A geometric algebra reformulation of 2x2 matrices: the dihedral group $\mathcal{D}_4$ in bra-ket notation

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Abstract. We represent vector rotation operators in terms of bras or kets of half-angle exponentials in Clifford (geometric) algebra $\mathcal{C}l_{3,0}$. We show that SO_3 is a rotation group and we define the dihedral group D_4 as its finite subgroup. We use the Euler-Rodrigues formulas to compute the multiplication table of $\mathcal{D}_4$ and derive its group algebra identities. We take the linear combination of rotation operators in $\mathcal{D}_4$ to represent the four Fermion matrices in Sakurai, which in turn we use to decompose any 2×2 matrix. We show that bra and ket operators generate left- and right-acting matrices, respectively. We also show that the Pauli spin matrices are not vectors but vector rotation operators, except for $\hat{\sigma}_2$ which requires a subsequent multiplication by the imaginary number i geometrically interpreted as the unit oriented volume.

1 Introduction

Clifford (geometric) algebra $\mathcal{C}l_{3,0}$ may be presented by replacing the three Cartesian basis vectors and the unit real number by their correponding 2×2 Pauli spin matrices[1]. For example,

$$1 \quad \equiv \quad \hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1}$$

$$\mathbf{e}_1 \quad \equiv \quad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2}$$

$$\mathbf{e}_2 \quad \equiv \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{3}$$

$$\mathbf{e}_3 \quad \equiv \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4}$$

From these definitions we can show that the orthonormality relation holds[2]:

$$\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j = 2\delta_{jk}, \tag{5}$$

for $j, k \in \{1, 2, 3\}$. And that the product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is governed by the Pauli identity[3]

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}), \tag{6}$$

where $i = \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$ is an imaginary number that commutes with vectors.

The representation of the unit vectors in terms of matrices presupposes that matrices are more fundamental than vectors. But what if we adopt the opposite view that vectors are more fundamental than matrices? That is, given only the rule for multiplying unit vectors in Eq. (5), can we arrive at a definition of a matrix and its corresponding algebra?

Yes, we can. To justify this claim requires three steps. First, we represent the symmetry group of a square, the dihedral group $\mathcal{D}_4$, by exponentials of imaginary half vectors as done in the following general form for vector rotation[4]:

$$\mathbf{r}' = e^{-i\boldsymbol{\theta}/2} \, \mathbf{r} \, e^{i\boldsymbol{\theta}/2}, \tag{7}$$

where $\boldsymbol{\theta}$ is the axis of rotation of vector $\mathbf{r}$ and $|\boldsymbol{\theta}|$ is the magnitude of counterclockwise angle of rotation. Second, we construct linear combinations of the $\mathcal{D}_4$ rotation operators to represent the four Fermion matrices in Sakurai[5] or the four standard basis for dioptric power space in Harris[6]:

$$\mathsf{e}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{e}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{8}$$

$$\mathsf{e}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{e}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{9}$$

And third, we express any matrix as a linear combination of the four Fermion matrices.

In Hestenes's representation of symmetry groups, the fundamental operation is not rotation but reflection. If $\mathbf{r}$ is a vector reflected with respect to the mirror with normal vector unit vector $\boldsymbol{\eta}$, then[7]

$$\mathbf{r}' = \underline{N}\mathbf{r} = -\boldsymbol{\eta}\mathbf{r}\boldsymbol{\eta}, \tag{10}$$

where $\underline{N}$ is a left-acting reflection operator. The product of two reflections is a rotation[8]:

$$\underline{N}_1\underline{N}_2\mathbf{r} = (\boldsymbol{\eta}_2\boldsymbol{\eta}_1)\,\mathbf{r}\,(\boldsymbol{\eta}_1\boldsymbol{\eta}_2) = e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2}, \qquad (11)$$

where $\boldsymbol{\theta}$ is parallel to $\boldsymbol{\eta}_1 \times \boldsymbol{\eta}_2$ and $|\boldsymbol{\theta}|/2$ is the angle between $\boldsymbol{\eta}_1$ and $\boldsymbol{\eta}_2$. (See Fig. 1)

In our case, we shall view rotations as more fundamental than reflections. To do this, we need to define a new operation similar to but distinct from reflection: flip. If $\mathbf{r}'$ is the vector $\mathbf{r}$ flipped with respect to the axis along the unit vector $\boldsymbol{\eta}$, then

$$\mathbf{r}' = \boldsymbol{\eta}\mathbf{r}\boldsymbol{\eta}, \qquad (12)$$

which differs from that of reflection by a sign. Notice that a flip is actually a 180° rotation about $\boldsymbol{\eta}$:

$$\mathbf{r}' = e^{-i\boldsymbol{\eta}\pi/2}\,\mathbf{r}\,e^{i\boldsymbol{\eta}\pi/2} = (-i\boldsymbol{\eta})\mathbf{r}(i\boldsymbol{\eta}) = \boldsymbol{\eta}\mathbf{r}\boldsymbol{\eta}. \qquad (13)$$
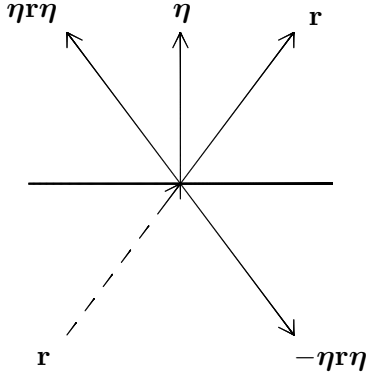

Figure 1: The vector $\boldsymbol{\eta}\mathbf{r}\boldsymbol{\eta}$ is the ray $\mathbf{r}$ flipped 180° about the mirror normal $\boldsymbol{\eta}$. The opposite of $\boldsymbol{\eta}\mathbf{r}\boldsymbol{\eta}$ is the reflected ray $-\boldsymbol{\eta}\mathbf{r}\boldsymbol{\eta}$.

Hestenes remarked that the pair of rotors (rotation operators) $\pm R$ distinguish equivalent rotations in opposite senses[7]. In Hestenes underbar notation, we may write this as

$$\mathbf{x}' = \mathbf{x}\underline{\pm R} = (\pm R)^{\dagger}\mathbf{x}(\pm R), \qquad (14)$$

where $R = e^{i\boldsymbol{\theta}/2}$ ($\underline{\pm R}$ is actually left-acting in Hestenes, but we changed its direction of action for consistency with our conventions). This theorem of Hestenes is very useful, so we shall rederive it here. His underline notation, however, we shall change to bras or kets, in order to suggest the direction of action:

$$\mathbf{x}' = \mathbf{x}\underline{R} = \langle R^{\dagger}|\mathbf{x} = \mathbf{x}|R\rangle. \qquad (15)$$

Notice that the bars suggest some similarity with the *absolute value* bars | |, which is what we intend. Note

that the $\langle R^{\dagger}|$ and $|R\rangle$ are the bra and ket representations of the rotation operator; these representations are only notations for they do not obey Dirac's bracket algebra as described in Simmons and Guttmann[9].

We shall divide this paper into five sections. The first section is the Introduction. In the second section, we shall give a brief summary of Clifford (geometric) algebra $\mathcal{Cl}_{3,0}$. We shall discuss the orthonormality axiom, the Pauli identity for vector products, the Euler's theorem for the exponential of an imaginary vector, and the general vector rotation formula. In the third section, we shall use the exponential rotation operators in geometric algebra to describe the elements of the general rotation group in three dimensions, SO_3. We shall rederive Rodrigues formulas for the composition of rotations to prove the group properties of SO_3. We shall also rederive Hestenes theorem on equivalent rotations, together with some theorems on bra and ket transformation. In the the fourth section, we discuss the rotation subgroup $\mathcal{D}_4$, which is a finite subgroup of SO_3. We shall construct the $\mathcal{D}_4$ group algebra and use this to define the Fermion and Pauli 2×2 matrices. The fifth section is Conclusions.

2 Geometric Algebra

2.1 Vectors and Imaginary Numbers

The Clifford (geometric) algebra $\mathcal{Cl}_{3,0}$ is an associative algebra generated by three vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ that satisfy the orthonormality relation in Eq. (5). That is,

$$\mathbf{e}_j^2 = 1, \qquad (16)$$
$$\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j, \quad j \neq k. \qquad (17)$$

We shall refer to Eqs. (16) and (17) as the normality and orthogonality axioms, respectively.

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors spanned by the three unit spatial vectors in $\mathcal{Cl}_{3,0}$. By the orthonormality axioms in Eqs. (16) and (17), we can show that the product of these two vectors is given by the Pauli identity in Eq. (6):

$$\mathbf{a}\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}). \qquad (18)$$

Notice that if $\mathbf{a}_{\parallel}$ and $\mathbf{a}_{\perp}$ are the components of $\mathbf{a}$ parallel and perpendicular to $\mathbf{b}$, then

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}_{\parallel}\mathbf{b} = \mathbf{b}\,\mathbf{a}_{\parallel}, \qquad (19)$$
$$i(\mathbf{a} \times \mathbf{b}) = \mathbf{a}_{\perp}\mathbf{b} = \mathbf{b}\,\mathbf{a}_{\perp}. \qquad (20)$$

That is, parallel vectors commute; perpendicular vectors anticommute.

