vectors, then $a_1 a_2 \cdots \check{a}_j \cdots a_r$ is the product of all the vectors except for a_j.

I define the *left inner product* of vector a and r-vector A_r (also called the inner product of a *into* A_r) to be

$$a \rfloor A_r := \frac{1}{2}\big[aA_r - (-1)^r A_r a\big], \tag{42}$$

and I also define the *right inner product* of A_r and a (or the inner product of A_r *by* a) to be

$$\begin{aligned} A_r \lfloor a &:= \frac{1}{2}\big[A_r a - (-1)^r a A_r\big] \\ &= (-1)^{r-1} a \rfloor A_r. \end{aligned} \tag{43}$$

(Just as in Section 3, the "floor" of the inner product always points toward the vector. Later I'll show how to calculate it the other way.) When $r = 1$, I recover the inner product of vectors defined earlier. An equivalent way to write the relation between these products under interchange is

$$a \rfloor A_+ = -A_+ \lfloor a \quad \text{while} \quad a \rfloor A_- = A_- \lfloor a. \tag{44}$$

Here's why I define inner products this way.

Theorem 7. *$a \rfloor A_r$ and $A_r \lfloor a$ are both $r-1$-vectors, so the left or right inner product with a vector is a grade lowering operation.*

Proof. To show this, I start by proving this relation: if a, a_1, a_2, …, a_r are vectors, then

$$\frac{1}{2}\big[aa_1a_2\cdots a_r - (-1)^r a_1a_2\cdots a_r a\big] = \sum_{j=1}^{r}(-1)^{j-1}(a \rfloor a_j)a_1a_2\cdots\check{a}_j\cdots a_r. \tag{45}$$

I proceed by induction. If $r = 1$ the result is true because it reduces to the definition of the inner product. Suppose the result holds for $r-1$, so

$$\frac{1}{2}(aa_1a_2\cdots a_{r-1}) = \frac{1}{2}(-1)^{r-1}(a_1a_2\cdots a_{r-1}a) + \sum_{j=1}^{r-1}(-1)^{j-1}(a \rfloor a_j)a_1a_2\cdots\check{a}_j\cdots a_{r-1}. \tag{46}$$

Then since

$$\begin{aligned} \frac{1}{2}(aa_1a_2\cdots a_r) &= \frac{1}{2}(aa_1a_2\cdots a_{r-1})a_r \\ &= \frac{1}{2}(-1)^{r-1}(a_1a_2\cdots a_{r-1}a)a_r + \sum_{j=1}^{r-1}(-1)^{j-1}(a \rfloor a_j)a_1a_2\cdots\check{a}_j\cdots a_{r-1}a_r, \end{aligned} \tag{47}$$

we find

$$\begin{aligned} &\frac{1}{2}\big[aa_1a_2\cdots a_r - (-1)^r a_1a_2\cdots a_r a\big] \\ &\quad = \frac{1}{2}(-1)^{r-1}(a_1a_2\cdots a_{r-1}aa_r + a_1a_2\cdots a_{r-1}a_r a) + \sum_{j=1}^{r-1}(-1)^{j-1}(a \rfloor a_j)a_1a_2\cdots\check{a}_j\cdots a_{r-1}a_r \\ &\quad = (-1)^{r-1}(a \rfloor a_r)a_1a_2\cdots a_{r-1} + \sum_{j=1}^{r-1}(-1)^{j-1}(a \rfloor a_j)a_1a_2\cdots\check{a}_j\cdots a_{r-1}a_r \end{aligned}$$

$$= \sum_{j=1}^{r} (-1)^{j-1} (a \rfloor a_j) a_1 a_2 \cdots \check{a}_j \cdots a_r, \tag{48}$$

which is the desired result.

Now let's look at the special case that the numbered vectors are an anticommuting set:

$$\frac{1}{2}\big[a e_1 e_2 \cdots e_r - (-1)^r e_1 e_2 \cdots e_r a\big] = \sum_{j=1}^{r} (-1)^{j-1} (a \rfloor e_j) e_1 e_2 \cdots \check{e}_j \cdots e_r. \tag{49}$$

The right hand side is a sum of $r-1$-blades, making it an $r-1$-vector. Now a generic r-vector is a sum of r-blades, and any r-blade can be written $e_1 e_2 \cdots e_r$, so it follows that for a vector a and r-vector A_r the quantity $\frac{1}{2}\big[a A_r - (-1)^r A_r a\big]$, which is the left inner product, is an $r-1$-vector. Since the right inner product differs only by a sign, it's an $r-1$-vector too. □

This begins to show why we have two inner products: the vector operates on the r-vector to lower its grade, not the other way around. Two products allow the vector to do this from either side. Notice that when $r = 0$ (so $A_r = \lambda$) both inner products reduce to

$$a \rfloor \lambda = \lambda \lfloor a = 0. \tag{50}$$

In retrospect this makes sense: these products lower grade, and the scalars have the lowest grade in the algebra, so there's nothing other than zero for them to be.

At this point it can't be much of a surprise that the *outer product* of vector a and r-vector A_r is defined to be

$$a \wedge A_r := \frac{1}{2}\big[a A_r + (-1)^r A_r a\big], \tag{51}$$

and the outer product with the order reversed is given by

$$\begin{aligned} A_r \wedge a &:= \frac{1}{2}\big[A_r a + (-1)^r a A_r\big] \\ &= (-1)^r a \wedge A_r. \end{aligned} \tag{52}$$

When $r = 1$, of course I recover the outer product of vectors. The behavior of the outer product under interchange is the opposite of the inner product, and it can also be written

$$a \wedge A_+ = A_+ \wedge a \quad \text{while} \quad a \wedge A_- = -A_- \wedge a. \tag{53}$$

This theorem is probably no surprise either.

Theorem 8. *$a \wedge A_r$ is an $r+1$-vector, so the outer product with a vector is a grade raising operation.*

Proof. To show this, I again need to prove a preliminary result:

$$\frac{1}{2}\big[a(a_1 \wedge a_2 \wedge \cdots \wedge a_r) + (-1)^r (a_1 \wedge a_2 \wedge \cdots \wedge a_r) a\big] = a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r. \tag{54}$$

Again I use induction. If $r = 1$ the expression reduces to the definition of the outer product of two vectors, so suppose it's true for $r-1$. Let $a_1 \wedge a_2 \wedge \cdots \wedge a_r = e_1 e_2 \cdots e_r$ where any two e_i anticommute; then

$$a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r = a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r \tag{55}$$

because the substitution of the e_i for the a_i yields a factor of $\det(R)=\pm 1$ which can be eliminated as before, so the preliminary result becomes

$$\frac{1}{2}\big[a(e_1 e_2 \cdots e_r) + (-1)^r (e_1 e_2 \cdots e_r) a\big] = a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r. \tag{56}$$

To prove it, let's begin by looking at the term on the right hand side, which we know is the sum of $(r+1)!$ permutations. I want to regroup it into $r+1$ terms, each of which consists of all permutations that put a particular one of the $r+1$ vectors in the first position. Let the permutations that put a first be called π, and the ones that put e_i first be called π_i; then

$$a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r = \frac{1}{r+1}\left[\frac{1}{r!}\sum_{\pi}(\operatorname{sgn}\pi) a e_{\pi(1)} e_{\pi(2)} \cdots e_{\pi(r)} + \right.$$
$$\left. \sum_{j=1}^{r} \frac{1}{r!}\sum_{\pi_j}(\operatorname{sgn}\pi_j) e_j a_{\pi_j(1)} a_{\pi_j(2)} \cdots a_{\pi_j(r)}\right] \tag{57}$$

where one of the $a_{\pi_j(i)}$ is a and the others are the e_i other than e_j. Now the $e_{\pi(i)}$ in the first term on the right hand side can be rearranged, canceling the sgn π factors just as before, so

$$\frac{1}{r+1}\left[\frac{1}{r!}\sum_{\pi}(\operatorname{sgn}\pi) a e_{\pi(1)} e_{\pi(2)} \cdots e_{\pi(r)}\right] = \frac{1}{r+1} a e_1 e_2 \cdots e_r. \tag{58}$$

As for the other terms, π_j as a permutation of a and all the e_i that puts e_j in the first spot has the same sign as the corresponding permutation of just a and the other e_i times a factor $(-1)^j$, because this is the factor gained by moving e_j from its original $j+1$ position to the front. Therefore with that factor added each π_j may be thought of as a permutation of only the remaining r vectors, or

$$\begin{aligned}
&\frac{1}{r!}\sum_{\pi_j}(\operatorname{sgn}\pi_j) e_j a_{\pi_j(1)} a_{\pi_j(2)} \cdots a_{\pi_j(r)} \\
&\qquad = (-1)^j e_j (a \wedge e_1 \wedge e_2 \wedge \cdots \wedge \check{e}_j \wedge \cdots \wedge e_r) \\
&\qquad = \frac{1}{2}(-1)^j e_j [a e_1 e_2 \cdots \check{e}_j \cdots e_r + (-1)^{r-1} e_1 e_2 \cdots \check{e}_j \cdots e_r a] \\
&\qquad = \frac{1}{2}\big[(-1)^j e_j a e_1 e_2 \cdots \check{e}_j \cdots e_r + (-1)^r e_1 e_2 \cdots e_r a\big].
\end{aligned} \tag{59}$$

In the second line I used the fact that the relation is assumed true for $r-1$, and in the second term on the third line I moved e_j past the first $j-1$ e_i. Now since

$$e_j a = 2a \rfloor e_j - a e_j, \tag{60}$$

the relation above becomes

$$\frac{1}{r!}\sum_{\pi_j}(\operatorname{sgn}\pi_j) e_j a_{\pi_j(1)} a_{\pi_j(2)} \cdots a_{\pi_j(r)} \tag{61}$$
$$\begin{aligned}
&= \frac{1}{2}(-1)^{j-1} a e_j e_1 e_2 \cdots \check{e}_j \cdots e_r + (-1)^j (a \rfloor e_j) e_1 e_2 \cdots \check{e}_j \cdots e_r + \frac{1}{2}(-1)^r e_1 e_2 \cdots e_r a \\
&= \frac{1}{2} a e_1 e_2 \cdots e_r + \frac{1}{2}(-1)^r e_1 e_2 \cdots e_r a + (-1)^j (a \rfloor e_j) e_1 e_2 \cdots \check{e}_j \cdots e_r.
\end{aligned} \tag{62}$$

Putting all this back together, I get

$$\begin{aligned}
&a \wedge e_1 \wedge e_2 \wedge \cdots \wedge e_r \\
&\quad = \frac{1}{r+1}\left[a e_1 e_2 \cdots e_r + \sum_{j=1}^{r}\left(\frac{1}{2} a e_1 e_2 \cdots e_r + \frac{1}{2}(-1)^r e_1 e_2 \cdots e_r a + \right.\right. \\
&\qquad\qquad \left.\left. (-1)^j (a \rfloor e_j) e_1 e_2 \cdots \check{e}_j \cdots e_r \right)\right] \\
&\quad = \left(\frac{1}{r+1} + \frac{r}{2r+2}\right) a e_1 e_2 \cdots e_r + \frac{r}{2r+2}(-1)^r e_1 e_2 \cdots e_r a -
\end{aligned}$$

$$\frac{1}{2r+2}\big[ae_1e_2\cdots e_r-(-1)^r e_1e_2\cdots e_r a\big]$$
$$=\frac{1}{2}\big[ae_1e_2\cdots e_r+(-1)^r e_1e_2\cdots e_r a\big], \tag{63}$$

which proves the preliminary result.

The right hand side of Eq. (54) is an $r+1$-blade. Now a generic r-vector is a sum of r-blades, and any r-blade is of the form $a_1\wedge a_2\wedge\cdots\wedge a_r$, so it follows that for a vector a and r-vector A_r the quantity $\frac{1}{2}\big[aA_r+(-1)^r A_r a\big]$, which is just the outer product, is an $r+1$-vector. And that's that. □

You may wonder why I didn't define two outer products. In this case, it makes equal sense to think of the vector raising the r-vector's grade by 1 or the r-vector raising the vector's grade by r.

So now we know that

$$aA_r = a\rfloor A_r + a\wedge A_r, \tag{64}$$

and further

$$\begin{aligned} a\rfloor A_r &= \langle aA_r\rangle_{r-1} \\ a\wedge A_r &= \langle aA_r\rangle_{r+1}\,, \end{aligned} \tag{65}$$

and similar results hold for $A_r a$:

$$A_r a = A_r\lfloor a + A_r\wedge a, \tag{66}$$

where

$$\begin{aligned} A_r\lfloor a &= \langle A_r a\rangle_{r-1} \\ A_r\wedge a &= \langle A_r a\rangle_{r+1}\,. \end{aligned} \tag{67}$$

These expressions generalize Eqs. (40) and (41) and reduce to them when $r=1$. In fact, summing over grades r in Eqs. (64) and (66) shows that they're true for any multivector A. So I've achieved my goal from the beginning of this section, at least for the special case of multiplying by a vector. The expressions for $\rfloor$ and $\lfloor$ in Eqs. (65) and (67) work even when $r=0$, because back in Section 2 I made a point of defining all negative-grade components of a multivector to vanish. This is why.

4.2. The general inner product, outer product, and geometric product

So far I have shown that the product of a vector and a multivector is the sum of two terms; if the multivector is grade r, the two terms have grades $r-1$ and $r+1$. I've also shown how to calculate each term separately. Now I'll use this information to characterize the product of any two multivectors, and after that I'll introduce the most general forms for the inner and outer products.

Let $A=\sum_r A_r$ and $B=\sum_s B_s$; then $AB=\sum_{r,s}A_rB_s$, so I'll consider each term separately.

Theorem 9. *A_rB_s consists of* $\min\{r,s\}+1$ *terms of grades* $|r-s|$, $|r-s|+2$, $|r-s|+4$, *…*, $r+s$, *or*

$$A_rB_s=\sum_{j=0}^{\min\{r,s\}}\langle A_rB_s\rangle_{|r-s|+2j}\,. \tag{68}$$

Proof. If $r=0$ or $s=0$ this expression is obviously true, so next I'll consider the case $0<r\le s$, assume A_r is an r-blade $\boldsymbol{A_r}$ (if it's true for a blade it's true for sums of blades), and proceed by induction on r. If $r=1$ the expression becomes Eq. (64), which I've proved already; so assume it's true for $r-1$. $\boldsymbol{A_r}$ can be written $a\boldsymbol{A_{r-1}}$ where a is a vector and $\boldsymbol{A_{r-1}}$ is an $r-1$-blade, so

$$\boldsymbol{A_r}B_s = a\boldsymbol{A_{r-1}}B_s$$

$$= \sum_{j=0}^{\min\{r-1,s\}} a \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{|r-1-s|+2j}$$

$$= \sum_{j=0}^{r-1} a \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j+1} \tag{69}$$

where the second line uses the fact that the relation is assumed true for $r-1$ and the last line follows from the inequality $r \leq s$. Now applying Eq. (64),

$$\begin{aligned} \boldsymbol{A_r} B_s &= \sum_{j=0}^{r-1} \Big[a \rfloor \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j+1} + a \wedge \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j+1} \Big] \\ &= a \rfloor \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+1} + \\ &\quad \sum_{j=1}^{r-1} \Big[a \rfloor \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j+1} + a \wedge \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j-1} \Big] + \\ &\quad a \wedge \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s+r-1}. \end{aligned} \tag{70}$$

Noting the grades of the various terms in the sum, I identify

$$\begin{aligned} \left\langle \boldsymbol{A_r} B_s \right\rangle_{s-r} &= a \rfloor \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+1} \\ \left\langle \boldsymbol{A_r} B_s \right\rangle_{s-r+2j} &= a \rfloor \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j+1} + a \wedge \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s-r+2j-1} \\ \left\langle \boldsymbol{A_r} B_s \right\rangle_{r+s} &= a \wedge \left\langle \boldsymbol{A_{r-1}} B_s \right\rangle_{s+r-1}. \end{aligned} \tag{71}$$

Since $s - r = |r - s|$, $\boldsymbol{A_r} B_s$ is now expressed as a sum of terms of grade $|r - s|$, $|r - s| + 2$, $|r - s| + 4$, $\ldots$, $r + s$, which proves the result for r. The remaining case is $0 < s \leq r$, which is proved by induction on s. □

This proof actually gives somewhat explicit formulas for the terms in the product. To illustrate this I'll consider Eqs. (71) for the special case $r = 2$, so $\boldsymbol{A_2} = e_1 e_2$:

$$\begin{aligned} \boldsymbol{A_2} B_s &= e_1 \rfloor \left\langle e_2 B_s \right\rangle_{s-1} + e_1 \rfloor \left\langle e_2 B_s \right\rangle_{s+1} + e_1 \wedge \left\langle e_2 B_s \right\rangle_{s-1} + e_1 \wedge \left\langle e_2 B_s \right\rangle_{s+1} \\ &= e_1 \rfloor (e_2 \rfloor B_s) + e_1 \rfloor (e_2 \wedge B_s) + e_1 \wedge (e_2 \rfloor B_s) + e_1 \wedge (e_2 \wedge B_s). \end{aligned} \tag{72}$$

I could have arrived at the same result by writing $\boldsymbol{A_2} B_s = e_1 e_2 B_s$, using Eq. (64) to expand $e_2 B_s$, and using Eq. (64) again to expand the product of e_1 with each term. The first term is of grade $s - 2$, the middle two terms are of grade s (one grade lowering and one grade raising operation applied to B_s), and the final term is of grade $s + 2$.

Theorem 9 tells us something important: while $A_r B_s$ is not an $r+s$-vector, every term has grade $r+s-2j$ for some j, so $A_r B_s$ is even if $r + s$ is even and odd if $r + s$ is odd. That means that the product of two even grade elements is itself an even grade element, so the even grade subspace of any geometric algebra (defined at the end of Section 2) is not just a subspace but a subalgebra. (This is not true of the odd subspace, because the product of two odd elements is also even.) Also, since $(-1)^{r+s-2j} = (-1)^{r+s}$, it follows that

$$\begin{aligned} a \rfloor (A_r B_s) &= \frac{1}{2}(a A_r B_s - (-1)^{r+s} A_r B_s a) \\ a \wedge (A_r B_s) &= \frac{1}{2}(a A_r B_s + (-1)^{r+s} A_r B_s a) \end{aligned} \tag{73}$$

for any vector a, which is kinda nice. Four identities follow from this.

Theorem 10.

$$\begin{aligned} a \rfloor (A_r B_s) &= (a \rfloor A_r) B_s + (-1)^r A_r (a \rfloor B_s) \\ &= (a \wedge A_r) B_s - (-1)^r A_r (a \wedge B_s) \\ a \wedge (A_r B_s) &= (a \wedge A_r) B_s - (-1)^r A_r (a \rfloor B_s) \\ &= (a \rfloor A_r) B_s + (-1)^r A_r (a \wedge B_s). \end{aligned} \tag{74}$$

Proof. These are all proved the same way, so I'll show only the first one. Starting with the first of Eqs. (73) and then adding and subtracting $\frac{1}{2}(-1)^r A_r a B_s$,

$$\begin{aligned} a \rfloor (A_r B_s) &= \frac{1}{2}(a A_r B_s - (-1)^{r+s} A_r B_s a) \\ &= \frac{1}{2}(a A_r B_s - (-1)^r A_r a B_s) + \frac{1}{2}(-1)^r (A_r a B_s - (-1)^s A_r B_s a) \\ &= (a \rfloor A_r) B_s + (-1)^r A_r (a \rfloor B_s) \end{aligned} \tag{75}$$

where I reassembled the terms into inner products using the first of Eqs. (73) again. □

By summing over grades s, you can see that these identities are valid even if B is a general multivector. (They're also valid for general A with a little tweaking; see Section 5.1.)

An obvious generalization of Eqs. (73) is

$$\begin{aligned} a \rfloor (a_1 a_2 \cdots a_r) &= \frac{1}{2}(a a_1 a_2 \cdots a_r - (-1)^r a_1 a_2 \cdots a_r a) \\ a \wedge (a_1 a_2 \cdots a_r) &= \frac{1}{2}(a a_1 a_2 \cdots a_r + (-1)^r a_1 a_2 \cdots a_r a) \end{aligned} \tag{76}$$

and the first of these equations can be used to prove another handy result. (I used this result to prove Theorem 6, you may recall.)

Theorem 11.

$$a \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = \sum_{j=1}^{r} (-1)^{j-1} (a \rfloor a_j)\, a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_r. \tag{77}$$

Proof. Using the first of Eqs. (76) and Eq. (45), I can write

$$\begin{aligned} a \rfloor (a_1 a_2 \cdots a_r) &= \frac{1}{2}(a a_1 a_2 \cdots a_r - (-1)^r a_1 a_2 \cdots a_r a) \\ &= \sum_{j=1}^{r} (-1)^{j-1} (a \rfloor a_j)\, a_1 a_2 \cdots \check{a}_j \cdots a_r. \end{aligned} \tag{78}$$

I'll prove just below that the grade-s term in the product of s vectors is their outer product (see Eq. (85)), so by taking the $r-1$-grade term of both sides and using that result, the identity follows. □

Here's a nice mnemonic for remembering the coefficients in this sum. In the jth term, a acts on a_j, so imagine that a_j first has to be moved to the far left side of the outer product, which requires $j-1$ interchanges of adjacent vectors and introduces a factor of $(-1)^{j-1}$.

I can use the method of proof of this theorem to prove a fact about versors.

Theorem 12. *An r-versor $a_1 a_2 \cdots a_r$ is a linear combination of terms, each of which is an outer product of some subset of $\{a_j\}_{j=1,\ldots,r}$. The number of factors in each term is even or odd as r is even or odd.*

Proof. As usual, the proof is by induction. The result is true if $r = 0$, 1, or 2, so assume it's true for $r-1$; then

$$\begin{aligned} a_1 a_2 \cdots a_r &= a_1 \rfloor (a_2 \cdots a_r) + a_1 \wedge (a_2 \cdots a_r) \\ &= \sum_{j=2}^{r} (-1)^{j-2} (a_1 \rfloor a_j) a_2 \cdots \check{a}_j \cdots a_r + a_1 \wedge (a_2 \cdots a_r), \end{aligned} \tag{79}$$

where I used Eq. (78) to go from the first to the second line. The first term is a linear combination of products of $r-2$ of the a_i, so by the $r-2$ result the first term is a linear combination of outer products. The number of factors in each term is even or odd as $r-2$ is even or odd, or as r is even or odd. The second term is the outer product of a_1 with the product of the remaining $r-1$ vectors. By the $r-1$ result, that product is a linear combination of outer products, and each term is even or odd as $r-1$ is even or odd. When you take its outer product with a_1, it's still a linear combination of outer products, and each term is even or odd as r is even or odd. □

Now I define for any $A = \sum_r A_r$ and $B = \sum_s B_s$

$$\begin{aligned} A \rfloor B &:= \sum_{r,s} \langle A_r B_s \rangle_{s-r} \\ A \lfloor B &:= \sum_{r,s} \langle A_r B_s \rangle_{r-s} \\ A \wedge B &:= \sum_{r,s} \langle A_r B_s \rangle_{r+s} . \end{aligned} \tag{80}$$

All previous expressions for the inner and outer products of two objects are special cases of these definitions. Further, this defintion for the outer product of two objects and the definition for the outer product of arbitrarily many vectors in Eq. (30) are consistent with each other thanks to Eq. (54), which shows that

$$a \wedge (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = a \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_r. \tag{81}$$

Some other facts are worth mentioning.

1. $A_r \rfloor B_r = A_r \lfloor B_r = \langle A_r B_r \rangle$.
2. If $r > s$ then $A_r \rfloor B_s = B_s \lfloor A_r = 0$ because all negative-grade multivectors vanish.
3. The lowest grade term in $A_r B_s$ is $A_r \rfloor B_s$ if $r \le s$ and $A_r \lfloor B_s$ if $r \ge s$.
4. The highest grade term in $A_r B_s$ is $A_r \wedge B_s$.
5. For any λ, $\lambda A = \lambda \rfloor A = \lambda \wedge A$ and $A\lambda = A \lfloor \lambda = A \wedge \lambda$. The product of a vector and any multivector is a sum of inner and outer products, as shown by Eqs. (64) and (66). In all other cases there are additional terms of intermediate grades.

These definitions make the inner and outer products much easier to work with, because in general the geometric product has nicer algebraic properties. The main advantage of the inner and outer products over the full product is nice behavior under interchange of the factors; as you'll see in Section 5.2, $A_r B_s$ and $B_s A_r$ are related, but not in a way that's easy to use, while going from $A_r \rfloor B_s$ to $B_s \lfloor A_r$ or from $A_r \wedge B_s$ to $B_s \wedge A_r$ is just a matter of a sign change (Eqs. (141) and (143)).

The definitions also allow me to deduce three more identities from Eqs. (74); I take the $r+s-1$-grade term of the first of Eqs. (74), the $r-s+1$-grade term of the third, and the $s-r+1$-grade term of the last, with the results

$$\begin{aligned} a \rfloor (A_r \wedge B_s) &= (a \rfloor A_r) \wedge B_s + (-1)^r A_r \wedge (a \rfloor B_s) \\ a \wedge (A_r \lfloor B_s) &= (a \wedge A_r) \lfloor B_s - (-1)^r A_r \lfloor (a \rfloor B_s) \\ a \wedge (A_r \rfloor B_s) &= (a \rfloor A_r) \rfloor B_s + (-1)^r A_r \rfloor (a \wedge B_s). \end{aligned} \tag{82}$$

(Taking an appropriate-grade term of the second identity only yields a special case of the third of Eqs. (83) below.) Again, these expressions are actually valid for general B, and also for general A when I use some results from Section 5.1.

Like the geometric product, the inner and outer products are distributive, and they obey these identities.

Theorem 13.

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C$$

$$
\begin{aligned}
A \rfloor (B \lfloor C) &= (A \rfloor B) \lfloor C \\
A \rfloor (B \rfloor C) &= (A \wedge B) \rfloor C \\
A \lfloor (B \wedge C) &= (A \lfloor B) \lfloor C
\end{aligned}
\tag{83}
$$

So the outer product and certain combinations of left and right inner products are associative, but neither left nor right inner products are associative by themselves. In the homogeneous case, the first relation above becomes

$$
A_r \wedge (B_s \wedge C_t) = \langle A_r B_s C_t \rangle_{r+s+t} \,. \tag{84}
$$

An important special case of this is

$$
a_1 \wedge a_2 \wedge \cdots \wedge a_r = \langle a_1 a_2 \cdots a_r \rangle_r \,. \tag{85}
$$

Associativity of the outer product plus its properties under interchange of factors leads to the result

$$
a \wedge A \wedge b = -b \wedge A \wedge a \tag{86}
$$

where a and b are any vectors and A is any multivector.

Proof. For the first relation, I note that

$$
\begin{aligned}
A_r \wedge (B_s \wedge C_t) &= A_r \wedge \langle B_s C_t \rangle_{s+t} \\
&= \left\langle A_r \langle B_s C_t \rangle_{s+t} \right\rangle_{r+s+t} \\
&= \langle A_r B_s C_t \rangle_{r+s+t} \\
&= \left\langle \langle A_r B_s \rangle_{r+s} C_t \right\rangle_{r+s+t} \\
&= (A_r \wedge B_s) \wedge C_t.
\end{aligned}
\tag{87}
$$

The crucial step is taken on the third line, where the $\langle\,\rangle_{s+t}$ is dropped. This can be done because the only term in $A_r B_s C_t$ that has grade $r+s+t$ is the term that comes from multiplying A_r by the highest grade term in $B_s C_t$. From this it follows that $A \wedge (B \wedge C) = (A \wedge B) \wedge C$ for any A, B, and C. The third line also gives me Eq. (84). For the second relation, consider

$$
\begin{aligned}
A_r \rfloor (B_s \lfloor C_t) &= A_r \rfloor \langle B_s C_t \rangle_{s-t} \\
&= \left\langle A_r \langle B_s C_t \rangle_{s-t} \right\rangle_{s-(r+t)} \,.
\end{aligned}
\tag{88}
$$

Now this vanishes automatically unless $r \leq s - t$, in which case

$$
\left\langle A_r \langle B_s C_t \rangle_{s-t} \right\rangle_{s-(r+t)} = \langle A_r B_s C_t \rangle_{s-(r+t)} \tag{89}
$$

because when the inequality is satisfied, the only term in $A_r B_s C_t$ that has grade $s-(r+t)$ is the term that comes from multiplying A_r by the lowest grade term in $B_s C_t$. Therefore

$$
\begin{aligned}
A_r \rfloor (B_s \lfloor C_t) &= \langle A_r B_s C_t \rangle_{s-(r+t)} \\
&= \langle A_r B_s C_t \rangle_{(s-r)-t} \\
&= \left\langle \langle A_r B_s \rangle_{s-r} C_t \right\rangle_{(s-r)-t} \\
&= \langle A_r B_s \rangle_{s-r} \lfloor C_t \\
&= (A_r \rfloor B_s) \lfloor C_t.
\end{aligned}
\tag{90}
$$

This expression also vanishes unless the inequality is satisfied, so $A \rfloor (B \lfloor C) = (A \rfloor B) \lfloor C$ in general. Finally,

$$
A_r \rfloor (B_s \rfloor C_t) = A_r \rfloor \langle B_s C_t \rangle_{t-s}
$$

$$= \left\langle A_r \left\langle B_s C_t \right\rangle_{t-s} \right\rangle_{t-(r+s)}. \tag{91}$$

Now this vanishes automatically unless $r + s \leq t$, in which case logic similar to that used above yields

$$\begin{aligned} A_r \rfloor (B_s \rfloor C_t) &= \left\langle A_r B_s C_t \right\rangle_{t-(r+s)} \\ &= \left\langle \left\langle A_r B_s \right\rangle_{r+s} C_t \right\rangle_{t-(r+s)} \\ &= \left\langle A_r B_s \right\rangle_{r+s} \rfloor C_t \\ &= (A_r \wedge B_s) \rfloor C_t. \end{aligned} \tag{92}$$

This final expression also vanishes unless the inequality is satisfied; therefore $A \rfloor (B \rfloor C) = (A \wedge B) \rfloor C$ for all A, B, and C. A similar proof shows that $(A \lfloor B) \lfloor C = A \lfloor (B \wedge C)$. □

It's useful to introduce an order of operations of these products to cut down on parentheses. The order is outer products, followed by inner products, followed by geometric products. Thus, for example,

$$A \rfloor B \wedge CD = \{A \rfloor (B \wedge C)\}\, D. \tag{93}$$

Despite this convention, I'll occasionally put the parentheses back in for clarity. However, I will use it a lot in Section 7 on projections, rotations, and reflections. (This is not the only convention in use; the other one reverses the order of inner and outer products. I picked this one.)

Now that I've defined two separate inner products, I should probably say why. Generally, we associate inner products with projections, but that turns out not to be quite right. Look back at Eq. (5) for the projection of v along u; noting that the inner product of a scalar into a vector is actually their product, you'll see Eq. (5) can also be written

$$P_u(v) = (v \rfloor u) \rfloor u^{-1}. \tag{94}$$

So orthogonal projection is actually a *double* inner product. The geometric meaning of a single inner product, as I'll show in Section 4.3, is this: for blades $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$, $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ is also a blade, and it represents the subspace of vectors *orthogonal to* $\boldsymbol{A_r}$ and *contained in* $\boldsymbol{B_s}$. So the roles played by the factors in the inner product are not the same; that's why the product is asymmetric and there are two of them, so either factor can play either role.

Finally, I can use the new definition of the left inner product to generalize Theorem 11.

Theorem 14. *If $r \leq s$ then*

$$B_r \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s) = \sum (-1)^{\sum_{j=1}^{r}(i_j - j)} (B_r \rfloor a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r})\, a_{i_{r+1}} \wedge \cdots \wedge a_{i_s}, \tag{95}$$

where the sum is performed over all possible choices of $\{a_{i_j}\}_{j=1,\ldots,r}$ out of $\{a_i\}_{i=1,\ldots,s}$, and in each term i_1 through i_r and i_{r+1} through i_s separately are in ascending order.

The coefficients in this sum can be remembered using the same trick used for Theorem 11. In that case, you imagine that you need to permute each vector to the far left in order to act on it with a. For this theorem, you imagine that you need to permute each distinct subset of r vectors to the far left, keeping them in their original order, in order to act on them with B_r. In both cases, permuting vectors to the far left introduces a power of -1 equal to the required number of interchanges of adjacent vectors.

Just as in Theorem 11, each inner product in the sum is a scalar. Since each term picks r elements out of a set s elements, the sum has $\binom{s}{r}$ terms.

Proof. If the result is true for an r-blade, it is true for any r-vector, so assume $\boldsymbol{B_r}$ is an r-blade. Now I'll proceed by induction on r. If $r = 1$ this becomes Theorem 11; so assume it's true for $r - 1$. Now $\boldsymbol{B_r} = a \wedge \boldsymbol{B_{r-1}}$ where a is a vector and $\boldsymbol{B_{r-1}}$ is an $r-1$-blade, so using the third of Eqs. (83) I can write

$$\boldsymbol{B_r} \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s) = (a \wedge \boldsymbol{B_{r-1}}) \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s)$$

$$
\begin{aligned}
&= a \rfloor [\boldsymbol{B_{r-1}} \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s)] \\
&= \sum (-1)^{\sum_{j=1}^{r-1}(i_j - j)} \boldsymbol{B_{r-1}} \rfloor (a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_{r-1}}) \times \\
&\qquad a \rfloor [a_{i_r} \wedge \cdots \wedge a_{i_s}] \qquad (96)
\end{aligned}
$$

where i_1 through i_{r-1} and i_r through i_s are in ascending order separately. Now I use Theorem 11 to get

$$
\begin{aligned}
\boldsymbol{B_r} \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s) = \sum (-1)^{\sum_{j=1}^{r-1}(i_j - j)} \boldsymbol{B_{r-1}} \rfloor (a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_{r-1}}) \times \\
\left[\sum_{k=r}^{s} (-1)^{k-r} a \rfloor a_{i_k} \, a_{i_r} \wedge \cdots \wedge \check{a}_{i_k} \wedge \cdots \wedge a_{i_s} \right]. \qquad (97)
\end{aligned}
$$

This expression has $(s-r+1)\binom{s}{r-1} = r\binom{s}{r}$ terms, which is too many by a factor of r, so it's time to do some grouping.

