Field Potentials

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The Field Multivector

The Field Multivector

We have introduced the multivector $\mathcal{F}$, composed by a vector and a bivector part

$$\mathcal{F} = \mathbf{E} + i\,\eta\,\mathbf{H}\,. \tag{1}$$

and we have written the four Maxwell equation as a single one:

$$\left(\nabla + \frac{1}{v}\partial_t\right)\mathcal{F} = \eta\,\left(v\rho - \mathbf{J}\right)\,. \tag{2}$$

By using the operator ∂^+ as

$$\partial^+ = \nabla + \frac{1}{v}\partial_t \tag{3}$$

and the source multivector $\mathcal{J}$

$$\mathcal{J} = \eta\,\left(v\rho - \mathbf{J}\right) \tag{4}$$

which allows to write (2) as

$$\partial^+\mathcal{F} = \mathcal{J}\,. \tag{5}$$

Operators Summary

Table 1: Table summarizing operator definitions. The symbols with a tilde refer to 2 by 2 matrices, while the symbols with a bar refer to combination of 4 by 4 Dirac matrices. The underlined symbols refer to frequency domain operators.

$$
\begin{aligned}
\partial^{\pm} &= \nabla \pm \tfrac{1}{v}\,\partial_t \\
\tilde{\partial}^{\pm} &= \tilde{\nabla} \pm \tfrac{1}{v}\,\partial_t\,\sigma_0 \\
\bar{\partial}^{\pm} &= \begin{pmatrix} \pm\frac{1}{v}\,\partial_t\,\sigma_0 & \tilde{\nabla} \\ \tilde{\nabla} & \pm\frac{1}{v}\,\partial_t\,\sigma_0 \end{pmatrix} \\
\bar{\partial}_d &= \begin{pmatrix} \tilde{\partial}^{+} & 0 \\ 0 & \tilde{\partial}^{-} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\underline{\partial}^{\pm} &= \nabla \pm j\,k \\
\underline{\tilde{\partial}}^{\pm} &= \tilde{\nabla} \pm j\,k\,\sigma_0 \\
\underline{\bar{\partial}}^{\pm} &= \begin{pmatrix} \pm j\,k\,\sigma_0 & \tilde{\nabla} \\ \tilde{\nabla} & \pm j\,k\,\sigma_0 \end{pmatrix} \\
\underline{\bar{\partial}}_d &= \begin{pmatrix} \underline{\tilde{\partial}}^{+} & 0 \\ 0 & \underline{\tilde{\partial}}^{-} \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\Box^2 &= \partial^{+}\partial^{-} = \nabla^2 - \tfrac{1}{v^2}\partial_t^2 \\
\underline{\partial}^2 &= \underline{\partial}^{+}\underline{\partial}^{-} = \nabla^2 + k^2
\end{aligned}
$$

Field Potentials: GA approach

Field Potentials

It is assumed that the field $\mathcal{F}$ can be recovered from a scalar potential ϕ and a vector potential $\mathbf{A}$, using the following expression:

$$\mathcal{F} = \left(\nabla - \frac{1}{v}\partial_t\right)(-\phi + v\,\mathbf{A}) \tag{6}$$

or, equivalently, by introducing the multivector $\mathcal{A}$ defined as

$$\mathcal{A} = -\phi + v\,\mathbf{A} \tag{7}$$

we can write

$$\mathcal{F} = \partial^{-}\mathcal{A} \tag{8}$$

Potential equation

When (6) is inserted into (2) we recover the following equation:

$$\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)(-\phi + v\,\mathbf{A}) = \frac{\rho}{\epsilon} - \eta\,\mathbf{J} \tag{9}$$

which may be synthetically expressed as

$$\Box\mathcal{A} = \mathcal{J}\,. \tag{10}$$

By separating the scalar and the vector part, we recover respectively the scalar and vector wave equations

$$\begin{aligned}\left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\phi \quad &= \quad -\frac{\rho}{\epsilon} && (11)\\ \left(\nabla^2 - \frac{1}{v^2}\partial_t^2\right)\mathbf{A} \quad &= \quad -\mu\mathbf{J}\,. && (12)\end{aligned}$$

Potential equations

Finally, by equating (6) and (1) we get:

$$\begin{aligned}
\mathcal{F} &= \mathbf{E} + \eta\,\hat{\mathbf{H}} = \partial^{-}\mathcal{A} \\
&= \left(\nabla - \frac{1}{v}\partial_t\right)(-\phi + v\,\mathbf{A}) \\
&= \frac{1}{v}\partial_t\phi + v\nabla\cdot\mathbf{A} - \nabla\wedge\phi - \partial_t\mathbf{A} + v\,\nabla\wedge\mathbf{A}\,. \qquad (13)
\end{aligned}$$