In general, we may express every element $\hat{A}$ in $\mathcal{Cl}_{3,0}$ as a linear combination of a scalar, a vector, an imaginary vector (bivector), and an imaginary scalar (trivector):

$$\hat{A} = A_0 + \mathbf{A}_1 + i\mathbf{A}_2 + iA_3. \qquad (21)$$

Following Jancewicz, we shall call $\hat{A}$ as a cliffor[10].

2.2 Rotations

Let $\boldsymbol{\theta}$ be a vector. Multiplying this by i results to the bivector $i\boldsymbol{\theta}$, which we may geometrically interpret as the oriented plane perpendicular to vector $\boldsymbol{\theta}$ (or in the language of forms, $\boldsymbol{\theta}$ is the Hodge map or dual of $i\boldsymbol{\theta}$). It is easy to see that the square of $i\boldsymbol{\theta}$ is negative:

$$(i\boldsymbol{\theta})^2 = i^2\boldsymbol{\theta}^2 = -|\boldsymbol{\theta}|^2. \tag{22}$$

Thus, we may use Euler's theorem in complex analysis to write

$$e^{i\boldsymbol{\theta}} = \cos|\boldsymbol{\theta}| + i\frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}\sin|\boldsymbol{\theta}|. \tag{23}$$

Notice that the exponential $e^{i\boldsymbol{\theta}}$, like the product $\mathbf{ab}$ in Eq. (18), is a sum of a scalar and an imaginary vector.
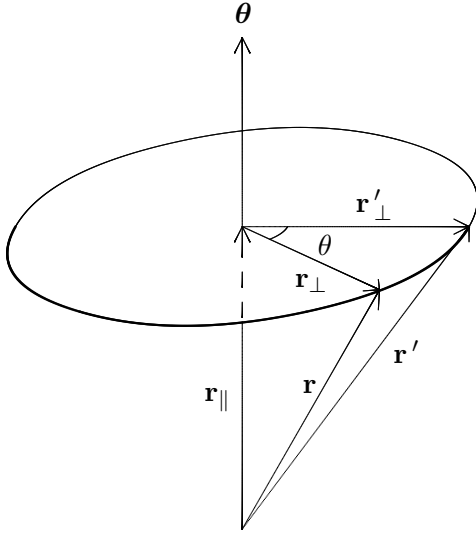

Figure 2: The vector $\mathbf{r}' = e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2}$ is the vector $\mathbf{r}$ rotated counterclockwise about the vector $\boldsymbol{\theta}$ by an angle $\theta = |\boldsymbol{\theta}|$.

If $\mathbf{r}'$ is the vector $\mathbf{r}$ rotated counterclockwise about the vector $\boldsymbol{\theta}$ by an angle $|\boldsymbol{\theta}|$ (Figure 2), we write

$$\mathbf{r} = e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2}. \tag{24}$$

If we expand $\mathbf{r}$ in terms of its components parallel and perpendicular to $\boldsymbol{\theta}$, then we can show that

$$\mathbf{r}_\parallel e^{i\boldsymbol{\theta}/2} = e^{i\boldsymbol{\theta}/2}\mathbf{r}_\parallel, \tag{25}$$
$$\mathbf{r}_\perp e^{i\boldsymbol{\theta}/2} = e^{-i\boldsymbol{\theta}/2}\mathbf{r}_\perp, \tag{26}$$

so that Eq. (24) reduces to

$$\mathbf{r}' = \mathbf{r}_\parallel + \mathbf{r}_\perp e^{i\boldsymbol{\theta}}. \tag{27}$$

Thus, the parallel component is unaffected by the rotation. (See Fig. 2)

The second term on the right side of Eq. (27) may be expanded as

$$\begin{aligned}
\mathbf{r}_\perp e^{i\boldsymbol{\theta}} &= \mathbf{r}_\perp \cos|\boldsymbol{\theta}| + i\mathbf{r}_\perp\frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}\sin|\boldsymbol{\theta}| \\
&= \mathbf{r}_\perp \cos|\boldsymbol{\theta}| - \mathbf{r}_\perp \times \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}\sin|\boldsymbol{\theta}|, \tag{28}
\end{aligned}$$

where we used the Pauli identity in Eq. (18). Equation (28) states that $\mathbf{r}_\perp e^{i\boldsymbol{\theta}}$ is the vector $\mathbf{r}_\perp$ rotated counterclockwise about the vector $\boldsymbol{\theta}$ by an angle $|\boldsymbol{\theta}|$.[11]

Rotation operators may also act on a cliffor $\hat{A}$ to yield a new cliffor $\hat{A}'$:

$$\hat{A}' = e^{-i\boldsymbol{\theta}/2}\hat{A}\,e^{i\boldsymbol{\theta}/2} \tag{29}$$

Using the expansion of $\hat{A}$ in Eq. (21), together with the commutation relations

$$A_0\,e^{i\boldsymbol{\theta}/2} = e^{i\boldsymbol{\theta}/2}A_0, \tag{30}$$
$$iA_3\,e^{i\boldsymbol{\theta}/2} = e^{i\boldsymbol{\theta}/2}iA_3, \tag{31}$$

we arrive at

$$\begin{aligned}
A_0' + iA_3' &= A_0 + iA_3, \tag{32} \\
\mathbf{A}_1' + i\mathbf{A}_2' &= e^{-i\boldsymbol{\theta}/2}(\mathbf{A}_1 + i\mathbf{A}_2)e^{i\boldsymbol{\theta}/2}, \tag{33}
\end{aligned}$$

after separating the linearly independent parts. Thus only the vector-bivector part is affected by the rotation; the scalar-trivector (complex number) remains unchanged.

3 The Rotation Group SO_3

3.1 Group Properties

Rotations form a group labelled as $SO(3)$ or the special orthogonal group in three dimensions (orthogonal matrices are traditionally used to represent the group elements). To prove that SO_3 is a group, we must show that it satisfies four properties: closure, associativity, existence of an inverse, and existence of an identity.[12]

a. Closure. The product of two rotations is also a rotation. That is, if $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are vectors, there exists a vector $\boldsymbol{\theta}_3$ such that

$$e^{-i\boldsymbol{\theta}_3/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}_3/2} = e^{-i\boldsymbol{\theta}_2/2}e^{-i\boldsymbol{\theta}_1/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}_1/2}e^{i\boldsymbol{\theta}_2/2}, \tag{34}$$

where $\mathbf{r}$ is a vector in $\mathcal{Cl}_{3,0}$. Equation (34) is known as Rodrigues's theorem.

To prove the claim in Eq. (34), it is sufficient to prove only the equality of the right exponentials,

$$e^{i\boldsymbol{\theta}_3/2} = e^{i\boldsymbol{\theta}_1/2}e^{i\boldsymbol{\theta}_2/2}, \tag{35}$$

because the equality of the left exponentials immediately follows by taking the inverse of Eq. (35):

$$e^{-i\boldsymbol{\theta}_3/2} = e^{-i\boldsymbol{\theta}_2/2}e^{-i\boldsymbol{\theta}_1/2}. \tag{36}$$

Expanding the exponentials in Eq. (35), distributing the terms, and separating the scalar and imaginary vector parts, we obtain

$$\begin{aligned}
\cos\frac{|\boldsymbol{\theta}_3|}{2} &= \cos\frac{|\boldsymbol{\theta}_1|}{2}\cos\frac{|\boldsymbol{\theta}_2|}{2} \\
&\quad - \frac{\boldsymbol{\theta}_1\cdot\boldsymbol{\theta}_2}{|\boldsymbol{\theta}_1||\boldsymbol{\theta}_2|}\sin\frac{|\boldsymbol{\theta}_1|}{2}\sin\frac{|\boldsymbol{\theta}_2|}{2},
\end{aligned} \tag{37}$$

$$\begin{aligned}
\frac{\boldsymbol{\theta}_3}{|\boldsymbol{\theta}_3|}\sin\frac{|\boldsymbol{\theta}_3|}{2} &= \frac{\boldsymbol{\theta}_1}{|\boldsymbol{\theta}_1|}\sin\frac{|\boldsymbol{\theta}_1|}{2}\cos\frac{|\boldsymbol{\theta}_2|}{2} \\
&\quad + \frac{\boldsymbol{\theta}_2}{|\boldsymbol{\theta}_2|}\sin\frac{|\boldsymbol{\theta}_2|}{2}\cos\frac{|\boldsymbol{\theta}_1|}{2} \\
&\quad - \frac{\boldsymbol{\theta}_1\times\boldsymbol{\theta}_2}{|\boldsymbol{\theta}_1||\boldsymbol{\theta}_2|}\sin\frac{|\boldsymbol{\theta}_2|}{2}\sin\frac{|\boldsymbol{\theta}_1|}{2},
\end{aligned} \tag{38}$$

where we factored out i in the second equation. We shall refer to these equations as the Euler-Rodrigues's formulas.[13][14]