First, notice that each term is a scalar calculated using r of the vectors, multiplied by the outer product of the remaining $s-r$ vectors arranged in ascending order, and that every possible choice of r vectors occurs. That means that for some choice of scalars $C(a_{i_1}, \ldots, a_{i_r})$,

$$
\boldsymbol{B_r} \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s) = \sum C(a_{i_1}, \ldots, a_{i_r}) \, a_{i_{r+1}} \wedge \ldots \wedge a_{i_s} \qquad (98)
$$

where the sum is over all choices of r vectors out of the set of s. All that remains is to figure out what the coefficients $C(a_{i_1}, \ldots, a_{i_r})$ are. Well, for a given choice of a_{i_1} through a_{i_r}, the coefficient will include terms in which one of the a_{i_j} is in the inner product with a while the others are in the inner product with $\boldsymbol{B_{r-1}}$, or

$$
C(a_{i_1}, \ldots, a_{i_r}) = \sum_{j=1}^{r} (-1)^{\epsilon_j} \boldsymbol{B_{r-1}} \rfloor (a_{i_1} \wedge \cdots \wedge \check{a}_{i_j} \wedge \cdots \wedge a_{i_r}) \, a \rfloor a_{i_j} \qquad (99)
$$

for some value of ϵ_j for each j. This sum has r terms, one for each of the a_{i_j}, which is exactly the number I need, so now I need to figure out the exponents ϵ_j. Remember the mnemonic I've been using: each vector's contribution to ϵ_j equals the difference between its position in the original outer product and the position to which it is moved to compute the inner product. (You can verify that this is true for every vector in Eq. (97) by inspection.) So all we have to do is figure out those positions. First consider every a_{i_k} in the inner product with $\boldsymbol{B_{r-1}}$ where $k < j$: each one is moved from position i_k to position k, so it contributes $i_k - k$ to ϵ_j. Now consider the a_{i_k} where $k > j$: each one is moved from position i_k to position $k-1$ (because position j is empty), so it contributes $i_k - k + 1$.

Finally, let's take a look at a_{i_j}. If its inner product is taken with a, then Eq. (97) tells me that it is part of the second group of vectors. Therefore vectors $a_{i_{j+1}}$ through a_{i_r} had to be moved to its left, moving it from its original position i_j ahead to $i_j - j + r$. It is then moved to position r for the inner product with a, so its contribution to ϵ_j is $i_j - j$. Therefore

$$
\begin{aligned}
\epsilon_j &= \sum_{k=1}^{j-1} (i_k - k) + \sum_{k=j+1}^{r} (i_k - k + 1) + i_j - j \\
&= \sum_{k=1}^{r} (i_k - k) + r - j. \qquad (100)
\end{aligned}
$$

Putting this in Eq. (99) gets me

$$
C(a_{i_1}, \ldots, a_{i_r}) = \sum_{j=1}^{r} (-1)^{\sum_{k=1}^{r}(i_k - k) + r - j} \boldsymbol{B_{r-1}} \rfloor (a_{i_1} \wedge \cdots \wedge \check{a}_{i_j} \wedge \cdots \wedge a_{i_r}) \, a \rfloor a_{i_j}
$$

$$
\begin{aligned}
&= (-1)^{\sum_{k=1}^r (i_k - k)} (-1)^{r-1} \boldsymbol{B_{r-1}} \rfloor \left[\sum_{j=1}^r (-1)^{1-j} (a \rfloor a_{i_j})\, a_{i_1} \wedge \cdots \wedge \check{a}_{i_j} \wedge \cdots \wedge a_{i_r} \right] \\
&= (-1)^{\sum_{k=1}^r (i_k - k)} (-1)^{r-1} \boldsymbol{B_{r-1}} \rfloor [a \rfloor (a_{i_1} \wedge \cdots \wedge a_{i_r})] \\
&= (-1)^{\sum_{k=1}^r (i_k - k)} (-1)^{r-1} (\boldsymbol{B_{r-1}} \wedge a) \rfloor (a_{i_1} \wedge \cdots \wedge a_{i_r}) \\
&= (-1)^{\sum_{k=1}^r (i_k - k)} (a \wedge \boldsymbol{B_{r-1}}) \rfloor (a_{i_1} \wedge \cdots \wedge a_{i_r}) \\
&= (-1)^{\sum_{k=1}^r (i_k - k)} \boldsymbol{B_r} \rfloor (a_{i_1} \wedge \cdots \wedge a_{i_r}).
\end{aligned} \tag{101}
$$

Comparing this expression for $C(a_{i_1}, \ldots, a_{i_r})$ with the statement of the theorem, I see that I've proved the result. □

It's easy to verify that

$$\sum_{j=1}^r (i_j - j) = \sum_{j=1}^r (i_j - 1) - \frac{r(r-1)}{2}, \tag{102}$$

so this is another way to write the exponent of -1 in the statement of the theorem. I'll use this later.

Now I can answer some questions left hanging in Section 1.3. I asked how you could tell what nvn^{-1} and $v \wedge nn^{-1}$ are without knowing how they were derived. (Notice I'm using order of operations to drop parentheses.) First let's do nvn^{-1}; since this is proportional to nvn, I'll look at that instead. We know now that the product of three vectors will in general by the sum of a vector and a trivector, and the trivector is the outer product of the factors. In this case the outer product is $n \wedge v \wedge n$, which vanishes because n appears twice, so the product must be pure vector.

Next let's look at $v \wedge nn^{-1}$; again I'll consider $v \wedge nn$ because the answer will be the same. The easiest thing to do here is expand the product with the final n into inner and outer products:

$$
\begin{aligned}
v \wedge nn &= v \wedge n \lfloor n + v \wedge n \wedge n \\
&= v \wedge n \lfloor n
\end{aligned} \tag{103}
$$

because the $v \wedge n \wedge n$ term vanishes. Even if you didn't know the remaining term had to be a vector, you could figure it out because it starts as a vector and has one grade raising and one grade lowering operation applied to it.

I also asked how to calculate $\left(v \wedge nn^{-1}\right) \rfloor n$ to verify that $v \wedge nn^{-1}$ really is perpendicular to n. (Remember that $\cdot$ has changed to $\rfloor$ since we got through Section 2.) Since the inner product of vectors is just the scalar part of their geometric product,

$$
\begin{aligned}
(v \wedge nn^{-1}) \rfloor n &= \left\langle v \wedge nn^{-1} n \right\rangle \\
&= \langle v \wedge n \rangle \\
&= 0
\end{aligned} \tag{104}
$$

since $v \wedge n$ has no scalar part. Easy, huh?

4.3. The geometric meaning of the inner and outer products

I've now accomplished all I set out to do in this section except for geometric interpretation. First I'll handle the outer product.

Theorem 15. *Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be nonzero blades where $r, s \geq 1$.*

(a) *$\boldsymbol{A_r} \wedge \boldsymbol{B_s} = 0$ iff $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ share nonzero vectors.*

(b) *$\boldsymbol{A_r} \wedge \boldsymbol{B_s}$, if nonzero, represents the direct sum of the corresponding subspaces.*

Proof. This is true because, by the same reasoning used in Theorem 3, $\boldsymbol{A_r} \wedge \boldsymbol{B_s} = 0$ iff the factors of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ form a linearly dependent set, which is true iff the subspaces share a nonzero vector. The fact that nonzero $\boldsymbol{A_r} \wedge \boldsymbol{B_s}$ represents the direct sum of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ follows immediately, since $a \wedge \boldsymbol{A_r} \wedge \boldsymbol{B_s} = 0$ iff a is a linear combination of the factors of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$. □

So forming the outer product is equivalent to taking the direct sum, and it's nonzero iff the direct sum can be taken.

Next let's move on to the inner product. I already said at the end of Section 4.2 that the inner product combines inclusion in one subspace and orthogonality to the other, and I need some terminology to more conveniently describe this. In Section 3 I defined the orthogonal complement of a subspace: it's the set of all vectors orthogonal to every vector in the subspace, or equivalently the orthogonal complement of $\boldsymbol{A_r}$ is the set of all vectors a satisfying $a \rfloor \boldsymbol{A_r} = 0$. Here's another definition: the *orthogonal complement of* $\boldsymbol{A_r}$ *in* $\boldsymbol{B_s}$ is the intersection of $\boldsymbol{B_s}$ and the orthogonal complement of $\boldsymbol{A_r}$. Algebraically, a is in the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$ iff $a \rfloor \boldsymbol{A_r} = 0$ and $a \wedge \boldsymbol{B_s} = 0$. Now I'm ready to prove the result.

Theorem 16. *Let* $\boldsymbol{A_r}$ *and* $\boldsymbol{B_s}$ *be nonzero blades where* $r, s \geq 1$.

(a) $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ *iff* $\boldsymbol{A_r}$ *contains a nonzero vector orthogonal to* $\boldsymbol{B_s}$.

(b) *If* $r < s$ *then* $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$*, if nonzero, is an* $s - r$*-blade representing the orthogonal complement of* $\boldsymbol{A_r}$ *in* $\boldsymbol{B_s}$.

Again, this is why two different inner products are defined; the geometric roles played by the two factors in the product aren't the same. In contrast, Theorem 15 shows that the roles played by the two factors in the outer product are the same, which is why there's only one outer product. This also explains geometrically why $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ when $r > s$; as I show in the proof, if one subspace is higher-dimensional than other, the larger subspace always contains a nonzero vector that is orthogonal to the smaller one.

Proof. First I'll consider the case $r \leq s$; try to be surprised that the proof is by induction on r. The $r = 1$ result is taken care of by the proof of Theorem 6 (see both the theorem and the discussion right after the proof), so assume the results have been proved for $r - 1$ and consider $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$. To prove part (a), let a_1 be any vector in $\boldsymbol{A_r}$; then for some $\boldsymbol{A_{r-1}}$ I can write $\boldsymbol{A_r} = \boldsymbol{A_{r-1}} \wedge a_1$, which means

$$\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = \boldsymbol{A_{r-1}} \wedge a_1 \rfloor \boldsymbol{B_s} = \boldsymbol{A_{r-1}} \rfloor (a_1 \rfloor \boldsymbol{B_s}). \tag{105}$$

Suppose $\boldsymbol{A_r}$ contains a vector orthogonal to $\boldsymbol{B_s}$; then let that vector be a_1, so $a_1 \rfloor \boldsymbol{B_s} = 0$, so $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$. For the converse, assume $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$; then it follows that $\boldsymbol{A_{r-1}} \rfloor (a_1 \rfloor \boldsymbol{B_s}) = 0$. There are now three possibilities. The first is $a_1 \rfloor \boldsymbol{B_s} = 0$, in which case a_1 is orthogonal to $\boldsymbol{B_s}$ and I'm done. If not, then by the $r - 1$ result $\boldsymbol{A_{r-1}}$ contains a vector a_2 orthogonal to $a_1 \rfloor \boldsymbol{B_s}$; if that vector happens to be orthogonal to all of $\boldsymbol{B_s}$ then I'm also done. Now for the third case: a_2 is orthogonal to $a_1 \rfloor \boldsymbol{B_s}$ but not $\boldsymbol{B_s}$. The proof of Theorem 6 showed that I can factor $\boldsymbol{B_s}$ as $b_1 \wedge \cdots \wedge b_s$ where $a_1 \rfloor b_1 \neq 0$ while the other b_j are orthogonal to a_1, so

$$a_1 \rfloor \boldsymbol{B_s} = (a_1 \rfloor b_1) b_2 \wedge \cdots \wedge b_s. \tag{106}$$

The only way a_2 can be orthogonal to $a_1 \rfloor \boldsymbol{B_s}$ but not $\boldsymbol{B_s}$ is if $a_2 \rfloor b_1 \neq 0$ while a_2 is orthogonal to all the other b_j. In that case consider

$$a = a_1 - \left(\frac{a_1 \rfloor b_1}{a_2 \rfloor b_1}\right) a_2. \tag{107}$$

This vector lies in $\boldsymbol{A_r}$; it's nonzero because a_1 and a_2 are linearly independent; and it's orthogonal to all the b_j, including b_1, by construction. Thus it's orthogonal to $\boldsymbol{B_s}$. So in all three cases $\boldsymbol{A_r}$ contains a vector orthogonal to $\boldsymbol{B_s}$. To prove part (b), assume $r < s$ and $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = \boldsymbol{A_{r-1}} \rfloor (a_1 \rfloor \boldsymbol{B_s}) \neq 0$. Then by the $r - 1$ result $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ is the space of all vectors in $a_1 \rfloor \boldsymbol{B_s}$ that are orthogonal to $\boldsymbol{A_{r-1}}$. However, by the $r = 1$ result $a_1 \rfloor \boldsymbol{B_s}$ is the orthogonal complement of a_1 in $\boldsymbol{B_s}$, so a vector lies in $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ iff it lies in $\boldsymbol{B_s}$ and is orthogonal both to a_1 and to $\boldsymbol{A_{r-1}}$, and thus to all of $\boldsymbol{A_r}$. This proves both parts for $r \le s$.

Now for the case $r > s$; $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ automatically, so I can forget about part (b) and I only need to show that $\boldsymbol{A_r}$ always contains a vector orthogonal to $\boldsymbol{B_s}$. Consider $\boldsymbol{B_s} \rfloor \boldsymbol{A_r}$; either it vanishes or it doesn't. If it doesn't, then it contains vectors in $\boldsymbol{A_r}$ orthogonal to $\boldsymbol{B_s}$ and I'm done. If it does vanish, then $\boldsymbol{B_s}$ contains vectors orthogonal to $\boldsymbol{A_r}$; let $\boldsymbol{B_p}$ represent the subspace of all such vectors. If $p = s$, then $\boldsymbol{B_s}$ is orthogonal to $\boldsymbol{A_r}$ and I'm also done. If $p < s$, then $\boldsymbol{B_s} = \boldsymbol{B_p} \wedge \boldsymbol{B_{s-p}}$ where $\boldsymbol{B_{s-p}}$ contains no vectors orthogonal to $\boldsymbol{A_r}$, which implies $\boldsymbol{B_{s-p}} \rfloor \boldsymbol{A_r} \ne 0$. Then any vector in $\boldsymbol{B_{s-p}} \rfloor \boldsymbol{A_r}$ lies in $\boldsymbol{A_r}$ and is orthogonal to $\boldsymbol{B_{s-p}}$; but by lying in $\boldsymbol{A_r}$ the vector is already orthogonal to $\boldsymbol{B_p}$, so it's orthogonal to all of $\boldsymbol{B_s}$ and the result is proved. □

Incidentally, I'll show in Section 5.2 that $A_r \lfloor B_s = (-1)^{r(s-1)} B_s \rfloor A_r$ for any r- and s-vectors, so the geometric interpretation of the right inner product is the same as the left product, but with the factors reversed, which certainly seems reasonable.

As a fun exercise, at this point you might look back at the identities in Eqs. (83) in the special case that A, B, and C are blades and try to figure out the geometrical meaning of each one.

To explain the next few theorems, I need some facts about blade $\boldsymbol{A}$ to be proved in Section 5.4.

1. $\boldsymbol{A}^2$ is a scalar.
2. $\boldsymbol{A}$ is invertible iff $\boldsymbol{A}^2 \ne 0$, and $\boldsymbol{A}^{-1} = \boldsymbol{A}/\boldsymbol{A}^2$. Therefore $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ represent the same subspace.
3. $\boldsymbol{A}$ is invertible iff the inner product is nondegenerate on $\boldsymbol{A}$.

The inner and outer products of blades can shed light on how their subspaces are related. For example, if one subspace lies inside another, their blades are related as follows.

Theorem 17. *Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be nonzero blades where $1 \le r \le s$.*

(a) *If $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$, then $\boldsymbol{A_r B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$.*

(b) *The converse is true if either (1) $r = 1$ or $s = 1$ or (2) $\boldsymbol{A_r}$ or $\boldsymbol{B_s}$ is invertible.*

Proof. First let $r = 1$; then $a\boldsymbol{B_s} = a \rfloor \boldsymbol{B_s}$ iff $a \wedge \boldsymbol{B_s} = 0$, which is true iff a belongs to $\boldsymbol{B_s}$. Now assume the result is true for $r-1$ and let $\boldsymbol{A_r}$ be a subspace of $\boldsymbol{B_s}$; I can write $\boldsymbol{A_r} = \boldsymbol{A_{r-1}}a$ for some vector a such that a and $\boldsymbol{A_{r-1}}$ are orthogonal, so the $r = 1$ result lets me write

$$\boldsymbol{A_r B_s} = \boldsymbol{A_{r-1}} a \boldsymbol{B_s} = \boldsymbol{A_{r-1}} a \rfloor \boldsymbol{B_s}. \tag{108}$$

Now $a \rfloor \boldsymbol{B_s}$ is the orthogonal complement of a in $\boldsymbol{B_s}$, which means $a \rfloor \boldsymbol{B_s}$ contains $\boldsymbol{A_{r-1}}$, so by the $r-1$ result

$$\begin{aligned}\boldsymbol{A_r B_s} &= \boldsymbol{A_{r-1}} a \rfloor \boldsymbol{B_s} = \boldsymbol{A_{r-1}} \rfloor (a \rfloor \boldsymbol{B_s})\\ &= (\boldsymbol{A_{r-1}} \wedge a) \rfloor \boldsymbol{B_s}\\ &= \boldsymbol{A_r} \rfloor \boldsymbol{B_s},\end{aligned} \tag{109}$$

which is the desired result. (If $a \rfloor \boldsymbol{B_s} = 0$ then $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ also by Theorem 16, so if one side vanishes then so does the other.) I've already shown the converse is true when $r = 1$, and $s = 1$ implies $r = 1$, so assume $\boldsymbol{A_r}$ is invertible; then $\boldsymbol{A_r B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ implies $\boldsymbol{B_s} = \boldsymbol{A_r^{-1} A_r} \rfloor \boldsymbol{B_s}$. By assumption $\boldsymbol{B_s} \ne 0$, so $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} \ne 0$ also. Therefore if $r = s$, then $\boldsymbol{B_s}$ is just a nonzero multiple of $\boldsymbol{A_r^{-1}}$. Since $\boldsymbol{A_r}$ and $\boldsymbol{A_r^{-1}}$ represent the same subspace, $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ also represent the same subspace. If $r < s$, then since $\boldsymbol{B_s}$ is an s-vector, $\boldsymbol{A_r^{-1} A_r} \rfloor \boldsymbol{B_s}$ must be too; but only its highest grade term, its outer product, has grade s, so for this relation to hold the product must equal the outer product, so $\boldsymbol{B_s} = \boldsymbol{A_r^{-1}} \wedge (\boldsymbol{A_r} \rfloor \boldsymbol{B_s})$. Therefore $\boldsymbol{B_s}$ is the direct sum of $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ and $\boldsymbol{A_r^{-1}}$;

but $\boldsymbol{A_r^{-1}}$ represents the same subspace as $\boldsymbol{A_r}$, so $\boldsymbol{A_r}$ is obviously a subspace of $\boldsymbol{B_s}$. The proof when $\boldsymbol{B_s}$ is invertible is similar; $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ implies $\boldsymbol{A_r} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}\boldsymbol{B_s^{-1}}$, so if $r = s$, $\boldsymbol{A_r}$ is a nonzero multiple of $\boldsymbol{B_s}$. If $r < s$, then since $\boldsymbol{A_r}$ is an r-vector, $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}\boldsymbol{B_s^{-1}}$ must be too; but only its lowest grade term, the inner product, has grade r, so the product must equal the inner product, so $\boldsymbol{A_r} = (\boldsymbol{A_r} \rfloor \boldsymbol{B_s}) \rfloor \boldsymbol{B_s^{-1}}$. Therefore, $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s^{-1}}$, and thus of $\boldsymbol{B_s}$. □

Another possible relationship is orthogonality: two subspaces are orthogonal if every vector in one is orthogonal to every vector in the other. In that case, their blades are related as follows.

Theorem 18. *Let $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ be nonzero blades where $r, s \geq 1$.*

(a) *If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$.*

(b) *The converse is true if either (1) $r = 1$ or $s = 1$ or (2) $\boldsymbol{A_r}$ or $\boldsymbol{B_s}$ is invertible.*

Proof. To begin, I note that $\boldsymbol{A_r}$ can be written $a_1 a_2 \cdots a_r$ where the a_i are orthogonal to each other; similarly, $\boldsymbol{B_s}$ can be expressed $b_1 b_2 \cdots b_s$ where the b_j are also orthogonal to each other. Now suppose that $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal; then all of the a_i and b_j are orthogonal to each other as well, so now I can use the rule that the product of orthogonal vectors equals their outer product to get

$$\begin{aligned}\boldsymbol{A_r}\boldsymbol{B_s} &= a_1 a_2 \cdots a_r b_1 b_2 \cdots b_s \\ &= a_1 \wedge a_2 \wedge \cdots \wedge a_r \wedge b_1 \wedge b_2 \wedge \cdots \wedge b_s \\ &= \boldsymbol{A_r} \wedge \boldsymbol{B_s}.\end{aligned} \tag{110}$$

To prove the converse, first let $r = 1$; then $a\boldsymbol{B_s} = a \wedge \boldsymbol{B_s}$ iff $a \rfloor \boldsymbol{B_s} = 0$, which is true iff a is orthogonal to $\boldsymbol{B_s}$. Now assume $\boldsymbol{A_r}$ is invertible and let $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$; then $\boldsymbol{B_s} = \boldsymbol{A_r^{-1}}\boldsymbol{A_r} \wedge \boldsymbol{B_s}$. Since $\boldsymbol{A_r^{-1}}$ represents the same subspace as $\boldsymbol{A_r}$, which is a subspace of $\boldsymbol{A_r} \wedge \boldsymbol{B_s}$, it follows that $\boldsymbol{B_s} = \boldsymbol{A_r^{-1}} \rfloor (\boldsymbol{A_r} \wedge \boldsymbol{B_s})$, so $\boldsymbol{B_s}$ is orthogonal to $\boldsymbol{A_r}$. The proof when $s = 1$ or $\boldsymbol{B_s}$ is invertible proceeds similarly. □

You may be surprised that the converse parts of these theorems aren't generally true; let me give an example to show why not. Consider an algebra with orthogonal vectors e_1, e_2, e_3, and e_4 such that e_2 is null but the others aren't. (These four vectors define only a subspace of the full space of vectors, or Axiom 5 would be violated.) Let $\boldsymbol{A_3} = e_1 e_2 e_3$ and $\boldsymbol{B_3} = e_2 e_3 e_4$; then $e_2^2 = 0$ implies $\boldsymbol{A_3}\boldsymbol{B_3} = 0$, so $\boldsymbol{A_3} \rfloor \boldsymbol{B_3} = \boldsymbol{A_3} \wedge \boldsymbol{B_3} = 0$ also. However, neither $\boldsymbol{A_3}$ nor $\boldsymbol{B_3}$ is a subspace of the other, so the converse part of Theorem 17 doesn't hold, and the subspaces are not orthogonal to each other (they both contain non-null e_3), so the converse part of Theorem 18 doesn't hold either. Null vectors make life hard sometimes.

Incidentally, $A_r B_s = A_r \rfloor B_s$ implies $B_s A_r = B_s \lfloor A_r$ for any r- and s-vectors, and the same is true if the inner product is replaced with the outer product (this follows from Eq. (140)), so the last two theorems don't depend on the order of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$. Really, it would be weird if they did.

Let me end with a result that combines the previous few theorems in an interesting way. Suppose $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$; then it seems plausible that $\boldsymbol{B_s}$ should be the direct sum of $\boldsymbol{A_r}$ and its orthogonal complement in $\boldsymbol{B_s}$. (For example, three-dimensional Euclidean space is the direct sum of the z axis and its orthogonal complement, the xy plane.) Using our theorems, that suggests something like $\boldsymbol{B_s} = \boldsymbol{A_r} \wedge (\boldsymbol{A_r} \rfloor \boldsymbol{B_s})$. Now that can't be right as it stands because the result shouldn't depend on the weight of $\boldsymbol{A_r}$, just its attitude. That's easy to fix, though: maybe $\boldsymbol{B_s} = \boldsymbol{A_r} \wedge (\boldsymbol{A_r^{-1}} \rfloor \boldsymbol{B_s})$ instead. That turns out to be right, but with some caveats. Let me prove a theorem I need first.

Theorem 19. *If $1 \leq r \leq s$ and nonzero blades $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ satisfy $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$, then $\boldsymbol{A_r^2}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge (\boldsymbol{A_r} \rfloor \boldsymbol{B_s})$.*

Proof. Believe it or not, this result can be proved without induction. Suppose the condition is true; then

$$\boldsymbol{A_r^2}\boldsymbol{B_s} = \boldsymbol{A_r}(\boldsymbol{A_r}\boldsymbol{B_s}) = \boldsymbol{A_r}(\boldsymbol{A_r} \rfloor \boldsymbol{B_s}). \tag{111}$$

Since $\boldsymbol{A_r}^2$ is a number, the left hand side is an s-vector; so the right hand side must be also. Using the same logic as in the proof of Theorem 17, the product on the right hand side must equal the outer product, and that proves the result. □

To interpret this theorem, suppose $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$, so $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$ by Theorem 17. Now either $\boldsymbol{A_r}$ is invertible or it's not; first suppose it is. Then $\boldsymbol{A_r}^2 \neq 0$, so taking the result of Theorem 19 and dividing by $\boldsymbol{A_r}^2$ yields $\boldsymbol{B_s} = \boldsymbol{A_r} \wedge (\boldsymbol{A_r}^{-1} \rfloor \boldsymbol{B_s})$, which is the result I was after. Notice I had to assume $\boldsymbol{A_r}$ was invertible to get this, though. What if it isn't? In that case $\boldsymbol{A_r}$ contains a nonzero vector orthogonal to all of $\boldsymbol{A_r}$, so one of two things can happen: either that vector is also orthogonal to all of $\boldsymbol{B_s}$, so $\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$, or it isn't, in which case that vector also lies in $\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$, so $\boldsymbol{A_r} \wedge (\boldsymbol{A_r} \rfloor \boldsymbol{B_s}) = 0$. Since $\boldsymbol{A_r}^2 = 0$, the theorem nicely includes both of those cases.

5. Other operations

Our computational powers have grown by leaps and bounds, but they're not yet complete. Some more operations will be useful to us later, so I'll describe them all here. Please be aware that different authors use different symbols for some of these operations; I've listed all my symbol choices in Appendix A.

5.1. Grade involution

The first has the formidable name *grade involution.* It is represented by an * and defined as follows.

$$\begin{aligned} \lambda^* &:= \lambda \\ a^* &:= -a \\ (AB)^* &:= A^* B^* \\ (A+B)^* &:= A^* + B^*. \end{aligned} \tag{112}$$

This operation takes reflection of vectors through the origin (the second line in the definition), sometimes called the parity operation, and extends it to the whole algebra. From these rules it follows that

$$(a_1 a_2 \cdots a_r)^* = (-1)^r a_1 a_2 \cdots a_r, \tag{113}$$

which implies

$$A_r^* = (-1)^r A_r, \tag{114}$$

so grade involution leaves even grades alone while changing the sign of odd-grade multivectors, or

$$A^* = \langle A \rangle_+ - \langle A \rangle_- \,. \tag{115}$$

This is equivalent to the occasionally handy result

$$\langle A \rangle_\pm = \frac{1}{2}(A \pm A^*). \tag{116}$$

Eq. (114) tells me that

$$\langle A \rangle_r^* = \langle A^* \rangle_r \,, \tag{117}$$

so grade involution commutes with taking the grade-r part, and

$$A^{**} = A \tag{118}$$

for any multivector A. (That's what makes it an involution.) Suppose A is invertible; then since $(A^{-1})^* A^* = (A^{-1}A)^* = 1^* = 1$,

$$(A^{-1})^* = (A^*)^{-1}. \tag{119}$$

By projecting onto terms of appropriate grade, the third rule in the definition becomes

$$(A \rfloor B)^* = A^* \rfloor B^*$$

$$
\begin{aligned}
(A \lfloor B)^* &= A^* \lfloor B^* \\
(A \wedge B)^* &= A^* \wedge B^*.
\end{aligned} \tag{120}
$$

Formulas with factors of $(-1)^r$ are usually simplified by grade involution. For example, Eqs. (42) and (51) for the inner and outer products of vector a with multivector A become

$$
\begin{aligned}
a \rfloor A &= \frac{1}{2}(aA - A^* a) \\
a \wedge A &= \frac{1}{2}(aA + A^* a),
\end{aligned} \tag{121}
$$

the definitions of $A \lfloor a$ and $A \wedge a$ from Eqs. (43) and (52) become

$$
\begin{aligned}
A \lfloor a &= -a \rfloor A^* \\
A \wedge a &= a \wedge A^*,
\end{aligned} \tag{122}
$$

and finally the identities from Eqs. (74) and (82) can now be written

$$
\begin{aligned}
a \rfloor (AB) &= (a \rfloor A)B + A^*(a \rfloor B) \\
&= (a \wedge A)B - A^*(a \wedge B) \\
a \wedge (AB) &= (a \wedge A)B - A^*(a \rfloor B) \\
&= (a \rfloor A)B + A^*(a \wedge B).
\end{aligned} \tag{123}
$$

$$
\begin{aligned}
a \rfloor (A \wedge B) &= (a \rfloor A) \wedge B + A^* \wedge (a \rfloor B) \\
a \wedge (A \lfloor B) &= (a \wedge A) \lfloor B - A^* \lfloor (a \rfloor B) \\
a \wedge (A \rfloor B) &= (a \rfloor A) \rfloor B + A^* \rfloor (a \wedge B).
\end{aligned} \tag{124}
$$

Every once in a while I'll use ^{*r} to indicate grade involution taken r times. A^{*r} equals A if r is even and A^* if r is odd.

This operation is called "inversion" by some authors, but that can be confused with the multiplicative inverse, which would be bad because both are important and are used frequently (sometimes at the same time).

A blade and its grade involution represent the same subspace since each is a multiple of the other by Eq. (114).

5.2. Reversion

The second operation is called *reversion* or *taking the reverse* and is represented by a †. It's a little more complicated, and it's defined as follows.

$$
\begin{aligned}
\lambda^\dagger &:= \lambda \\
a^\dagger &:= a \\
(AB)^\dagger &:= B^\dagger A^\dagger \\
(A+B)^\dagger &:= A^\dagger + B^\dagger.
\end{aligned} \tag{125}
$$

From this it follows that, for example,

$$
(a_1 a_2 \cdots a_r)^\dagger = a_r \cdots a_2 a_1, \tag{126}
$$

which shows that the reverse of any multivector is found by writing it as a sum of blades and reversing the order of the vectors in each blade. Hence the name. This also implies

$$
A^{\dagger\dagger} = A \tag{127}
$$

for any multivector A. So reversion is also an involution.

Let $\{e_i\}_{i=1,\ldots,r}$ be an anticommuting set; then

$$\begin{aligned}(e_1e_2\cdots e_r)^\dagger &= e_r\cdots e_2e_1\\ &= (-1)^{r(r-1)/2}\,e_1e_2\cdots e_r\end{aligned} \tag{128}$$

because $r(r-1)/2$ interchanges are needed to return the vectors to their original order; therefore

$$A_r^\dagger = (-1)^{r(r-1)/2}A_r. \tag{129}$$

If you evaluate this expression for different r, you quickly find (a) multivectors two grades apart behave oppositely under reversion, and (b) two adjacent grades behave the same under reversion iff the lower grade is even (scalars and vectors, for example). Since the effect of either reversion or grade involution is to change signs of a multivector grade by grade, these operations commute:

$$A^{*\dagger} = A^{\dagger *}. \tag{130}$$

Eq. (129) also shows that

$$\langle A\rangle_r^\dagger = \left\langle A^\dagger\right\rangle_r, \tag{131}$$

so taking the reverse commutes with taking the grade-r part, just as grade involution does. Suppose A is invertible; then since $(A^{-1})^\dagger A^\dagger = (AA^{-1})^\dagger = 1^\dagger = 1$,

$$(A^{-1})^\dagger = (A^\dagger)^{-1}. \tag{132}$$

By projecting onto terms of appropriate grade, the third rule in the definition becomes

$$\begin{aligned}(A\rfloor B)^\dagger &= B^\dagger\lfloor A^\dagger\\ (A\lfloor B)^\dagger &= B^\dagger\rfloor A^\dagger\\ (A\wedge B)^\dagger &= B^\dagger\wedge A^\dagger.\end{aligned} \tag{133}$$

Among other things, this shows that the right inner product can be defined in terms of the left inner product and reversion and is thus technically redundant. Oh well.

Here's an easy application of Eq. (129):

$$\begin{aligned}\langle AB\rangle_r &= (-1)^{r(r-1)/2}\left\langle (AB)^\dagger\right\rangle_r\\ &= (-1)^{r(r-1)/2}\left\langle B^\dagger A^\dagger\right\rangle_r\end{aligned} \tag{134}$$

with the nice special case

$$\langle AB\rangle = \left\langle B^\dagger A^\dagger\right\rangle. \tag{135}$$

The even more special case where A and B are homogeneous is also useful. The product of two homogeneous multivectors of different grades doesn't have a scalar part, so $\langle A_rB_s\rangle = \langle B_sA_r\rangle = 0$ if $r\neq s$, and when $r=s$ we get

$$\begin{aligned}\langle A_rB_r\rangle &= \left\langle B_r^\dagger A_r^\dagger\right\rangle\\ &= (-1)^{r(r-1)/2}\,(-1)^{r(r-1)/2}\,\langle B_rA_r\rangle\\ &= \langle B_rA_r\rangle.\end{aligned} \tag{136}$$

Therefore $\langle A_rB_s\rangle = \langle B_sA_r\rangle$ in general. Since the geometric product is distributive, I can go all the way to

$$\langle AB\rangle = \langle BA\rangle \tag{137}$$

for any A and B, or better yet

$$\langle AB\cdots CD\rangle = \langle DAB\cdots C\rangle, \tag{138}$$

so the scalar part of a product is cyclic in its factors. This is very useful. Eqs. (135) and (137) together also imply

$$\langle AB\rangle = \left\langle A^\dagger B^\dagger\right\rangle . \tag{139}$$

Almost every identity involving reverses is proved by successively applying Eq. (129) and using the rules in Eqs. (125). For example, let's examine a generic term in $A_r B_s$:

$$\begin{aligned}
\langle A_r B_s\rangle_{r+s-2j} &= (-1)^{(r+s-2j)(r+s-2j-1)/2}\left\langle (A_r B_s)^\dagger\right\rangle_{r+s-2j} \\
&= (-1)^{(r+s-2j)(r+s-2j-1)/2}\left\langle B_s^\dagger A_r^\dagger\right\rangle_{r+s-2j} \\
&= (-1)^{(r+s-2j)(r+s-2j-1)/2}(-1)^{r(r-1)/2}(-1)^{s(s-1)/2}\left\langle B_s A_r\right\rangle_{r+s-2j} \\
&= (-1)^{rs-j}\left\langle B_s A_r\right\rangle_{r+s-2j} .
\end{aligned} \tag{140}$$

So multiplication may not commute, but $A_r B_s$ and $B_s A_r$ aren't totally unrelated; term by term, they're actually equal up to signs. (Also, successive terms of $A_r B_s$, whose grades differ by 2, have opposite behavior under reversion, which I expected given what I said after Eq. (129).) This result also has two important special cases. First, suppose $r \le s$; then Eq. (140) with $j = r$ refers to the lowest grade term, so

$$A_r \rfloor B_s = (-1)^{r(s-1)} B_s \lfloor A_r. \tag{141}$$

Notice that this expression also holds when $r > s$ because both sides vanish. So the inner product of an odd-grade multivector into an even-grade multivector anticommutes (with left changing to right and vice versa), as in

$$A_- \rfloor B_+ = -B_+ \lfloor A_-, \tag{142}$$

but in all other cases it commutes. Without any restrictions on r and s, Eq. (140) when $j = 0$ gives for the highest grade term

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r, \tag{143}$$

so the outer product of two odd-grade multivectors anticommutes like so,

$$A_- \wedge B_- = -B_- \wedge A_-, \tag{144}$$

with all other cases commuting. (These last few results are equivalent to Eq. (133), by the way.)