We can solve for the magnitude $|\boldsymbol{\theta}_3|$ by taking the inverse cosine of Eq. (37). The value of $|\boldsymbol{\theta}_3|$ is unique in the range $0 \le |\boldsymbol{\theta}_3| \le \pi$. After solving for $|\boldsymbol{\theta}_3|$, we then substitute this result in Eq. (38) to solve for the unique direction angle $\boldsymbol{\theta}_3/|\boldsymbol{\theta}_3|$. Thus, given vectors $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$, there exists a vector $\boldsymbol{\theta}_3$ defined by Eqs. (37) and (38), so that Eq. (35) holds, which is what we wish to show.

b. Associativity. Because products in $\mathcal{C}l_{3,0}$ are associative, then

$$\begin{aligned}
\mathbf{r}' &= (e^{-i\boldsymbol{\theta}_3/2}e^{-i\boldsymbol{\theta}_2/2})e^{-i\boldsymbol{\theta}_1/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}_1/2}(e^{i\boldsymbol{\theta}_2/2}e^{i\boldsymbol{\theta}_3/2}), \\
&= e^{-i\boldsymbol{\theta}_3/2}(e^{-i\boldsymbol{\theta}_2/2}e^{-i\boldsymbol{\theta}_1/2})\,\mathbf{r}\,(e^{i\boldsymbol{\theta}_1/2}e^{i\boldsymbol{\theta}_2/2})e^{i\boldsymbol{\theta}_3/2}
\end{aligned} \tag{39}$$

Thus, products of rotations are associative.

c. Identity. The identity rotation exists:

$$\mathbf{r} = e^{-0}\,\mathbf{r}\,e^0. \tag{40}$$

d. Inverse. Every rotation has an inverse. That is, if $\boldsymbol{\theta}$ is a vector, then

$$\begin{aligned}
\mathbf{r} &= e^{i\boldsymbol{\theta}/2}e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2}e^{-i\boldsymbol{\theta}/2} \\
&= e^{-i\boldsymbol{\theta}/2}e^{i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{-i\boldsymbol{\theta}/2}e^{i\boldsymbol{\theta}/2}.
\end{aligned} \tag{41}$$

3.2 Bra and Ket Representations

Let us rewrite the rotation expression in Eq. (24) as

$$\mathbf{r} = e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2} = \langle e^{-i\boldsymbol{\theta}/2}|\mathbf{r} = \mathbf{r}|e^{i\boldsymbol{\theta}/2}\rangle, \tag{42}$$

where

$$\langle\, e^{-i\boldsymbol{\theta}/2}\,| = \left\langle \cos\frac{\boldsymbol{\theta}}{2} - i\frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}\sin\frac{|\boldsymbol{\theta}|}{2}\,\right|, \tag{43}$$

$$|\,e^{i\boldsymbol{\theta}/2}\,\rangle = \left|\, \cos\frac{\boldsymbol{\theta}}{2} + i\frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}\sin\frac{|\boldsymbol{\theta}|}{2}\,\right\rangle. \tag{44}$$

We shall refer to the angle braces expressions as the bra and ket representations for an element of a rotation group, respectively. The representations on the left side of the equations are the polar form; those on the right side are the Cartesian form. Note that the bra is a left-acting operator, while the ket is a right-acting one.

The use of bra and ket representations can be tricky. So let us summarize some useful identities and comment on some likely algebraic traps:

a. Action on Cliffors. The bra and ket operators can also act on a general cliffor $\hat{A}$:

$$\langle\, e^{-i\boldsymbol{\theta}/2}\,|\,\hat{A} = \hat{A}\,|\,e^{i\boldsymbol{\theta}/2}\,\rangle. \tag{45}$$

If $\hat{A} = A_0 + iA_3$, then $\hat{A}$ is unaffected by the rotation:

$$A_0 + iA_3 = \langle e^{-i\boldsymbol{\theta}/2}|(A_0 + iA_3) = (A_0 + iA_3)|\,e^{i\boldsymbol{\theta}/2}\,\rangle. \tag{46}$$

Now, since the ket $|e^{i\boldsymbol{\theta}/2}\rangle$ is a right-acting operator and the bra $\langle e^{-i\boldsymbol{\theta}/2}|$ is a left-acting one, we must not put the cliffor operand $\hat{A}$ on the right side of a ket operator or on the left side of a bra operator. But if we do so on purpose, the result is a new operator:

$$\mathbf{r}\,|\,e^{i\boldsymbol{\theta}/2}\,\rangle\hat{A} = e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2}\hat{A}, \tag{47}$$

$$\hat{A}\langle\, e^{-i\boldsymbol{\theta}/2}\,|\,\mathbf{r} = \hat{A}\,e^{-i\boldsymbol{\theta}/2}\,\mathbf{r}\,e^{i\boldsymbol{\theta}/2}. \tag{48}$$

That is, the action of the new operator is as follows: rotate the vector $\mathbf{r}$ about the vector $\boldsymbol{\theta}$ counterclockwise by an angle $|\boldsymbol{\theta}|$ and the result multiply by $\hat{A}$.

What if we have a chain of cliffors and kets:

$$\mathbf{r}\,\hat{A}_1\,|K_1\rangle\hat{A}_2\,|K_2\rangle\cdots\hat{A}_n\,|K_n\rangle? \tag{49}$$

How shall we interpret this? Let us adopt the following convention: if $\hat{A}\,|K\rangle$ is *explicitly* defined as a right-acting operator acting on $\mathbf{r}$, then $\hat{A}$ acts first on $\mathbf{r}$ and the result is acted on by $|K\rangle$:

$$\mathbf{r}\,\hat{A}\,|K\rangle = (\mathbf{r}\hat{A})|K\rangle. \tag{50}$$

We shall call this the left-most precedence convention for right-acting operators (for left-acting operators it is right-most precedence). Hence,

$$\begin{aligned}
\mathbf{r}\,\hat{A}_1\,&|K_1\rangle\hat{A}_2\,|K_2\rangle\cdots\hat{A}_n\,|K_n\rangle \\
&= (\cdots(((\mathbf{r}\hat{A}_1)|K_1\rangle)\hat{A}_2)\cdots\hat{A}_n)|K\rangle.
\end{aligned} \tag{51}$$

b. Equivalent Kets. We may rewrite each counterclockwise rotation operator in terms of its clockwise counterpart:

$$| e^{i\boldsymbol{\theta}/2} \rangle = | e^{-i\mathbf{e}_\theta (2\pi - |\boldsymbol{\theta}|)/2} \rangle, \qquad (52)$$

where

$$\mathbf{e}_\theta = \frac{\boldsymbol{\theta}}{|\boldsymbol{\theta}|}, \qquad (53)$$

which should not be confused with the unit vector in polar coordinates. Because

$$e^{\pm i\mathbf{e}_\theta \pi} = -1, \qquad (54)$$

then Eq. (52) reduces to

$$| e^{i\boldsymbol{\theta}/2} \rangle = | -e^{i\boldsymbol{\theta}/2} \rangle. \qquad (55)$$

Thus, two representations are equivalent if their corresponding ket arguments are additive inverses of each other. (We may think of the ket bars as absolute value operators.) We shall refer to Eq. (55) as the Hestenes's rotation representation equivalence.

c. Products of Kets. The product of two kets is the ket of the product of their arguments:

$$| e^{i\boldsymbol{\theta}_1/2} \rangle | e^{i\boldsymbol{\theta}_2/2} \rangle = | e^{i\boldsymbol{\theta}_1/2} e^{i\boldsymbol{\theta}_1/2} \rangle, \qquad (56)$$

because

$$\mathbf{r} \, | e^{i\boldsymbol{\theta}_1/2} \rangle | e^{i\boldsymbol{\theta}_2/2} \rangle = e^{-i\boldsymbol{\theta}_2/2} e^{-i\boldsymbol{\theta}_1/2} \, \mathbf{r} \, e^{i\boldsymbol{\theta}_1/2} e^{i\boldsymbol{\theta}_2/2}. \quad (57)$$

For example, the factorization

$$| e^{i\mathbf{e}_2 \pi/2} e^{i\mathbf{e}_1 \pi/4} \rangle = | i\mathbf{e}_2 \rangle | e^{i\mathbf{e}_1 \pi/4} \rangle \qquad (58)$$

is allowed, because $i\mathbf{e}_2$ is expressible as an exponential of a imaginary vector. On the other hand,

$$| e^{i\mathbf{e}_2 \pi/2} e^{i\mathbf{e}_1 \pi/4} \rangle \neq | i \rangle | \mathbf{e}_2 e^{i\mathbf{e}_1 \pi/4} \rangle, \qquad (59)$$

because i and $\mathbf{e}_2 e^{i\mathbf{e}_1 \pi/4}$ cannot be similarly expressed.

To avoid confusion, we shall not write bra and ket operators in the same side of an equation as in the expectation value expression in Quantum Mechanics,

$$\mathbf{r}' = \langle e^{-i\boldsymbol{\theta}_1/2} | \mathbf{r} | e^{i\boldsymbol{\theta}_2/2} \rangle \overset{?}{=} \mathbf{r} | e^{i\boldsymbol{\theta}_1/2} \rangle | e^{i\boldsymbol{\theta}_2/2} \rangle$$
$$\overset{?}{=} \mathbf{r} | e^{i\boldsymbol{\theta}_2/2} \rangle | e^{i\boldsymbol{\theta}_1/2} \rangle, \quad (60)$$

for we would not know which operator comes first. As a convention, we shall use ket operators in our computations and convert our final results in terms of bra operators, but only if necessary.

d. Sum of Kets. The rule for the sum of two kets is simple: do not combine or split their arguments. That is,

$$| e^{i\boldsymbol{\theta}_1} \rangle + | e^{i\boldsymbol{\theta}_2} \rangle \neq | e^{i\boldsymbol{\theta}_2} + e^{i\boldsymbol{\theta}_2} \rangle, \qquad (61)$$

unless, of course, if one of the terms is zero. The reason for this rule is in the original definition of rotations in Eq. (34), where we derived our half-angle representation for right acting operators. That is, Eq. (61) is not allowed because

$$e^{-i\boldsymbol{\theta}_1/2} \, \mathbf{r} \, e^{i\boldsymbol{\theta}_1/2} + e^{-i\boldsymbol{\theta}_2/2} \, \mathbf{r} \, e^{i\boldsymbol{\theta}_2/2}$$
$$\neq (e^{-i\boldsymbol{\theta}_2/2} + e^{-i\boldsymbol{\theta}_1/2}) \mathbf{r} \, (e^{i\boldsymbol{\theta}_1/2} + e^{i\boldsymbol{\theta}_2/2}). (62)$$

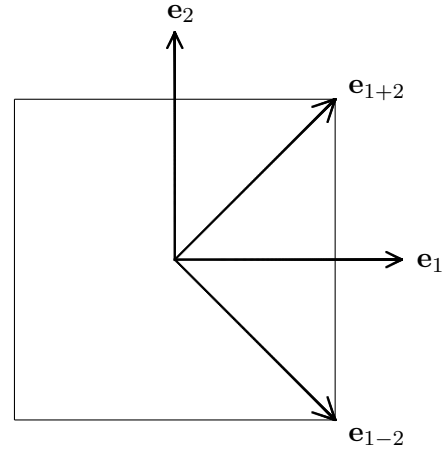# 4 Rotation Group $\mathcal{D}_4$

4.1 Group Elements

Figure 3: The symmetry axes of a unit square in the $xy-$plane. The vector $\mathbf{e}_3$ is pointing out of the paper.

Let us define the positions of the vertices of a square to be the set

$$\mathcal{V}_4 = \{\mathbf{e}_{1+2}, \mathbf{e}_{-1+2}, \mathbf{e}_{-1+2}, \mathbf{e}_{1-2}\}, \qquad (63)$$

where

$$\mathbf{e}_{\pm 1 \pm 2} = \frac{1}{\sqrt{2}}(\pm \mathbf{e}_1 \pm \mathbf{e}_2). \qquad (64)$$

Notice that the symmetry axes of the square are

$$\mathcal{S}_4 = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_{1+2}, \mathbf{e}_{1-2}\}. \qquad (65)$$

(See Figure 3)

If we define the dihedral group $\mathcal{D}_4$ to be the set of symmetry operations on a square defined in Eq. (63),

then we may represent the elements of $\mathcal{D}_4$ as

$$1 = |1\rangle, \tag{66}$$
$$|e^{i\mathbf{e}_1\pi/2}\rangle = |i\mathbf{e}_1\rangle = |e^{-i\mathbf{e}_1\pi/2}\rangle, \tag{67}$$
$$|e^{i\mathbf{e}_2\pi/2}\rangle = |i\mathbf{e}_2\rangle = |e^{-i\mathbf{e}_2\pi/2}\rangle, \tag{68}$$
$$|e^{i\mathbf{e}_3\pi/2}\rangle = |i\mathbf{e}_3\rangle = |e^{-i\mathbf{e}_3\pi/2}\rangle, \tag{69}$$
$$|e^{i\mathbf{e}_3\pi/4}\rangle = |\mathbf{e}_{0+3i}\rangle = |e^{-i\mathbf{e}_3 3\pi/4}\rangle, \tag{70}$$
$$|e^{-i\mathbf{e}_3\pi/4}\rangle = |\mathbf{e}_{0-3i}\rangle = |e^{i\mathbf{e}_3 3\pi/4}\rangle, \tag{71}$$
$$|e^{i(\mathbf{e}_{1+2})\pi/2}\rangle = |i\mathbf{e}_{1+2}\rangle = |e^{-i(\mathbf{e}_{1+2})\pi/2}\rangle, \tag{72}$$
$$|e^{i(\mathbf{e}_{1-2})\pi/2}\rangle = |i\mathbf{e}_{1-2}\rangle = |e^{-i(\mathbf{e}_{1-2})\pi/2}\rangle, \tag{73}$$

where

$$\mathbf{e}_{0\pm3i} = \frac{1}{\sqrt{2}}(1 \pm i\mathbf{e}_3). \tag{74}$$

Note that we have removed the $\pm$ sign in the arguments of the representations, because the two representations are equivalent by Hestenes's theorem in Eq. (55). For example, we could have written Eqs. (66) and (67) as

$$1 = |\pm1\rangle, \tag{75}$$
$$|\pm e^{i\mathbf{e}_1\pi/2}\rangle = |\pm i\mathbf{e}_1\rangle = |\pm e^{-i\mathbf{e}_1\pi/2}\rangle. \tag{76}$$

There are only two generators for $\mathcal{D}_4$:

$$G_{\mathcal{D}4} = \{\,|i\mathbf{e}_1\rangle, |\mathbf{e}_{0+3i}\rangle\,\}. \tag{77}$$

The first corresponds to a flip about $\mathbf{e}_1$; the second, to a $\pi/2$ counterclockwise rotation about $\mathbf{e}_3$. The other group elements can be expressed in terms of these two:

$$|1\rangle = |i\mathbf{e}_1\rangle^2 = |\mathbf{e}_{0+3i}\rangle^4, \tag{78}$$
$$|i\mathbf{e}_3\rangle = |\mathbf{e}_{0+3i}\rangle^2 = |i\mathbf{e}_3\rangle^{-1}, \tag{79}$$
$$|\mathbf{e}_{0-3i}\rangle = |\mathbf{e}_{0+3i}\rangle^3 = |\mathbf{e}_{0+3i}\rangle^{-1}, \tag{80}$$
$$|i\mathbf{e}_2\rangle = |i\mathbf{e}_1\rangle|\mathbf{e}_{0+3i}\rangle^2 = |i\mathbf{e}_2\rangle^{-1}, \tag{81}$$
$$|i\mathbf{e}_{1+2}\rangle = |i\mathbf{e}_1\rangle|\mathbf{e}_{0+3i}\rangle = |i\mathbf{e}_{1+2}\rangle^{-1}, \tag{82}$$
$$|i\mathbf{e}_{1-2}\rangle = |i\mathbf{e}_1\rangle|\mathbf{e}_{0+3i}\rangle^3 = |i\mathbf{e}_{1-2}\rangle^{-1}. \tag{83}$$

(See Table 1)

4.2 Identities

In actual computations, it is simpler to work with the following five group elements: $|i\mathbf{e}_1\rangle$, $|i\mathbf{e}_2\rangle$, $|i\mathbf{e}_3\rangle$, $|i\mathbf{e}_{0+3i}\rangle$, and $|i\mathbf{e}_{0-3i}\rangle$. The first three correspond to $180°$ flips about a coordinate axis; the next two correspond to counterclockwise and clockwise quarter rotations about the $z-$axis direction $\mathbf{e}_3$.

a. Products. The flip operators $|i\mathbf{e}_1\rangle$, $|i\mathbf{e}_2\rangle$, $|i\mathbf{e}_3\rangle$ are generators of an Abelian (commuting) quasi-quaternion algebra:

$$|i\mathbf{e}_1\rangle^2 = |i\mathbf{e}_2\rangle^2 = |i\mathbf{e}_3\rangle^2 = |1\rangle, \tag{84}$$

		$\lvert 1\rangle$	$\lvert i\mathbf{e}_1\rangle$	$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_3\rangle$
$\lvert 1\rangle$		$\lvert 1\rangle$	$\lvert i\mathbf{e}_1\rangle$	$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_3\rangle$
$\lvert i\mathbf{e}_1\rangle$		$\lvert i\mathbf{e}_1\rangle$	$\lvert 1\rangle$	$\lvert i\mathbf{e}_3\rangle$	$\lvert i\mathbf{e}_2\rangle$
$\lvert i\mathbf{e}_2\rangle$		$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_3\rangle$	$\lvert 1\rangle$	$\lvert i\mathbf{e}_1\rangle$
$\lvert i\mathbf{e}_3\rangle$		$\lvert i\mathbf{e}_3\rangle$	$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_1\rangle$	$\lvert 1\rangle$
$\lvert\mathbf{e}_{0+3i}\rangle$		$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$
$\lvert\mathbf{e}_{0-3i}\rangle$		$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$	$\lvert\mathbf{e}_{0+3i}\rangle$
$\lvert i\mathbf{e}_{1+2}\rangle$		$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$
$\lvert i\mathbf{e}_{1-2}\rangle$		$\lvert i\mathbf{e}_{1-2}\rangle$	$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$