The properties of objects under reversion are sometimes helpful in sorting out their grades. As an example, let me reconsider the product nvn of three vectors from Section 1.3. Notice that $(nvn)^\dagger = nvn$. Now vectors don't change sign under reversion but trivectors do. Therefore nvn has no trivector component and is pure vector.

A blade and its reverse represent the same subspace since each is a multiple of the other by Eq. (129).

5.3. Clifford conjugation

The third involution in a geometric algebra is called *Clifford conjugation* or *taking the Clifford conjugate*. It's represented by a ‡ and defined as follows:

$$\begin{aligned}
\lambda^\ddagger &:= \lambda \\
a^\ddagger &:= -a \\
(AB)^\ddagger &:= B^\ddagger A^\ddagger \\
(A+B)^\ddagger &:= A^\ddagger + B^\ddagger.
\end{aligned} \tag{145}$$

If this looks like a mixture of grade involution and reversion, that's because it is; in fact,

$$A^\ddagger = A^{*\dagger}. \tag{146}$$

This immediately tells us that Clifford conjugation really is an involution,

$$A^{\ddagger\ddagger} = A, \tag{147}$$

that it commutes with taking the grade-r part,

$$\langle A\rangle_r^{\ddagger} = \left\langle A^{\ddagger}\right\rangle_r, \tag{148}$$

that when A is invertible

$$(A^{-1})^{\ddagger} = (A^{\ddagger})^{-1}, \tag{149}$$

and finally

$$\begin{aligned}(A \rfloor B)^{\ddagger} &= B^{\ddagger} \lfloor A^{\ddagger} \\ (A \lfloor B)^{\ddagger} &= B^{\ddagger} \rfloor A^{\ddagger} \\ (A \wedge B)^{\ddagger} &= B^{\ddagger} \wedge A^{\ddagger}.\end{aligned} \tag{150}$$

The Clifford conjugate of an r-vector is given by

$$\begin{aligned}A_r^{\ddagger} &= A_r^{*\dagger} \\ &= (-1)^r(-1)^{r(r-1)/2} A_r \\ &= (-1)^{r(r+1)/2} A_r.\end{aligned} \tag{151}$$

This looks a lot like reversion. If you evaluate this for different r you find that (a) multivectors two grades apart behave oppositely under Clifford conjugation, just as with reversion, but (b) two adjacent grades behave the same under Clifford conjugation iff the lower grade is *odd*, not even (vectors and bivectors, for example). So Clifford conjugation resembles reversion with grades shifted by 1, so to speak.

A blade and its Clifford conjugate represent the same subspace since each is a multiple of the other by Eq. (151).

5.4. The scalar product

Next is the *scalar product*, defined by

$$A * B := \left\langle A^{\dagger} B\right\rangle. \tag{152}$$

(Some authors define $A * B = \langle AB\rangle$. I'll tell you why I don't shortly.) First consider the scalar product of homogeneous multivectors. Only the lowest-grade term in the product (the inner product) has any chance of being a scalar, so it's certainly true that

$$A_r * B_s = \left\langle A_r^{\dagger} \rfloor B_s\right\rangle = \left\langle A_r^{\dagger} \lfloor B_s\right\rangle. \tag{153}$$

Since the scalar product and inner products are distributive by construction, it follows that

$$A * B = \left\langle A^{\dagger} \rfloor B\right\rangle = \left\langle A^{\dagger} \lfloor B\right\rangle \tag{154}$$

for any multivectors. Now of course we actually know a little more than Eq. (153) lets on. Only the product of two equal-grade homogeneous multivectors has a scalar part, so

$$A_r * B_s = (A_r^{\dagger} \rfloor B_s)\, \delta_{rs} = (A_r^{\dagger} \lfloor B_s)\, \delta_{rs}. \tag{155}$$

Therefore, homogeneous multivectors of different grades are orthogonal under the scalar product. That means the scalar product of two general multivectors may also be written

$$\begin{aligned}A * B &= \sum_r A_r * B_r \\ &= \sum_r A_r^{\dagger} \rfloor B_r = \sum_r A_r^{\dagger} \lfloor B_r.\end{aligned} \tag{156}$$

This also makes it clear that

$$A * B = A^* * B^*. \tag{157}$$

My results for reversion and Clifford conjugation also establish some properties of this product; for example,

$$A * B = B * A = A^\dagger * B^\dagger = A^\ddagger * B^\ddagger \tag{158}$$

The next to last equality shows that an equivalent definition of the scalar product is $\langle AB^\dagger \rangle$. The scalar product interacts with the other products we know this way.

Theorem 20.

$$\begin{aligned} A * (BC) &= (B^\dagger A) * C \\ A * (B \lfloor C) &= (B^\dagger \lfloor A) * C \\ A * (B \rfloor C) &= (B^\dagger \wedge A) * C \\ A * (B \wedge C) &= (B^\dagger \rfloor A) * C \end{aligned} \tag{159}$$

Proof. The first identity is proved as follows:

$$\begin{aligned} A * (BC) &= \langle A^\dagger BC \rangle \\ &= (A^\dagger B)^\dagger * C \\ &= (B^\dagger A) * C. \end{aligned} \tag{160}$$

The remaining three are proved roughly the same way, so I'll prove only the first one. Using Eqs. (156) and (133) and the second of Eqs. (83),

$$\begin{aligned} A * (B \lfloor C) &= \langle A^\dagger \rfloor (B \lfloor C) \rangle \\ &= \langle (A^\dagger \rfloor B) \lfloor C \rangle \\ &= (A^\dagger \rfloor B)^\dagger * C \\ &= (B^\dagger \lfloor A) * C. \end{aligned} \tag{161}$$

□

The last of Eqs. (159) is the basis for a different approach to geometric algebra, followed for example in [5]. You start by defining the outer product and the scalar product; then you decide you'd like to be able to factor the term B out of expressions like $A * (B \wedge C)$. You do this by defining an inner product that obeys the last of Eqs. (159). Then you define the geometric product of two vectors to be the sum of their inner and outer products, and you're off and running. This has the advantage that it starts with two products that have clearly separated geometric functions: the outer product builds subspaces out of vectors, and the scalar product carries all the metric information. It's thus more congenial to the point of view inherent in differential forms, which are built using only an outer product and which clearly separate metric and non-metric properties. Personally, I think it's cleaner to start with the fundamental product and define every other product directly in terms of it, which is why I follow the approach given here.

I use the scalar product to define the *magnitude* or *norm* of a multivector by

$$|A|^2 := A * A. \tag{162}$$

A is said to be *null* if $|A|^2 = 0$ and a *unit* multivector if $|A|^2 = \pm 1$. (Despite the notation, $|A|^2$ can be negative. In fact, $|A|^2$ can be all sorts of things, since the scalars aren't necessarily real numbers.) Eqs. (157) and (158) imply

$$|A|^2 = |A^*|^2 = |A^\dagger|^2 = |A^\ddagger|^2. \tag{163}$$

I define $|A|^n$ for other powers n in the obvious way as a power of $|A|^2$, but due care should be taken that the power in question is well-defined (for example, be careful if $|A|^2$ is negative).

The squared magnitude of a scalar or vector is just its square, and that result can be generalized a bit. Suppose A is an r-versor, so it is a product $a_1 a_2 \cdots a_r$; then

$$\begin{aligned}|A|^2 &= A * A = \left\langle A^\dagger A \right\rangle \\ &= \left\langle (a_1 a_2 \cdots a_r)^\dagger a_1 a_2 \cdots a_r \right\rangle \\ &= \langle a_r \cdots a_2 a_1 a_1 a_2 \cdots a_r \rangle \\ &= a_1^2 a_2^2 \cdots a_r^2. \end{aligned} \tag{164}$$

Therefore $|A|^2$ is the product of the squares of its factors. (This is why I included the reverse in the definition.) Notice also that if A is a versor then $A^\dagger A$ also equals $|A|^2$. This gives me a couple of useful results.

First, versors can be factored out of scalar products in an interesting way.

Theorem 21. *Versor A and general multivectors B and C obey*

$$(AB) * (AC) = (BA) * (CA) = |A|^2\, B * C. \tag{165}$$

Therefore if either A or B is a versor,

$$|AB|^2 = |A|^2 |B|^2. \tag{166}$$

Proof.

$$\begin{aligned}(AB) * (AC) &= \left\langle (AB)^\dagger AC \right\rangle \\ &= \left\langle B^\dagger A^\dagger AC \right\rangle \\ &= \left\langle |A|^2\, B^\dagger C \right\rangle \\ &= |A|^2\, B * C \end{aligned} \tag{167}$$

and a similar argument using Eq. (138) proves the other part of the equation. The second part follows by setting $B = C$. □

Second, versors are easy to invert.

Theorem 22. *Versor A is invertible iff it's non-null, its inverse is given by*

$$A^{-1} = \frac{A^\dagger}{|A|^2}, \tag{168}$$

and the squared norm of the inverse is given by

$$|A^{-1}|^2 = |A|^{-2}. \tag{169}$$

Proof. If $|A| \neq 0$, then clearly Eq. (168) gives an inverse of A, so A must be invertible. Conversely, suppose A is invertible; then there exists a B such that $AB = 1$. Then it follows that $B^\dagger A^\dagger = 1$ also, so

$$\begin{aligned} 1 &= B^\dagger A^\dagger AB \\ &= |A|^2 B^\dagger B, \end{aligned} \tag{170}$$

so $|A| \neq 0$; thus a product of vectors is invertible iff it's non-null, and its inverse is given by the above expression. For the squared norm, just calculate $|A^{-1}|^2$ using Eq. (168). □

An r-blade $\boldsymbol{A_r}$ is a special type of r-versor, so these theorems apply to blades too. But for blades, a few more results are also true. Since $\boldsymbol{A_r}^\dagger = (-1)^{r(r-1)/2}\boldsymbol{A_r}$, $|A|^2 = A^\dagger A$ becomes $|\boldsymbol{A_r}|^2 = (-1)^{r(r-1)/2}\boldsymbol{A_r}^2$. Therefore the norm of an r-blade differs from its square at most by a sign. That means unit r-blades also satisfy $\boldsymbol{A_r}^2 = \pm 1$, although that ± 1 may not be the blade's squared norm. It also follows that the inverse of $\boldsymbol{A_r}$ equals the additional expressions

$$\boldsymbol{A_r}^{-1} = (-1)^{r(r-1)/2}\frac{\boldsymbol{A_r}}{|\boldsymbol{A_r}|^2} = \frac{\boldsymbol{A_r}}{\boldsymbol{A_r}^2}. \tag{171}$$

So the inverse of an r-blade is a multiple of the original r-blade, just as with vectors. Therefore they represent the same subspace.

In Section 1.3, I asked how you would calculate the inverse of 2-blade $a \wedge b$. Well, now we know: divide the original blade by its square. I actually calculated $(a \wedge b)^2$ in Section 1.1, and the result was $-a^2b^2\sin^2\theta$. Therefore

$$(a \wedge b)^{-1} = \frac{b \wedge a}{a^2b^2\sin^2\theta}. \tag{172}$$

By the way, this is also the reverse of $a \wedge b$ divided by its norm squared, as it should be.

Next, I give a geometric property of null blades.

Theorem 23. *A nonzero blade is null (and thus noninvertible) iff the inner product is degenerate on the subspace it represents.*

> *Proof.* $\boldsymbol{A_r} = e_1e_2\cdots e_r$ is null iff $e_i^2 = 0$ for at least one i, in which case e_i is orthogonal to every vector in the span of $\{e_j\}_{i=1,\ldots,r}$, which is just $\boldsymbol{A_r}$. That means that either (a) $e_i = 0$ or (b) $e_i \neq 0$ but the inner product is degenerate on $\boldsymbol{A_r}$. Since $\boldsymbol{A_r} \neq 0$, none of the e_i vanish, so that leaves case (b): the inner product must be degenerate. Therefore nonzero $\boldsymbol{A_r}$ is null iff the inner product is degenerate on $\boldsymbol{A_r}$. □

So every nonzero blade is invertible in a Euclidean space, while in non-Euclidean spaces things aren't as simple.

And here's an interesting property of products of versors.

Theorem 24. *If $r, s \geq 1$ and nonzero versors A_r and B_s satisfy $A_r\,B_s = 0$, then both versors are null.*

> *Proof.* This one is easy: if $A_r\,B_s = 0$, then $|A_r|^2B_s = A_r^\dagger A_r\,B_s = 0$ also. Now since B_s is assumed nonzero, it follows that $|A_r|^2 = 0$, or A_r is null. Going back to $A_r\,B_s = 0$ and multiplying from the right by $B_s^\dagger$ establishes that B_s is also null. □

This also means that the product of a non-null versor and any versor is nonzero.

A special case of this theorem arises if a vector a both lies in blade $\boldsymbol{A_r}$ ($a \wedge \boldsymbol{A_r} = 0$) and is orthogonal to it ($a \rfloor \boldsymbol{A_r} = 0$). In that case $a\boldsymbol{A_r} = 0$, and we say *a annihilates* $\boldsymbol{A_r}$. The theorem tells us that a must be a null vector and $\boldsymbol{A_r}$ must be a null blade, which is clear from Theorem 23 since the existence of such a vector makes the inner product degenerate on $\boldsymbol{A_r}$.

To conclude this section, I assume the scalars are real so I can define the weight of an r-blade, as I said I would back in Section 3. The weight is supposed to be a higher-dimensional generalization of volume, and one way to get that is the following: express $\boldsymbol{A_r}$ as a product of r orthogonal vectors, and define the weight to be the product of the lengths of those vectors. Then the weight is the volume of an r-dimensional parallelepiped that spans the correct subspace. That's what the norm gives us, as Eq. (164) shows, so I define

$$\text{weight}(\boldsymbol{A_r}) := \sqrt{||\boldsymbol{A_r}|^2|}. \tag{173}$$

The extra $|\,|$ is under the square root because, as I've repeatedly mentioned, the squared norm can be negative. By this definition when $r = 0$, the weight of a scalar is its absolute value. This definition only works on scalars for which an absolute value and square root are defined, which is why I'm defining it only for real algebras.

When I get to integral calculus on geometric algebras, I'll be using the weights of blades not to define the theory but to interpret parts of it. Thus integration will be defined on any geometric algebra, but some of its meaning will apply only to real algebras. Since all of our applications will be on real algebras, I think we'll be fine.

5.5. The dual

The next operation is called a *duality transformation* or *taking the dual*. Let $\boldsymbol{A_r}$ be an invertible r-blade; then the dual of any multivector B by $\boldsymbol{A_r}$ is $B \rfloor \boldsymbol{A_r}^{-1}$. (Duality gets its own symbol only in a special case, which I'll describe below.) To understand what taking the dual does, let B be a s-blade $\boldsymbol{B_s}$.

1. If $s > r$, the dual of $\boldsymbol{B_s}$ vanishes.

2. If $s = r$, the dual of $\boldsymbol{B_s}$ is a scalar which is zero iff $\boldsymbol{B_s}$ contains a vector orthogonal to $\boldsymbol{A_r}$ (Theorem 16).

3. If $s < r$, the dual of $\boldsymbol{B_s}$ is either zero or an $r - s$-blade representing the orthogonal complement of $\boldsymbol{B_s}$ in $\boldsymbol{A_r}$ (Theorem 16 again). If $\boldsymbol{B_s}$ was inside $\boldsymbol{A_r}$ to begin with, the dual of $\boldsymbol{B_s}$ is just $\boldsymbol{B_s}\boldsymbol{A_r}^{-1}$ (Theorem 17).

The dual of an arbitrary B is a sum of these results. Duality transformations are useful both for taking orthogonal complements of blades (based on the observations above) and for performing orthogonal projections into subspaces, as I'll show in Section 7.1.

Although one can take the dual by any invertible blade, one class of blades is by far the most important: those that represent the entire vector space. The dual by these blades is very useful and also has simpler properties than the dual in general.

Let the dimension of the vector space be n; then all n-blades either vanish identically (if the factors are dependent) or represent the same subspace (namely the whole space); therefore Theorem 4 says that all n-blades are multiples of one another. Since the inner product on the whole space is nondegenerate by Axiom 5, Theorem 23 says that all nonzero n-blades are also invertible and thus non-null, so I define a *volume element* $\boldsymbol{I}$ to be a unit n-blade. This determines $\boldsymbol{I}$ to within a sign. (Some people call a volume element a *pseudoscalar*, but I won't.) In fact, I can calculate $|\boldsymbol{I}|^2$ explicitly. Let $\{e_i\}_{i=1,\ldots,r}$ be an orthonormal basis, and suppose p of the e_i square to -1 while the rest square to 1. Let $\boldsymbol{I} = e_1 e_2 \cdots e_n$; then using Eq. (164),

$$|\boldsymbol{I}|^2 = e_1^2 e_2^2 \cdots e_n^2 = (-1)^p. \tag{174}$$

Therefore $|\boldsymbol{I}|^2 = 1$ in any Euclidean space, while $|\boldsymbol{I}|^2 = -1$ in Minkowski space. Since $|\boldsymbol{I}|^2 = \boldsymbol{I}^\dagger \boldsymbol{I}$, this implies

$$\boldsymbol{I}^2 = (-1)^{n(n-1)/2+p}. \tag{175}$$

Given Theorem 23, we can now see that Axiom 5 is just another way to say "volume elements are invertible." I could have used instead a weaker axiom that implies only "volume elements are nonzero," and that would have been enough to prove some foundational results, like this one that I've been promising for some time.

Theorem 25. *The outer product of linearly independent vectors is nonzero.*

> *Proof.* Let $\boldsymbol{A_r}$ be the outer product of linearly independent vectors. Since volume elements are nonzero, $\boldsymbol{A_r}$ must lie in a subspace represented by a nonzero blade $\boldsymbol{A_s}$; then by Theorem 5 there exists a blade $\boldsymbol{A_{s-r}}$ such that $\boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}} = \boldsymbol{A_s}$. Thus $\boldsymbol{A_r}$ is a factor of a nonzero blade, so $\boldsymbol{A_r}$ is nonzero too. □

This theorem is actually equivalent to "volume elements are nonzero" because each implies the other. Because of this, some authors take this weaker statement as an axiom instead of my Axiom 5. I still like my axiom, though, because if $\boldsymbol{I}$ is invertible, taking the dual by $\boldsymbol{I}$ is also invertible. This makes the dual much more useful, as you'll see below.

Unless otherwise specified, "the dual of A" means "the dual of A by $\boldsymbol{I}$" and is denoted $A^\perp$. Let's reconsider the three ways the dual of a blade can turn out when we're taking the dual by $\boldsymbol{I}$.

1. There are no s-blades for $s > n$, so the first option can't happen.

2. Any n-blade $\boldsymbol{B_n} = \lambda \boldsymbol{I}$ for some λ, in which case the dual of $\boldsymbol{B_n}$ is just λ.

3. If $s < n$, then $\boldsymbol{B_s}$ represents a subspace of the full space, so the dual of $\boldsymbol{B_s}$ is just $\boldsymbol{B_s}\boldsymbol{I}^{-1}$. It cannot be zero; if it were, then Theorem 16 would say that $\boldsymbol{B_s}$ contains a nonzero vector orthogonal to the whole space, which Axiom 5 doesn't allow. The theorem also tells me that the dual of $\boldsymbol{B_s}$ represents the orthogonal complement of $\boldsymbol{B_s}$.

So the general formula for the dual of multivector A is

$$A^{\perp} := A \rfloor \boldsymbol{I}^{-1} = A\boldsymbol{I}^{-1} \tag{176}$$

and the dual of a blade represents its orthogonal complement. (Hence the choice of symbol.) Taking the dual by $\boldsymbol{I}^{-1}$ instead of $\boldsymbol{I}$ is the inverse operation; it's denoted by $A^{-\perp}$. Since $\boldsymbol{I}^{-1} = \boldsymbol{I}/\boldsymbol{I}^2$, $A^{\perp}$ and $A^{-\perp}$ differ only by a factor of $\boldsymbol{I}^2$, so the duality transformation is its own inverse up to at most a sign.

Since the product of any multivector with $\boldsymbol{I}$ is an inner product, it's true for any A_r that

$$\begin{aligned} A_r\boldsymbol{I} &= (-1)^{r(n-1)}\boldsymbol{I}A_r \\ &= \boldsymbol{I}A_r^{*(n-1)}, \end{aligned} \tag{177}$$

so for any multivector A,

$$A\boldsymbol{I} = \boldsymbol{I}A^{*(n-1)}. \tag{178}$$

This has several consequences.

1. Even multivectors commute with $\boldsymbol{I}$ regardless of the value of n, so their duals can be taken be taken from either side with no difference.
2. In odd-dimensional spaces, the dual of any multivector can be taken from either side.
3. In even-dimensional spaces, the dual of an odd multivector can still be taken from either side, and the results differ only by a sign.
4. The first of Eqs. (177) is true even if A_r is an r-versor because all terms in A_r are even or odd as r is even or odd.
5. In even-dimensional spaces Eqs. (116) and (178) can be used to separate the pure even and pure odd parts of a multivector:
$$\langle A\rangle_{\pm} = \frac{1}{2}(A \pm \boldsymbol{I}A\boldsymbol{I}^{-1}) \quad \text{if } n \text{ is even.} \tag{179}$$

Duality lets me prove a surprising result.

Theorem 26. *If the vector space is n-dimensional, every $n-1$-vector is an $n-1$-blade.*

Proof. Let A_{n-1} by an $n-1$-vector. The dual of A_{n-1} is a vector, so A_{n-1} is the dual of a vector. But vectors are 1-blades, and the dual of a blade is also a blade, so A_{n-1} is an $n-1$-blade. □

One corollary of this is that in dimensions below four, all r-vectors are actually r-blades. 0-vectors and 1-vectors are always blades, n-vectors are always blades (which takes care of bivectors in two dimensions and trivectors in three), and bivectors in three dimensions are $n-1$-vectors and thus blades.

The dual is distributive over addition, and it's easy to show that

$$(AB)^{\perp} = AB^{\perp}. \tag{180}$$

Taking appropriate-grade terms also shows that

$$\begin{aligned} (A \wedge B)^{\perp} &= A \rfloor B^{\perp} \\ (A \rfloor B)^{\perp} &= A \wedge B^{\perp}. \end{aligned} \tag{181}$$

Thus the dual relates the inner and outer products. (Here's another way to prove these results: start with the third of Eqs. (83) and set $C = \boldsymbol{I}^{-1}$. That gets you the first equation. Then replace B with $B^\perp$ and take the inverse dual of both sides; that gets you the other equation.) A special case of this is $(a \rfloor \boldsymbol{A_r})^\perp = a \wedge \boldsymbol{A_r}^\perp$, which means vector a is orthogonal to subspace $\boldsymbol{A_r}$ iff a lies in $\boldsymbol{A_r}^\perp$. That's further confirmation that duals represent orthogonal complements. It also shows that any subspace has a *direct* representation (all a such that $a \wedge \boldsymbol{A_r} = 0$) and a *dual* representation (all a such that $a \rfloor \boldsymbol{A_r}^\perp = 0$). These two representations are both useful in different situations.

In the discussion around Theorem 19, I said that if $\boldsymbol{A_r}$ is invertible, then any space that contains $\boldsymbol{A_r}$ is the direct sum of $\boldsymbol{A_r}$ and its orthogonal complement. This is certainly true for the whole space, and it's nicely expressed in terms of duals.

Theorem 27. *The whole space is the direct sum of $\boldsymbol{A_r}$ and $\boldsymbol{A_r}^\perp$ iff $\boldsymbol{A_r}$ is invertible.*

Proof.

$$\boldsymbol{A_r} \wedge \boldsymbol{A_r}^\perp = (\boldsymbol{A_r} \rfloor \boldsymbol{A_r})^\perp = \boldsymbol{A_r}^2 \boldsymbol{I}^{-1} = \frac{\boldsymbol{A_r}^2}{\boldsymbol{I}^2}\boldsymbol{I}. \tag{182}$$

Now Theorem 22 and Eq. (171) tell me $\boldsymbol{A_r}$ is invertible iff $\boldsymbol{A_r}^2 \neq 0$. So if $\boldsymbol{A_r}$ is invertible, Eq. (182) shows that $\boldsymbol{I}$ is the direct sum of $\boldsymbol{A_r}$ and its orthogonal complement; and if $\boldsymbol{A_r}$ is not invertible, the equation shows that $\boldsymbol{A_r}$ and its dual have vectors in common, so they don't even have a direct sum. □

If A is invertible, so is $A^\perp$:

$$\begin{aligned}(A^\perp)^{-1} &= (A\boldsymbol{I}^{-1})^{-1} \\ &= \boldsymbol{I}A^{-1}.\end{aligned} \tag{183}$$

The dual of a grade involution is given by

$$\begin{aligned}(A^*)^\perp &= A^*\boldsymbol{I}^{-1} \\ &= (-1)^n A^*(\boldsymbol{I}^{-1})^* \\ &= (-1)^n (A\boldsymbol{I}^{-1})^* \\ &= (-1)^n (A^\perp)^*.\end{aligned} \tag{184}$$

The dual of a reverse is

$$\begin{aligned}(A^\dagger)^\perp &= \left[(A^\perp \boldsymbol{I})^\dagger\right]^\perp \\ &= \left[\boldsymbol{I}^\dagger (A^\perp)^\dagger\right]^\perp.\end{aligned} \tag{185}$$

Combining these results, the dual of a Clifford conjugate is

$$(A^\ddagger)^\perp = \left[\boldsymbol{I}^\ddagger (A^\perp)^\ddagger\right]^\perp. \tag{186}$$

Finally, Theorem 21 and the second part of Theorem 22 show that the dual almost preserves scalar products:

$$\begin{aligned}A^\perp * B^\perp &= (A\boldsymbol{I}^{-1}) * (B\boldsymbol{I}^{-1}) \\ &= |\boldsymbol{I}^{-1}|^2 \, A * B \\ &= |\boldsymbol{I}|^{-2} \, A * B.\end{aligned} \tag{187}$$

So taking the dual preserves scalar products up to a scale factor.

Occasionally it's convenient to take the dual by volume elements that aren't normalized. In that case, the dual and its inverse may differ by more than a sign, but the difference is still only a scalar multiple. All the results in this section are valid regardless of the normalization of $\boldsymbol{I}$.

5.6. The commutator

The final operation is called the *commutator*, defined as follows.

$$A \times B := \frac{1}{2}(AB - BA). \tag{188}$$

Notice the factor of $\frac{1}{2}$, which is not present in the usual definition of the commutator, used for example in quantum mechanics. The commutator obeys the identity

$$A \times (BC) = (A \times B)C + B(A \times C), \tag{189}$$

which is easily verified by expanding out the commutators. This shows that the commutator is a derivation on the algebra (it obeys the Leibnitz rule). Use this identity to expand $A \times (BC)$ and $A \times (CB)$ and take half the difference; the result is the *Jacobi identity*

$$A \times (B \times C) = (A \times B) \times C + B \times (A \times C). \tag{190}$$

The presence of the second term on the right hand side shows that the commutator is not associative. This identity is often given in the cyclic form

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0. \tag{191}$$

From the defining properties of the three involutions it's easy to see that

$$\begin{aligned} (A \times B)^* &= A^* \times B^* \\ (A \times B)^\dagger &= B^\dagger \times A^\dagger \\ (A \times B)^\ddagger &= B^\ddagger \times A^\ddagger. \end{aligned} \tag{192}$$

The commutator of any multivector with a scalar clearly vanishes, and the commutator with a vector can be expressed nicely by decomposing a general multivector as $A = \langle A \rangle_+ + \langle A \rangle_-$ and recalling Eqs. (42) and (51) for the inner and outer products. The result is

$$\begin{aligned} a \times A &= a \rfloor \langle A \rangle_+ + a \wedge \langle A \rangle_- \\ A \times a &= \langle A \rangle_+ \lfloor a + \langle A \rangle_- \wedge a. \end{aligned} \tag{193}$$

This lets me prove an important result about commuting multivectors.

Theorem 28. *The following statements are equivalent.*

1. *A commutes with all multivectors.*
2. *A commutes with all vectors.*
3. *$A = \lambda + \mu \langle \boldsymbol{I} \rangle_-$.*

Item 3 is my sneaky way of saying A equals λ in even-dimensional vector spaces and $\lambda + \mu\boldsymbol{I}$ in odd-dimensional spaces.

Proof. Since scalars commute with everything, I won't mention them again. If A commutes with all multivectors then it obviously commutes with all vectors. On the other hand, if A commutes with all vectors then it commutes with all blades, since these are products of vectors. Therefore A commutes with all sums of blades, and thus all multivectors.

Now for item 3. The first of Eqs. (193) tells me that $a \times \boldsymbol{I} = a \wedge \boldsymbol{I}$ if the vector space is odd-dimensional and $a \rfloor \boldsymbol{I}$ if the space is even-dimensional. Now $a \wedge \boldsymbol{I} = 0$ and $a \rfloor \boldsymbol{I} \neq 0$ for all a, because every vector lies in $\boldsymbol{I}$ and no vector is orthogonal to it; therefore all vectors commute with $\boldsymbol{I}$ in odd-dimensional spaces and no vectors commute with $\boldsymbol{I}$ in even-dimensional spaces. To finish off, let $\boldsymbol{A_r}$ be an r-blade where $0 < r < n$. If r is even, then $a \times \boldsymbol{A_r} = a \rfloor \boldsymbol{A_r}$, and if this vanished for all a then $\boldsymbol{A_r}$ would be orthogonal to the whole space, in violation of Axiom 5. If r is odd, then $a \times \boldsymbol{A_r} = a \wedge \boldsymbol{A_r}$. Since $r < n$ there certainly exists a vector a outside $\boldsymbol{A_r}$, in which case $a \wedge \boldsymbol{A_r} \neq 0$. □

The most interesting of all is the commutator with a bivector.

Theorem 29.

$$A_2 \times A_r = \langle A_2 A_r \rangle_r\,, \tag{194}$$

so the commutator of a bivector and an r-vector is an r-vector; commutation with a bivector is a grade preserving operation.

Proof. To show this, I note that

$$\begin{aligned} A_2 A_r &= A_2 \rfloor A_r + \langle A_2 A_r \rangle_r + A_2 \wedge A_r \\ A_r A_2 &= A_2 \rfloor A_r - \langle A_2 A_r \rangle_r + A_2 \wedge A_r. \end{aligned} \tag{195}$$

The first equation is obvious when $r \geq 2$; for $r < 2$, recall that in such cases $A_2 \rfloor A_r = 0$. The second equation follows from the first because of the properties of the inner and outer products under interchange and Eq. (140) when $j = 1$. Subtracting these equations yields

$$A_2 \times A_r = \frac{1}{2}(A_2 A_r - A_r A_2) = \langle A_2 A_r \rangle_r\,. \tag{196}$$

□

In particular, the set of bivectors is closed under commutation. That means the bivectors form a Lie algebra with the commutator serving as the Lie product. That will be important later when I show how to use geometric algebra to describe Lie groups and Lie algebras.

Since commutation with a bivector is grade preserving, the identity in Eq. (189) still holds if $A = A_2$ and I replace all geometric products with either inner or outer products:

$$\begin{aligned} A_2 \times (B \rfloor C) &= (A_2 \times B) \rfloor C + B \rfloor (A_2 \times C) \\ A_2 \times (B \lfloor C) &= (A_2 \times B) \lfloor C + B \lfloor (A_2 \times C) \\ A_2 \times (B \wedge C) &= (A_2 \times B) \wedge C + B \wedge (A_2 \times C). \end{aligned} \tag{197}$$

The last of these relations can be generalized in this way.

Theorem 30.

$$A_2 \times (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = \sum_{j=1}^{r} a_1 \wedge a_2 \wedge \cdots \wedge (A_2 \lfloor a_j) \wedge \cdots \wedge a_r. \tag{198}$$

Proof. As usual, I use induction. The result is true when $r = 1$ because the commutator with a vector is the same as the right inner product, and the $r = 2$ result follows from the last of Eqs. (197), so let's assume the result is true for $r - 1$. Then by associativity of the outer product

$$A_2 \times (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = A_2 \times (\boldsymbol{B_{r-1}} \wedge a_r) \tag{199}$$

where $\boldsymbol{B_{r-1}} = a_1 \wedge a_2 \wedge \cdots \wedge a_{r-1}$. Applying the last of Eqs. (197) and the $r - 1$ result yields

$$\begin{aligned} A_2 \times (a_1 \wedge a_2 \wedge \cdots \wedge a_r) &= (A_2 \times \boldsymbol{B_{r-1}}) \wedge a_r + \boldsymbol{B_{r-1}} \wedge (A_2 \times a_r) \\ &= \sum_{j=1}^{r-1} \big[a_1 \wedge a_2 \wedge \cdots \wedge (A_2 \lfloor a_j) \wedge \cdots \wedge a_{r-1} \wedge a_r\big] + \\ &\qquad a_1 \wedge a_2 \wedge \cdots \wedge a_{r-1} \wedge (A_2 \lfloor a_r) \\ &= \sum_{j=1}^{r} a_1 \wedge a_2 \wedge \cdots \wedge (A_2 \lfloor a_j) \wedge \cdots \wedge a_r, \end{aligned} \tag{200}$$

which completes the proof. □

I expand the order of operations to include all of these new operations as follows: perform the involutions, then duals, then outer, then inner, then geometric products, then scalar products, and finally commutators. Following this convention, the parentheses in Eqs. (159) and the left hand sides of Eqs. (189), (197), and (198) (but not the right hand sides) may be omitted.

6. Geometric algebra in Euclidean space

Now let's apply everything I've done so far to some familiar cases. I'll work through the algebras of two- and three-dimensional real Euclidean space explicitly, revealing some neat surprises along the way.

6.1. Two dimensions and complex numbers

First I'll consider the real plane $\mathbb{R}^2$ with the Euclidean scalar product; this is often denoted $\mathbb{E}^2$. It has an orthonormal basis $\{e_1, e_2\}$, which produces a geometric algebra spanned by the elements 1, e_1, e_2, and e_1e_2. That last element satisfies

$$|e_1e_2|^2 = (e_1e_2)^\dagger e_1e_2 = e_2e_1e_1e_2 = 1. \tag{201}$$

Therefore it qualifies as a volume element $\boldsymbol{I}$. Since bivectors change sign under reversion, it also satisfies $\boldsymbol{I}^2 = -1$. It defines a right-handed orientation, and a few examples show that all vectors anticommute with $\boldsymbol{I}$. This is consistent with Eq. (178).