		$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$
$\lvert 1\rangle$		$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$
$\lvert i\mathbf{e}_1\rangle$		$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$	$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$
$\lvert i\mathbf{e}_2\rangle$		$\lvert i\mathbf{e}_{1-2}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$	$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert\mathbf{e}_{0+3i}\rangle$
$\lvert i\mathbf{e}_3\rangle$		$\lvert\mathbf{e}_{0-3i}\rangle$	$\lvert\mathbf{e}_{0+3i}\rangle$	$\lvert i\mathbf{e}_{1-2}\rangle$	$\lvert i\mathbf{e}_{1+2}\rangle$
$\lvert\mathbf{e}_{0+3i}\rangle$		$\lvert i\mathbf{e}_3\rangle$	$\lvert 1\rangle$	$\lvert i\mathbf{e}_1\rangle$	$\lvert i\mathbf{e}_2\rangle$
$\lvert\mathbf{e}_{0-3i}\rangle$		$\lvert 1\rangle$	$\lvert i\mathbf{e}_3\rangle$	$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_1\rangle$
$\lvert i\mathbf{e}_{1+2}\rangle$		$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_1\rangle$	$\lvert 1\rangle$	$\lvert i\mathbf{e}_3\rangle$
$\lvert i\mathbf{e}_{1-2}\rangle$		$\lvert i\mathbf{e}_1\rangle$	$\lvert i\mathbf{e}_2\rangle$	$\lvert i\mathbf{e}_3\rangle$	$\lvert 1\rangle$

Table 1: Multiplication tables for the dihedral group $\mathcal{D}_4$.

and

$$|i\mathbf{e}_1\rangle|i\mathbf{e}_2\rangle = |i\mathbf{e}_3\rangle = |i\mathbf{e}_2\rangle|i\mathbf{e}_1\rangle, \tag{85}$$
$$|i\mathbf{e}_2\rangle|i\mathbf{e}_3\rangle = |i\mathbf{e}_1\rangle = |i\mathbf{e}_3\rangle|i\mathbf{e}_2\rangle, \tag{86}$$
$$|i\mathbf{e}_3\rangle|i\mathbf{e}_1\rangle = |i\mathbf{e}_2\rangle = |i\mathbf{e}_1\rangle|i\mathbf{e}_3\rangle. \tag{87}$$

Equation (84) states that a $180°-$rotation applied twice leaves an object unchanged; Eqs. (85) to (87) state that two consecutive $180°-$rotations about two mutually orthogonal axes is equal to a $180°-$rotation about an axis perpendicular to the previous two.

The product of the flip operator $|i\mathbf{e}_k\rangle$ and the quarter rotation operator $|i\mathbf{e}_{0\pm3i}\rangle$ satisfies the following conjugation-commutation relations:

$$|i\mathbf{e}_1\rangle|\mathbf{e}_{0\pm3i}\rangle = \frac{1}{\sqrt{2}}|i\mathbf{e}_1(1 \pm i\mathbf{e}_3)\rangle$$
$$= \frac{1}{\sqrt{2}}|(1 \mp i\mathbf{e}_3)i\mathbf{e}_1\rangle$$
$$= |\mathbf{e}_{0\mp3i}\rangle|i\mathbf{e}_1\rangle, \tag{88}$$
$$|i\mathbf{e}_2\rangle|\mathbf{e}_{0\pm3i}\rangle = |\mathbf{e}_{0\mp3i}\rangle|i\mathbf{e}_2\rangle, \tag{89}$$
$$|i\mathbf{e}_3\rangle|\mathbf{e}_{0\pm3i}\rangle = |\mathbf{e}_{0\pm3i}\rangle|i\mathbf{e}_3\rangle. \tag{90}$$

The passing over of $|i\mathbf{e}_1\rangle$ or $|i\mathbf{e}_1\rangle$ transforms $|\mathbf{e}_{0+3i}\rangle$ into its conjugate $|\mathbf{e}_{0-3i}\rangle$. But the passing over of $|i\mathbf{e}_3\rangle$ leaves $|\mathbf{e}_{0+3i}\rangle$ unchanged.

Alternatively, we can impose that $|\mathbf{e}_{0\pm3i}\rangle$ remain invariant and let $|i\mathbf{e}_j\rangle$ mutate, for $j = 1, 2$. That is, we

6

write
$$|\mathbf{e}_{0\pm3i}\rangle|i\mathbf{e}_j\rangle = |i\mathbf{e}_j'\rangle|\mathbf{e}_{0\pm3i}\rangle. \qquad (91)$$

Our problem is to express $|i\mathbf{e}_j'\rangle$ in terms of $|i\mathbf{e}_j\rangle$.

Let us first consider $|\mathbf{e}_{0\pm3i}\rangle$. From Eqs. (88) and (90), we have

$$|\mathbf{e}_{0+3i}\rangle|i\mathbf{e}_j\rangle = |i\mathbf{e}_j\rangle|\mathbf{e}_{0-3i}\rangle. \qquad (92)$$

Since

$$|\mathbf{e}_{0-3i}\rangle = |\mathbf{e}_{0+3i}\rangle^2|\mathbf{e}_{0+3i}\rangle = |i\mathbf{e}_3\rangle|\mathbf{e}_{0+3i}\rangle, \qquad (93)$$

then Eq. (92) becomes

$$|\mathbf{e}_{0+3i}\rangle|i\mathbf{e}_j\rangle = |i\mathbf{e}_j\rangle|i\mathbf{e}_3\rangle|\mathbf{e}_{0+3i}\rangle, \qquad (94)$$

so that

$$|i\mathbf{e}_j'\rangle = |i\mathbf{e}_j\rangle|i\mathbf{e}_3\rangle. \qquad (95)$$

Similarly, for $|\mathbf{e}_{0-3i}\rangle$, we have

$$|\mathbf{e}_{0-3i}\rangle|i\mathbf{e}_j\rangle = |i\mathbf{e}_j\rangle|\mathbf{e}_{0+3i}\rangle. \qquad (96)$$

Since

$$\begin{aligned} |\mathbf{e}_{0+3i}\rangle &= |\mathbf{e}_{0+3i}\rangle|\mathbf{e}_{0-3i}\rangle^{-1}|\mathbf{e}_{0-3i}\rangle \qquad &(97)\\ &= |\mathbf{e}_{0+3i}\rangle^2|\mathbf{e}_{0-3i}\rangle = |i\mathbf{e}_3\rangle|\mathbf{e}_{0-3i}\rangle, \quad &(98) \end{aligned}$$

then Eq. (96) becomes

$$|\mathbf{e}_{0-3i}\rangle|i\mathbf{e}_j\rangle = |i\mathbf{e}_j\rangle|i\mathbf{e}_3\rangle|\mathbf{e}_{0-3i}\rangle, \qquad (99)$$

so that Eq. (95) still holds.

We may now combine Eqs. (94) and (99) into one:

$$|\mathbf{e}_{0\pm3i}\rangle|i\mathbf{e}_j\rangle = |i\mathbf{e}_j\rangle|i\mathbf{e}_3\rangle|\mathbf{e}_{0\pm3i}\rangle. \qquad (100)$$

That is,

$$\begin{aligned} |\mathbf{e}_{0\pm3i}\rangle|i\mathbf{e}_1\rangle &= |i\mathbf{e}_2\rangle|\mathbf{e}_{0\pm3i}\rangle, \qquad &(101)\\ |\mathbf{e}_{0\pm3i}\rangle|i\mathbf{e}_2\rangle &= |i\mathbf{e}_1\rangle|\mathbf{e}_{0\pm3i}\rangle. \qquad &(102) \end{aligned}$$

Thus, the passing over of $|\mathbf{e}_{0\pm3i}\rangle$ changes $|i\mathbf{e}_1\rangle$ to $|i\mathbf{e}_2\rangle$ and vice versa.

As a check, let us expand both sides of Eq. (101):

$$\begin{aligned} |\mathbf{e}_{0\pm3i}\rangle|i\mathbf{e}_1\rangle &= |\frac{1}{\sqrt{2}}(1\pm i\mathbf{e}_3)(i\mathbf{e}_1)\rangle, \\ &= |\frac{1}{\sqrt{2}}(i\mathbf{e}_1 \mp i\mathbf{e}_2)\rangle, \qquad &(103) \end{aligned}$$

$$\begin{aligned} |i\mathbf{e}_2\rangle|\mathbf{e}_{0\pm3i}\rangle &= |\frac{1}{\sqrt{2}}(i\mathbf{e}_2)(1\pm i\mathbf{e}_3)\rangle, \\ &= |\frac{1}{\sqrt{2}}(i\mathbf{e}_2 \mp i\mathbf{e}_1)\rangle. \qquad &(104) \end{aligned}$$

If we take the $+$ case, then we have verified our claim. If we choose the $-$ case, the arguments of the resulting expansions are additive inverses of each other, which means

that the ket representations are equivalent by Hestenes's theorem in Eq. (55). This ends the proof.

The proof for Eq. (102) is similar.

c. Sums. Let $\mathbf{r}$ be the position of a point in three-dimensional Cartesian space:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \qquad (105)$$

The actions of the $\mathcal{D}_4$ elements on $\mathbf{r}$ are

$$\begin{aligned} \mathbf{r}\,|1\rangle &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \qquad &(106)\\ \mathbf{r}\,|i\mathbf{e}_1\rangle &= x_1\mathbf{e}_1 - x_2\mathbf{e}_2 - x_3\mathbf{e}_3, \qquad &(107)\\ \mathbf{r}\,|i\mathbf{e}_2\rangle &= -x_1\mathbf{e}_1 + x_2\mathbf{e}_2 - x_3\mathbf{e}_3, \qquad &(108)\\ \mathbf{r}\,|i\mathbf{e}_3\rangle &= -x_1\mathbf{e}_1 - x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \qquad &(109)\\ \mathbf{r}\,|\mathbf{e}_{0+3i}\rangle &= x_1\mathbf{e}_2 - x_2\mathbf{e}_1 + x_3\mathbf{e}_3, \qquad &(110)\\ \mathbf{r}\,|\mathbf{e}_{0-3i}\rangle &= -x_1\mathbf{e}_2 + x_2\mathbf{e}_1 + x_3\mathbf{e}_3, \qquad &(111)\\ \mathbf{r}\,|i\mathbf{e}_{1+2}\rangle &= x_1\mathbf{e}_2 + x_2\mathbf{e}_1 - x_3\mathbf{e}_3, \qquad &(112)\\ \mathbf{r}\,|i\mathbf{e}_{1-2}\rangle &= -x_1\mathbf{e}_2 - x_2\mathbf{e}_1 - x_3\mathbf{e}_3. \qquad &(113) \end{aligned}$$

We may add the actions of the $\mathcal{D}_4$ elements to arrive at null results, depending on the definition of the position vector $\mathbf{r}$. If we define

$$\begin{aligned} \mathbf{r}_1 &= x_1\mathbf{e}_1, \qquad &(114)\\ \mathbf{r}_{1+2} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2, \qquad &(115)\\ \mathbf{r}_{1+2+3} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3, \qquad &(116) \end{aligned}$$

then we have the following quasi-quaternion identities:

$$\begin{aligned} 0 &= \mathbf{r}_1(|1\rangle - |i\mathbf{e}_1\rangle), \qquad &(117)\\ 0 &= \mathbf{r}_{1+2}(|1\rangle + |i\mathbf{e}_3\rangle), \qquad &(118)\\ 0 &= \mathbf{r}_{1+2}(|i\mathbf{e}_1\rangle + |i\mathbf{e}_2\rangle), \qquad &(119)\\ 0 &= \mathbf{r}_{1+2}(|\mathbf{e}_{0+3i}\rangle + |\mathbf{e}_{0-3i}\rangle), \qquad &(120)\\ 0 &= \mathbf{r}_{1+2}(|i\mathbf{e}_{1+2}\rangle + |i\mathbf{e}_{1-2}\rangle), \qquad &(121)\\ 0 &= \mathbf{r}_{1+2+3}(|1\rangle + |i\mathbf{e}_1\rangle + |i\mathbf{e}_2\rangle + |i\mathbf{e}_3\rangle), \quad &(122) \end{aligned}$$

which we may rewrite as

$$\begin{aligned} |1\rangle - |i\mathbf{e}_1\rangle &\equiv 0; \quad \mathbf{r} \in \mathcal{C}l_{1,0}, &(123)\\ |1\rangle + |i\mathbf{e}_3\rangle &\equiv 0; \quad \mathbf{r} \in \mathcal{C}l_{2,0}, &(124)\\ |i\mathbf{e}_1\rangle + |i\mathbf{e}_2\rangle &\equiv 0; \quad \mathbf{r} \in \mathcal{C}l_{2,0}, &(125)\\ |\mathbf{e}_{0+3i}\rangle + |\mathbf{e}_{0-3i}\rangle &\equiv 0; \quad \mathbf{r} \in \mathcal{C}l_{2,0}, &(126)\\ |i\mathbf{e}_{1+2}\rangle + |i\mathbf{e}_{1-2}\rangle &\equiv 0; \quad \mathbf{r} \in \mathcal{C}l_{2,0}, &(127)\\ |1\rangle + |i\mathbf{e}_1\rangle + |i\mathbf{e}_2\rangle + |i\mathbf{e}_3\rangle &\equiv 0; \quad \mathbf{r} \in \mathcal{C}l_{3,0}. &(128) \end{aligned}$$

Note that the ket operator identities that hold for a higher dimensional definition of the position vector $\mathbf{r}$ also hold for the lower dimensional definition.

c. Products of Sums. Using the quasiquaternion algebra for orthogonal flips in Eqs. (84) to (87), we can show that

$$(|1\rangle + |i\mathbf{e}_k\rangle)^2 = 2(|1\rangle + |i\mathbf{e}_k\rangle), \qquad (129)$$

for $k = 1, 2, 3$. We can also show that

$$
\begin{aligned}
0 &= (|1\rangle + |i\mathbf{e}_1\rangle)(|1\rangle + |i\mathbf{e}_2\rangle), & (130)\\
&= (|1\rangle + |i\mathbf{e}_1\rangle)(|1\rangle + |i\mathbf{e}_3\rangle), & (131)\\
&= (|1\rangle + |i\mathbf{e}_2\rangle)(|1\rangle + |i\mathbf{e}_3\rangle). & (132)
\end{aligned}
$$

where we used the null identity in Eq. (128). Note that these products are commutative.

Another set of products that will become useful in the next subsection on dyadic operators are the following:

$$
\begin{aligned}
|\mathbf{e}_{0\pm3i}\rangle(|1\rangle + |i\mathbf{e}_1\rangle) &= (|1\rangle + |i\mathbf{e}_2\rangle)|\mathbf{e}_{0\pm3i}\rangle, & (133)\\
|\mathbf{e}_{0\pm3i}\rangle(|1\rangle + |i\mathbf{e}_2\rangle) &= (|1\rangle + |i\mathbf{e}_1\rangle)|\mathbf{e}_{0\pm3i}\rangle, & (134)\\
|\mathbf{e}_{0\pm3i}\rangle(|1\rangle + |i\mathbf{e}_3\rangle) &= (|1\rangle + |i\mathbf{e}_3\rangle)|\mathbf{e}_{0\pm3i}\rangle, & (135)
\end{aligned}
$$

where we used the identities in Eqs. (90), (101), and (102). Notice that only the binomial involving $|i\mathbf{e}_3\rangle$ is unaffected.

4.3 Dyadics

a. Right-Acting Operators. We may add the actions of the $\mathcal{D}_4$ elements to arrive at new transformations, which we shall denote using the $\cdot\mathbf{e}_{\mu\nu}$ operators:

$$
\begin{aligned}
\mathbf{r} \cdot \mathbf{e}_{11} &= \frac{1}{2}\mathbf{r}(|1\rangle + |i\mathbf{e}_1\rangle) = x_1\mathbf{e}_1, & (136)\\
\mathbf{r} \cdot \mathbf{e}_{12} &= \frac{1}{2}\mathbf{r}(|\mathbf{e}_{0+3i}\rangle + |i\mathbf{e}_{1+2}\rangle)\\
&= \frac{1}{2}\mathbf{r}(|1\rangle + |i\mathbf{e}_1\rangle)|\mathbf{e}_{0+3i}\rangle = x_1\mathbf{e}_2, & (137)\\
\mathbf{r} \cdot \mathbf{e}_{21} &= \frac{1}{2}\mathbf{r}(|\mathbf{e}_{0-3i}\rangle + |i\mathbf{e}_{1+2}\rangle)\\
&= \frac{1}{2}\mathbf{r}(|1\rangle + |i\mathbf{e}_2\rangle)|\mathbf{e}_{0-3i}\rangle = x_2\mathbf{e}_1, & (138)\\
\mathbf{r} \cdot \mathbf{e}_{22} &= \frac{1}{2}\mathbf{r}(|1\rangle + |i\mathbf{e}_2\rangle) = x_2\mathbf{e}_2. & (139)
\end{aligned}
$$

Thus, the right-acting operator $\cdot\mathbf{e}_{\mu\nu}$ extracts the μ-component of $\mathbf{r}$ and replaces $\mathbf{e}_\mu$ by $\mathbf{e}_\nu$.

Formally, let us define the $\cdot\mathbf{e}_{\mu\nu}$ operators in terms of the flip and quarter rotation operators:

$$
\begin{aligned}
\cdot\mathbf{e}_{11} &= \frac{1}{2}(|1\rangle + |i\mathbf{e}_1\rangle), & (140)\\
\cdot\mathbf{e}_{12} &= \frac{1}{2}(|1\rangle + |i\mathbf{e}_1\rangle)|\mathbf{e}_{0+3i}\rangle, & (141)\\
\cdot\mathbf{e}_{21} &= \frac{1}{2}(|1\rangle + |i\mathbf{e}_2\rangle)|\mathbf{e}_{0-3i}\rangle, & (142)\\
\cdot\mathbf{e}_{22} &= \frac{1}{2}(|1\rangle + |i\mathbf{e}_2\rangle). & (143)
\end{aligned}
$$

We shall refer to the $\cdot\mathbf{e}_{\mu\nu}$ operators as the unit extraction-replacement operators.