Now for the geometric products. We know what a scalar times anything and a vector times a vector look like; all that remains is the product of a vector and a bivector, or equivalently the product of a vector and $\boldsymbol{I}$. To see what that does, notice that

$$\begin{aligned} \boldsymbol{I}e_1 &= -e_2 \\ \boldsymbol{I}e_2 &= e_1, \end{aligned} \tag{202}$$

so multiplication of a unit vector by $\boldsymbol{I}$ results in an orthogonal unit vector. (Which it should, since multiplying by $\boldsymbol{I}$ takes the dual to within a sign.) Eq. (202) actually tells us a bit more: left multiplication by $\boldsymbol{I}$ rotates a vector clockwise through $\pi/2$. Similarly, right multiplication rotates a vector counterclockwise through the same angle. So $\boldsymbol{I}^2 = -1$ means that two rotations in the same sense through $\pi/2$ have the same effect as multiplying by -1. Of course, this is true only in two dimensions.

The even subalgebra of any geometric algebra is always of interest, so let's take a moment to look at it. A generic even multivector can be written $Z = x + \boldsymbol{I}y$ where x and y are real numbers and $\boldsymbol{I}^2 = -1$; in other words, the even subalgebra of $\mathbb{E}^2$ is isomorphic to the algebra of complex numbers. Now this may be a little bit of a surprise, because the even subalgebra represents scalars and areas, while we normally think of complex numbers as vectors in the Argand plane. But there's another way to think of complex numbers: the polar form $z = re^{i\theta}$ reminds us that z also represents a rotation through angle θ followed by a dilatation by r. How do these two interpretations of complex numbers relate?

It works out because we're in two dimensions. Then and only then, the even subalgebra is isomorphic to the space of vectors; a generic vector in $\mathbb{E}^2$ takes the form $z = xe_1 + ye_2$ where x and y are real numbers, and there's a natural isomorphism between the vectors and the even subalgebra of the form

$$\begin{aligned} Z &= e_1 z \\ z &= e_1 Z. \end{aligned} \tag{203}$$

This isomorphism maps a vector in the e_1 direction onto a pure "real" number, so e_1 plays the role of the real axis. It also maps a vector in the e_2 direction onto a pure "imaginary" number, so e_2 is the imaginary axis. Now think about complex conjugation: it leaves the real part alone while changing the sign of the imaginary part. Therefore complex conjugation is a reflection along the e_2 axis, which takes z to $z' = -e_2ze_2$. What happens to the corresponding even element? It gets mapped to

$$\begin{aligned} Z' &= e_1 z' \\ &= -e_1(e_2ze_2) \\ &= -e_1e_2(e_1Z)e_2 \\ &= -\boldsymbol{I}e_1(x + \boldsymbol{I}y)e_2 \\ &= x - \boldsymbol{I}y \\ &= Z^\dagger, \end{aligned} \tag{204}$$

where I used the fact that $\boldsymbol{I}$ anticommutes with all vectors. Therefore complex conjugation corresponds to taking the reverse. Now let w and z be vectors with corresponding even elements W and Z; it follows that

$$\begin{aligned} wz &= e_1 W e_1 Z \\ &= e_1(x + \boldsymbol{I}y)e_1 Z \\ &= (x - \boldsymbol{I}y)Z \\ &= W^\dagger Z. \end{aligned} \tag{205}$$

Now let's look at this. The right hand side is the product of one complex number with the conjugate of another. That has two terms: the real part equals the dot product of the corresponding vectors, while the magnitude of the imaginary part equals the magnitude of the cross product of the vectors. The left hand side is the geometric product of the vectors, which is exactly the same thing. One of the goals of geometric algebra was to take the complex product, which combines the two-dimensional dot and cross products naturally, and generalize it to any number of dimensions. (That was a goal of quaternions too. I'll show how that worked out in the next section.)

Now for rotations. An element W of the even subalgebra has a polar form $r\exp(-\boldsymbol{I}\theta)$ for some r and θ. Letting $r = 1$, multiplication by a vector z produces the vector

$$\begin{aligned} z' &= Wz \\ &= \exp(-\boldsymbol{I}\theta)z \\ &= \exp(-\boldsymbol{I}\theta/2)\exp(-\boldsymbol{I}\theta/2)z \\ &= \exp(-\boldsymbol{I}\theta/2)z\exp(\boldsymbol{I}\theta/2) \\ &= RzR^{-1} \end{aligned} \tag{206}$$

where I defined $R = \exp(-\boldsymbol{I}\theta/2)$. Clearly R is a rotor, so multiplication by W performs a counterclockwise rotation through θ. What is the corresponding transformation of Z?

$$\begin{aligned} Z' &= e_1 z' \\ &= e_1 W z \\ &= wz \\ &= W^\dagger Z \\ &= \exp(\boldsymbol{I}\theta)Z. \end{aligned} \tag{207}$$

Thus vector z is rotated counterclockwise through θ when the corresponding even element Z is multiplied by $\exp(\boldsymbol{I}\theta)$, exactly as you'd expect.

In conclusion, the complex numbers are the even subalgebra of the geometric algebra of the Euclidean plane; the identification with vectors is just an accident in two dimensions, just as identifying planes with normal vectors works only in three dimensions. Now while complex algebra is useful, so is complex analysis; we use its techniques to perform many ostensibly real integrals, for example. If geometric algebra generalizes complex algebra to any dimension, then perhaps calculus of geometric algebras could generalize complex analysis too. I'll describe geometric calculus later, and I'll show how it generalizes the Cauchy integral theorem and other useful results.

6.2. Three dimensions, Pauli matrices, and quaternions

Much that is true in two dimensions carries over to three: $\mathbb{E}^3$ has an orthonormal basis $\{e_1, e_2, e_3\}$, so its geometric algebra is spanned by 1, e_1, e_2, e_3, e_1e_2, e_1e_3, e_2e_3, and the volume element $e_1e_2e_3$. This volume element is also denoted $\boldsymbol{I}$, defines a right-handed orientation, satisfies $|\boldsymbol{I}|^2 = \boldsymbol{I}^\dagger\boldsymbol{I} = 1$, and squares to -1. Unlike the two-dimensional case, a few examples show that all vectors and bivectors commute with $\boldsymbol{I}$, as required by Theorem 28. Now I've repeatedly mentioned that only in three dimensions can you identify planes with normal vectors, which is why the cross product works there. The map between planes and normal vectors should be a duality transformation, so the cross product should be the dual of something. Well, it is. If a and b are vectors, then

$$a \times b = (a \wedge b)^{\perp}. \tag{208}$$

So cross products are easily converted into outer products and vice versa. Yay. This shows why the cross product is not associative even though the outer product is; the dual gets in the way. Since duality is just multiplication by $-\boldsymbol{I}$ and vectors and bivectors commute with $\boldsymbol{I}$, I can use Eq. (208) to write

$$ab = a \rfloor b + \boldsymbol{I} a \times b, \tag{209}$$

which is the three-dimensional analog of the product $W^{\dagger}Z$ of complex numbers W and Z. Another popular product in traditional vector algebra is the triple product $a \cdot b \times c$, which relates to geometric algebra by

$$a \cdot b \times c = (a \wedge b \wedge c)^{\perp}. \tag{210}$$

This form makes the cyclic property of the triple product obvious. The triple cross product $a \times (b \times c)$ is also pretty common, and it can be expressed in geometric algebra as

$$a \times (b \times c) = -a \rfloor (b \wedge c). \tag{211}$$

From here, it's easy to see that the BAC-CAB rule for expanding this product is really just a special case of Theorem 11.

I'd like to say a little more about cross products. Since $(a \wedge b)^{\perp} = a \rfloor b^{\perp}$ and in three dimensions $b^{\perp}$ is a bivector, it follows that $a \times b$ is also the inner product of a and a bivector. You may recall that in classical mechanics, linear and angular velocity are related by a cross product: $\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$. The geometric algebra equivalent is $v = \Omega \lfloor r$, where $\Omega = \boldsymbol{\omega}^{\perp}$ is an angular velocity bivector in the instantaneous plane of rotation. (Bivectors figure prominently in rotational dynamics, as I'll show in Section 10.1.) You may also recall that the magnetic part of the Lorentz force on a point charge is $\boldsymbol{F} = q\boldsymbol{v} \times \boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic field vector. In geometric algebra this becomes $F = qv \rfloor B$, where $B = \boldsymbol{B}^{\perp}$ is the magnetic field bivector. I'll show later on that the bivector representation of B is more physically motivated than the vector version.

A consequence of Theorem 26, which I mentioned at the time, is that in dimensions under four, every r-vector is actually an r-blade. In two dimensions that was obviously true; we had only scalars, vectors, and multiples of $\boldsymbol{I}$. In three dimensions the scalars, vectors, and trivectors are obviously blades (the trivectors are multiples of $\boldsymbol{I}$), and I can show using geometry that all bivectors are 2-blades. Consider two 2-blades $\boldsymbol{A_2}$ and $\boldsymbol{B_2}$; each represents a plane passing through the origin, and any two such planes in three dimensions share a common line. Therefore $\boldsymbol{A_2} = a \wedge b$ and $\boldsymbol{B_2} = a \wedge c$ where a is a vector along the line shared by the planes. This means that

$$\boldsymbol{A_2} + \boldsymbol{B_2} = a \wedge b + a \wedge c = a \wedge (b + c) \tag{212}$$

is also a 2-blade. Thus any bivector in three dimensions is a 2-blade, as Theorem 26 demands.

Now for the products. As before, we know what a scalar times anything or a vector times a vector looks like; next I'll do a vector times a bivector. Let a be a vector and $\boldsymbol{B}$ be a bivector; then $a = a_{\parallel} + a_{\perp}$ where $a_{\parallel}$ lies in the plane determined by $\boldsymbol{B}$ and $a_{\perp}$ is perpendicular to it. In that case there exists a vector b perpendicular to $a_{\parallel}$ such that $\boldsymbol{B} = a_{\parallel} b$, so

$$\begin{aligned} a\boldsymbol{B} &= (a_{\parallel} + a_{\perp}) a_{\parallel} b \\ &= a_{\parallel}^2 b + a_{\perp} a_{\parallel} b \\ &= a_{\parallel}^2 b + a_{\perp} \wedge a_{\parallel} \wedge b. \end{aligned} \tag{213}$$

So the product of a and $\boldsymbol{B}$ is the sum of two terms: a vector in the plane of $\boldsymbol{B}$ perpendicular to a, and the trivector defined by $\boldsymbol{B}$ and the component of a perpendicular to it. Clearly the vector is $a \rfloor \boldsymbol{B}$ and the trivector is $a \wedge \boldsymbol{B}$. The trivector can also be written

$$\pm\, |a_{\perp}||a_{\parallel}||b|\boldsymbol{I} = \pm |a_{\parallel}||\boldsymbol{B}|\boldsymbol{I} \tag{214}$$

where $|\,|$ is the magnitude of a multivector defined in Section 5.4. The $\pm$ is there because we don't know the orientation of the system defined by the three vectors.

Vector times trivector is even easier. If a is a vector and $\boldsymbol{T}$ is a trivector, then $\boldsymbol{T} = abc$ where b and c are perpendicular to each other and to a, so

$$a\boldsymbol{T} = a^2bc = a^2 b \wedge c. \tag{215}$$

So $a\boldsymbol{T}$ is a bivector representing the plane to which a is perpendicular. This is clearly $a \rfloor \boldsymbol{T}$.

The product of two bivectors looks like this: since all bivectors are 2-blades representing planes, let vector a lie along the direction shared by bivectors $\boldsymbol{A_2}$ and $\boldsymbol{B_2}$, so $\boldsymbol{A_2} = ba$ and $\boldsymbol{B_2} = ac$ where b and c are perpendicular to a but not necessarily to each other; then

$$\begin{aligned} \boldsymbol{A_2 B_2} &= baac \\ &= a^2 bc \\ &= a^2 b \rfloor c + a^2 b \wedge c. \end{aligned} \tag{216}$$

So the product of two bivectors is a scalar plus a bivector representing the plane normal to their intersection line. The first term is $\boldsymbol{A_2} \rfloor \boldsymbol{B_2} = \boldsymbol{A_2} \lfloor \boldsymbol{B_2}$ and the second term is $\boldsymbol{A_2} \times \boldsymbol{B_2}$.

Next, a bivector times a trivector: if bivector $\boldsymbol{B} = ab$ where a and b are perpendicular, then there exists vector c perpendicular to a and b such that trivector $\boldsymbol{T} = bac$, in which case

$$\boldsymbol{BT} = baabc = a^2 b^2 c, \tag{217}$$

so the product of a bivector and a trivector is a vector perpendicular to the plane of the bivector. This is also $\boldsymbol{B} \rfloor \boldsymbol{T}$.

The product of two trivectors is just a number. In fact, it's the product of the volumes defined by the two trivectors, with the sign determined by their relative orientations.

The general multiplication rule for the basis vectors can be written as

$$e_i e_j = \delta_{ij} + \sum_k \boldsymbol{I} \epsilon_{ijk} e_k, \tag{218}$$

which is exactly the multiplication rule for the Pauli matrices. Therefore the Pauli matrices are just a matrix representation of the basis vectors of three dimensional space. It is well known that the Pauli matrices form a Euclidean Clifford algebra, but the idea that they are literally matrix representations of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$ is not so familiar.

Finally, the even subalgebra of the geometric algebra on $\mathbb{E}^3$ has some surprises for us too. Let the unit bivectors be labeled

$$\boldsymbol{B_1} = e_2 e_3, \ \boldsymbol{B_2} = e_1 e_3, \ \text{and } \boldsymbol{B_3} = e_1 e_2. \tag{219}$$

Notice that this definition is not consistently right-handed because of $\boldsymbol{B_2}$. These objects satisfy the relations

$$\boldsymbol{B_1}^2 = \boldsymbol{B_2}^2 = \boldsymbol{B_3}^2 = -1 \tag{220}$$

and

$$\boldsymbol{B_1 B_2 B_3} = -1, \tag{221}$$

so the even subalgebra of the algebra on $\mathbb{E}^3$, which is spanned by 1 and the $\boldsymbol{B_i}$, is isomorphic to the quaternions. The quaternions were created to generalize the complex numbers to three dimensions, of course, so something like this was expected; but the quaternions as Hamilton conceived them were intended to correspond to the three unit directions, not three planes. The map between them works differently in two dimensions and three, so while complex numbers can be thought of consistently as either vectors or bivectors, quaternions can be mapped from one to the other only by introducing an inconsistency in the handedness, as I've done here.

7. More on projections, reflections, and rotations

In Sections 1.1 and 1.2 I introduced projections along vectors, reflections along vectors, and rotations in planes. My purpose was to get you interested in geometric algebra by showing how well it handled all three operations compared to traditional vector algebra. Well, there's more. It turns out that these operations can be defined on subspaces just as well as vectors; for example, rotating a subspace means rotating all the vectors in it. As I'll show, in geometric algebra this is very easy, and the resulting formulas are almost the same as the formulas for vectors.

7.1. Orthogonal projections and rejections

Let's restate what I did in Sections 1.1 and 1.2 a little differently. Let u and v be vectors; then the orthogonal projection of v along u is given by

$$P_u(v) = v \rfloor u u^{-1} = (v \rfloor u) \rfloor u^{-1} \tag{222}$$

and the orthogonal rejection of v from u is given by

$$R_u(v) = v \wedge u u^{-1} = v \wedge u \lfloor u^{-1}. \tag{223}$$

(The second parts of each equation are easy to verify.) $P_u(v)$ is parallel to u, $R_u(v)$ is orthogonal to u, and $P_u(v) + R_u(v) = v$. These operations require u to be invertible, so it can't be a null vector. I promised in Section 1.2 that this would have geometrical meaning, and now we're about to see what it is.

7.1.1. Projecting a vector into a subspace

Let's take a moment to consider the general notion of projection into a subspace. Let S be a subspace (S is not a blade this time; it really is the subspace itself) and let a be a vector not in S. Then for any $v \in S$ I can write $a = v + (a - v)$, which is the sum of a vector in S and a vector not in S. So which v is the "projection" of a into S? We can't say without further information. For example, consider two subspaces S_1 and S_2 that share only the zero vector; then if a vector lies in their direct sum, it can be expressed only one way as a vector from S_1 plus a vector from S_2, and thus has a unique projection into either subspace. Projection into a subspace is specified not only by the subspace itself but also by the subspace the rest of the vector will belong to, and the operation is well-defined only if the two subspaces share only the zero vector.

Now consider orthogonal projection as an example of this. The idea is to express a vector as a sum of two terms, one in subspace S and one in $S^\perp$, the orthogonal complement of S. This works only if S and $S^\perp$ have no nonzero vectors in common, which is true iff the inner product is nondegenerate on S. Thus orthogonal projection is well-defined only for a subspace with an invertible blade. In that case, I get this result.

Theorem 31. *If a is a vector and $\boldsymbol{A_r}$ is an invertible blade, then the orthogonal projection of a into and the orthogonal rejection of a from subspace $\boldsymbol{A_r}$ are given by*

$$\begin{aligned} P_{\boldsymbol{A_r}}(a) &= a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} = (a \rfloor \boldsymbol{A_r}) \rfloor \boldsymbol{A_r}^{-1} \\ R_{\boldsymbol{A_r}}(a) &= a \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1} = a \wedge \boldsymbol{A_r} \lfloor \boldsymbol{A_r}^{-1}. \end{aligned} \tag{224}$$

Proof. First, it's clear that $P_{\boldsymbol{A_r}}(a) + R_{\boldsymbol{A_r}}(a) = a$. Now, $a \rfloor \boldsymbol{A_r}$ is the dual of a by $\boldsymbol{A_r}^{-1}$; it is zero if a is orthogonal to $\boldsymbol{A_r}$, and otherwise it is an $r-1$-blade representing the subspace of $\boldsymbol{A_r}$ orthogonal to a. In that case its product with $\boldsymbol{A_r}^{-1}$ equals its inner product with $\boldsymbol{A_r}^{-1}$, which is just the dual by $\boldsymbol{A_r}$; the result is a vector that lies in $\boldsymbol{A_r}$. On the other hand, $a \wedge \boldsymbol{A_r}$ is zero if a lies in $\boldsymbol{A_r}$, and otherwise it is an $r+1$-blade that contains $\boldsymbol{A_r}$. In that case the product with $\boldsymbol{A_r}^{-1}$ equals the right inner product, and is the dual of $\boldsymbol{A_r}^{-1}$ by $(a \wedge \boldsymbol{A_r})^{-1}$, so the result is a vector orthogonal to $\boldsymbol{A_r}$. Both formulas give vectors, they sum to a, the first lies in $\boldsymbol{A_r}$ and vanishes iff a is orthogonal to $\boldsymbol{A_r}$, and the second is orthogonal to $\boldsymbol{A_r}$ and vanishes iff a lies in $\boldsymbol{A_r}$. Therefore the two expressions are obviously the orthogonal projection of a into and the orthogonal rejection of a from $\boldsymbol{A_r}$. □

So projecting into a subspace is the same as projecting onto a vector; you just replace the vector with the blade representing the subspace. We'll see several more examples of this idea below.

I can demonstrate directly that $P_{\boldsymbol{A_r}}(a)$ lies in $\boldsymbol{A_r}$:

$$P_{\boldsymbol{A_r}}(a) \wedge \boldsymbol{A_r} = (a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1}) \wedge \boldsymbol{A_r} = \left\langle a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \boldsymbol{A_r} \right\rangle_{r+1} = \langle a \rfloor \boldsymbol{A_r} \rangle_{r+1} = 0. \tag{225}$$

Similarly, I can show that $R_{\boldsymbol{A_r}}(a)$ is orthogonal to $\boldsymbol{A_r}$ as follows:

$$R_{\boldsymbol{A_r}}(a) \rfloor \boldsymbol{A_r} = (a \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1}) \rfloor \boldsymbol{A_r} = \left\langle a \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \boldsymbol{A_r} \right\rangle_{r-1} = \langle a \wedge \boldsymbol{A_r} \rangle_{r-1} = 0. \tag{226}$$

This result applies to the Gram-Schmidt process for producing an orthogonal set of vectors from a linearly independent set with the same span. Let $\{a_j\}_{j=1,\ldots,r}$ be linearly independent; then we build the orthogonal set $\{b_j\}_{j=1,\ldots,r}$ as follows. Let $b_1 = a_1$ to start with. Then b_2 equals a_2 minus its projection onto b_1, or equivalently the orthogonal rejection of a_2 from b_1. Next, b_3 equals the orthogonal rejection of a_3 from the span of b_1 and b_2, and so on through all of the a_j. Therefore we proceed as follows.

1. Let $b_1 = a_1$.
2. For each j starting with 1, let $\boldsymbol{B_j} = b_1 \wedge \cdots \wedge b_j$.
3. Then let $b_{j+1} = a_{j+1} \wedge \boldsymbol{B_j} \boldsymbol{B_j}^{-1}$.

This procedure will work only if each $\boldsymbol{B_j}$ is invertible, which is why it is normally used only in Euclidean spaces.

If $\boldsymbol{A_r}$ is a blade, then $\boldsymbol{A_r}^{\perp}$ represents the orthogonal complement of $\boldsymbol{A_r}$. That means that orthogonal projection into $\boldsymbol{A_r}^{\perp}$ should equal orthogonal rejection from $\boldsymbol{A_r}$. Using Eqs. (181) and (183), this is easy to show directly.

$$\begin{aligned} a \rfloor \boldsymbol{A_r}^{\perp} (\boldsymbol{A_r}^{\perp})^{-1} &= (a \wedge \boldsymbol{A_r})^{\perp} (\boldsymbol{A_r}^{\perp})^{-1} \\ &= a \wedge \boldsymbol{A_r} \boldsymbol{I}^{-1} \boldsymbol{I} \boldsymbol{A_r}^{-1} \\ &= a \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1}. \end{aligned} \tag{227}$$

If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, then the projection of a vector into their direct sum should be the sum of the projections into the subspaces individually. (For example, the projection of a vector into the Euclidean xy plane should be the sum of the projections onto the x and y axes separately.) This can also be shown directly. By Theorem 18, $\boldsymbol{A_r} \wedge \boldsymbol{B_s} = \boldsymbol{A_r} \boldsymbol{B_s}$, so using the first of Eqs. (74) I find

$$\begin{aligned} P_{\boldsymbol{A_r} \wedge \boldsymbol{B_s}}(a) &= P_{\boldsymbol{A_r} \boldsymbol{B_s}}(a) \\ &= a \rfloor (\boldsymbol{A_r} \boldsymbol{B_s})(\boldsymbol{A_r} \boldsymbol{B_s})^{-1} \\ &= [(a \rfloor \boldsymbol{A_r}) \boldsymbol{B_s} + (-1)^r \boldsymbol{A_r} (a \rfloor \boldsymbol{B_s})] \boldsymbol{B_s}^{-1} \boldsymbol{A_r}^{-1} \\ &= a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} + (-1)^r \boldsymbol{A_r} (a \rfloor \boldsymbol{B_s}) \boldsymbol{B_s}^{-1} \boldsymbol{A_r}^{-1}. \end{aligned} \tag{228}$$

Now let's work on that last term. If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, $\boldsymbol{A_r}$ and $a \rfloor \boldsymbol{B_s}$ are too, so their product is an outer product, so I can interchange them and pick up a factor of $(-1)^{r(s-1)}$. And since $\boldsymbol{A_r}^{-1}$ and $\boldsymbol{B_s}^{-1}$ are multiples of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$, their product is also an outer product, so I can interchange them and pick up a factor of $(-1)^{rs}$. Putting all this in Eq. (228),

$$\begin{aligned} P_{\boldsymbol{A_r} \wedge \boldsymbol{B_s}}(a) &= a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} + (-1)^{r + r(s-1) + rs} (a \rfloor \boldsymbol{B_s}) \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \boldsymbol{B_s}^{-1} \\ &= a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} + a \rfloor \boldsymbol{B_s} \boldsymbol{B_s}^{-1} \\ &= P_{\boldsymbol{A_r}}(a) + P_{\boldsymbol{B_s}}(a). \end{aligned} \tag{229}$$

7.1.2. Projecting a multivector into a subspace

Now that I can project a vector into a subspace, how about projecting one subspace into another? As I suggested above, this seems simple enough: project subspace $\boldsymbol{B_s}$ into subspace $\boldsymbol{A_r}$ by taking every vector in $\boldsymbol{B_s}$, projecting it into $\boldsymbol{A_r}$, and seeing what subspace you get. The rejection should be similar: just reject all the vectors individually. However, if I am a bit more precise, I discover a wrinkle. I define

$$\begin{aligned} P_{\boldsymbol{A_r}}(b_1 \wedge \cdots \wedge b_s) &:= P_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge P_{\boldsymbol{A_r}}(b_s) \\ R_{\boldsymbol{A_r}}(b_1 \wedge \cdots \wedge b_s) &:= R_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge R_{\boldsymbol{A_r}}(b_s). \end{aligned} \tag{230}$$

Now suppose the set $\{b_j\}_{j=1,\cdots,s}$ is linearly independent but their projections are not. That would happen necessarily if, for example, I projected a plane into a line. In that case, the projection defined this way vanishes. Instead of objecting to this wrinkle, I decide that it provides useful extra information. If by chance the projections of the members of $\boldsymbol{B_s}$ do not form an s-dimensional space, so be it; I accept that the projection is zero.

These formulas make geometric sense, but they aren't very easy to use. However, they can be made simpler, and extended to all multivectors to boot. Here's how.

Theorem 32. *For any invertible blade $\boldsymbol{A_r}$ and vectors $\{b_j\}_{j=1,\ldots,s}$,*

$$\begin{aligned} P_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge P_{\boldsymbol{A_r}}(b_s) &= (b_1 \wedge \cdots \wedge b_s) \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \\ R_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge R_{\boldsymbol{A_r}}(b_s) &= (b_1 \wedge \cdots \wedge b_s) \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1}. \end{aligned} \tag{231}$$

Proof. I start with the first equation. Since each $b_j = P_{\boldsymbol{A_r}}(b_j) + R_{\boldsymbol{A_r}}(b_j)$, the outer product $b_1 \wedge \cdots \wedge b_s$ can be written as a sum of terms, one of which equals $P_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge P_{\boldsymbol{A_r}}(b_s)$ while each of the others contains at least one $R_{\boldsymbol{A_r}}(b_j)$. Consider what happens to each term when you take the inner product with $\boldsymbol{A_r}$ and multiply by $\boldsymbol{A_r}^{-1}$. The term $P_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge P_{\boldsymbol{A_r}}(b_s)$ lies inside $\boldsymbol{A_r}$, so by Theorem 17 the inner product becomes a product, so the $\boldsymbol{A_r}$ and $\boldsymbol{A_r}^{-1}$ cancel out and you're left with $P_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge P_{\boldsymbol{A_r}}(b_s)$. On the other hand, each of the other terms contains a factor orthogonal to $\boldsymbol{A_r}$, so the inner product with $\boldsymbol{A_r}$ vanishes. Thus the first equation is valid.

For the second equation, I again write $b_1 \wedge \cdots \wedge b_s$ as a sum of terms, but this time I note that one of them equals $R_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge R_{\boldsymbol{A_r}}(b_s)$ while each of the others contains at least one $P_{\boldsymbol{A_r}}(b_j)$. Consider what happens to each term when you take the outer product with $\boldsymbol{A_r}$ and multiply by $\boldsymbol{A_r}^{-1}$. The term $R_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge R_{\boldsymbol{A_r}}(b_s)$ is orthogonal to $\boldsymbol{A_r}$, so by Theorem 18 the outer product becomes a product, so the $\boldsymbol{A_r}$ and $\boldsymbol{A_r}^{-1}$ cancel out and you're left with $R_{\boldsymbol{A_r}}(b_1) \wedge \cdots \wedge R_{\boldsymbol{A_r}}(b_s)$. On the other hand, each of the other terms contains a factor that lies in $\boldsymbol{A_r}$, so the outer product with $\boldsymbol{A_r}$ vanishes. Thus the second equation is valid too. □

Therefore $P_{\boldsymbol{A_r}}(\boldsymbol{B_s}) = \boldsymbol{B_s} \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1}$ and $R_{\boldsymbol{A_r}}(\boldsymbol{B_s}) = \boldsymbol{B_s} \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1}$ for any blade $\boldsymbol{B_s}$. Taking the obvious step, I define the orthogonal projection and rejection of any multivector to be

$$\begin{aligned} P_{\boldsymbol{A_r}}(B) &:= B \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} = (B \rfloor \boldsymbol{A_r}) \rfloor \boldsymbol{A_r}^{-1} \\ R_{\boldsymbol{A_r}}(B) &:= B \wedge \boldsymbol{A_r} \boldsymbol{A_r}^{-1} = B \wedge \boldsymbol{A_r} \lfloor \boldsymbol{A_r}^{-1}. \end{aligned} \tag{232}$$

You might be surprised that both projection and rejection leave scalars untouched:

$$P_{\boldsymbol{A_r}}(\lambda) = R_{\boldsymbol{A_r}}(\lambda) = \lambda. \tag{233}$$

This had to happen for reasons I'll explain in Section 9.6. Projecting into and rejecting from $\boldsymbol{I}$ do what you think they should (except for that odd bit with scalars):

$$\begin{aligned} P_{\boldsymbol{I}}(B) &= B \\ R_{\boldsymbol{I}}(B) &= \langle B \rangle . \end{aligned} \tag{234}$$

Running it the other way around, here's what happens when you project and reject $\boldsymbol{I}$:

$$\begin{aligned} P_{\boldsymbol{A_r}}(\boldsymbol{I}) &= \boldsymbol{I}\,\delta_{rn} \\ R_{\boldsymbol{A_r}}(\boldsymbol{I}) &= \boldsymbol{I}\,\delta_{r0}. \end{aligned} \tag{235}$$

Again, this makes sense; only the whole space is big enough to project $\boldsymbol{I}$ into, and only zero-dimensional spaces are small enough to reject $\boldsymbol{I}$ from.

With a little relabeling and rearranging, the first parts of Eqs. (232) become

$$\begin{aligned} A \rfloor \boldsymbol{B_s} &= P_{\boldsymbol{B_s}}(A)\boldsymbol{B_s} \\ A \wedge \boldsymbol{B_s} &= R_{\boldsymbol{B_s}}(A)\boldsymbol{B_s}. \end{aligned} \tag{236}$$

This shows that the inner or outer product of a multivector and a blade can also be expressed as a geometric product, as long as the blade is invertible so projection is defined. Using Theorem 21, this also shows that the norm squared of $A \rfloor \boldsymbol{B_s}$ equals the norm squared of $P_{\boldsymbol{B_s}}(A)$ times the norm squared of $\boldsymbol{B_s}$, and a similar result holds for the outer product.

Comparing Eqs. (222) and (223) with Eqs. (232), the level of generality achieved is astounding. Starting with the projection of one vector along another, I've shown that the projection of any multivector into a subspace is meaningful and is given by the *same expression*, with the multivector and blade put in place of the two vectors. The rejection of one vector from another follows the same pattern. It is true that we've lost one property: we no longer have $P_{\boldsymbol{A_r}}(B) + R_{\boldsymbol{A_r}}(B) = B$ in general. This makes geometric sense, however, if you look at the proof of Theorem 32: neither a projected blade nor a rejected blade includes all the terms that are partly projected and partly rejected, so to speak.

7.2. Reflections

To start, I'll review reflections from Section 1.2. I defined the reflection of vector v along axis n as follows: the projection of v along n gets a minus sign, while the rejection of v from n is unchanged. If the reflection is denoted v', then

$$v' = -nvn^{-1}. \tag{237}$$

What I didn't show in Section 1.2 is that reflections preserve inner products, which I'll show now. Using Eq. (237), the definition of the inner product, and the cyclic property of the scalar part of a product,

$$\begin{aligned} a' \rfloor b' &= \langle a'b' \rangle \\ &= \left\langle nan^{-1}nbn^{-1} \right\rangle \\ &= \left\langle abn^{-1}n \right\rangle \\ &= \langle ab \rangle \\ &= a \rfloor b. \end{aligned} \tag{238}$$

7.2.1. Reflecting a vector in a subspace

Just as I used projection and rejection along an axis to define reflection along an axis, I can use projection and rejection in a subspace to define reflection in a subspace. The reflection of a in $\boldsymbol{A_r}$ is constructed by giving the projection of a into $\boldsymbol{A_r}$ a minus sign and leaving the rejection of a from $\boldsymbol{A_r}$ alone. Using Theorem 31, Eq. (141), and Eq. (143), I find

$$\begin{aligned} a' &:= -P_{\boldsymbol{A_r}}(a) + R_{\boldsymbol{A_r}}(a) \\ &= -a \rfloor \boldsymbol{A_r}\boldsymbol{A_r}^{-1} + a \wedge \boldsymbol{A_r}\boldsymbol{A_r}^{-1} \\ &= -(-1)^{r-1}\boldsymbol{A_r} \lfloor a\boldsymbol{A_r}^{-1} + (-1)^r\boldsymbol{A_r} \wedge a\boldsymbol{A_r}^{-1} \\ &= (-1)^r\boldsymbol{A_r}a\boldsymbol{A_r}^{-1} \\ &= \boldsymbol{A_r}a^{*r}\boldsymbol{A_r}^{-1}. \end{aligned} \tag{239}$$

In the last line, a^{*r} means a is grade involuted r times; I introduced the notation back in Section 5.1. (You may wonder why I did this instead of just leaving in the $(-1)^r$. It will make sense in the next section.) Another way to arrive at this formula is to write $\boldsymbol{A_r} = a_1 a_2 \cdots a_r$ and reflect a along each of the a_j in succession. Once again, an expression in terms of vectors generalizes to subspaces with only minimal change. Reflections in subspaces also preserve inner products; the proof is very similar to Eqs. (238).

7.2.2. Reflecting a multivector in a subspace

Now that I can reflect vectors, I can reflect subspaces too: the reflection of subspace $\boldsymbol{B_s}$ in subspace $\boldsymbol{A_r}$ is found by taking every vector from $\boldsymbol{B_s}$, reflecting it in $\boldsymbol{A_r}$, and seeing what subspace you get. That would mean something like

$$\begin{aligned}(b_1 \wedge \cdots \wedge b_s)' &:= b_1' \wedge \cdots \wedge b_s' \\ &= \left(\boldsymbol{A_r} b_1^{*r} \boldsymbol{A_r}^{-1}\right) \wedge \cdots \wedge \left(\boldsymbol{A_r}^{-1} b_s^{*r} \boldsymbol{A_r}^{-1}\right). \end{aligned} \tag{240}$$

Again, this is geometrically sensible but not easy to use. Fear not; I can fix that. To start with, notice what Eq. (239) shows: conjugating a vector by an invertible r-blade gives you back a vector. A more general version of that is also true.

Theorem 33. *If A is a versor and B_s is an s-vector, then*

$$AB_sA^\dagger = \left\langle AB_sA^\dagger \right\rangle_s, \tag{241}$$

so conjugation by an invertible versor is grade preserving.