The action of $\cdot\mathbf{e}_{\mu\nu}$ on a vector $\mathbf{e}_\lambda$ and on another operator $\cdot\mathbf{e}_{\mu'\nu'}$ are given by

$$
\begin{aligned}
\mathbf{e}_\lambda \cdot \mathbf{e}_{\mu\nu} &= \delta_{\lambda\mu}\mathbf{e}_\nu, & (144)\\
\cdot\mathbf{e}_{\mu'\nu'} \cdot \mathbf{e}_{\mu\nu} &= \delta_{\nu'\mu} \cdot \mathbf{e}_{\mu'\nu}. & (145)
\end{aligned}
$$

The proof of Eq. (144) is obvious from Eqs. (136) to (139). For products involving $\cdot\mathbf{e}_{11}$ and $\cdot\mathbf{e}_{22}$ with other $\cdot\mathbf{e}_{\mu\nu}$ operators, the identities in Eqs. (129) and (132) are sufficient for the proof. But for products involving $\cdot\mathbf{e}_{12}$ and $\cdot\mathbf{e}_{21}$, we need the additional identities in Eq. (133) and (134), together with the theorem $|\mathbf{e}_{0-3i}\rangle = |\mathbf{e}_{0+3i}\rangle^{-1}$ in Eq. (80).

Notice the similarity of the action of the $\cdot\mathbf{e}_{\mu\nu}$ operators to Symon's unit dyadic operators[15]. In fact, we could make the formal replacement

$$
\cdot\mathbf{e}_{\mu\nu} = \cdot\mathbf{e}_\mu\mathbf{e}_\nu \tag{146}
$$

and rederive the results in Eq. (144) and (145), assuming that the dot product takes precedence over the geometric product.

b. Left-Acting Operators. The bra-to-ket conversion in Eq. (42), together with the equivalent ket representations in Eq. (66) to (73), lets us write

$$
\begin{aligned}
\mathbf{r}\,|1\rangle &= \langle (1)^{-1}|\,\mathbf{r} &= \langle 1|\,\mathbf{r}, & (147)\\
\mathbf{r}\,|i\mathbf{e}_1\rangle &= \langle (i\mathbf{e}_1)^{-1}|\,\mathbf{r} &= \langle i\mathbf{e}_1|\,\mathbf{r}, & (148)\\
\mathbf{r}\,|i\mathbf{e}_2\rangle &= \langle (i\mathbf{e}_2)^{-1}|\,\mathbf{r} &= \langle i\mathbf{e}_2|\,\mathbf{r}, & (149)\\
\mathbf{r}\,|i\mathbf{e}_3\rangle &= \langle (i\mathbf{e}_3)^{-1}|\,\mathbf{r} &= \langle i\mathbf{e}_3|\,\mathbf{r}, & (150)\\
\mathbf{r}\,|\mathbf{e}_{0+3i}\rangle &= \langle (e^{i\mathbf{e}_3\pi/4})^{-1}|\,\mathbf{r} &= \langle \mathbf{e}_{0-3i}|\,\mathbf{r}, & (151)\\
\mathbf{r}\,|\mathbf{e}_{0-3i}\rangle &= \langle (e^{-i\mathbf{e}_3\pi/4})^{-1}|\,\mathbf{r} &= \langle \mathbf{e}_{0+3i}|\,\mathbf{r}, & (152)\\
\mathbf{r}\,|i\mathbf{e}_{1+2}\rangle &= \langle (e^{i(\mathbf{e}_{1+2})\pi/2})^{-1}|\,\mathbf{r} &= \langle i\mathbf{e}_{1+2}|\,\mathbf{r}, & (153)\\
\mathbf{r}\,|i\mathbf{e}_{1-2}\rangle &= \langle (e^{i(\mathbf{e}_{1-2})\pi/2})^{-1}|\,\mathbf{r} &= \langle i\mathbf{e}_{1-2}|\,\mathbf{r}. & (154)
\end{aligned}
$$

The right side of Eqs. (147) to (154) are the bra representations of the $\mathcal{D}_4$ group elements.

Using these ket-bra translations, we may transform Eqs. (136) to (139) into

$$
\begin{aligned}
\mathbf{e}_{11} \cdot \mathbf{r} &= x_1\mathbf{e}_1, & (155)\\
\mathbf{e}_{21} \cdot \mathbf{r} &= x_1\mathbf{e}_2, & (156)\\
\mathbf{e}_{12} \cdot \mathbf{r} &= x_2\mathbf{e}_1, & (157)\\
\mathbf{e}_{22} \cdot \mathbf{r} &= x_2\mathbf{e}_2, & (158)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{e}_{11}\cdot &= \frac{1}{2}(\langle 1| + \langle i\mathbf{e}_1|), & (159)\\
\mathbf{e}_{21}\cdot &= \frac{1}{2}\langle \mathbf{e}_{0-3i}|(\langle 1| + \langle i\mathbf{e}_1|), & (160)\\
\mathbf{e}_{12}\cdot &= \frac{1}{2}\langle \mathbf{e}_{0+3i}|(\langle 1| + \langle i\mathbf{e}_2|), & (161)\\
\mathbf{e}_{22}\cdot &= \frac{1}{2}(\langle 1| + \langle i\mathbf{e}_2|) & (162)
\end{aligned}
$$

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are the four left-acting dyadic operators.

Using similar theorems as those used to prove Eqs. (144) and (145), we can show that

$$
\begin{aligned}
\mathsf{e}_{\nu\mu} \cdot \mathbf{e}_\lambda &= \delta_{\mu\lambda}\mathbf{e}_\nu, & (163) \\
\mathsf{e}_{\nu\mu} \cdot \mathsf{e}_{\nu'\mu'}\cdot &= \delta_{\mu\nu'}\mathsf{e}_{\nu\mu'}\cdot & (164)
\end{aligned}
$$

Notice that the left- and right-acting diadics are related by transposition:

$$
(\cdot\, \mathsf{e}_{\mu\nu})^T = \mathsf{e}_{\nu\mu}\cdot \tag{165}
$$

4.4 Matrices

a. Right-Acting Matrices. Let $\mathbf{r}$ and $\mathbf{r}'$ be vectors on the $xy-$plane,

$$
\begin{aligned}
\mathbf{r} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathcal{C}l_{2,0}, & (166) \\
\mathbf{r}' &= x_1'\mathbf{e}_1 + x_2'\mathbf{e}_2 \in \mathcal{C}l_{2,0}, & (167)
\end{aligned}
$$

and let $\cdot\,\mathsf{M}$ be a linear combination of right-acting dyadics,

$$
\cdot\,\mathsf{M} = \cdot\,\mathsf{e}_{11}M_{11} + \cdot\,\mathsf{e}_{12}M_{12} + \cdot\,\mathsf{e}_{21}M_{21} + \cdot\,\mathsf{e}_{22}M_{22}, \tag{168}
$$

where

$$
\cdot\,\mathsf{e}_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \cdot\,\mathsf{e}_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{169}
$$

$$
\cdot\,\mathsf{e}_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \cdot\,\mathsf{e}_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{170}
$$

are the four Fermion matrices.

If $\mathbf{r}'$ is vector $\mathbf{r}$ right-acted by $\cdot\,\mathsf{M}$,

$$
\mathbf{r}' = \mathbf{r} \cdot \mathsf{M}, \tag{171}
$$

then by Eq. (144), we have

$$
\begin{aligned}
x_1' &= x_1M_{11} + x_2M_{21}, & (172) \\
x_2' &= x_1M_{12} + x_2M_{22}. & (173)
\end{aligned}
$$

In matrix form, this is

$$
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{174}
$$

Notice that to get x_1' we multiply the $\mathbf{r}$ column with the left column of $\cdot\,\mathsf{M}$, as prescribed by Eq. (172); for x_2', we multiply $\mathbf{r}$ with the right column of $\cdot\,\mathsf{M}$, as prescribed by Eq. (173).

On the other hand, to derive the rules for the matrix products, let $\cdot\,\mathsf{M}$, $\cdot\,\mathsf{M}'$, and $\cdot\,\mathsf{M}''$ be 2×2 right-acting matrices such that

$$
\cdot\,\mathsf{M}'' = \cdot\,\mathsf{M} \cdot \mathsf{M}'. \tag{175}
$$

where M is defined in Eq. (168); M' and M'' are defined similarly. That is,

$$
\begin{pmatrix} M_{11}'' & M_{12}'' \\ M_{21}'' & M_{22}'' \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M_{11}' & M_{12}' \\ M_{21}' & M_{22}' \end{pmatrix} \tag{176}
$$

Using the rule for dyadic products in Eq. (145) and separating the matrix components, we obtain

$$
\begin{aligned}
M_{11}'' &= M_{11}M_{11}' + M_{12}M_{21}', & (177) \\
M_{12}'' &= M_{11}M_{12}' + M_{12}M_{22}', & (178) \\
M_{21}'' &= M_{21}M_{11}' + M_{22}M_{21}', & (179) \\
M_{22}'' &= M_{21}M_{12}' + M_{22}M_{22}', & (180)
\end{aligned}
$$

which agrees with the traditional definition of matrix multiplication.

b. Left-Acting Matrices. If M is the right acting matrix defined in Eq. (168), then by Eqs. (165), we may convert M into left-acting by taking its transpose, as noted in Symon[16]:

$$
\mathbf{r}' = \mathsf{M}^T \cdot \mathbf{r} = \mathbf{r} \cdot \mathsf{M}, \tag{181}
$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the two-dimensional vectors defined in Eq. (166) to (167), and

$$
\mathsf{M}^T = M_{11}\mathsf{e}_{11} + M_{12}\mathsf{e}_{21} + M_{21}\mathsf{e}_{12} + M_{22}\mathsf{e}_{22}. \tag{182}
$$

That is,

$$
\begin{aligned}
\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} &= \begin{pmatrix} M_{11} & M_{21} \\ M_{12} & M_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} & (183) \\
&= \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. & (184)
\end{aligned}
$$

As expected, the first component x_1' is the product of the first row of the left acting matrix M^T and the column vector $\mathbf{x}$; for the second component x_2', the product is with the second row. Notice that the column-column multiplication for the action of right-acting matrices on column vectors is pedagogically simpler than the row-column multiplication for left-acting matrices.

Applying another right-acting operator M' on Eq. (181) yields

$$
\mathsf{M}'^T \cdot \mathsf{M}^T \cdot \mathbf{r} = \mathbf{r} \cdot \mathsf{M} \cdot \mathsf{M}'. \tag{185}
$$

Because $\mathsf{M}\mathsf{M}'$ is a matrix, then by Eq. (181), we have

$$
(\cdot\,\mathsf{M} \cdot \mathsf{M}')^T = \mathsf{M}'^T \cdot \mathsf{M}^T\cdot, \tag{186}
$$

which is the known theorem for the transpose of a product of matrices[17]. But unlike the traditional expansion, we clarify the handedness of the matrices.

The product of two left-acting matrices is the same as if the matrices were right acting, by virtue of the dyadic

product relation in Eq. (164). That is, if we rewrite Eq. (175) as

$$\mathsf{M}'' \cdot = \mathsf{M} \cdot \mathsf{M}' \cdot, \qquad (187)$$

then we will still arrive at Eqs. (176) to (180).

c. $\mathcal{D}_4$ Matrices. The 2×2 matrix representations of the dihedral group $\mathcal{D}_4$ elements may be readily obtained by setting $x_3 = 0$ in Eqs. (106) to (113), and using the rules for the actions of the Fermion dyadics on vectors:

$$\mathbf{r}|1\rangle = \mathbf{r}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{r} \cdot (\mathbf{e}_{11} + \mathbf{e}_{22}), \qquad (188)$$

$$\mathbf{r}|i\mathbf{e}_1\rangle = \mathbf{r}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{r} \cdot (\mathbf{e}_{11} - \mathbf{e}_{22}), \qquad (189)$$

$$\mathbf{r}|i\mathbf{e}_2\rangle = \mathbf{r}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{r} \cdot (-\mathbf{e}_{11} + \mathbf{e}_{22}), \qquad (190)$$

$$\mathbf{r}|i\mathbf{e}_3\rangle = \mathbf{r}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{r} \cdot (-\mathbf{e}_{11} - \mathbf{e}_{22}), \qquad (191)$$

$$\mathbf{r}|\mathbf{e}_{0+3i}\rangle = \mathbf{r}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{r} \cdot (\mathbf{e}_{12} - \mathbf{e}_{21}), \qquad (192)$$

$$\mathbf{r}|\mathbf{e}_{0-3i}\rangle = \mathbf{r}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \mathbf{r} \cdot (-\mathbf{e}_{12} + \mathbf{e}_{21}), \qquad (193)$$

$$\mathbf{r}|i\mathbf{e}_{1+2}\rangle = \mathbf{r}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \mathbf{r} \cdot (\mathbf{e}_{12} + \mathbf{e}_{21}), \qquad (194)$$

$$\mathbf{r}|i\mathbf{e}_{1-2}\rangle = \mathbf{r}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \mathbf{r} \cdot (-\mathbf{e}_{12} - \mathbf{e}_{21}). \qquad (195)$$

Note that the bra representations of these operators have the same arguments as those of the kets, except for the ket $|\mathbf{e}_{0\pm3i}\rangle$ whose bra representation is $\langle\mathbf{e}_{0\mp3i}|$, as noted in Eqs. (147) to (154). Note also that the transpose of these right-acting matrices are the left-acting matrix representations of the dihedral group $\mathcal{D}_4$ as given by Lakshminarayanan and Viana[18].

d. Pauli Matrices. In terms of leftt-acting bra operators, Campbell's four primary matrices $|\mathbf{I}|$, $|+|$, $|\mathbf{x}|$, and $|\mathbf{J}|$ and their negatives may be expressed as[19]

$$\langle 1| = |\mathbf{I}| = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \equiv -\langle i\mathbf{e}_3|, \qquad (196)$$

$$\langle i\mathbf{e}_1| = |+| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \equiv -\langle i\mathbf{e}_2|, \qquad (197)$$

$$\langle i\mathbf{e}_{1+2}| = |\mathbf{x}| = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv -\langle i\mathbf{e}_{1-2}|, \qquad (198)$$

$$\langle \mathbf{e}_{0-3i}| = |\mathbf{J}| = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv -\langle i\mathbf{e}_{0+3i}|. \qquad (199)$$

Note that these equivalence relations are only valid if their operand $\mathbf{r} \in \mathcal{C}l_{2,0}$, as required by the ket identities in Eqs. (124) to (127).

The Pauli $\hat{\sigma}$−matrices may be defined similarly:

$$\hat{\sigma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (\mathbf{e}_{11} + \mathbf{e}_{22}) \cdot \equiv \langle 1|, \qquad (200)$$

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = (\mathbf{e}_{12} + \mathbf{e}_{21}) \cdot \equiv \langle i\mathbf{e}_{1+2}|, \qquad (201)$$

$$\hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = (-i\mathbf{e}_{12} + i\mathbf{e}_{21}) \cdot \equiv i\langle \mathbf{e}_{0-3i}| \qquad (202)$$

$$\hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (\mathbf{e}_{11} - \mathbf{e}_{22}) \cdot \equiv \langle i\mathbf{e}_1|, \qquad (203)$$

Note the extra imaginary number $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ in $\hat{\sigma}_2$.

To geometrically interpret the Pauli sigma matrices, we express it in terms of ket exponentials:

$$\hat{\sigma}_0 \cdot \mathbf{r} = \mathbf{r}|1\rangle, \qquad (204)$$
$$\hat{\sigma}_1 \cdot \mathbf{r} = \mathbf{r}|i\mathbf{e}_{1+2}\rangle = \mathbf{r}|e^{i\mathbf{e}_{1+2}\pi/2}\rangle, \qquad (205)$$
$$\hat{\sigma}_2 \cdot \mathbf{r} = (i\mathbf{r})|\mathbf{e}_{0-3i}\rangle = (i\mathbf{r})|e^{-i\mathbf{e}_3\pi/4}\rangle, \qquad (206)$$
$$\hat{\sigma}_3 \cdot \mathbf{r} = \mathbf{r}|i\mathbf{e}_1\rangle = \mathbf{r}|e^{i\mathbf{e}_1\pi/2}\rangle. \qquad (207)$$

Because the arguments of the exponentials are half-angles, then $\hat{\sigma}_0$ is an identity rotation of the vector

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad (208)$$

$\hat{\sigma}_1$ is a flip (rotation by π) about the the diagonal axis $\mathbf{e}_{1+2}$ and $\hat{\sigma}_3$ is a flip about the $\mathbf{e}_1$ axis; the matrix σ_2 is a $\pi/2$ rotation about $\mathbf{e}_3$ and the result multiplied by i.

5   Conclusions

We represented vector rotation operators in terms of bras or kets of half-angle exponentials in Clifford (geometric) algebra $\mathcal{C}l_{3,0}$. The bras are left-acting operators with negative exponential arguments; the kets are right-acting operators with postive arguments. We showed that SO_3 is a rotation group and the dihedral group $\mathcal{D}_4$ as its finite subgroup. We derived several theorems: Euler-Rodrigues formulas, Hestenes's representation equivalence, bra-to-ket transformation, and ket commutation-conjugation identities. We computed the group table of $\mathcal{D}_4$ by illustrating a square on the xy−plane, labeling its several symmetry axis, and defining its half-angle exponential ket operators. We took linear combinations of the elements of $\mathcal{D}_4$ to represent the four Fermion matrices, which in turn we used to decompose any 2×2 matrix. We showed that bra and ket operators generate left- and right-acting matrices, respectively, which are related by transposition. We also showed that the Pauli spin matrices are not vectors but vector rotation operators, except for $\hat{\sigma}_2$ which required a subsequent multiplication by the imaginary number i geometrically interpreted as the unit oriented volume.

The bra and ket representations may be used to describe the elements of other crystallographic point groups: the half-angle exponential representation would provide us immediate information on the direction of the symmetry axis and its corresponding rotation angles. So in the computation of group tables, we need not worry about matrix representations of group elements: as long as we can imagine the products of composite rotations using palm-twisting arguments or compute them using the rules of geometric algebra, we would be able to arrive at our answers in the least time, even mentally.

We may also extend our geometric algebra reformulation to 3×3 matrices. In this case, the symmetry group that we must use is that of a cube.

Acknowledgments

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