Proof. The theorem is true for versors if it's true for vectors, so I'll look at aB_sa. Using Eqs. (64) and (66), I can write

$$aB_sa = (a \rfloor B_s) \lfloor a + (a \rfloor B_s) \wedge a + (a \wedge B_s) \lfloor a + a \wedge B_s \wedge a. \tag{242}$$

The first term is grade $s-2$, the middle two terms are grade s, and the last term is grade $s+2$, so I'm done if I can ditch the first and last terms. The last term vanishes because a appears twice in the outer product (compare Eq. (86)), and the first term vanishes because it can be rewritten as $(-1)^{s-1}(B_s \lfloor a) \lfloor a = (-1)^{s-1} B_s \lfloor (a \wedge a) = 0$. Since Theorem 22 tells me that the inverse of a versor, if it has one, is its reverse divided by its norm squared, conjugation by an invertible versor preserves grade too. ☐

I'll use this to get the result I really want.

Theorem 34. *If A is a versor, then*

$$(ABA^\dagger) \wedge (ACA^\dagger) = |A|^2 \, A(B \wedge C)A^\dagger. \tag{243}$$

Therefore if A is invertible, $(ABA^{-1}) \wedge (ACA^{-1}) = A(B \wedge C)A^{-1}$.

Proof. The result is true for general B and C if it's true for B_s and C_t, and I've already shown that versor conjugation preserves grades, so

$$\begin{aligned}(AB_sA^\dagger) \wedge (AC_tA^\dagger) &= \left\langle AB_sA^\dagger AC_tA^\dagger \right\rangle_{s+t} \\ &= |A|^2 \left\langle AB_sC_tA^\dagger \right\rangle_{s+t} \\ &= |A|^2 \, A \left\langle B_sC_t \right\rangle_{s+t} A^\dagger \\ &= |A|^2 \, A(B_s \wedge C_t)A^\dagger. \end{aligned} \tag{244}$$

If A is invertible, then $|A|^2 \neq 0$, so dividing both sides by $|A|^4$ yields the desired result. ☐

Now for reflections. If $\boldsymbol{B_s} = b_1 \wedge \cdots \wedge b_s$, then

$$\begin{aligned}
\boldsymbol{B}'_{\boldsymbol{s}} &:= b'_1 \wedge \cdots \wedge b'_s \\
&= \left(\boldsymbol{A_r} b_1^{*r} \boldsymbol{A_r}^{-1}\right) \wedge \cdots \wedge \left(\boldsymbol{A_r} b_s^{*r} \boldsymbol{A_r}^{-1}\right) \\
&= \boldsymbol{A_r}(b_1^{*r} \wedge \cdots \wedge b_s^{*r})\boldsymbol{A_r}^{-1} \\
&= \boldsymbol{A_r}(b_1 \wedge \cdots \wedge b_s)^{*r}\boldsymbol{A_r}^{-1} \\
&= \boldsymbol{A_r}\boldsymbol{B_s}^{*r}\boldsymbol{A_r}^{-1}.
\end{aligned} \tag{245}$$

Taking the obvious next step, I define the reflection of multivector B in subspace $\boldsymbol{A_r}$ to be

$$B' := \boldsymbol{A_r} B^{*r} \boldsymbol{A_r}^{-1}. \tag{246}$$

So reflection in $\boldsymbol{A_r}$ is done by grade involuting r times and then conjugating by $\boldsymbol{A_r}$. This is a little more complicated than the reflection of a vector along an axis that we started with, Eq. (237), but not much. And of course it reduces to Eq. (237) when $\boldsymbol{A_r}$ and B are vectors.

The reflection of $\boldsymbol{I}$ in a subspace is

$$\begin{aligned}
\boldsymbol{I}' &= \boldsymbol{A_r}\boldsymbol{I}^{*r}\boldsymbol{A_r}^{-1} \\
&= (-1)^{nr}\boldsymbol{A_r}\boldsymbol{I}\boldsymbol{A_r}^{-1} \\
&= (-1)^{nr}(-1)^{r(n-1)}\boldsymbol{I}\boldsymbol{A_r}\boldsymbol{A_r}^{-1} \\
&= (-1)^r\boldsymbol{I}.
\end{aligned} \tag{247}$$

This makes sense because r directions in the space were reflected. So the orientation changes iff r is odd.

You may have noticed that I now have two ways to reflect a vector around the origin. The first is grade involution, and the second is to reflect the vector in a volume element. Since both operations have been extended to the whole algebra in a way that respects products, they ought to be equal not just for vectors but for any multivector, or

$$A^* = \boldsymbol{I} A^{*n} \boldsymbol{I}^{-1}. \tag{248}$$

To show that this really is true, start with Eq. (178), grade involute both sides, and use $\boldsymbol{I}^* = (-1)^n\boldsymbol{I}$. Then multiply both sides by $\boldsymbol{I}^{-1}$ on the right and voilà.

Finally, I can relate reflection in $\boldsymbol{A_r}$ and reflection in $\boldsymbol{A_r}^\perp$. Reflection of vector a in $\boldsymbol{A_r}$ gives the component in $\boldsymbol{A_r}$ a minus sign and leaves the component in $\boldsymbol{A_r}^\perp$ alone, while reflection in $\boldsymbol{A_r}^\perp$ does the opposite. Therefore one reflection should be the negative of the other, or

$$\boldsymbol{A_r}^\perp a^{*(n-r)}(\boldsymbol{A_r}^\perp)^{-1} = -\boldsymbol{A_r} a^{*r}\boldsymbol{A_r}^{-1}. \tag{249}$$

Extending this to general multivectors, I expect

$$\boldsymbol{A_r}^\perp B^{*(n-r)}(\boldsymbol{A_r}^\perp)^{-1} = (\boldsymbol{A_r} B^{*r}\boldsymbol{A_r}^{-1})^*. \tag{250}$$

And indeed that's what I find:

$$\begin{aligned}
\boldsymbol{A_r}^\perp B_s^{*(n-r)}(\boldsymbol{A_r}^\perp)^{-1} &= (-1)^{s(n-r)}\boldsymbol{A_r}\boldsymbol{I}^{-1}B_s(\boldsymbol{A_r}\boldsymbol{I}^{-1})^{-1} \\
&= (-1)^{s(n-r)}(-1)^{s(n-1)}\boldsymbol{A_r}B_s\boldsymbol{I}^{-1}\boldsymbol{I}\boldsymbol{A_r}^{-1} \\
&= (-1)^s(-1)^{rs}\boldsymbol{A_r}B_s\boldsymbol{A_r}^{-1} \\
&= (\boldsymbol{A_r}B_s^{*r}\boldsymbol{A_r}^{-1})^*.
\end{aligned} \tag{251}$$

7.3. Rotations

After all this work, rotations are fairly anticlimactic. Once again, I start with a review of Section 1.2. I showed there that a rotation in a plane is the product of two reflections along vectors in that plane, so

$$v' = RvR^{-1} \tag{252}$$

where R is the product of two invertible vectors, also called a biversor or a rotor. A rotation clearly preserves inner products since it's just two reflections in succession, but you can show it directly by an argument very much like Eqs. (238).

7.3.1. *Rotating a multivector in a plane*

The rotation of a subspace is as simple to understand as the reflection. In fact, it's the example I started this whole section with: you rotate a subspace by rotating all the vectors in it. The argument is identical to the argument for reflections: if $\boldsymbol{A_r} = a_1 \wedge \cdots \wedge a_r$, then $\boldsymbol{A'_r} = a'_1 \wedge \cdots \wedge a'_r$. Therefore the rotation by R is

$$\begin{aligned}\boldsymbol{A'_r} &:= a'_1 \wedge \cdots \wedge a'_r \\ &= \left(Ra_1R^{-1}\right) \wedge \cdots \wedge \left(Ra_rR^{-1}\right) \\ &= R(a_1 \wedge \cdots \wedge a_r)R^{-1} \\ &= R\boldsymbol{A_r}R^{-1}.\end{aligned} \tag{253}$$

The grade inversion of $\boldsymbol{A_r}$ is absent because it is performed twice, once for each factor in the rotor. Therefore the rule for rotating any multivector is

$$A' := RAR^{-1}, \tag{254}$$

which is exactly the same as the formula for vectors.

Since $\boldsymbol{I}$ commutes with even multivectors (Eq. (178)), rotations leave $\boldsymbol{I}$ alone,

$$R\boldsymbol{I}R^{-1} = \boldsymbol{I}, \tag{255}$$

as expected.

When I first discussed rotations in Section 1.2, I said that any two axes in the same plane separated by the same angle would generate the same rotation. That means that if I take the two vectors in R and rotate them the same amount in the plane of R, the resulting rotor should perform the same rotation. Therefore, if R and S are rotors in the same plane, SRS^{-1} should represent the same rotation as R. You can show this directly: R and S are both scalars plus multiples of the same area element, so they commute. Therefore $SRS^{-1} = RSS^{-1} = R$.

Every linear transformation of vectors can be extended to the entire geometric algebra; I'll describe that process later. These three transformations extend in a particularly compact way, but not all transformations do. Rotations and reflections behave as well as they do because they are orthogonal transformations, and geometric algebra is particularly well-suited to represent them. In fact, it's a good idea to pause and notice just how good a job it does; compare Eq. (254) to the increasingly complicated expressions you get when you rotate tensors of ever-increasing rank. One of the great strengths of geometric algebra is its ability to extend orthogonal transformations to the whole algebra in such a simple fashion.

7.3.2. *Rotations in three dimensions*

In a real three dimensional space, rotations have an interesting property that is easy to understand using geometric algebra: the product of two rotations is another rotation. If R_1 represents the first rotation and R_2 the second, then their product is $R = R_2R_1$. We lose no generality by demanding that both R_1 and R_2 are unit rotors; and that means R is a unit even versor. In three dimensions the only even grades are zero and two, so R is actually a scalar plus a bivector: $R = \langle R\rangle + \langle R\rangle_2$. Therefore $|R|^2 = 1$ becomes $\langle R\rangle^2 + |\langle R\rangle_2|^2 = 1$. That tells me that $R = \cos(\theta/2) - B\sin(\theta/2)$ for some θ and unit bivector B. And in three dimensions every bivector is a 2-blade, so $\boldsymbol{B}$ represents some plane, and thus $R = \exp(-\boldsymbol{B}\theta/2)$, which is a rotation through θ in plane $\boldsymbol{B}$. As soon as I climb the ladder to four dimensions, though, I lose this result, because R_2R_1 could have a 4-vector part.

8. Frames and bases

Now I'll consider a geometric algebra $\mathcal{G}^n$ in which the space of vectors has finite dimension n. Let $\{a_i\}_{i=1,\ldots,n}$ be a basis for the vector space, which I will also call a *frame*. (The a_i are not assumed orthogonal.) Then a generic element of the algebra will be the sum of a scalar and terms of the form $a_{i_1}a_{i_2}\cdots a_{i_r}$ for $r \leq n$. Theorem 12 tells me that any such element is a linear combination of blades made up of the $\{a_{i_j}\}$; therefore the scalar 1 and the blades $a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r}$ generate the whole geometric algebra. I'll now show that they actually form a basis, and I'll also show how to calculate the components of an arbitrary multivector in this basis.

8.1. Reciprocal frames

Given a frame $\{a_i\}_{i=1,\ldots,n}$, another frame $\{a^i\}_{i=1,\ldots,n}$ is called a *reciprocal frame* to the first if it satisfies

$$a^i \rfloor a_j = \delta^i_j. \tag{256}$$

If such a set of vectors exists, it is a frame because Eq. (256) guarantees that the a^i are linearly independent, so they form a basis. To construct such vectors, consider their definition: a^j should be orthogonal to all of the a_i except for a_j, so an obvious way to make it is to take the outer product of all of the a_i except for a_j and then take its dual, which is what I'll do.

Let $a_N = a_1 \wedge a_2 \wedge \cdots \wedge a_n$; then a_N is a (possibly unnormalized) volume element. (Even though a_N is a blade, I'm not denoting it with capital letters or boldface; you'll see why in the next section.) Then I define

$$a^i := (-1)^{i-1}(a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_n)a_N^{-1}. \tag{257}$$

$\{a^i\}$ is a reciprocal frame because, using the first of Eqs. (181),

$$\begin{aligned} a_i \rfloor a^j &= (-1)^{j-1} a_i \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_n a_N^{-1}) \\ &= (-1)^{j-1}(a_i \wedge a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_n)a_N^{-1} \end{aligned} \tag{258}$$

Now if $i \neq j$ then a_i equals one of the other vectors in the outer product, so the whole thing vanishes. If $i = j$, I move a_i past the first $i-1$ vectors to its original spot, which cancels out the $(-1)^{j-1}$ prefactor. Therefore

$$\begin{aligned} a_i \rfloor a^j &= (a_1 \wedge \cdots \wedge a_n)a_N^{-1}\delta^j_i \\ &= \delta^j_i. \end{aligned} \tag{259}$$

This definition exactly expresses the geometrical idea I started with; a_N was chosen to perform the duality transform because it gets the normalization right.

Since both $\{a_i\}$ and $\{a^j\}$ are bases for the vectors, any vector v can be written $v = \sum v^i a_i$ or $v = \sum v_j a^j$. In fact, it's obvious that $v^i = v \rfloor a^i$ and $v_j = v \rfloor a_j$, so the components of v on either basis are easily calculated using the other basis. Using the definition of a^i and the first of Eqs. (181) again, I find that

$$\begin{aligned} v^i &= v \rfloor a^i \\ &= (-1)^{i-1} v \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_n a_N^{-1}) \\ &= (-1)^{i-1}(v \wedge a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_n)a_N^{-1} \\ &= (a_1 \wedge \cdots \wedge a_{i-1} \wedge v \wedge a_{i+1} \wedge \cdots \wedge a_n)a_N^{-1}. \end{aligned} \tag{260}$$

Compare this with Eqs. (16) and (17) back in Section 1.2.

Since v is a vector, the expressions for its components can be written $v^i = v * a^i$ and $v_j = v * a_j$, where $*$ is the scalar product. These forms for the components can be generalized a long way, as I'll show in the next section.

The inner product of any two vectors follows easily from their components:

$$b \rfloor c = \sum_{i,j} b_i c^j (a^i \rfloor a_j) = \sum_i b_i c^i, \tag{261}$$

and switching the frames on which I expand b and c gives me an equally valid result in terms of the components b^i and c_i.

A frame and its reciprocal satisfy a useful identity.

Theorem 35.

$$\sum_i a_i\, a^i = \sum_i a^i\, a_i = n. \tag{262}$$

Proof.

$$\sum_i a_i\, a^i = \sum_i a_i \rfloor a^i + \sum_i a_i \wedge a^i$$
$$= n + \sum_i a_i \wedge a^i. \tag{263}$$

To evaluate the second term, expand a^i on the original frame to get $a^i = \sum_j (a^i \rfloor a^j) a_j$, so

$$\sum_i a_i \wedge a^i = \sum_i a_i \wedge \left(\sum_j a^i \rfloor a^j a_j \right)$$
$$= \sum_{i,j} (a_i \wedge a_j)(a^i \rfloor a^j)$$
$$= 0 \tag{264}$$

because $a^i \rfloor a^j$ is symmetric in i and j while $a_i \wedge a_j$ is antisymmetric. The proof that $\sum_i a^i\, a_i = n$ is the same except for exchanging superscripts and subscripts. □

8.2. Multivector bases

Before I continue, I need some fancy new notation. Let I be a string of indices $i_1, i_2, \ldots, i_r$, and given a string I let a_I be defined by

$$a_I := a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r}, \tag{265}$$

and similarly for a^I. I will use the symbol N only to refer to the string $1, 2, \ldots, n$, to be consistent with a_N in the previous section. I also allow I to be the "empty" sequence, in which case I define $a_I = a^I = 1$. Then I immediately know several things:

1. $a_I = a^I = 0$ iff the string I contains at least one index twice.
2. If I and J contain the same elements but in a different order, then $a_I = (\operatorname{sgn}\sigma) a_J$ and $a^I = (\operatorname{sgn}\sigma) a^J$, where σ is the permutation that changes I to J.
3. Theorem 12 tells me that given a frame $\{a_i\}$ for the vectors, the set $\{a_I\}$ (or $\{a^I\}$) where I ranges over all *increasing* sequences generates $\mathcal{G}^n$. (I is an increasing sequence if $i_1 < i_2 < \cdots < i_r$.)

To show either set forms a basis, I'll use this result.

Theorem 36.

$$a^I * a_J = \delta^I_J \tag{266}$$

where δ^I_J vanishes if either I or J repeats indices or if I is not a permutation of J (including having a different length), and otherwise it equals the sign of the permutation that takes I to J.

Proof. If either string repeats indices then both sides vanish, and both sides also vanish when the lengths of I and J are different (the right side by definition, the left side because a^I and a_J have different grades); to get the other results, I let $I = i_1 < i_2 < \cdots < i_r$ and $J = j_1 < j_2 < \cdots < j_r$ and use Eq. (156) to find

$$a^I * a_J = \big\langle (a^{i_1} \wedge a^{i_2} \wedge \cdots \wedge a^{i_r})^\dagger (a_{j_1} \wedge a_{j_2} \wedge \cdots \wedge a_{j_r}) \big\rangle$$
$$= \big\langle (a^{i_r} \wedge \cdots \wedge a^{i_2} \wedge a^{i_1})(a_{j_1} \wedge a_{j_2} \wedge \cdots \wedge a_{j_r}) \big\rangle$$
$$= (a^{i_r} \wedge \cdots \wedge a^{i_2} \wedge a^{i_1}) \rfloor (a_{j_1} \wedge a_{j_2} \wedge \cdots \wedge a_{j_r}). \tag{267}$$

Now consider the case where the i_k equal the j_k; using the third of Eqs. (83) and Eq. (77),

$$(a^{i_r} \wedge \cdots \wedge a^{i_2} \wedge a^{i_1}) * (a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r})$$

$$
\begin{aligned}
&= (a^{i_r} \wedge \cdots \wedge a^{i_2}) \rfloor \left[a^{i_1} \rfloor (a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r})\right] \\
&= (a^{i_r} \wedge \cdots \wedge a^{i_2}) \rfloor \left[\sum_{j=1}^{r} (-1)^{j-1} a^{i_1} \rfloor a_{i_j}\, a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge \check{a}_{i_j} \wedge \cdots \wedge a_{i_r}\right] \\
&= (a^{i_r} \wedge \cdots \wedge a^{i_2}) \rfloor (a_{i_2} \wedge \cdots \wedge a_{i_r}), \qquad (268)
\end{aligned}
$$

which can be repeated for the i_2 term and for each successive term until the final result

$$
(a^{i_r} \wedge \cdots \wedge a^{i_2} \wedge a^{i_1}) * (a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r}) = 1 \qquad (269)
$$

is reached. Now suppose that i_k equals none of the j_l; then when the evaluation of the scalar product as shown above reaches the kth iteration, all of the $a^{i_k} \rfloor a_{j_l}$ terms will vanish, and so will the scalar product. This establishes the result for I and J increasing; the general result follows from the properties of a^I and a_J under rearrangement of elements. □

From this result it's pretty obvious that for any multivector A,

$$
A = \sum_I A^I a_I \quad \text{where} \quad A^I = A * a^I \qquad (270)
$$

and the sum extends over all increasing sequences I (including the null sequence to pick up the scalar part). Therefore, given a frame $\{a_i\}$, the elements a_I form a true basis for the geometric algebra, and the equation above shows how to expand any multivector on this basis. (Incidentally, the roles of the frame $\{a_i\}$ and the reciprocal frame $\{a^j\}$ can be exchanged in this expansion, just as vectors can be expanded on either set with the other used to compute the coefficients.) Since the number of distinct r-blades in a basis for each r is $\binom{n}{r}$, it follows that

$$
\dim \mathcal{G}^n = \sum_{r=0}^{n} \dim \mathcal{G}^n_r = \sum_{r=0}^{n} \binom{n}{r} = 2^n. \qquad (271)
$$

I can also express the scalar product of any two multivectors B and C in terms of their components:

$$
B * C = \sum_{I,J} B_I C^J (a^I * a_J) = \sum_I B_I C^I, \qquad (272)
$$

and switching the bases on which I expand B and C gives me an equally valid result in terms of the components B^I and C_I.

A consequence of all this is the following theorem.

Theorem 37. *Multivector A is uniquely determined by either of the following.*

1. *$A * B$ for every multivector B.*
2. *$\langle A \rangle$ and $a \rfloor A$ for every vector a.*

Proof. Part 1 is obvious. In fact, it's overkill; $A * B$ for all B in a basis for the algebra will do. By the distributive property, part 2 is equivalent to this statement: if $\langle A \rangle = 0$ and all $a \rfloor A = 0$, then $A = 0$. So that's what I'll prove. To do this, I assume A is an r-vector A_r; if the result is true for r-vectors then it's true for general multivectors too. If $r = 0$ or 1 then I'm done, so let $r > 1$. Let $\{a_i\}$ be a frame and $\{a^i\}$ its reciprocal frame; then a component of A_r on the basis defined by $\{a_i\}$ is $(a^{i_r} \wedge \cdots \wedge a^{i_2} \wedge a^{i_1}) \rfloor A_r$ for some strictly ascending choice of i_1 through i_r. However,

$$
\begin{aligned}
(a^{i_r} \wedge \cdots \wedge a^{i_2} \wedge a^{i_1}) \rfloor A_r &= (a^{i_r} \wedge \cdots \wedge a^{i_2}) \rfloor (a^{i_1} \rfloor A_r) \\
&= 0 \qquad (273)
\end{aligned}
$$

since $a^{i_1} \rfloor A_r = 0$. So all the components of A_r vanish, so $A_r = 0$.

This proves the theorem in every finite-dimensional algebra, but it's usually true in infinite-dimensional spaces too. In fact, extra structures are usually imposed on infinite-dimensional spaces for exactly this purpose, and I will happily assume henceforth that this has always been done. □

This result can be extended *ad nauseum*: A is uniquely determined by $\langle A\rangle$, $\langle A\rangle_1$, and $A_2 \rfloor A$ for every bivector A_2, and so on.

While any r-vector can be expanded using the frame r-vectors, it can also be expanded using only the frame vectors; but now the coefficients aren't necessarily scalars.

Theorem 38.

$$\sum_i a^i\, a_i \rfloor A_r = \sum_i a^i \wedge (a_i \rfloor A_r) = r A_r, \tag{274}$$

and the same is true if the frame and its reciprocal are interchanged.

Proof. For the first equality, note that

$$\begin{aligned}\sum_i a^i\, a_i \rfloor A_r &= \sum_i a^i \rfloor (a_i \rfloor A_r) + \sum_i a^i \wedge (a_i \rfloor A_r)\\ &= \sum_i (a^i \wedge a_i) \rfloor A_r + \sum_i a^i \wedge (a_i \rfloor A_r)\\ &= \left(\sum_i a^i \wedge a_i\right) \rfloor A_r + \sum_i a^i \wedge (a_i \rfloor A_r)\\ &= \sum_i a^i \wedge (a_i \rfloor A_r)\end{aligned} \tag{275}$$

because I showed in the proof of Theorem 35 that $\sum_i a^i \wedge a_i = 0$.

The second equality is true for all r-vectors if it's true for all members of a basis for r-vectors, so I have to prove it only on a basis; and I know just the basis to use. Let $A_r = a^{j_1} \wedge \cdots \wedge a^{j_r}$ for some increasing sequence of indices; then

$$\begin{aligned}\sum_i a^i \wedge (a_i \rfloor A_r) &= \sum_i a^i \wedge \left[a_i \rfloor (a^{j_1} \wedge \cdots \wedge a^{j_r})\right]\\ &= \sum_{i,k} (-1)^{k-1} (a_i \rfloor a^{j_k})\, a^i \wedge a^{j_1} \wedge \cdots \wedge \check{a}^{j_k} \wedge \cdots \wedge a^{j_r}\\ &= \sum_k (-1)^{k-1} a^{j_k} \wedge a^{j_1} \wedge \cdots \wedge \check{a}^{j_k} \wedge \cdots \wedge a^{j_r}.\end{aligned} \tag{276}$$

In each term of the sum, I move a^{j_k} past $k-1$ other vectors to return it to its original spot, which cancels the $(-1)^{k-1}$ factor, so

$$\begin{aligned}\sum_i a^i \wedge (a_i \rfloor A_r) &= \sum_k a^{j_1} \wedge \cdots \wedge a^{j_r}\\ &= r A_r.\end{aligned} \tag{277}$$

That completes the first half of the proof, and exchanging superscripts and subscripts provides the other half. □

Just as the original frame has a volume element a_N, the reciprocal frame has a volume element a^N defined in an analogous way: $a^N = a^1 \wedge \cdots \wedge a^n$. Now a_N and a^N have to be scalar multiples of each other, and since Theorem 36 shows that $a_N * a^N = 1$, I conclude that

$$a^N = \frac{a_N}{|a_N|^2}. \tag{278}$$

A quick calculation shows

$$|a^N|^2 = |a_N|^{-2}, \tag{279}$$

so for real algebras, the weights of the volume elements of a frame and its reciprocal are themselves reciprocals.

Given a frame $\{a_i\}$, I have Eq. (257) for the members a^i of the reciprocal frame, but I don't have an equally direct formula for the reciprocal multivectors a^I; all I can do right now is take the outer product of Eq. (257) several times. However, I can get a nicer formula for a^I using the same logic that got me the reciprocal vectors in the first place. By construction, a^i is orthogonal to the outer product of all the frame vectors except a_i, and similarly for a^j; therefore $a^i \wedge a^j$ is orthogonal to the outer product of all the frame vectors except a_i and a_j. Therefore $a^i \wedge a^j$ is dual to $a_1 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_n$. To make this easier to write out, for any string of indices I let me define I^c to be its ascending complement, so I^c includes exactly the indices not in I in ascending order. In these terms, a^I is dual to a_{I^c}. To be more precise, I have

Theorem 39. *If I represents ascending i_1 through i_r,*

$$a^I = (-1)^{\sum_{j=1}^{r}(i_j-1)} a_{I^c}\, a_N^{-1}. \tag{280}$$

Notice that this includes Eq. (257) as a special case when I has only one index i.

Proof. To prove this, I'll calculate $a^I a_N$.

$$\begin{aligned} a^I a_N &= a^I \rfloor a_N \\ &= (a^{i_1} \wedge \cdots \wedge a^{i_r}) \rfloor a_N \\ &= (a^{i_1} \wedge \cdots \wedge a^{i_{r-1}}) \rfloor (a^{i_r} \rfloor a_N). \end{aligned} \tag{281}$$

To calculate $a^{i_r} \rfloor a_N$ I use Eq. (77):

$$\begin{aligned} a^{i_r} \rfloor a_N &= a^{i_r} \rfloor (a_1 \wedge \cdots \wedge a_n) \\ &= \sum_j (-1)^{j-1} (a^{i_r} \rfloor a_j) a_1 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_n \\ &= (-1)^{i_r-1} a_1 \wedge \cdots \wedge \check{a}_{i_r} \wedge \cdots \wedge a_n \\ &= (-1)^{i_r-1} a_{i_r^c}. \end{aligned} \tag{282}$$

Now Eq. (281) becomes

$$\begin{aligned} a^I a_N &= (-1)^{i_r-1} (a^{i_1} \wedge \cdots \wedge a^{i_{r-1}}) \rfloor a_{i_r^c} \\ &= (-1)^{i_r-1} (a^{i_1} \wedge \cdots \wedge a^{i_{r-2}}) \rfloor (a^{i_{r-1}} \rfloor a_{i_r^c}). \end{aligned} \tag{283}$$

When I evaluate $a^{i_{r-1}} \rfloor a_{i_r^c}$ using Eq. (77) again, I remove $a_{i_{r-1}}$ and multiply by $(-1)^{i_{r-1}-1}$. (This is why I put the indices of I in ascending order; i_r is later than i_{r-1}, so $a_{i_{r-1}}$ is still in position i_{r-1} in $a_{i_r^c}$.) Thus each step removes a factor a_{i_j} from a_N and multiplies by $(-1)^{i_j-1}$, with the final result

$$a^I a_N = (-1)^{\sum_{j=1}^{r}(i_j-1)} a_{I^c}. \tag{284}$$

Now I'll take care of the special cases on each extreme: I is empty and $I = N$. When I is empty, Eq. (280) reduces to $a^I = a_N a_N^{-1} = 1$, which is correct, and when $I = N$, I^c is empty so $a_{I^c} = 1$, so Eq. (280) becomes

$$\begin{aligned} a^N &= (-1)^{\sum_{j=1}^{n}(j-1)} a_N^{-1} \\ &= (-1)^{n(n-1)/2} \frac{a_N}{a_N{}^2} \\ &= \frac{a_N}{|a_N|^2}, \end{aligned} \tag{285}$$

where I used Eq. (171) in the last step. Since this matches Eq. (278), it too is correct and the theorem is proved. □

To wrap up this part, let me consider the special case where $\{a_i\}$ is an orthonormal frame, which I'll denote $\{e_i\}$. Then the reciprocal frame is a lot easier to find: it's clear on inspection that $e^i = e_i^{-1} = e_i^{-2} e_i$ fits the bill. Since the frame vectors are normalized, $e_i^2 = \pm 1$, so $e^i = \pm e_i$. For any I, let I_m be the number of elements in the product for e_I that have negative square; then $e^I = (-1)^{I_m} e_I$, so the multivector basis and its reciprocal differ at most by signs. On such a basis, Eq. (272) for the scalar product becomes

$$B * C = \sum_I (-1)^{I_m} B_I \, C_I = \sum_I (-1)^{I_m} B^I C^I. \tag{286}$$

If the space of vectors is Euclidean, then $I_m = 0$ for any I, so the magnitude is positive definite, so the entire geometric algebra is also a Euclidean space under the scalar product. If the space of vectors is non-Euclidean, then the algebra has (very) mixed signature under the scalar product.

8.3. Orthogonal projections using frames

In traditional vector algebra, the orthogonal projection of a vector into a subspace is given as a sum of projections onto a basis for the subspace. Although we don't need to do that in geometric algebra, we still can. Let $\boldsymbol{A_r} = a_1 \wedge \cdots \wedge a_r$; then

$$\begin{aligned} P_{\boldsymbol{A_r}}(a) &= a \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \\ &= a \rfloor (a_1 \wedge \cdots \wedge a_r) \boldsymbol{A_r}^{-1} \\ &= \sum_{j=1}^{r} (-1)^{j-1} a \rfloor a_j (a_i \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_r) \boldsymbol{A_r}^{-1}. \end{aligned} \tag{287}$$

Now $\boldsymbol{A_r}$ is a volume element for its subspace, so comparing with Eq. (257) shows me that the vectors in the sum above are the reciprocal frame to $\{a_j\}$, or

$$P_{a_1 \wedge \cdots \wedge a_r}(a) = \sum_{j=1}^{r} (a \rfloor a_j) a^j. \tag{288}$$

Since the reciprocal frame volume element $\boldsymbol{A^r} = a^1 \wedge \cdots \wedge a^r$ equals $\boldsymbol{A_r}/|\boldsymbol{A_r}|^2$ (cf. Eq. (285)), projection using either volume element gives the same result; had I used $\boldsymbol{A^r}$, I'd have ended up with Eq. (288) with a_j and a^j interchanged.

I can do the same thing with any s-vector B_s. Let $\boldsymbol{A_r}$ be defined as before; then using Eq. (95) from Theorem 14 I find that

$$\begin{aligned} P_{\boldsymbol{A_r}}(B_s) &= B_s \rfloor \boldsymbol{A_r} \boldsymbol{A_r}^{-1} \\ &= B_s \rfloor (a_1 \wedge \cdots \wedge a_r) \boldsymbol{A_r}^{-1} \\ &= \sum (-1)^{\sum_{j=1}^{s} (i_j - j)} (B_s \rfloor a_{i_1} \wedge \cdots \wedge a_{i_s})(a_{i_{s+1}} \wedge \cdots \wedge a_{i_r}) \boldsymbol{A_r}^{-1} \\ &= (-1)^{s(s-1)/2} \sum (-1)^{\sum_{j=1}^{s} (i_j - 1)} (B_s \rfloor a_{i_1} \wedge \cdots \wedge a_{i_s})(a_{i_{s+1}} \wedge \cdots \wedge a_{i_r}) \boldsymbol{A_r}^{-1} \\ &= \sum (-1)^{\sum_{j=1}^{s} (i_j - 1)} (B_s^{\dagger} \rfloor a_{i_1} \wedge \cdots \wedge a_{i_s})(a_{i_{s+1}} \wedge \cdots \wedge a_{i_r}) \boldsymbol{A_r}^{-1}, \end{aligned} \tag{289}$$

where in the next to last line I used Eq. (102). If I now let I be the sequence i_1 through i_s and use Eq. (280) for the reciprocal multivector basis, I find

$$\begin{aligned} P_{\boldsymbol{A_r}}(B_s) &= \sum_I (B_s^{\dagger} \rfloor a_I)(-1)^{\sum_{j=1}^{s} (i_j - 1)} \, a_{I^c} \, \boldsymbol{A_r}^{-1} \\ &= \sum_I (B_s * a_I) a^I. \end{aligned} \tag{290}$$

This expression is still true if I let the sum run over increasing sequences of any length, since all the additional terms vanish. Therefore for any multivector B

$$P_{a_1 \wedge \cdots \wedge a_r}(B) = \sum_I (B * a_I) a^I, \tag{291}$$

where the sum runs over all increasing sequences, and the expression is still true if the bases $\{a_I\}$ and $\{a^I\}$ are interchanged.

9. Linear algebra

Now that I've said so much about linear spaces, let's take the next step and put some linear functions on them. If U and V are vector spaces with the same set of scalars, a function $F : U \to V$ is said to be *linear* if $F(\alpha u + \beta v) = \alpha F(u) + \beta F(v)$, so linear functions respect linear combinations. Linear functions have a very well-developed theory, and they're important all over applied mathematics; in fact, when a function isn't linear, one of the first things we do is consider its local linear approximation, the derivative.

In this section I'll hit the highlights of linear algebra using the tools and perspective of geometric algebra. I'll start by reviewing some basic properties of linear functions, and then I'll introduce the adjoint. I'll use it to describe three special types of functions: symmetric, skew symmetric, and orthogonal, each of which relates to its adjoint in a certain way. All three have special forms in geometric algebra, which I'll consider in detail. After that, I'll take a giant step into geometric algebra proper by showing how to take a linear function on vectors and extend it in a very natural way to every multivector in the whole algebra. This is where geometric algebra really starts to shine, because it lets me see old topics in new and useful ways. For example, our old friend the eigenvector will be joined by eigenplanes, eigenvolumes, and more, and I'll show how to use them to describe linear functions. (It's immediately clear that a rotation has an eigenplane with eigenvalue 1, for example.) I'll also give a very easy and intuitive definition of the determinant, and I'll show how easy determinants are to calculate in geometric algebra.

I'm going to focus on functions that take vectors to vectors, and their extensions to the whole algebra will be grade-preserving. To some of you that might seem rather timid; since a geometric algebra is itself a (big) vector space, why not just jump in with both feet and go right for linear functions from multivectors to multivectors, whether they preserve grade or not? Well, of course you can, and we already have; duality does that, for example. General tensors will also do that, and I'll consider them in due course.

9.1. Preliminaries

If F and G are linear, then so are their linear combinations, and so are their inverses if they exist. If $F : U \to V$, then U and V are called the *domain* and *codomain* of F respectively. (Some authors call the codomain the *range*.) F singles out two special subspaces: the *kernel* of F, or $\mathrm{Ker}(F)$, is a subspace of the domain consisting of all the vectors that F maps to 0, and the *range* of F, or $\mathrm{Range}(F)$, is a subspace of the codomain containing all the vectors that F maps something to. (The range is sometimes called the *image*, presumably by the same folks who've already used the word range to mean the codomain.) It's suggestive to think of $\mathrm{Ker}(F)$ as $F^{-1}(0)$ and $\mathrm{Range}(F)$ as $F(U)$. F is one-to-one iff $\mathrm{Ker}(F) = \{0\}$, and F is onto iff $\mathrm{Range}(F) = V$. The dimension of $\mathrm{Ker}(F)$ is called the *nullity* of F, or $\mathrm{null}(F)$, and the dimension of $\mathrm{Range}(F)$ is called the *rank* of F, or $\mathrm{rank}(F)$. If the dimension of the domain U is finite, the *rank-nullity theorem* says

$$\mathrm{rank}(F) + \mathrm{null}(F) = \dim U. \tag{292}$$

So if both domain and codomain have the same finite dimension, F is one-to-one iff it's also onto. Therefore to show invertibility, you only have to show either one-to-one or onto, and the other part follows automatically. If W is a subspace of the domain of F, then the restriction of F to W is well-defined and also linear; it's denoted F_W. Since blades represent subspaces, I'll sometimes write $F_{\boldsymbol{A}}$ for the restriction of F to the subspace $\boldsymbol{A}$ represents.

Often we care specifically about linear functions from U to itself, which I'll call *linear transformations* or *linear operators.* A pretty popular linear operator on any space is the identity; I denote the identity on U by Id_U.

Since our subject is geometric algebra, I will assume that all vector spaces have inner products and belong to geometric algebras. With that, recall that Theorem 37 in Section 8 shows that any multivector is determined uniquely by its scalar products with all multivectors. Looking only at vectors, that means u is uniquely fixed if one knows $u * v$ (or equivalently $u \rfloor v$) for all v. This has two useful consequences. The first lets me reconstruct linear functions.

Theorem 40. *A linear function $F : U \to V$ is completely determined by knowledge of $F(u) * v$ for all $u \in U$ and $v \in V$.*

Proof. In the finite-dimensional case, F can be constructed explicitly: $F(u) = \sum_i \left[a^i * F(u)\right] a_i$, where $\{a_i\}_{i=1,\ldots,n}$ is any frame in V and $\{a^j\}_{j=1,\ldots,n}$ is its reciprocal frame. Since we know all the $a^i * F(u)$, we know $F(u)$. We can use any frame for this construction since the set of all inner products determines $F(u)$ uniquely for each u by Theorem 37.

In the infinite-dimensional case, it's not obvious we can perform this construction, but for all applications I know of, the space is rigged in some way to allow something like this to be done. So I'll assume I can do it. □

Any linear transformation F defines a new bilinear product on vectors by $a \rfloor F(b)$. The second consequence of Theorem 37 lets me go the other way: I start with the product and define F.

Theorem 41. *If $\circ$ is a bilinear function from vectors to scalars, there's unique a linear transformation F such that $u \circ v = u \rfloor F(v)$ for all u and v.*

Proof. Again, in finite dimensions the proof is by construction: Let $F(u) = \sum_i (a^i \circ u) a_i$. This defines a linear function because the product $\circ$ is bilinear, and it satisfies

$$\begin{aligned} u \rfloor F(v) &= u \rfloor \left(\sum_i (a^i \circ v)\, a_i \right) \\ &= \sum_i (a^i \circ v)\, (u \rfloor a_i) \\ &= \left(\sum_i (u \rfloor a_i)\, a^i \right) \circ v \\ &= u \circ v. \end{aligned} \tag{293}$$

Since F is determined by $u \rfloor F(v) = u \circ v$ for all u and v, this function is unique.

In the infinite-dimensional case, I will assume that whatever structure is needed to make this result true has been added. □

This is useful, because any given vector space can support many different inner products, but only one inner product at a time can be encoded into a geometric algebra. This theorem tells me I have a way to use the other products if I decide I need to. Also, there are bilinear products that I want to use that can't be inner products because they aren't symmetric. This theorem lets me include those products too, although of course the corresponding function F will have different properties. What those properties are will be the subject of future sections.

9.2. The adjoint

If $F : U \to V$ is linear, then its *adjoint* is the unique linear function $\overline{F} : V \to U$ defined by

$$\overline{F}(v) * u := v * F(u) \quad \text{for all } u \in U \text{ and } v \in V. \tag{294}$$

Notice that $\overline{F}$ switches domain and codomain compared to F and that $*$ is interchangeable with either $\rfloor$ or $\lfloor$ in this definition. The adjoint of the identity is pretty easy: $\overline{\mathrm{Id}_U} = \mathrm{Id}_U$. Theorem 40 tells me how to construct $\overline{F}$ explicitly in the finite-dimensional case:

$$\begin{aligned} \overline{F}(v) &= \sum_i \left[a^i * \overline{F}(v)\right] a_i \\ &= \sum_i \left[F(a^i) * v\right] a_i. \end{aligned} \tag{295}$$

(Notice that for this to make sense, $v \in V$ while the frame $\{a_i\} \subset U$.) The bilinearity of the inner product shows that taking the adjoint is itself a linear operation:

$$\overline{\alpha F + \beta G} = \alpha \overline{F} + \beta \overline{G}. \tag{296}$$

The relationship between F and $\overline{F}$ is symmetric, so each is the adjoint of the other, or equivalently

$$\overline{\overline{F}} = F. \tag{297}$$

Suppose $F : U \to V$ and $G : V \to W$, and let $GF : U \to W$ denote the composition of F and G. Then

$$\begin{aligned} w * GF(u) &= \overline{G}(w) * F(u) \\ &= \overline{F}\,\overline{G}(w) * u, \end{aligned} \tag{298}$$

which tells me that

$$\overline{GF} = \overline{F}\,\overline{G}. \tag{299}$$

A special case of this arises if F is a operator on U, in which case F^n is defined for any n and

$$\overline{F^n} = (\overline{F})^n. \tag{300}$$

Now suppose $F : U \to V$ is invertible, so there's an $F^{-1} : V \to U$ such that $F^{-1}\,F = \mathrm{Id}_U$ and $F\,F^{-1} = \mathrm{Id}_V$. In that case, since $\overline{F}\,\overline{F^{-1}} = \overline{F^{-1}\,F} = \overline{\mathrm{Id}_U} = \mathrm{Id}_U$, and similarly $\overline{F^{-1}}\,\overline{F} = \mathrm{Id}_V$,

$$(\overline{F})^{-1} = \overline{F^{-1}}. \tag{301}$$

The special subspaces defined by a linear function and its adjoint are related in interesting ways.

Theorem 42. *If $F : U \to V$ is linear,*

$$\mathrm{Ker}(F) = \mathrm{Range}(\overline{F})^{\perp}. \tag{302}$$

If in addition U is finite-dimensional,

$$\mathrm{rank}(F) = \mathrm{rank}(\overline{F}). \tag{303}$$

Proof. For the first part,

$$\begin{aligned} u \in \mathrm{Ker}(F) \quad &\text{iff} \quad F(u) = 0 \\ &\text{iff} \quad F(u) \rfloor v = 0 \quad \text{for all } v \in V \\ &\text{iff} \quad u \rfloor \overline{F}(v) = 0 \quad \text{for all } v \in V \\ &\text{iff} \quad u \in \mathrm{Range}(\overline{F})^{\perp}. \end{aligned}$$

For the second part, we start with the rank-nullity theorem and the result of the first part:

$$\begin{aligned} \mathrm{rank}(F) &= \dim U - \mathrm{null}(F) \\ &= \dim U - \dim \mathrm{Ker}(F) \\ &= \dim U - \dim \mathrm{Range}(\overline{F})^{\perp}. \end{aligned}$$

Now $\mathrm{Range}(\overline{F})$ and $\mathrm{Range}(\overline{F})^{\perp}$ are duals, so their dimensions add up to the dimension of U; so picking up where I left off,

$$\begin{aligned} \mathrm{rank}(F) &= \dim U - \dim \mathrm{Range}(\overline{F})^{\perp} \\ &= \dim \mathrm{Range}(\overline{F}) \\ &= \mathrm{rank}(\overline{F}). \end{aligned} \tag{304}$$

□

9.3. Normal operators

In the next few sections I'll be considering operators that commute with their adjoints: $F\overline{F} = \overline{F}F$. These are called *normal operators*, and they have properties that I'll describe here so I can use them later.

Theorem 43. *Any power of a normal operator is also normal.*

Proof. If F is normal, then $F^n(\overline{F})^n$ is easily transformed to $(\overline{F})^n F^n$ by moving all the $\overline{F}$ factors past all the factors of F. □

Theorem 44. *F is normal iff $F(u) \rfloor F(v) = \overline{F}(u) \rfloor \overline{F}(v)$ for any u and v.*

Proof. First assume F is normal. Then

$$\begin{aligned} F(u) \rfloor F(v) &= \overline{F}F(u) \rfloor v \\ &= F\overline{F}(u) \rfloor v \\ &= \overline{F}(u) \rfloor \overline{F}(v). \end{aligned} \tag{305}$$

Now assume the relation holds. Then

$$\begin{aligned} F\overline{F}(u) \rfloor v &= \overline{F}(u) \rfloor \overline{F}(v) \\ &= F(u) \rfloor F(v) \\ &= \overline{F}F(u) \rfloor v, \end{aligned} \tag{306}$$

so by Theorem 40 $F\overline{F} = \overline{F}F$ and F is normal. □

Theorem 45. *F is normal iff $F(u)^2 = \overline{F}(u)^2$ for all u.*

Proof. Since

$$F(u) \rfloor F(v) = \frac{1}{2}\left[F(u+v)^2 - F(u)^2 - F(v)^2\right], \tag{307}$$

$F(u)^2 = \overline{F}(u)^2$ for all u implies $F(u) \rfloor F(v) = \overline{F}(u) \rfloor \overline{F}(v)$ for all u and v. On the other hand, $F(u) \rfloor F(v) = \overline{F}(u) \rfloor \overline{F}(v)$ for all u and v implies $F(u)^2 = \overline{F}(u)^2$ for all u just by considering the case $u = v$. So squares are equal iff inner products are equal, which takes us back to the previous theorem. □

Theorem 46. *If F is normal and the inner product is nondegenerate on both* $\mathrm{Range}(F)$ *and* $\mathrm{Range}(\overline{F})$, *then* $\mathrm{Ker}(F) = \mathrm{Ker}(\overline{F})$. *If in addition the domain of F is finite-dimensional,* $\mathrm{Range}(F) = \mathrm{Range}(\overline{F})$.

Proof. For the first part,

$$\begin{aligned} u \in \mathrm{Ker}(F) &\Longrightarrow F(u) = 0 \\ &\Longrightarrow \overline{F}F(u) = 0 \\ &\Longrightarrow F\overline{F}(u) = 0 \\ &\Longrightarrow \overline{F}(u) \in \mathrm{Ker}(F). \end{aligned} \tag{308}$$

By Theorem 42, $\mathrm{Ker}(F) = \mathrm{Range}(\overline{F})^\perp$, so $\overline{F}(u) \in \mathrm{Range}(\overline{F})^\perp$. But wait a second: $\overline{F}(u) \in \mathrm{Range}(\overline{F})$ by definition, and $\mathrm{Range}(\overline{F})$ is nondegenerate, so it must be that $\overline{F}(u) = 0$, so $u \in \mathrm{Ker}(\overline{F})$. Therefore $\mathrm{Ker}(F) \subset \mathrm{Ker}(\overline{F})$. The same argument with F and $\overline{F}$ interchanged shows $\mathrm{Ker}(\overline{F}) \subset \mathrm{Ker}(F)$, so $\mathrm{Ker}(F) = \mathrm{Ker}(\overline{F})$.

For the second part, since the domain of F is finite-dimensional any subspace and its orthogonal complement are duals, so each is the orthogonal complement of the other. That and Theorem 42 tell me that $\mathrm{Range}(F) = \mathrm{Ker}(\overline{F})^\perp$ and $\mathrm{Range}(\overline{F}) = \mathrm{Ker}(F)^\perp$. But I just showed that $\mathrm{Ker}(F) = \mathrm{Ker}(\overline{F})$, so $\mathrm{Range}(F) = \mathrm{Range}(\overline{F})$ too. □

So if the conditions of this theorem are satisfied, normal F and $\overline{F}$ are both one-to-one (or onto) or neither one is.

9.4. Symmetric and skew symmetric operators

A linear operator is *symmetric* if it equals its adjoint, $\overline{F} = F$, and *skew symmetric* or *skew* if it is the negative of its adjoint, $\overline{F} = -F$. The names come from this theorem.

Theorem 47. *The bilinear product $a \circ b := a \rfloor F(b)$ is (anti)symmetric iff F is (skew) symmetric.*

Proof. Since $a \rfloor F(b) = b \rfloor \overline{F}(a)$, it follows that $a \circ b = b \circ a$ iff $\overline{F} = F$ and $a \circ b = -b \circ a$ iff $\overline{F} = -F$. □

Recall that every bilinear product has this form for some F (Theorem 41).

Both types of operator are normal, so all the results of Section 9.3 apply to them. (They're all pretty trivial in these cases, I have to admit.) Further, every linear operator is the sum of a symmetric and a skew symmetric operator, because

$$F = \frac{1}{2}\left(F + \overline{F}\right) + \frac{1}{2}\left(F - \overline{F}\right). \tag{309}$$

Also, for any linear operator F, both $\overline{F}F$ and $F\overline{F}$ are symmetric.

Powers of symmetric and skew symmetric operators are themselves symmetric or skew symmetric.

Theorem 48. *Any power of a symmetric operator is symmetric. Any even power of a skew symmetric operator is symmetric, and any odd power is skew symmetric.*

Proof. Since $\overline{F^n} = (\overline{F})^n$, $\overline{F} = F$ implies $\overline{F^n} = F^n$, so F^n is symmetric also, and $\overline{F} = -F$ implies $\overline{F^n} = (-1)^n F^n$, so F^n is symmetric or skew as n is even or odd. □

The spectral theorem says that every symmetric F has a frame $\{a_i\}$ of eigenvectors with eigenvalues $\{\lambda_i\}$, which means $F(a)$ for any a is given by

$$F(a) = \sum_i \lambda_i (a * a^i) a_i. \tag{310}$$

Conversely, every F of this form is symmetric. Analogous results hold on infinite-dimensional spaces with various additional restrictions.

Skew symmetric operators also have a canonical form, which is expressed very nicely in geometric algebra. As motivation, notice that if F is skew, then $a \rfloor F(a) = 0$, so F maps any vector to an orthogonal vector. Well, I know something else that does that: taking the dual by a bivector. In fact, the function $F(a) = a \rfloor A_2$ for any bivector A_2 is skew, because the resulting bilinear product is antisymmetric:

$$\begin{aligned} a \circ b &= a \rfloor (b \rfloor A_2) \\ &= (a \wedge b) \rfloor A_2 \\ &= -(b \wedge a) \rfloor A_2 \\ &= -b \rfloor (a \rfloor A_2) \\ &= -b \circ a. \end{aligned} \tag{311}$$

It turns out all skew functions are of this form.

Theorem 49. *F is skew iff $F(a) = a \rfloor A_2$ for a unique bivector A_2.*

Proof. I just finished showing that any F of this form is skew. Knowing F, I can reconstruct A_2 uniquely using any frame $\{a_i\}$ and Theorem 38 when $r = 2$:

$$\begin{aligned} A_2 &= \frac{1}{2}\sum_i a^i \wedge (a_i \rfloor A_2) \\ &= \frac{1}{2}\sum_i a^i \wedge F(a_i). \end{aligned} \tag{312}$$

Now assume F is skew and let A_2 be defined as above. I find that for any a,

$$\begin{aligned}
a \rfloor A_2 &= \frac{1}{2} a \rfloor \left(\sum_i a^i \wedge F(a_i) \right) \\
&= \frac{1}{2} \sum_i (a \rfloor a^i) F(a_i) - \frac{1}{2} \sum_i a^i (a \rfloor F(a_i)) \\
&= \frac{1}{2} \sum_i (a \rfloor a^i) F(a_i) + \frac{1}{2} \sum_i a^i (F(a) \rfloor a_i) \\
&= \frac{1}{2} F \left(\sum_i (a \rfloor a^i)\, a_i \right) + \frac{1}{2} \sum_i (F(a) \rfloor a_i)\, a^i \\
&= \frac{1}{2} F(a) + \frac{1}{2} F(a) \\
&= F(a).
\end{aligned} \tag{313}$$

□

Therefore every antisymmetric bilinear product is of the form $a \wedge b \rfloor A_2 = a \rfloor A_2 \lfloor b$ for some A_2.

9.5. Isometries and orthogonal transformations

The final special linear operator is an *isometry*, which preserves inner products: $F(u) \rfloor F(v) = u \rfloor v$. (Equivalently, isometries preserve squares of vectors.) Isometries are always one-to-one, because

$$\begin{aligned}
F(u) = 0 &\Longrightarrow F(u) \rfloor F(v) = 0 \quad \text{for all } v \\
&\Longrightarrow u \rfloor v = 0 \quad \text{for all } v \\
&\Longrightarrow u = 0.
\end{aligned} \tag{314}$$

So in finite dimensions, isometries are also onto and thus invertible. An invertible isometry is called an *orthogonal transformation*. The two are distinct only on infinite-dimensional spaces, but most of the results I'll show don't actually require invertibility, so I'll continue to make the distinction.

Any power of an isometry is also an isometry, as is clear from the definition. An isometry satisfies

$$u \rfloor v = F(u) \rfloor F(v) = \overline{F} F(u) \rfloor v, \tag{315}$$

so if F is an isometry then $\overline{F} F = \mathrm{Id}$. If F is also invertible, then its inverse has to be $\overline{F}$, so we also have $F \overline{F} = \mathrm{Id}$. Therefore orthogonal transformations satisfy $\overline{F} = F^{-1}$ and are also normal, and as a bonus F^{-1} is orthogonal too. And as with isometries, any power of an orthogonal transformation is also orthogonal.

9.5.1. Isometries and versors

So far I've described three isometries: the parity operation (which was extended to the whole algebra as grade involution in Section 5.1), reflections, and rotations (both in Section 7). Now a rotation is two reflections, and as I showed in Section 7.2.2, the parity operation is reflection in a volume element, which amounts to n reflections in succession. So every isometry I've shown so far is a composition of reflections. That's no accident: the *Cartan-Dieudonné theorem* shows that every isometry in an n-dimensional space is the composition of at most n reflections along axes. That's fantastic news, because reflections are easy to do in geometric algebra; so now we have powerful tools to perform and analyze any isometry at all.

So what does a general isometry look like? Remembering Eq. (237), I find that the isometry F that takes vector u and reflects it along axes $a_1, a_2, \ldots, a_r$ in succession is

$$\begin{aligned}
F(u) &= (-1)^r (a_r \cdots a_2\, a_1) u (a_1^{-1} a_2^{-1} \cdots a_r^{-1}) \\
&= (a_r \cdots a_2\, a_1) u^{*r} (a_r \cdots a_2\, a_1)^{-1}
\end{aligned}$$

$$= A_r u^{*r} A_r^{-1} \tag{316}$$

where $A_r = a_r \cdots a_2\, a_1$ is an invertible r-versor. Thus a general isometry in finite dimensions is grade involution followed by conjugation with an invertible versor. (This is why I defined versors in the first place, and it's also why I've been proving so many results not just for blades but for versors in general.) Now this looks a lot like Eq. (239) for reflecting a vector in a subspace; in fact, Eq. (239) is just a special case of this result, since a blade is a special type of versor. Therefore this operation extends to the whole algebra the same way reflection in subspaces did in Section 7.2.2: a general isometry on multivectors takes the form

$$F(B) = A_r B^{*r} A_r^{-1} \tag{317}$$

and it reduces to reflection in a subspace iff the versor A_r is an r-blade. This also makes it clear that the isometries generated by A_r and A_r^{-1} are inverses of each other.

Even though every versor is associated with an isometry, the association isn't exactly one-to-one. After all, A_r and λA_r generate the same isometry for any $\lambda \neq 0$. (The ultimate reason for this is that a and λa represent the same axis, and thus the same reflection.) We can eliminate most of that ambiguity, however, by composing our versors out of unit vectors; in that case, A_r is a unit versor. That doesn't eliminate the sign ambiguity, but we can live with that. Is there any further ambiguity? Amazingly, no. I'll show this in two steps. First, basically the same argument used to derive Eq. (247) shows that

$$A_r \boldsymbol{I}^{*r} A_r^{-1} = (-1)^r \boldsymbol{I}, \tag{318}$$

so isometries divide into two classes: *even* isometries, which leave $\boldsymbol{I}$ alone and are represented by even versors, and *odd* isometries, which change the sign of $\boldsymbol{I}$ and are represented by odd versors. This also shows that an isometry is odd iff it's the composition of an even isometry and one reflection. As I said I would back in Section 2, I'll now start referring to any even invertible versor as a rotor, so rotors represent even isometries.

Now I'll prove the result.

Theorem 50. *Versors A_r and B_s represent the same isometry iff $A_r = \lambda B_s$ for some $\lambda \neq 0$.*

So if we consider only unit versors, the association of versors to isometries is exactly two-to-one.

Proof. If A_r is a nonzero multiple of B_s, we know they represent the same isometry, so let's prove it the other way. Suppose $A_r u^{*r} A_r^{-1} = B_s u^{*s} B_s^{-1}$ for all u. Since r and s are both even or both odd, I can drop the grade involutions and I'm left with $A_r u A_r^{-1} = B_s u B_s^{-1}$, which can be sneakily rewritten

$$B_s^{-1} A_r \times u = 0 \quad \text{for all } u. \tag{319}$$

Then Theorem 28 tells me that $B_s^{-1} A_r = \lambda + \mu \langle \boldsymbol{I} \rangle_-$. But $B_s^{-1} A_r$ is even, so its odd part vanishes and I'm left with $B_s^{-1} A_r = \lambda$. Now if $\lambda = 0$, by Theorem 24 both A_r and B_s would be null, which they aren't since I've been inverting both of them. Therefore $\lambda \neq 0$ and $A_r = \lambda B_s$. □

You might be tempted at this point to associate an isometry with a unique sequence of reflections, but you can't. That's because the factorization of a versor into vectors isn't unique. For example, suppose a and b are orthogonal Euclidean unit vectors; then

$$ab = \left(\frac{a-b}{\sqrt{2}}\right)\left(\frac{a+b}{\sqrt{2}}\right), \tag{320}$$

so in this case two different sequences of reflections give the same isometry. In fact, reflections along any orthonormal basis for a subspace will result in reflection in that subspace, so in that case infinitely many reflection sequences produce the same isometry. But that's only because they all produce the same versor to within a sign.

9.5.2. *Rotors and biversors*

Nothing I've done in this section up to now has made any assumptions about the scalars, but for this last part I assume the scalars are real. Every rotor is a product of biversors, so I want to take a moment to examine them. Consider ab where a and b are unit vectors; it represents the composition of reflections in the b and a directions in succession, so the resulting isometry acts in the $a \wedge b$ plane. My plan to analyze ab is to expand it as $a \rfloor b + a \wedge b$ and figure out each piece separately. I'll do that by starting with Eq. (8) from back in Section 1.1,

$$a^2 b^2 = (a \rfloor b)^2 - (a \wedge b)^2. \tag{321}$$

Since a and b are unit vectors, the left hand side of Eq. (321) is ± 1. In what follows, I will set $\mu = a \rfloor b$, and I will set $a \wedge b = \lambda \boldsymbol{B}$ where $\boldsymbol{B}$ is a 2-blade. In the cases when $(a \wedge b)^2 \neq 0$, I'll choose λ so $\boldsymbol{B}$ is a unit blade; otherwise I'll come up with some other way to choose λ.

Before I get into the general cases, I'll handle a special case that I'll need to refer back to later: let a and b be orthogonal. Since $a \rfloor b = 0$, $(a \wedge b)^2 = \pm 1$. So $ab = a \wedge b$ is already a unit blade $\boldsymbol{B}$. To see what isometry it generates, let u lie in $\boldsymbol{B}$; then its product with $\boldsymbol{B}$ is an inner product, so they anticommute, so

$$\boldsymbol{B} u \boldsymbol{B}^{-1} = -u \boldsymbol{B} \boldsymbol{B}^{-1} = -u. \tag{322}$$

So versor $\boldsymbol{B}$ generates a reflection in the plane it represents. (Which we already knew from Section 7.2.1.)

The result in the general case depends on the sign of $\boldsymbol{B}^2$.

1. Suppose first that $\boldsymbol{B}^2 = -1$. Then Eq. (175) when $n = 2$ tells me that the number of negative-square vectors in a frame for $\boldsymbol{B}$ is either 0 or 2, so the inner product on $\boldsymbol{B}$ is either positive definite or negative definite, which I call the *Euclidean* or *elliptic* case. Then a^2 and b^2 have the same sign, so $a^2 b^2 = 1$. Putting all this in Eq. (321), I find
$$1 = \mu^2 + \lambda^2. \tag{323}$$
Therefore $\mu = \cos(\theta/2)$ and $\lambda = -\sin(\theta/2)$ for some θ, so
$$\begin{aligned} ab &= a \rfloor b + a \wedge b \\ &= \cos(\theta/2) - \boldsymbol{B} \sin(\theta/2) \\ &= \exp(-\boldsymbol{B}\theta/2) \end{aligned} \tag{324}$$
where the exponential is defined by its power series. You may remember this from the end of Section 1.2: it's a rotation through angle θ in the plane defined by $\boldsymbol{B}$. When $\theta = \pi$, I recover the special case I solved above: a rotation by π in a Euclidean plane equals a reflection in the plane.

2. Now suppose $\boldsymbol{B}^2 = 1$. In this case the inner product is indefinite, which is called the *hyperbolic* case. Now I have to give some thought to a^2 and b^2. First let them have the same sign, so Eq. (321) becomes
$$1 = \mu^2 - \lambda^2. \tag{325}$$
Therefore $\mu = \pm\cosh(\phi/2)$ and $\lambda = \mp\sinh(\phi/2)$ for some ϕ, so
$$\begin{aligned} ab &= a \rfloor b + a \wedge b \\ &= \pm\left(\cosh(\phi/2) - \boldsymbol{B} \sinh(\phi/2)\right) \\ &= \pm\exp(-\boldsymbol{B}\phi/2) \end{aligned} \tag{326}$$
where again the exponential is defined by its power series. (This isometry, by the way, is a rotation in the hyperbolic plane, and in special relativity it's a boost to velocity $c \tanh\phi$.) This time I couldn't absorb the sign of $a \rfloor b$ into a choice for the parameter, because cosh is always positive; but that affects only the rotor, not the corresponding isometry. So aside from that, the rotors for the last two cases have the same polar form, and the difference in their expansions as scalar plus bivector is due to the different behaviors of the area element.

3. Sticking with $\boldsymbol{B}^2 = 1$, now I consider $a^2 = -b^2$. Eq. (321) becomes

$$-1 = \mu^2 - \lambda^2, \tag{327}$$

so μ and λ change roles: $\mu = \mp \sinh(\phi/2)$ and $\lambda = \pm \cosh(\phi/2)$ for some ϕ, so for the rotor I get

$$\begin{aligned} ab &= a \rfloor b + a \wedge b \\ &= \mp \sinh(\phi/2) \pm \boldsymbol{B} \cosh(\phi/2) \\ &= \pm \boldsymbol{B} \left(\cosh(\phi/2) - \boldsymbol{B} \sinh(\phi/2)\right) \\ &= \pm \boldsymbol{B} \exp\left(-\boldsymbol{B}\phi/2\right). \end{aligned} \tag{328}$$

This rotor is the product of the previous one and the area element. As I showed in the special case above, this extra factor generates a reflection in the plane. Why is it showing up as a separate factor? Because unlike the Euclidean case, there is no hyperbolic rotation that performs a reflection in the plane, so it has to be included separately.

4. Finally, suppose $\boldsymbol{B}^2 = 0$, which means by Theorem 23 that the inner product is degenerate. (This also has a name: the *parabolic* case.) Now Eq. (321) reduces to $\pm 1 = \mu^2$. This doesn't make sense if the left hand side can be -1, so let me show that it can't. The inner product may be degenerate, but it can't be identically zero, or every vector would be null and there would be no axes in the plane to reflect along. Therefore there's a non-null vector somewhere in there which I'll call v, and the direction orthogonal to it is a null vector which together with v spans the plane. Because of this, the length squared of any vector in the plane is just v^2 times the square of its component along v, so they all have the same sign. Thus $a^2 b^2 = 1$ and $\mu = \pm 1$. Let $\boldsymbol{B}$ be any 2-blade that's convenient to use to represent the plane; then $a \wedge b = \mp \lambda \boldsymbol{B}$ for some λ, so the rotor becomes

$$\begin{aligned} ab &= a \rfloor b + a \wedge b \\ &= \pm \left(1 - \frac{\lambda}{2}\boldsymbol{B}\right) \\ &= \pm \exp\left(-\boldsymbol{B}\lambda/2\right) \end{aligned} \tag{329}$$

where once again the exponential is defined by its power series.

You may wonder what this rotor does. Its inverse is $\pm(1 + \frac{\lambda}{2}\boldsymbol{B})$, so for any vector u

$$\begin{aligned} (ab)u(ab)^{-1} &= \left(1 - \frac{\lambda}{2}\boldsymbol{B}\right) u \left(1 + \frac{\lambda}{2}\boldsymbol{B}\right) \\ &= u + \frac{\lambda}{2}(u\boldsymbol{B} - \boldsymbol{B}u) - \frac{\lambda^2}{4}\boldsymbol{B}u\boldsymbol{B} \\ &= u + \lambda u \rfloor \boldsymbol{B} - \frac{\lambda^2}{4}\boldsymbol{B}u\boldsymbol{B}. \end{aligned} \tag{330}$$

Each term on the right hand side is a vector, and you can directly verify that the square of the whole thing really is u^2. The verification is an interesting exercise; you find that $u \rfloor \boldsymbol{B}$ is orthogonal to u, $\boldsymbol{B}u\boldsymbol{B}$ is orthogonal to $u \rfloor \boldsymbol{B}$, $\boldsymbol{B}u\boldsymbol{B}$ is null, and the inner product of u and $\boldsymbol{B}u\boldsymbol{B}$ cancels the square of $u \rfloor \boldsymbol{B}$.

Putting all this together, I've shown that a general even isometry on a real vector space consists of any number of rotations in planes (elliptic, hyperbolic, or parabolic) and reflections in hyperbolic planes. An odd isometry is the same thing plus one reflection along an axis. All the rotations can be represented in the same polar form; the properties of the different area elements produce different types of rotations. If the whole space is Euclidean, things simplify further because there are no hyperbolic or parabolic planes: every isometry is a sequence of rotations in planes, preceded (or followed) by one reflection if it's odd.

The set of all isometries on a finite-dimensional real vector space forms a group called the *orthogonal group* on that space. All of this analysis tells us two main things about such groups:

1. The subset of even isometries forms a group of its own (it includes the identity and is closed under products), and the subset of odd isometries is a one-to-one copy of the even subgroup. A reflection along any axis provides the relation between the two subsets.
2. Aside from reflections in hyperbolic planes, all elements of the even subgroup are functions of parameters that can be continuously varied to zero, which results in the identity transformation.

These properties tell me that a space's orthogonal group is an example of a *Lie group*. This is a group which can be divided into a finite number of isomorphic subsets, and within each subset the elements can be labeled with a finite number of continuously-variable parameters. Whichever subset is lucky enough to contain the identity is a subgroup in its own right, so a Lie group is a continuously-parameterised group together with some isomorphic copies. (In our case, reflections along axes and in hyperbolic planes move us back and forth between the copies.) After I learn how to use geometric algebra to study Lie groups in general, I believe I'll be showing that they form even subalgebras which are generated by exponentiating bivectors. But this is more than enough on isometries for now.

9.6. Extending linear functions to the whole algebra

So far, I have considered only linear functions defined on vectors; but functions on vectors have an obvious extension to the whole algebra. For example, consider the r-dimensional space spanned by $\{a_i\}$; this is mapped by linear function F to the space spanned by $\{F(a_i)\}$. Since blades represent subspaces, it seems very natural to define F not just on vectors but on blades too; I set

$$F(a_1 \wedge \cdots \wedge a_r) := F(a_1) \wedge \cdots \wedge F(a_r). \tag{331}$$

If I then require this formal extension of F to be linear over multivectors, I get $F(A \wedge B) = F(A) \wedge F(B)$ for any A and B. Well, almost any A and B; my picture doesn't tell me what F should do to scalars. Assuming I figure that out, then I've found a way to naturally extend any linear function on vectors to a function on the whole geometric algebra that is not only linear but also respects outer products. As a matter of fact, I've actually done this already no fewer than six times. Four of the extensions were orthogonal projections, orthogonal rejections, reflections, and rotations in Section 7, and the fifth was general isometries in Section 9.5. I defined all five on vectors to start with, and I extended them in exactly the way I just suggested: I had them respect outer products. The sixth extension was grade involution in Section 5.1; I started with the parity operation $u \to -u$, but I extended it by making it respect not outer products but the full geometric product. (It ended up respecting outer products too, as the third equality in Eqs. (120) shows.) In retrospect, that looks a little daring; after all, a product of vectors contains terms of many different grades, and the rules I imposed on, say, three-fold and five-fold products could have put conflicting requirements on trivectors. So far, though, it looks like everything worked out. Whew. While my other five extensions look safer, because I didn't have different grades crossing over, they do raise a question: how *do* they act on products? Do you operate on each factor separately and then multiply them back together? That looks like it'd work for general isometries (and thus reflections and rotations) because the internal factors would cancel out, but I'm not too sure about projections and rejections. And then there's one property that all six extensions have in common: they leave scalars alone. What's up with that? Grade involution does it by definition, but the others were found to do so after their final definitions were stated. How come?

9.6.1. Outermorphisms

To answer these questions, I want to lay some groundwork by describing something a little more general. A linear function on geometric algebras that preserves outer products is called an *outermorphism*. It's not too hard to show that the composition of outermorphisms is also an outermorphism and that the inverse of an outermorphism, if it exists, is an outermorphism too. However, a linear combination of outermorphisms is not an outermorphism. To show this, let $\mathcal{F} = \alpha\mathcal{F}_1 + \beta\mathcal{F}_2$ where $\mathcal{F}_1$ and $\mathcal{F}_2$ are outermorphisms, and try evaluating both $\mathcal{F}(A \wedge B)$ and $\mathcal{F}(A) \wedge \mathcal{F}(B)$. You'll see the problem pretty quickly. For this reason, compositions and inverses of outermorphisms pop up frequently, but linear combinations don't.

Next, outermorphisms are very restricted in how they handle scalars. First linearity has a say: an outermorphism $\mathcal{F}$ has to be linear over everything, not just vectors, which means $\mathcal{F}(\lambda) = \mathcal{F}(\lambda 1) = \lambda\mathcal{F}(1)$.

So $\mathcal{F}(1)$ determines $\mathcal{F}$ for all scalars. But scalar multiplication is also an outer product, so $1 = 1 \wedge 1$. Thus the outermorphism property requires $\mathcal{F}(1) = \mathcal{F}(1) \wedge \mathcal{F}(1)$. So like 1, $\mathcal{F}(1)$ must equal its outer product with itself. Something that equals its own square is said to be *idempotent*; since we have several different products, we have several different types of idempotency. I've pointed out that 1 is an *outer idempotent*, and thanks to the outermorphism property, $\mathcal{F}(1)$ has to be an outer idempotent too. It turns out there aren't too many of those.

Theorem 51. *If $A = A \wedge A$, then $A = 0$ or 1.*

Proof. Let $A = \sum_r A_r$; then $A = A \wedge A$ becomes

$$A = \sum_{s,t} \langle A_s A_t \rangle_{s+t} \,. \tag{332}$$

I'll look at this one grade at a time. The grade-r part of this expression is

$$A_r = \sum_{s=0}^{r} \langle A_s A_{r-s} \rangle_r \,. \tag{333}$$

When $r = 0$, this becomes

$$A_0 = A_0^2, \tag{334}$$

so A_0 is either 0 or 1. When $r = 1$, I find

$$A_1 = 2 A_0 A_1. \tag{335}$$

Whether A_0 is 0 or 1, this equation requires $A_1 = 0$.

To show that all remaining A_r vanish, I proceed by induction. Suppose it's true for $r - 1$; then most of the terms in the sum for A_r drop out, leaving

$$A_r = 2 A_0 A_r. \tag{336}$$

Whether A_0 is 0 or 1, this gives me $A_r = 0$, and that completes the proof. □

So $\mathcal{F}(1) = 0$ or 1. That means outermorphisms can do only two things to scalars.

Theorem 52. *If $\mathcal{F}$ is an outermorphism, then either $\mathcal{F} = 0$ or $\mathcal{F}(\lambda) = \lambda$.*

This is why all six extensions left scalars alone; they had to.

Proof. Either $\mathcal{F}(1) = 0$ or $\mathcal{F}(1) = 1$. In the former case, for any A

$$\begin{aligned} \mathcal{F}(A) &= \mathcal{F}(1A) \\ &= \mathcal{F}(1 \wedge A) \\ &= \mathcal{F}(1) \wedge \mathcal{F}(A) \\ &= 0, \end{aligned} \tag{337}$$

so $\mathcal{F} = 0$. In the latter case, $\mathcal{F}(\lambda) = \lambda$ for all λ by linearity. □

Next, I'll take a passing look at adjoints. Linear functions on geometric algebras have adjoints just as they do on any other vector spaces: if $\mathcal{F} : \mathcal{G}^1 \to \mathcal{G}^2$ is linear, $\overline{\mathcal{F}} : \mathcal{G}^2 \to \mathcal{G}^1$ is a linear function given by

$$\overline{\mathcal{F}}(B) * A := B * \mathcal{F}(A) \quad \text{for all } A \in \mathcal{G}^1 \text{ and } B \in \mathcal{G}^2. \tag{338}$$

Outermorphisms are linear, so they have adjoints which are linear. I can't say more than that, though, until I consider a special class of outermorphisms, which is what's next.

9.6.2. *Outermorphism extensions*

Now, back to the reason we're here. My goal is to start with a linear function from vectors to vectors, define it on scalars by having it leave them alone (as Theorem 52 says I have to do), and then extend it to the rest of the algebra by linearity and respecting outer products. The result is an outermorphism that matches the original function on vectors. Can I start with any linear function and do this? Why sure; in fact, I can do it exactly one way.

Theorem 53. *Let F be a nonzero linear function that maps vectors to vectors; then there exists a unique outermorphism $[F]$ that reproduces F when applied to vectors.*

Proof. Existence is obvious, because definition on scalars and vectors, plus the outermorphism property, plus linearity is enough to define $[F]$ on any multivector. Uniqueness follows for the same reason. □

I had to specify $F \neq 0$ because technically if $F = 0$ then Theorem 52 allows two extensions: one is $[F] = 0$, and the other is $[F](A) = \langle A \rangle$. Both extensions vanish on all vectors and all blades; they differ only in the scalar option from Theorem 52 they use.

It's clear that $[F]$ maps r-blades into r-blades, so

$$\langle [F](A) \rangle_r = [F](\langle A \rangle_r). \tag{339}$$

So by requiring $[F]$ to preserve grade 1, I find that it preserves all grades. It also follows that

$$\begin{aligned} [F](A)^* &= [F](A^*) \\ [F](A)^\dagger &= [F](A^\dagger) \\ [F](A)^\ddagger &= [F](A^\ddagger). \end{aligned} \tag{340}$$

If $\mathcal{G}_1$ is the vector space of $\mathcal{G}$, then it's also pretty clear that

$$[\mathrm{Id}_{\mathcal{G}_1}] = \mathrm{Id}_{\mathcal{G}}\,. \tag{341}$$

Now let's see how outermorphism extensions behave under composition, inverses, and adjoints; I'll show compositions and inverses first.

Theorem 54. *If U, V, and W are the vector spaces of $\mathcal{G}^1$, $\mathcal{G}^2$, and $\mathcal{G}^3$ respectively, and $F : U \to V$ and $G : V \to W$ are linear, then*

(a) $[GF] = [G]\,[F]$.

(b) if F is invertible, $[F]^{-1} = \left[F^{-1}\right]$.

Proof. For part (a), there's no problem with the action on scalars or vectors, so all I need to check is the outermorphism property. I will check it for the product of vectors, which covers all the higher-grade cases too.

$$\begin{aligned} [G]\,[[F](u_1 \wedge \cdots \wedge u_r)] &= [G]\,[F(u_1) \wedge \cdots \wedge F(u_r)] \\ &= GF(u_1) \wedge \cdots \wedge GF(u_r) \\ &= [GF](u_1 \wedge \cdots \wedge u_r). \end{aligned} \tag{342}$$

Part (b) follows from part (a):

$$\begin{aligned} \left[F^{-1}\right]\,[F] &= \left[F^{-1}F\right] \\ &= [\mathrm{Id}_U] \\ &= \mathrm{Id}_{\mathcal{G}^1}, \end{aligned} \tag{343}$$

and similarly $[F]\,\left[F^{-1}\right] = \mathrm{Id}_{\mathcal{G}^2}$, so $\left[F^{-1}\right]$ is the inverse of $[F]$. □

As for adjoints, outermorphism extensions obey these relations, which (for the moment) use an amazing number of brackets.

Theorem 55.

$$
\begin{aligned}
A \rfloor \left\{ \overline{[F]}\,(B) \right\} &= \overline{[F]}\,(\{[F]\,(A)\} \rfloor B) \\
\{[F]\,(A)\} \lfloor B &= [F]\left(A \lfloor \left\{ \overline{[F]}\,(B) \right\}\right)
\end{aligned} \tag{344}
$$

Therefore $\overline{[F]} = \overline{[F]}$.

Because this theorem is hard to read, I'm going to state it in words. The purpose of the adjoint is to let you move F from one side of the scalar product to the other, as long as you change F to $\overline{F}$ along the way. This theorem says you can also move $[F]$ from the "high side" of the inner product to the "low side," as long as (a) you change $[F]$ to $\overline{[F]}$ and (b) you then act on the whole thing with $[F]$.

Take a look at two extreme cases. If the multivector on the low side is a scalar, the inner products become products, $[F]$ leaves the scalar alone, and it factors out. That's why there's an extra $[F]$ (or $\overline{[F]}$) acting on the whole thing. On the other extreme, if both sides have the same grade then the inner products are scalars and the extra $[F]$ or $\overline{[F]}$ drops out because it leaves scalars alone. With a little tweaking, that gets us back to the definition of the adjoint in Eq. (338). I'll do that tweaking in the proof.

Proof. I'll prove only the first relation; the second relation is the reverse of the first with a few substitutions. The result is true for general A and B if it's true for A_r and B_s. If $r > s$ both sides vanish identically, so let $r \le s$. If $r = 0$, then $A_r = \lambda$ and both sides reduce to $\lambda\,\overline{[F]}\,(B_s)$. If $s = 0$, then $r = 0$ so we're back to the previous case. For the remaining cases, I consider blades $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$. Next I'll prove $r = 1$ and any $s \ge 1$. It's true for $s = 1$, so assume it's true for $s - 1$, let $\boldsymbol{B_s} = b \wedge \boldsymbol{B_{s-1}}$, and consider

$$
\begin{aligned}
a \rfloor \overline{[F]}\,(\boldsymbol{B_s}) &= a \rfloor \overline{[F]}\,(b \wedge \boldsymbol{B_{s-1}}) \\
&= a \rfloor \left[\overline{F}(b) \wedge \overline{[F]}\,(\boldsymbol{B_{s-1}})\right] \\
&= \left[a \rfloor \overline{F}(b)\right]\,\overline{[F]}\,(\boldsymbol{B_{s-1}}) - \overline{F}(b) \wedge \left[a \rfloor \overline{[F]}\,(\boldsymbol{B_{s-1}})\right] \\
&= \left[F(a) \rfloor b\right]\,\overline{[F]}\,(\boldsymbol{B_{s-1}}) - \overline{F}(b) \wedge \overline{[F]}\,(F(a) \rfloor \boldsymbol{B_{s-1}}) \\
&= \overline{[F]}\,\left[(F(a) \rfloor b)\,\boldsymbol{B_{s-1}}\right] - \overline{[F]}\,\left[b \wedge (F(a) \rfloor \boldsymbol{B_{s-1}})\right] \\
&= \overline{[F]}\,\left[(F(a) \rfloor b)\,\boldsymbol{B_{s-1}} - b \wedge (F(a) \rfloor \boldsymbol{B_{s-1}})\right] \\
&= \overline{[F]}\,\left[F(a) \rfloor (b \wedge \boldsymbol{B_{s-1}})\right] \\
&= \overline{[F]}\,(F(a) \rfloor \boldsymbol{B_s}).
\end{aligned} \tag{345}
$$

Now for general $r \le s$. Fix s and assume the result is true for $r - 1$; then let $\boldsymbol{A_r} = \boldsymbol{A_{r-1}} \wedge a$ and consider

$$
\begin{aligned}
\boldsymbol{A_r} \rfloor \overline{[F]}\,(\boldsymbol{B_s}) &= (\boldsymbol{A_{r-1}} \wedge a) \rfloor \overline{[F]}\,(\boldsymbol{B_s}) \\
&= \boldsymbol{A_{r-1}} \rfloor (a \rfloor \overline{[F]}\,(\boldsymbol{B_s})) \\
&= \boldsymbol{A_{r-1}} \rfloor \overline{[F]}\,(F(a) \rfloor \boldsymbol{B_s}) \\
&= \overline{[F]}\,\left[[F]\,(\boldsymbol{A_{r-1}}) \rfloor (F(a) \rfloor \boldsymbol{B_s})\right] \\
&= \overline{[F]}\,\left[([F]\,(\boldsymbol{A_{r-1}}) \wedge F(a)) \rfloor \boldsymbol{B_s}\right] \\
&= \overline{[F]}\,([F]\,(\boldsymbol{A_{r-1}} \wedge a) \rfloor \boldsymbol{B_s}) \\
&= \overline{[F]}\,([F]\,(\boldsymbol{A_r}) \rfloor \boldsymbol{B_s})\,.
\end{aligned} \tag{346}
$$

And that takes care of all cases.

Finally, I'll show that $\overline{[F]} = \overline{[F]}$.

$$
A * \overline{[F]}\,(B) = \left\langle A^\dagger \rfloor \left\{ \overline{[F]}\,(B) \right\} \right\rangle
$$

$$
\begin{aligned}
&= \left\langle \overline{[F]} \left(\left\{ [F]\left(A^\dagger\right) \right\} \rfloor B \right) \right\rangle \\
&= \overline{[F]} \left(\left\langle \left\{ [F]\left(A^\dagger\right) \right\} \rfloor B \right\rangle \right) \\
&= \left\langle \left\{ [F]\left(A^\dagger\right) \right\} \rfloor B \right\rangle \\
&= \left\langle \left\{ [F]\left(A\right) \right\}^\dagger \rfloor B \right\rangle \\
&= [F]\left(A\right) * B. \qquad (347)
\end{aligned}
$$

Thus $\overline{[F]}$ satisfies Eq. (338) with $\mathcal{F} = [F]$, so $\overline{[F]} = \overline{[F]}$. □

Given the uniqueness of the outermorphism extension and its good behavior under composition, inverses, and adjoints, I will now happily drop the $[F]$ notation and let F refer either to the linear function on vectors or the resulting outermorphism. That certainly makes Theorem 55 easier to read: I'll take $A \rfloor F(B) = F\left(\overline{F}(A) \rfloor B\right)$ any day.

Recall that the restriction of F to the subspace represented by $\boldsymbol{A_r}$ is denoted $F_{\boldsymbol{A_r}}$. This restriction and $F(\boldsymbol{A_r})$ are related in an important way.

Theorem 56. *$F(\boldsymbol{A_r}) = 0$ iff $\operatorname{rank}(F_{\boldsymbol{A_r}}) < r$, which is true iff $F_{\boldsymbol{A_r}}$ is not one-to-one.*

> *Proof.* $F(\boldsymbol{A_r}) = 0$ iff F maps $\boldsymbol{A_r}$ to a subspace of dimension smaller than r, which means $\operatorname{rank}(F_{\boldsymbol{A_r}}) < r$. Since $\boldsymbol{A_r}$ is finite-dimensional, by the rank-nullity theorem this is true iff $\operatorname{null}(F_{\boldsymbol{A_r}}) > 0$, so F is not one-to-one. □

So $\boldsymbol{A_r} \in \operatorname{Ker} F$ iff $\operatorname{Ker}(F_{\boldsymbol{A_r}}) \neq \{0\}$. Note the two different meanings of F in these two statements.

Now only one mystery remains unsolved: why did grade involution turn out to be an outermorphism even though I made it preserve geometric products instead of outer products? Because the parity operation on which it's based has a special property.

Theorem 57. *The following conditions on F are equivalent.*

1. *F is an isometry.*
2. *$F(AB) = F(A)\,F(B)$ for all A and B.*
3. *$F(A \rfloor B) = F(A) \rfloor F(B)$ and $F(A \lfloor B) = F(A) \lfloor F(B)$ for all A and B.*

> *Proof.* I first assume F is an isometry. That means $F(a) \rfloor F(b) = a \rfloor b$ for all vectors; but $a \rfloor b$ is a scalar, so $F(a \rfloor b) = a \rfloor b$, so $F(a \rfloor b) = F(a) \rfloor F(b)$. Since $F(a \wedge b) = F(a) \wedge F(b)$ by the outermorphism property, it follows that $F(ab) = F(a)F(b)$. I can extend this to $F(a_1 a_2 \cdots a_r) = F(a_1)F(a_2)\cdots F(a_r)$ by induction: the result is true for $r = 2$, so assume it's true for $r-1$ and consider

$$
\begin{aligned}
F(a_1 a_2 \cdots a_r) &= F[a_1 \rfloor (a_2 \cdots a_r)] + F[a_1 \wedge (a_2 \cdots a_r)] \\
&= F\left[\sum_{j=2}^{r} (-1)^{j-2} a_1 \rfloor a_j\, a_2 \cdots \check{a}_j \cdots a_r\right] + F(a_1) \wedge F[a_2 \cdots a_r] \\
&= \sum_{j=2}^{r} (-1)^{j-2} a_1 \rfloor a_j F(a_2) \cdots \check{F}(a_j) \cdots F(a_r) + F(a_1) \wedge \{F(a_2) \cdots F(a_r)\} \\
&= \sum_{j=2}^{r} (-1)^{j-2} F(a_1) \rfloor F(a_j) F(a_2) \cdots \check{F}(a_j) \cdots F(a_r) + F(a_1) \wedge \{F(a_2) \cdots F(a_r)\} \\
&= F(a_1) \rfloor \{F(a_2) \cdots F(a_r)\} + F(a_1) \wedge \{F(a_2) \cdots F(a_r)\} \\
&= F(a_1) F(a_2) \cdots F(a_r). \qquad (348)
\end{aligned}
$$

Now let's look at $F(AB)$; by linearity it's a sum of terms of the form $F(\boldsymbol{A_r B_s})$. If $r = 0$ or $s = 0$ then $F(\boldsymbol{A_r B_s}) = F(\boldsymbol{A_r})\, F(\boldsymbol{B_s})$ by linearity and Theorem 52, so let $\boldsymbol{A_r} = a_1 a_2 \cdots a_r$ and $\boldsymbol{B_s} = b_1 b_2 \cdots b_s$; then Eq. (348) lets me show

$$\begin{aligned} F(\boldsymbol{A_r B_s}) &= F(a_1 a_2 \cdots a_r b_1 b_2 \cdots b_s) \\ &= F(a_1) F(a_2) \cdots F(a_r) F(b_1) F(b_2) \cdots F(b_s) \\ &= F(a_1 a_2 \cdots a_r) F(b_1 b_2 \cdots b_s) \\ &= F(\boldsymbol{A_r})\, F(\boldsymbol{B_s}). \end{aligned} \tag{349}$$

Therefore $F(AB) = F(A)\, F(B)$. Notice that the products among the a_i equal outer products, and the same is true of products among the b_j; it is the product of a_r and b_1 that does not equal an outer product, and this is the reason I needed $F(ab) = F(a)F(b)$, not just $F(a \wedge b) = F(a) \wedge F(b)$, for the proof.

Next, assume $F(AB) = F(A)\, F(B)$ for all A and B. Then, since F commutes with all grade operators,

$$\begin{aligned} F(A_r \rfloor B_s) &= F(\langle A_r B_s \rangle_{s-r}) \\ &= \langle F(A_r B_s) \rangle_{s-r} \\ &= \langle F(A_r)\, F(B_s) \rangle_{s-r} \\ &= F(A_r) \rfloor F(B_s), \end{aligned} \tag{350}$$

with the last line following because F preserves grades. Thus

$$F(A \rfloor B) = F(A) \rfloor F(B) \tag{351}$$

in general. Replacing $s - r$ with $r - s$ in the proof yields the same result for the right inner product.

Finally, assume $F(A \rfloor B) = F(A) \rfloor F(B)$ for all A and B; then $F(a) \rfloor F(b) = F(a \rfloor b) = a \rfloor b$ for any two vectors a and b, where the last equality follows because $a \rfloor b$ is a scalar. Therefore F is an isometry, and all three results listed above are equivalent.

If I had assumed F was not just an isometry but orthogonal, I could have used $\overline{F} = F^{-1}$ and Theorem 55 to prove the third part, but this way is better because it assumes less (on infinite-dimensional spaces at least). □

Thus an isometry may be extended by having it respect either outer products or geometric products, with the same result. That won't work for anything else, though. This is why grade involution came out fine, and that's why I could indeed have extended reflections and rotations by respecting products instead of outer products. Projections and rejections had to be done the way I did them, however.

9.7. Eigenblades and invariant subspaces

Vector a is an *eigenvector* of F with eigenvalue λ if $F(a) = \lambda a$, or equivalently $a \in \operatorname{Ker}(F - \lambda \operatorname{Id})$. In this definition a can't be zero, but λ can. Now suppose a_1 and a_2 are eigenvectors with eigenvalues λ_1 and λ_2 and let $\boldsymbol{A} = a_1 \wedge a_2$; then

$$\begin{aligned} F(\boldsymbol{A}) &= F(a_1 \wedge a_2) \\ &= F(a_1) \wedge F(a_2) \\ &= \lambda_1 \lambda_2\, a_1 \wedge a_2 \\ &= \lambda_1 \lambda_2\, \boldsymbol{A}. \end{aligned} \tag{352}$$

So $\boldsymbol{A}$ is an *eigenblade* of F; it's mapped by F to a multiple of itself.

An eigenblade is defined generally by $F(\boldsymbol{A}) = \lambda \boldsymbol{A}$, regardless of what the factors of $\boldsymbol{A}$ do. Rotation operators nicely illustrate the different ways eigenblades can arise, and how they are related (or not) to eigenvectors. Let $F(A) = \exp(-\boldsymbol{B}\theta/2) A \exp(\boldsymbol{B}\theta/2)$, so F is a rotation in plane $\boldsymbol{B}$ through angle θ. Then

$$F(\boldsymbol{B}) = \exp(-\boldsymbol{B}\theta/2) \boldsymbol{B} \exp(\boldsymbol{B}\theta/2)$$

$$
\begin{aligned}
&= \exp(-\boldsymbol{B}\theta/2)\exp(\boldsymbol{B}\theta/2)\boldsymbol{B} \\
&= \boldsymbol{B},
\end{aligned} \tag{353}
$$

so $\boldsymbol{B}$ is an eigenplane of the rotation operator with eigenvalue 1. However, since every vector in $\boldsymbol{B}$ gets rotated, in general none of them are eigenvectors. (The exception is $\boldsymbol{B}^2 = -1$ and $\theta = \pi$, which is a reflection: every vector in the plane is an eigenvector with eigenvalue -1.) You can check that $\boldsymbol{B}^\perp$ is also an eigenblade with eigenvalue 1, but that's for a different reason: every vector in $\boldsymbol{B}^\perp$ is left alone by the rotation. So $F(\boldsymbol{A}) = \boldsymbol{A}$ is consistent with $F(a) = a$ for every vector in $\boldsymbol{A}$, but it's consistent with many other things too.

If $\lambda = 0$, then Theorem 56 tells me that F is not one-to-one on $\boldsymbol{A}$, but that's all it tells me. On the other hand, $\lambda \neq 0$ tells me all sorts of things. First, Theorem 56 says F is one-to-one on $\boldsymbol{A}$, so $F_{\boldsymbol{A}}$ is invertible. But it also hints at what $\mathrm{Range}(F_{\boldsymbol{A}})$ is. In fact, it's hard to see how F could map $\boldsymbol{A}$ to a multiple of itself unless it also mapped all members of $\boldsymbol{A}$ back into $\boldsymbol{A}$. That would make $\boldsymbol{A}$ an *invariant subspace* of F; that is, a subspace that is mapped to itself by F. Put that together with Theorem 56 and you get this result.

Theorem 58. *Any eigenblade of F with nonzero eigenvalue is an invariant subspace of F on which F is invertible.*

Proof. Suppose $F(\boldsymbol{A}) = \lambda\boldsymbol{A}$ and $\lambda \neq 0$. We already know that F is invertible on $\boldsymbol{A}$, so I'll prove the first part. If a lies in $\boldsymbol{A}$, then $\boldsymbol{A} \wedge a = 0$, in which case

$$
\begin{aligned}
\boldsymbol{A} \wedge F(a) &= \lambda^{-1}\lambda\boldsymbol{A} \wedge F(a) \\
&= \lambda^{-1}F(\boldsymbol{A}) \wedge F(a) \\
&= \lambda^{-1}F(\boldsymbol{A} \wedge a) \\
&= 0,
\end{aligned} \tag{354}
$$

so $F(a)$ lies in $\boldsymbol{A}$ too. □

So F maps $\boldsymbol{A}$ invertibly onto $\boldsymbol{A}$. Now this is true for any $\lambda \neq 0$; but what does the actual value of λ tell us? Well, if $F(\boldsymbol{A}) = \lambda\boldsymbol{A}$ then $|F(\boldsymbol{A})|^2 = \lambda^2|\boldsymbol{A}|^2$, so the value of λ determines how much the norm squared of $\boldsymbol{A}$ changes. If the scalars are real, I can interpret this further. Recalling Eq. (173) for the weight of a blade, I find that

$$
\mathrm{weight}(F(\boldsymbol{A})) = |\lambda|\,\mathrm{weight}(\boldsymbol{A}), \tag{355}
$$

so F multiplies the weight associated with $\boldsymbol{A}$ by $|\lambda|$; and F changes the orientation of $\boldsymbol{A}$ if λ is negative. This suggests to me that λ is actually the determinant of $F_{\boldsymbol{A}}$, since it seems to be the factor by which the volume of $\boldsymbol{A}$ changes. The idea that the determinant of a linear transformation is actually an eigenvalue, not of the original transformation but of its outermorphism extension, is worth following up on and has general validity, even if the scalars aren't real. So I'll do that next.

9.8. The determinant

The *determinant* of a linear transformation is the factor by which it multiplies the volume element of the space it acts on. Finding it in geometric algebra is easy; we consider $F(\boldsymbol{I})$. This is an n-blade, so it has to be a multiple of $\boldsymbol{I}$. (Put another way, $\boldsymbol{I}$ is an eigenblade of all linear transformations.) That multiple is the determinant, or

$$
F(\boldsymbol{I}) =: \det(F)\boldsymbol{I}. \tag{356}
$$

An equivalent definition is

$$
\det(F) = F(\boldsymbol{I})^\perp. \tag{357}
$$

This way of defining the determinant is very intuitive and also very easy to use in calculations, as I'll show.

First, it's obvious that $\det(\mathrm{Id}) = 1$. Now let F and G be linear transformations; since

$$
\begin{aligned}
\det(FG)\boldsymbol{I} &= FG(\boldsymbol{I}) \\
&= F\left(\det(G)\boldsymbol{I}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \det(G) F(\boldsymbol{I}) \\
&= \det(G) \det(F) \boldsymbol{I}, \qquad (358)
\end{aligned}
$$

I find with minimum fuss that

$$
\det(FG) = \det(F)\det(G). \qquad (359)
$$

Therefore if F is invertible,

$$
\begin{aligned}
\det(F^{-1})\det(F) &= \det(F^{-1}F) \\
&= \det(\mathrm{Id}) \\
&= 1 \qquad (360)
\end{aligned}
$$

so

$$
\det(F^{-1}) = \det(F)^{-1}. \qquad (361)
$$

That tells me that $\det(F) \neq 0$ if F is invertible. I'll use that later.

Now for adjoints. From the definition in Eq. (338),

$$
\begin{aligned}
\overline{F}(\boldsymbol{I}) * \boldsymbol{I} &= \boldsymbol{I} * F(\boldsymbol{I}) \\
\det(\overline{F})\boldsymbol{I} * \boldsymbol{I} &= \det(F)\boldsymbol{I} * \boldsymbol{I} \\
\det(\overline{F}) &= \det(F). \qquad (362)
\end{aligned}
$$

It's easy when you know how.

Next I'll calculate the determinants of some specific operators.

- From Eq. (235) I get the determinants of orthogonal projections and rejections:

$$
\begin{aligned}
\det(P_{\boldsymbol{A_r}}) &= \delta_{rn} \\
\det(R_{\boldsymbol{A_r}}) &= \delta_{r0}. \qquad (363)
\end{aligned}
$$

- Let F be a symmetric operator with a frame $\{a_i\}$ of eigenvectors with eigenvalues $\{\lambda_i\}$. Then $a_1 \wedge \cdots \wedge a_n$ is a volume element, so

$$
\begin{aligned}
\det(F)\, a_1 \wedge \cdots \wedge a_n &= F(a_1 \wedge \cdots \wedge a_n) \\
&= F(a_1) \wedge \cdots \wedge F(a_n) \\
&= \lambda_1 \cdots \lambda_n\, a_1 \wedge \cdots \wedge a_n \qquad (364)
\end{aligned}
$$

  so

$$
\det(F) = \lambda_1 \cdots \lambda_n. \qquad (365)
$$

- If F is orthogonal, then $F(v) = A_r v^{*r} A_r^{-1}$ for some rotor A_r, so Eq. (318) shows that

$$
\det(F) = (-1)^r. \qquad (366)
$$

In real matrix algebra, there's a well-known relationship between determinants, invertibility, and adjoints: a matrix is invertible iff its determinant is nonzero, in which case its inverse is its adjugate (adjoint of the cofactor matrix) divided by its determinant. The geometric algebra equivalent, as you would expect, applies to the linear operator itself, not its matrix representation on some basis. On top of that, it's valid for any multivector.

Theorem 59. *F is invertible iff $\det(F) \neq 0$, and for any multivector A*

$$
F^{-1}(A) = \frac{\overline{F}(A^{-\perp})^{\perp}}{\det(F)}. \qquad (367)
$$

Recall that $^\perp$ is the duality transform and $^{-\perp}$ is its inverse.

Proof. I've already proven that if F is invertible, $\det(F) \neq 0$, so let's go the other way. Suppose $\det(F) \neq 0$ and let G be defined by $G(A) = \overline{F}(A^{-\perp})^\perp$. Then I use the result $A \rfloor \overline{F}(B) = \overline{F}(F(A) \rfloor B)$ from Theorem 55 to get

$$\begin{aligned}
GF(A) &= G[F(A)] \\
&= \overline{F}[F(A)^{-\perp}]^\perp \\
&= \overline{F}[F(A) \rfloor \boldsymbol{I}]\boldsymbol{I}^{-1} \\
&= [A \rfloor \overline{F}(\boldsymbol{I})]\boldsymbol{I}^{-1} \\
&= \det(\overline{F})(A \rfloor \boldsymbol{I})\boldsymbol{I}^{-1} \\
&= \det(F)A\boldsymbol{I}\boldsymbol{I}^{-1} \\
&= \det(F)A. \qquad (368)
\end{aligned}$$

A similar argument shows $FG(A) = \det(F)A$, so $F^{-1} = G/\det(F)$. □

You can see that you need the full apparatus of geometric algebra to do this: I take the dual, which is a geometric product with a volume element, and I need to use the outermorphism extension of F, because calculating $F^{-1}(a)$ involves calculating $\overline{F}(a^{-\perp})$, and $a^{-\perp}$ is not a vector. (Unless we're in a two-dimensional space, I suppose.)

10. Applications

10.1. Classical particle mechanics

The most obvious place to apply geometric algebra is classical mechanics, since it relies heavily on vector algebra already. In this section only I'll adopt the notational conventions of classical mechanics, so vectors are denoted $\boldsymbol{a}$, $\boldsymbol{b}$, and so on, the magnitude of vector $\boldsymbol{a}$ is denoted a, unit vectors are indicated with an overhat $\hat{}$, and the derivative of any quantity with respect to time is indicated by an overdot $\dot{}$. Since boldface means something else, I will not use boldface for blades in this section. The material in this section is largely drawn from [3] and [4].

10.1.1. Angular momentum as a bivector

As a particle moves over time, its position vector $\boldsymbol{r}$ sweeps out area at a rate that a picture easily shows to be

$$\dot{A} = \frac{1}{2}\boldsymbol{r} \wedge \boldsymbol{v} \qquad (369)$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$ is the particle's velocity vector. Unsurprisingly, the rate at which area is swept out is a 2-blade. This blade is proportional to the dynamical quantity

$$L := \boldsymbol{r} \wedge \boldsymbol{p} = m\boldsymbol{r} \wedge \boldsymbol{v} = 2m\dot{A}, \qquad (370)$$

called the *angular momentum*. In standard vector algebra, angular momentum is defined to be the vector $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$, the cross product of $\boldsymbol{r}$ and $\boldsymbol{p}$; the definition given here is the dual of that vector (see Eq. (208)), which is more natural given the association with areas. Nonetheless, since the algebraic properties of the outer and cross products are so similar, much of what one knows from the standard treatment holds without change; for example,

$$\begin{aligned}
\dot{L} &= m\boldsymbol{v} \wedge \boldsymbol{v} + m\boldsymbol{r} \wedge \dot{\boldsymbol{v}} \\
&= m\boldsymbol{r} \wedge \dot{\boldsymbol{v}} \\
&= \boldsymbol{r} \wedge \boldsymbol{F}, \qquad (371)
\end{aligned}$$

so L is conserved iff the force $\boldsymbol{F}$ is central (parallel or antiparallel to $\boldsymbol{r}$). Since L is conserved iff $\dot{A} = 0$, I have Kepler's Second Law: the position vector of a particle subject to central forces sweeps out equal areas in equal times. Further, the plane in which central force motion takes place is L itself. Writing $\boldsymbol{r} = r\hat{\boldsymbol{r}}$, which implies $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}$, I find

$$\begin{aligned} L &= \boldsymbol{r} \wedge \boldsymbol{p} = m\boldsymbol{r} \wedge \boldsymbol{v} \\ &= mr\hat{\boldsymbol{r}} \wedge (\dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}) \\ &= mr^2\hat{\boldsymbol{r}} \wedge \dot{\hat{\boldsymbol{r}}} \end{aligned} \tag{372}$$

since $\hat{\boldsymbol{r}} \wedge \hat{\boldsymbol{r}} = 0$. But I know a bit more than that; $\hat{\boldsymbol{r}}$ is a unit vector, or $\hat{\boldsymbol{r}} \rfloor \hat{\boldsymbol{r}} = 1$, the time derivative of which is $\hat{\boldsymbol{r}} \rfloor \dot{\hat{\boldsymbol{r}}} = 0$. This is just the familiar fact that a constant-length vector and its time derivative must always be perpendicular. (Incidentally, this shows that $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\hat{\boldsymbol{r}}}$ is a decomposition into radial and tangential components.) If $\hat{\boldsymbol{r}} \rfloor \dot{\hat{\boldsymbol{r}}} = 0$, then $\hat{\boldsymbol{r}} \wedge \dot{\hat{\boldsymbol{r}}} = \hat{\boldsymbol{r}}\dot{\hat{\boldsymbol{r}}}$, so

$$L = mr^2\hat{\boldsymbol{r}}\dot{\hat{\boldsymbol{r}}}. \tag{373}$$

Now this is nice because the geometric product has better properties than the outer product, and this is the first algebraic feature of this treatment that is genuinely new. Since L is a bivector,

$$L = -L^\dagger = -mr^2\dot{\hat{\boldsymbol{r}}}\hat{\boldsymbol{r}} \tag{374}$$

so the scalar l, the magnitude of L, is given by

$$l^2 := |L|^2 = L^\dagger L = -L^2 = m^2r^4\dot{\hat{\boldsymbol{r}}}^2. \tag{375}$$

Notice that l equals the magnitude of the angular momentum vector from standard treatments. It should, of course, since bivector L and the angular momentum vector are duals.

10.1.2. The Kepler problem

The Kepler problem is to determine the motion of a point particle of mass m moving in a potential of the form $V = -k/r$, where r is the particle's distance from some fixed origin. The particle experiences a force

$$\boldsymbol{F} = -\frac{k}{r^2}\hat{\boldsymbol{r}} \tag{376}$$

where the constant k is positive for an attractive force and negative for a repulsive force, so the particle's acceleration is given by

$$\dot{\boldsymbol{v}} = -\frac{k}{mr^2}\hat{\boldsymbol{r}}. \tag{377}$$

Now take a look at Eqs. (373) and (377). One is proportional to r^2 while the other is inversely proportional to r^2, so their product is independent of r. In fact, let me calculate the product:

$$\begin{aligned} L\dot{\boldsymbol{v}} &= \left(-mr^2\dot{\hat{\boldsymbol{r}}}\hat{\boldsymbol{r}}\right)\left(-\frac{k}{mr^2}\hat{\boldsymbol{r}}\right) \\ &= k\dot{\hat{\boldsymbol{r}}}, \end{aligned}$$

and since L is conserved and k is a constant, this implies

$$\frac{d}{dt}\left(L\boldsymbol{v} - k\hat{\boldsymbol{r}}\right) = 0. \tag{378}$$

Well, look at that: another constant of motion. The second term in the constant, $k\hat{\boldsymbol{r}}$, is clearly a vector, and the first term can be written

$$L\boldsymbol{v} = L \lfloor \boldsymbol{v} + L \wedge \boldsymbol{v}$$

$$
\begin{aligned}
&= L \lfloor \boldsymbol{v} + m\boldsymbol{r} \wedge \boldsymbol{v} \wedge \boldsymbol{v} \\
&= L \lfloor \boldsymbol{v}. \qquad (379)
\end{aligned}
$$

This is hardly a surprise; $\boldsymbol{v}$ is a vector in the plane defined by L, so by Theorems 16 and 17, $L\boldsymbol{v} = L \lfloor \boldsymbol{v}$ is a nonzero vector in the plane of L perpendicular to $\boldsymbol{v}$. Thus the conserved quantity is a vector in the plane of motion; it is often called the "Laplace-Runge-Lenz vector," and in traditional vector algebra treatments of the Kepler problem it typically appears at the end as the result of a great deal of work. Here it was the first thing I found.

I would actually prefer to define a dimensionless conserved vector, and this quantity clearly has dimensions of k, so I define a conserved vector $\boldsymbol{e}$ by

$$
\boldsymbol{e} := \frac{L\boldsymbol{v}}{k} - \hat{\boldsymbol{r}}. \qquad (380)
$$

I'd like to use this equation to get further expressions describing the motion of the particle; first is the polar equation, r as a function of direction. Since the expression for $\boldsymbol{e}$ has $\hat{\boldsymbol{r}}$ in it and $\hat{\boldsymbol{r}}\boldsymbol{r} = r$, it follows that I can get an equation for r by multiplying Eq. (380) by $k\boldsymbol{r}$, with the result

$$
L\boldsymbol{v}\boldsymbol{r} = k(\hat{\boldsymbol{r}}\boldsymbol{r} + \boldsymbol{e}\boldsymbol{r}). \qquad (381)
$$

The left hand side equals

$$
\begin{aligned}
L\boldsymbol{v}\boldsymbol{r} &= L(\boldsymbol{v} \rfloor \boldsymbol{r} + \boldsymbol{v} \wedge \boldsymbol{r}) \\
&= (\boldsymbol{r} \rfloor \boldsymbol{v})L - \frac{L^2}{m} \\
&= \frac{l^2}{m} + (\boldsymbol{r} \rfloor \boldsymbol{v})L \qquad (382)
\end{aligned}
$$

while $\hat{\boldsymbol{r}}\boldsymbol{r} = r$ and $\boldsymbol{e}\boldsymbol{r} = er\cos\theta + \boldsymbol{e} \wedge \boldsymbol{r}$, so putting it all together

$$
\frac{l^2}{m} + (\boldsymbol{r} \rfloor \boldsymbol{v})L = k(r + er\cos\theta + \boldsymbol{e} \wedge \boldsymbol{r}), \qquad (383)
$$

or on separating the scalar and bivector parts,

$$
\begin{aligned}
\frac{l^2}{m} &= k(r + er\cos\theta) \\
(\boldsymbol{r} \rfloor \boldsymbol{v})L &= \boldsymbol{e} \wedge \boldsymbol{r}. \qquad (384)
\end{aligned}
$$

The scalar equation can be solved for r with the result

$$
r = \frac{l^2/mk}{1 + e\cos\theta}, \qquad (385)
$$

which is the equation for a conic section with eccentricity e and one focus at the origin. Since the length of $\boldsymbol{e}$ is the eccentricity of the orbit, $\boldsymbol{e}$ is naturally called the *eccentricity vector*, which is the name I'll use for it henceforth.

The direction of $\boldsymbol{e}$ also has a geometrical meaning, but it's different in the attractive and repulsive cases, so I'll do one at a time. First I assume $k > 0$ and I note that r equals its minimum and maximum values when $\theta = 0$ and π respectively, which means that $\boldsymbol{e}$ points toward the particle's point of closest approach, called its *periapsis*, and away from its point of farthest retreat, called the *apoapsis*. (Fun fact: these two points are called the perigee and apogee if you're orbiting the earth, the perihelion and aphelion if you're orbiting the sun, and the pericynthion and apocynthion if you're orbiting the moon. So now you know.)

Now the repulsive case. If $k < 0$, we run into a problem: r has to be non-negative, so we have to have $1 + e\cos\theta \le 0$ for at least some values of θ, and the orbit may include only those values. This is possible iff $e > 1$, with the result that the orbit is a hyperbola. In this case, r takes on its smallest value when $\theta = \pi$, so in the repulsive case the eccentricity vector points away from the periapsis.

The motion in the Kepler problem is completely determined by two vectors, the initial position and velocity, which are themselves determined by six parameters. The conserved angular momentum supplies three parameters because it's a bivector, and the conserved eccentricity vector supplies two more (only two because the angular momentum fixes the plane of the motion), so the motion is completely determined by these two constants plus one further parameter, which may be taken to be the initial value of θ. If that's the case, then anything that doesn't depend on the starting point, such as any other constants of motion, should be a function of only L and $\boldsymbol{e}$. I'll now show that this is the case for the energy by finding the magnitude of the eccentricity vector.

$$\begin{aligned}(L\boldsymbol{v} - k\hat{\boldsymbol{r}})^2 &= k^2 e^2 \\ (L\boldsymbol{v})^2 - 2k(L\boldsymbol{v}) \rfloor \hat{\boldsymbol{r}} + k^2 &= k^2 e^2\end{aligned} \tag{386}$$

Using the fact that a vector equals its own reverse, the first term on the left hand side can be calculated as

$$\begin{aligned}(L\boldsymbol{v})^2 &= L\boldsymbol{v}L\boldsymbol{v} \\ &= (L\boldsymbol{v})^\dagger L\boldsymbol{v} \\ &= \boldsymbol{v}L^\dagger L\boldsymbol{v} \\ &= l^2 v^2.\end{aligned} \tag{387}$$

The second term on the left is $-2k$ times

$$\begin{aligned}(L\boldsymbol{v}) \rfloor \hat{\boldsymbol{r}} &= \frac{(L\boldsymbol{v}) \rfloor \boldsymbol{r}}{r} \\ &= \frac{\langle L\boldsymbol{v}\boldsymbol{r}\rangle}{r} \\ &= \frac{l^2}{mr},\end{aligned} \tag{388}$$

where in the last line I used Eq. (382), so now Eq. (386) becomes

$$\begin{aligned}l^2 v^2 - \frac{2kl^2}{mr} &= k^2(e^2 - 1) \\ \frac{2l^2}{m}\left(\frac{1}{2}mv^2 - \frac{k}{r}\right) &= k^2(e^2 - 1)\end{aligned} \tag{389}$$

or

$$E = \frac{mk^2}{2l^2}(e^2 - 1). \tag{390}$$

This gives the energy in terms of l and e.

I have derived all the main results of the Kepler problem (except for the time evolution) a whole lot more easily than standard treatments do. In fact, many textbooks don't even get to the eccentricity vector. Here geometric algebra is clearly superior to standard vector algebra both for solving the equations and for understanding the results.

A. Summary of definitions and formulas

A.1. Notation

$\mathcal{G}$	Geometric algebra
$\mathcal{G}_r$	Grade-r subspace of $\mathcal{G}$ (space of r-vectors)
$\mathcal{G}^n$	Geometric algebra of an n-dimensional vector space
A, B, etc.	General multivector
λ, μ, etc.	Scalar

a, b, u, v, etc.	Vector
A_r	r-vector (sometimes grade-r part of A)
$\boldsymbol{A_r}$	r-blade
A_+	Even-grade multivector
A_-	Odd-grade multivector
$\boldsymbol{I}$	Volume element
$\langle A \rangle_r$	Grade-r part of A
$\langle A \rangle$	Scalar (grade-0) part of A
$\langle A \rangle_+$	Even-grade part of A
$\langle A \rangle_-$	Odd-grade part of A
A^{-1}	Inverse of A
A^*	Grade involution of A
A^{*r}	r times grade involuted A
$A^\dagger$	Reverse of A
$A^\ddagger$	Clifford conjugate of A
$\lvert A\rvert^2$	Squared norm of A
$A^\perp$	Dual of A
$A^{-\perp}$	Inverse dual of A
AB	Geometric product of A and B
$A \rfloor B$	Left inner product of A into B
$A \lfloor B$	Right inner product of A by B
$A \wedge B$	Outer product of A and B
$A * B$	Scalar product of A and B
$A \times B$	Commutator of A and B
$P_{\boldsymbol{A_r}}(B)$	Orthogonal projection of B into $\boldsymbol{A_r}$
$R_{\boldsymbol{A_r}}(B)$	Orthogonal rejection of B from $\boldsymbol{A_r}$
U, V, W, etc.	Vector space
F, G, etc.	Linear function of vectors (or its outermorphism extension)
F_U	Restriction of F to subspace U
$F_{\boldsymbol{A}}$	Restriction of F to subspace represented by $\boldsymbol{A}$
Id	Identity function
$\mathrm{Ker}(F)$	Kernel of F
$\mathrm{Range}(F)$	Range of F
$\mathrm{null}(F)$	Nullity of F
$\mathrm{rank}(F)$	Rank of F
$\overline{F}$	Adjoint of F
$\det(F)$	Determinant of F
$\mathcal{F}$	Outermorphism

A.2. Axioms

A geometric algebra $\mathcal{G}$ is a set with two composition laws, addition and multiplication, that satisfy these axioms.

Axiom 1. *$\mathcal{G}$ is a ring with unit. The additive identity is called* 0 *and the multiplicative identity is called* 1.

Axiom 2. *$\mathcal{G}$ contains a field $\mathcal{G}_0$ of characteristic zero which includes* 0 *and* 1.

Axiom 3. *$\mathcal{G}$ contains a subset $\mathcal{G}_1$ closed under addition, and $\lambda \in \mathcal{G}_0, v \in \mathcal{G}_1$ implies $\lambda v = v\lambda \in \mathcal{G}_1$.*

Axiom 4. *The square of every vector is a scalar.*

Axiom 5. *The inner product is nondegenerate.*

Axiom 6. *If $\mathcal{G}_0 = \mathcal{G}_1$, then $\mathcal{G} = \mathcal{G}_0$. Otherwise, $\mathcal{G}$ is the direct sum of all the $\mathcal{G}_r$.*

A.3. Contents of a geometric algebra

An r-blade $\boldsymbol{A_r}$ is the outer product of r vectors, $a_1 \wedge \cdots \wedge a_r$. It represents the subspace spanned by $\{a_j\}_{j=1,\ldots,r}$, with a weight and orientation if the scalars are real.

$\boldsymbol{A_r} = 0$ iff the a_j are linearly dependent.

$\boldsymbol{A_r}$ and $\boldsymbol{B_r}$ define the same subspace iff $\boldsymbol{A_r} = \lambda \boldsymbol{B_r}$.

If $\boldsymbol{A_r}$ is a proper subspace of $\boldsymbol{A_s}$, then $\boldsymbol{A_r}$ can be factored out of $\boldsymbol{A_s}$ from either the left ($\boldsymbol{A_s} = \boldsymbol{A_r} \wedge \boldsymbol{A_{s-r}}$) or the right ($\boldsymbol{A_s} = \boldsymbol{A_{s-r}} \wedge \boldsymbol{A_r}$). The grade-$s-r$ factors in each case may be chosen to be the same except for at most a sign.

$a \wedge \boldsymbol{A_r} = 0$ iff a lies in $\boldsymbol{A_r}$.

$a \rfloor \boldsymbol{A_r} = 0$ iff a is orthogonal to $\boldsymbol{A_r}$.

The reflection of multivector B in subspace $\boldsymbol{A_r}$ is $\boldsymbol{A_r} B^{*r} \boldsymbol{A_r}^{-1}$.

$\boldsymbol{A_r}$, $\boldsymbol{A_r}^*$, $\boldsymbol{A_r}^\dagger$, $\boldsymbol{A_r}^\ddagger$, and $\boldsymbol{A_r}^{-1}$ (if it exists) represent the same subspace.

A.4. The inner, outer, and geometric products

$$A_r B_s = \sum_{j=0}^{\min\{r,s\}} \langle A_r B_s \rangle_{|r-s|+2j} \tag{1}$$

$$\langle A_r B_s \rangle_{r+s-2j} = (-1)^{rs-j} \langle B_s A_r \rangle_{r+s-2j} \tag{2}$$

$$\begin{aligned} \langle AB \rangle &= \langle BA \rangle && (3)\\ &= \langle A^* B^* \rangle && (4)\\ &= \left\langle A^\dagger B^\dagger \right\rangle && (5)\\ &= \left\langle A^\ddagger B^\ddagger \right\rangle && (6) \end{aligned}$$

$$\begin{aligned} A \rfloor B &= \sum_{r,s} \langle A_r B_s \rangle_{s-r} && (7)\\ A \lfloor B &= \sum_{r,s} \langle A_r B_s \rangle_{r-s} && (8)\\ A \wedge B &= \sum_{r,s} \langle A_r B_s \rangle_{r+s} && (9) \end{aligned}$$

$$\begin{aligned} A_r \rfloor B_s &= (-1)^{r(s-1)} B_s \lfloor A_r && (10)\\ A_r \wedge B_s &= (-1)^{rs} B_s \wedge A_r && (11) \end{aligned}$$

$$a_1 \wedge a_2 \wedge \cdots \wedge a_r = \langle a_1 a_2 \cdots a_r \rangle_r \tag{12}$$

$$a \rfloor A = \frac{1}{2}(aA - A^* a) \tag{13}$$

$$a \wedge A = \frac{1}{2}(aA + A^*a) \tag{14}$$

$$A \lfloor a = -a \rfloor A^* \tag{15}$$

$$A \wedge a = \quad a \wedge A^* \tag{16}$$

$$a \wedge A \wedge b = -b \wedge A \wedge a \tag{17}$$

$$a \rfloor (AB) = (a \rfloor A)B + A^*(a \rfloor B) \tag{18}$$

$$= (a \wedge A)B - A^*(a \wedge B) \tag{19}$$

$$a \wedge (AB) = (a \wedge A)B - A^*(a \rfloor B) \tag{20}$$

$$= (a \rfloor A)B + A^*(a \wedge B) \tag{21}$$

$$a \rfloor (A \wedge B) = (a \rfloor A) \wedge B + A^* \wedge (a \rfloor B) \tag{22}$$

$$a \wedge (A \lfloor B) = (a \wedge A) \lfloor B - A^* \lfloor (a \rfloor B) \tag{23}$$

$$a \wedge (A \rfloor B) = (a \rfloor A) \rfloor B + A^* \rfloor (a \wedge B) \tag{24}$$

$$a \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = \sum_{j=1}^{r} (-1)^{j-1} (a \rfloor a_j)\, a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_j \wedge \cdots \wedge a_r \tag{25}$$

$$a_1 \wedge (a_2 \wedge \cdots \wedge a_r) = a_1 \wedge a_2 \wedge \cdots \wedge a_r \tag{26}$$

If $r \leq s$ then

$$B_r \rfloor (a_1 \wedge a_2 \wedge \cdots \wedge a_s) = \sum (-1)^{\sum_{j=1}^{r}(i_j - j)} (B_r \rfloor a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r})\, a_{i_{r+1}} \wedge \cdots \wedge a_{i_s} \tag{27}$$

where the sum is performed over all possible choices of $\{a_{i_j}\}_{j=1,\ldots,r}$ out of $\{a_i\}_{i=1,\ldots,s}$, and in each term i_1 through i_r and i_{r+1} through i_s separately are in ascending order.

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C \tag{28}$$

$$A \rfloor (B \lfloor C) = (A \rfloor B) \lfloor C \tag{29}$$

$$A \rfloor (B \rfloor C) = (A \wedge B) \rfloor C \tag{30}$$

$$A \lfloor (B \wedge C) = (A \lfloor B) \lfloor C \tag{31}$$

If $A = a_1 a_2 \cdots a_r$, then

$$AB_sA^\dagger = \left\langle AB_sA^\dagger \right\rangle_s \tag{32}$$

and

$$(ABA^\dagger) \wedge (ACA^\dagger) = |A|^2 A(B \wedge C)A^\dagger \tag{33}$$

A.5. The geometric meaning of the inner and outer products

$\boldsymbol{A_r} \wedge \boldsymbol{B_s} = 0$ iff $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ share nonzero vectors.

$\boldsymbol{A_r} \wedge \boldsymbol{B_s}$, if nonzero, represents the direct sum of $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$.

$\boldsymbol{A_r} \rfloor \boldsymbol{B_s} = 0$ iff $\boldsymbol{A_r}$ contains a nonzero vector orthogonal to $\boldsymbol{B_s}$.

$\boldsymbol{A_r} \rfloor \boldsymbol{B_s}$, if nonzero, represents the orthogonal complement of $\boldsymbol{A_r}$ in $\boldsymbol{B_s}$.

If $\boldsymbol{A_r}$ and $\boldsymbol{B_s}$ are orthogonal, then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \wedge \boldsymbol{B_s}$.

If $\boldsymbol{A_r}$ is a subspace of $\boldsymbol{B_s}$, then $\boldsymbol{A_r}\boldsymbol{B_s} = \boldsymbol{A_r} \rfloor \boldsymbol{B_s}$.

The converses of the previous two statements are true if (1) $r = 1$ or $s = 1$ or (2) $\boldsymbol{A_r}$ or $\boldsymbol{B_s}$ is invertible.

$$P_{\boldsymbol{A_r}}(B) = B \rfloor \boldsymbol{A_r}\boldsymbol{A_r}^{-1} = (B \rfloor \boldsymbol{A_r}) \rfloor \boldsymbol{A_r}^{-1} \tag{34}$$
$$R_{\boldsymbol{A_r}}(B) = B \wedge \boldsymbol{A_r}\boldsymbol{A_r}^{-1} = B \wedge \boldsymbol{A_r} \lfloor \boldsymbol{A_r}^{-1} \tag{35}$$

A.6. Grade involution

$$\lambda^* = \ \ \lambda \tag{36}$$
$$a^* = -a \tag{37}$$
$$(AB)^* = A^*B^* \tag{38}$$
$$(A+B)^* = A^* + B^* \tag{39}$$

$$A_r^* = (-1)^r A_r \tag{40}$$

$$A^* = \langle A \rangle_+ - \langle A \rangle_- \tag{41}$$

$$A^* = \boldsymbol{I} A^{*n} \boldsymbol{I}^{-1} \tag{42}$$

$$\langle A \rangle_\pm = \frac{1}{2}(A \pm A^*) \tag{43}$$

$$A^{**} = A \tag{44}$$

$$(A^{-1})^* = (A^*)^{-1} \tag{45}$$

$$(A \rfloor B)^* = A^* \rfloor B^* \tag{46}$$
$$(A \lfloor B)^* = A^* \lfloor B^* \tag{47}$$
$$(A \wedge B)^* = A^* \wedge B^* \tag{48}$$

$$A\boldsymbol{I} = \boldsymbol{I} A^{*(n-1)} \tag{49}$$

A.7. Reversion

$$\lambda^\dagger = \lambda \tag{50}$$

$$a^\dagger = a \tag{51}$$

$$(AB)^\dagger = B^\dagger A^\dagger \tag{52}$$

$$(A+B)^\dagger = A^\dagger + B^\dagger \tag{53}$$

$$A_r^\dagger = (-1)^{r(r-1)/2} A_r \tag{54}$$

$$A^{\dagger\dagger} = A \tag{55}$$

$$(A^{-1})^\dagger = (A^\dagger)^{-1} \tag{56}$$

$$(A \rfloor B)^\dagger = B^\dagger \lfloor A^\dagger \tag{57}$$

$$(A \lfloor B)^\dagger = B^\dagger \rfloor A^\dagger \tag{58}$$

$$(A \wedge B)^\dagger = B^\dagger \wedge A^\dagger \tag{59}$$

A.8. Clifford conjugation

$$\lambda^\ddagger = \quad \lambda \tag{60}$$

$$a^\ddagger = -a \tag{61}$$

$$(AB)^\ddagger = B^\ddagger A^\ddagger \tag{62}$$

$$(A+B)^\ddagger = A^\ddagger + B^\ddagger \tag{63}$$

$$A^\ddagger = A^{*\dagger} = A^{\dagger *} \tag{64}$$

$$A_r^\ddagger = (-1)^{r(r+1)/2} A_r \tag{65}$$

$$A^{\ddagger\ddagger} = A \tag{66}$$

$$(A^{-1})^\ddagger = (A^\ddagger)^{-1} \tag{67}$$

$$(A \rfloor B)^\ddagger = B^\ddagger \lfloor A^\ddagger \tag{68}$$

$$(A \lfloor B)^\ddagger = B^\ddagger \rfloor A^\ddagger \tag{69}$$

$$(A \wedge B)^\ddagger = B^\ddagger \wedge A^\ddagger \tag{70}$$

A.9. The scalar product and norm

$$
\begin{aligned}
A * B &= \left\langle A^\dagger B \right\rangle &(71)\\
&= \left\langle A^\dagger \rfloor B \right\rangle &(72)\\
&= \left\langle A^\dagger \lfloor B \right\rangle &(73)
\end{aligned}
$$

$$
\begin{aligned}
A * B &= \sum_r A_r * B_r &(74)\\
&= \sum_r A_r^\dagger \rfloor B_r &(75)\\
&= \sum_r A_r^\dagger \lfloor B_r &(76)
\end{aligned}
$$

$$
\begin{aligned}
A * B &= B * A &(77)\\
&= A^* * B^* &(78)\\
&= A^\dagger * B^\dagger &(79)\\
&= A^\ddagger * B^\ddagger &(80)
\end{aligned}
$$

$$
\begin{aligned}
A * (BC) &= (B^\dagger A) * C &(81)\\
A * (B \lfloor C) &= (B^\dagger \lfloor A) * C &(82)\\
A * (B \rfloor C) &= (B^\dagger \wedge A) * C &(83)\\
A * (B \wedge C) &= (B^\dagger \rfloor A) * C &(84)
\end{aligned}
$$

Multivector A is uniquely determined by either of the following:

1. $A * B$ for every multivector B.
2. $\langle A \rangle$ and $a \rfloor A$ for every vector a.

$$
\begin{aligned}
|A|^2 &= A * A &(85)\\
&= |A^*|^2 &(86)\\
&= |A^\dagger|^2 &(87)\\
&= |A^\ddagger|^2 &(88)
\end{aligned}
$$

If $A = a_1 a_2 \cdots a_r$, then

(a) $|A|^2 = A^\dagger A = a_1^2 a_2^2 \cdots a_r^2$.

(b) A^{-1} exists iff $|A|^2 \neq 0$, in which case $A^{-1} = A^\dagger / |A|^2$ and $|A^{-1}|^2 = |A|^{-2}$.

(c) $(AB) * (AC) = (BA) * (CA) = |A|^2 \, B * C$.

For any blade $\boldsymbol{A_r}$,

(a) $|\boldsymbol{A_r}|^2 = 0$ iff the inner product is degenerate on $\boldsymbol{A_r}$.

(b) $\boldsymbol{A_r}^{-1}$ exists iff $|\boldsymbol{A_r}|^2 \neq 0$, in which case $\boldsymbol{A_r}^{-1} = (-1)^{r(r-1)/2} \boldsymbol{A_r} / |\boldsymbol{A_r}|^2 = \boldsymbol{A_r} / \boldsymbol{A_r}^2$.

A.10. The dual

A volume element $\boldsymbol{I}$ is a unit n-blade.

$$\begin{aligned} A^{\perp} &= A \rfloor \boldsymbol{I}^{-1} && (89) \\ &= A\boldsymbol{I}^{-1} && (90) \end{aligned}$$

$$\begin{aligned} A^{-\perp} &= A \rfloor \boldsymbol{I} && (91) \\ &= A\boldsymbol{I} && (92) \\ &= \boldsymbol{I}^2 A^{\perp} && (93) \end{aligned}$$

$\boldsymbol{A}^{\perp}$ is the orthogonal complement of $\boldsymbol{A}$.

$$\begin{aligned} (AB)^{\perp} &= AB^{\perp} && (94) \\ (A \wedge B)^{\perp} &= A \rfloor B^{\perp} && (95) \\ (A \rfloor B)^{\perp} &= A \wedge B^{\perp} && (96) \end{aligned}$$

$$(A^{\perp})^{-1} = \boldsymbol{I} A^{-1} \tag{97}$$

$$(A^*)^{\perp} = (-1)^n (A^{\perp})^* \tag{98}$$

$$(A^{\dagger})^{\perp} = \left[\boldsymbol{I}^{\dagger}(A^{\perp})^{\dagger}\right]^{\perp} \tag{99}$$

$$(A^{\ddagger})^{\perp} = \left[\boldsymbol{I}^{\ddagger}(A^{\perp})^{\ddagger}\right]^{\perp} \tag{100}$$

$$A^{\perp} * B^{\perp} = |\boldsymbol{I}|^{-2}\, A * B \tag{101}$$

$$\boldsymbol{A_r}^{\perp} B^{*(n-r)} (\boldsymbol{A_r}^{\perp})^{-1} = (\boldsymbol{A_r} B^{*r} \boldsymbol{A_r}^{-1})^* \tag{102}$$

$$R_{\boldsymbol{A}}(a) = P_{\boldsymbol{A}^{\perp}}(a) \tag{103}$$

A.11. The commutator

$$A \times B = \frac{1}{2}(AB - BA) \tag{104}$$

$$A \times (BC) = (A \times B)C + B(A \times C) \tag{105}$$

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0. \tag{106}$$

$$\begin{aligned} (A \times B)^* &= A^* \times B^* && (107) \\ (A \times B)^{\dagger} &= B^{\dagger} \times A^{\dagger} && (108) \\ (A \times B)^{\ddagger} &= B^{\ddagger} \times A^{\ddagger} && (109) \end{aligned}$$

$$\lambda \times A = 0 \tag{110}$$

$$a \times A = a \rfloor \langle A \rangle_+ + a \wedge \langle A \rangle_- \tag{111}$$
$$A \times a = \langle A \rangle_+ \lfloor a + \langle A \rangle_- \wedge a \tag{112}$$

$$A_2 \times A_r = \langle A_2 A_r \rangle_r \tag{113}$$

$$A_2 \times (B \rfloor C) = (A_2 \times B) \rfloor C + B \rfloor (A_2 \times C) \tag{114}$$
$$A_2 \times (B \lfloor C) = (A_2 \times B) \lfloor C + B \lfloor (A_2 \times C) \tag{115}$$
$$A_2 \times (B \wedge C) = (A_2 \times B) \wedge C + B \wedge (A_2 \times C) \tag{116}$$

$$A_2 \times (a_1 \wedge a_2 \wedge \cdots \wedge a_r) = \sum_{j=1}^{r} a_1 \wedge a_2 \wedge \cdots \wedge (A_2 \lfloor a_j) \wedge \cdots \wedge a_r \tag{117}$$

A commutes with all multivectors iff A commutes with all vectors iff $A = \lambda + \mu \langle \boldsymbol{I} \rangle_-$.

A.12. Frames and bases

If $\{a_i\}_{i=1,\ldots,n}$ is a frame with volume element $a_N = a_1 \wedge \cdots \wedge a_n$, the reciprocal frame is given by

$$a^i = (-1)^{i-1}(a_1 \wedge a_2 \wedge \cdots \wedge \check{a}_i \wedge \cdots \wedge a_n) a_N^{-1}. \tag{118}$$

It satisfies

$$a^i \rfloor a_j = \delta^i_j. \tag{119}$$

Let I be an increasing string of indices $i_1, i_2, \ldots, i_r$; then a_I and a^I are

$$a_I = a_{i_1} \wedge a_{i_2} \wedge \cdots \wedge a_{i_r} \tag{120}$$
$$a^I = a^{i_1} \wedge a^{i_2} \wedge \cdots \wedge a^{i_r}. \tag{121}$$

They satisfy

$$a_I * a^J = \delta^J_I, \tag{122}$$

and for any multivector A,

$$A = \sum_I A^I a_I \quad \text{where} \quad A^I = A * a^I \tag{123}$$
$$= \sum_I A_I a^I \quad \text{where} \quad A_I = A * a_I. \tag{124}$$

If I is increasing and I^c is the increasing string of indices complementary to I, then

$$a^I = (-1)^{\sum_{j=1}^{r}(i_j - 1)} a_{I^c}\, a_N^{-1}. \tag{125}$$

A frame and its reciprocal satisfy these identities:

$$\sum_i a^i\, a_i \rfloor A_r = \sum_i a^i \wedge (a_i \rfloor A_r) = r A_r \quad \text{for any } A_r. \tag{126}$$

$$\sum_i a_i\, a^i \rfloor A_r = \sum_i a_i \wedge (a^i \rfloor A_r) = r A_r \quad \text{for any } A_r. \tag{127}$$

$$\sum_i a_i\, a^i = \sum_i a^i\, a_i = n. \tag{128}$$

The volume element of the reciprocal frame, $a^N = a^1 \wedge \cdots \wedge a^n$, is also given by

$$a^N = \frac{a_N}{|a_N|^2}. \tag{129}$$

$$P_{a_1 \wedge \cdots \wedge a_r}(B) = \sum_I (B * a_I) a^I \tag{130}$$

$$= \sum_I (B * a^I) a_I \tag{131}$$

A.13. The adjoint of a linear operator

$$\overline{F}(B) * A = B * F(A) \tag{132}$$

$$\begin{aligned} A \rfloor \overline{F}(B) &= \overline{F}\left(F(A) \rfloor B\right) \\ F(A) \lfloor B &= F\left(A \lfloor \overline{F}(B)\right) \end{aligned} \tag{133}$$

$$\overline{\overline{F}} = F \tag{134}$$

$$\overline{GF} = \overline{F}\,\overline{G} \tag{135}$$

$$\overline{F}^{-1} = \overline{F^{-1}} \tag{136}$$

$$\det(\overline{F}) = \det(F) \tag{137}$$

A.14. Symmetric and skew symmetric operators

F is symmetric if $\overline{F} = F$ and skew symmetric (or skew) if $\overline{F} = -F$.

F is (skew) symmetric iff $a \rfloor F(b)$ is (anti)symmetric.

F is symmetric iff $F(a) = \sum_i \lambda_i (a * a^i) a_i$ for some frame $\{a_i\}$ of eigenvectors with eigenvalues $\{\lambda_i\}$.

F is skew iff $F(a) = a \rfloor A_2$ for some bivector A_2.

A.15. Isometries and orthogonal transformations

F is an isometry if $F(u) \rfloor F(v) = u \rfloor v$ for all u and v.

F is an isometry iff $\overline{F}F = \mathrm{Id}$.

F is an orthogonal transformation if F is an invertible isometry.

F is orthogonal iff $\overline{F} = F^{-1}$.

F is an isometry on a finite-dimensional space iff $F(a) = A_r a^{*r} A_r^{-1}$ for some invertible r-versor A_r.

The extension of orthogonal F on a finite-dimensional space to all multivectors is $F(B) = A_r B^{*r} A_r^{-1}$.

A.16. Eigenvalues, invariant subspaces, and determinants

$\boldsymbol{A} \neq 0$ is an eigenblade of F if $F(\boldsymbol{A}) = \lambda \boldsymbol{A}$.

An eigenblade of F is an invariant subspace on which F is invertible. The eigenvalue is $\det(F_{\boldsymbol{A}})$.

$$\det F = F(\boldsymbol{I})^{\perp} \tag{138}$$

$$F^{-1}(A) = \frac{\overline{F}(A^{-\perp})^{\perp}}{\det(F)} \tag{139}$$

B. Topics for future versions

These are the subjects I plan to add to the notes next, in no particular order. The items with asterisks are most interesting to me at the moment.

- Linear algebra
  - More on invariant subspaces and determinants*
  - Representing a general linear operator as a sequence of multivector multiplications*
- Differential and integral calculus
  - The directed integral of a multivector
  - The derivative of a multivector-valued function defined in terms of the directed integral*
  - Recovering traditional vector calculus
  - The fundamental theorem of calculus and its corollaries (Gauss' theorem, Stokes' theorem, Green's theorem, etc.)*
  - Taylor series*
  - Generalizations of Cauchy's integral formula*
  - The invertibility of the derivative (cf. the exterior derivative)
  - Solutions to standard ODEs and PDEs (simple harmonic oscillator, wave equation, etc.)*
  - Fourier analysis
  - Manifold theory
  - Lie groups and Lie algebras
  - Curvature*
- Geometry
  - Meet and join of subspaces
  - Projective splits (e.g. Minkowski spacetime into any observer's space + time)*
  - Different models of space (Euclidean, projective, conformal)
  - Geometric algebra on a vector space without an inner product
- Physics
  - Rotational dynamics and the inertia tensor*
  - Relativistic particle mechanics*
  - Electricity and magnetism in 3D and 4D*
  - Lagrangian and Hamiltonian mechanics*
  - Continuum mechanics and elasticity theory*
  - The Dirac equation
  - General relativity
  - The Galilei and Lorentz groups and their Lie algebras

References

[1] David Hestenes, *Space-Time Algebra* (New York: Gordon and Breach, 1966).

[2] David Hestenes and Garret Sobczyk, *Clifford Algebra to Geometric Calculus* (Dordrecht: D. Reidel Publishing Company, 1984).

[3] David Hestenes, *New Foundations for Classical Mechanics*, 2nd ed. (Dordrecht: Kluwer Academic Publishers, 1999).

[4] Chris Doran and Anthony Lasenby, *Geometric Algebra for Physicists* (Cambridge: Cambridge University Press, 2003).

[5] Leo Dorst, Daniel Fontijne, and Stephen Mann, *Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry*, rev. ed. (Amsterdam: Morgan Kaufmann, 2007).

[6] Alan Macdonald, *Linear and Geometric Algebra* (Charleston: CreateSpace, 2011).

[7] Pertti Lounesto, *Clifford Algebras and Spinors*, London Mathematical Society Lecture Note Series 286, 2nd ed. (Cambridge: Cambridge University Press, 2001).

[8] Alan Macdonald, "An Elementary Construction of the Geometric Algebra," Adv. Appl. Cliff. Alg. **12**, 1-6 (2002). An improved version is available at `http://faculty.luther.edu/~macdonal/`.