Considering the scalar part one gets

$$0 = \partial_t\phi + v^2\,\nabla\cdot\mathbf{A} \qquad (14)$$

$$\mathbf{E} = -\nabla\wedge\phi - \partial_t\mathbf{A} \qquad (15)$$

$$\eta\,\hat{\mathbf{H}} = v\,\nabla\wedge\mathbf{A} \qquad (16)$$

Lorenz gauge

We have obtained

$$0 = \partial_t \phi + v \nabla \cdot \mathbf{A} \tag{17}$$

This is the Lorenz gauge and has been obtained simply by equating the grades.

To summarize we have expressed Maxwell's equations as

$$\partial^+ \mathcal{F} = \mathcal{J} \,, \tag{18}$$

and we have then expressed the field in terms of the potential as

$$\mathcal{F} = \partial^- \mathcal{A} \tag{19}$$

obtaining the following equation

$$\mathcal{J} = \partial^+ \mathcal{F} = \partial^+ \partial^- \mathcal{A} = \Box \mathcal{A} \tag{20}$$

Potentials in spinor form in time–domain

Potentials in spinor form in time–domain

We start by rewriting (15) as

$$\mathbf{E} = -\nabla \wedge \phi - \partial_t \mathbf{A} = -\nabla\phi - \partial_0 v\, \mathbf{A} \tag{21}$$

where we have used (52) and denoted with ∂_0

$$\partial_0 = \frac{1}{v}\partial_t\,. \tag{22}$$

It is also convenient to express (16) in a different form noting that

$$\nabla \wedge \mathbf{A} = \nabla\mathbf{A} - \nabla\cdot\mathbf{A} = \nabla\,\mathbf{A} + \frac{1}{v}\partial_0\phi \tag{23}$$

and therefore

$$i\,\eta\,\mathbf{H} = v\,\nabla \wedge \mathbf{A} = v\,\nabla\mathbf{A} + \partial_0\phi\,. \tag{24}$$

Potentials in terms of Dirac matrices

So far we have therefore expressed the field in terms of the potentials, taking into account Lorenz condition, as

$$\begin{aligned} \mathbf{E} &= \partial_0 \left(-v\mathbf{A}\right) + \nabla \left(-\phi\right) \\ i\,\eta\,\mathbf{H} &= -\nabla \left(-v\mathbf{A}\right) - \partial_0 \left(-\phi\right) \end{aligned} \tag{25}$$

Is apparent that (25) exhibit a symmetry suitable for expressing it by means of Dirac matrices. Let us introduce the following notation:

$$\begin{aligned} \mathbf{e} &= \begin{pmatrix} E_z \\ i\,E_y + E_x \end{pmatrix} \\ \mathbf{h} &= \begin{pmatrix} \eta\, i\, H_z \\ \eta\, \left(i\, H_x - H_y\right) \end{pmatrix} \end{aligned} \tag{26}$$

Field from Potentials in matrix form

By using (26) we have

$$\begin{aligned}\begin{pmatrix}\mathbf{e}\\ \mathbf{h}\end{pmatrix} &= \begin{pmatrix}\partial_0\sigma_0 & \tilde{\nabla}\\ -\tilde{\nabla} & -\partial_0,\sigma_0\end{pmatrix}\begin{pmatrix}-v\tilde{A}\\ -\tilde{\phi}\end{pmatrix}\\ &= -\begin{pmatrix}\partial_0\sigma_0 & \tilde{\nabla}\\ -\tilde{\nabla} & -\partial_0,\sigma_0\end{pmatrix}\begin{pmatrix}v\tilde{A}\\ \tilde{\phi}\end{pmatrix}\end{aligned} \tag{27}$$

with the position

$$\begin{aligned}\tilde{A} &= \begin{pmatrix}A_z\\ A_x + i\,A_y\end{pmatrix}\\ \tilde{\phi} &= \begin{pmatrix}\phi\\ 0\end{pmatrix}.\end{aligned} \tag{28}$$

Potential quadrivector

By introducing the quadrivector for the potential

$$\bar{A} \quad = \quad \begin{pmatrix} vA_z \\ v\left(i\,A_y + A_x\right) \\ \phi \\ 0 \end{pmatrix} \tag{29}$$

and noting that the matrix appearing in (27) is the same appearing in Maxwell's equations we have

$$\bar{F} = -\not{\partial}\bar{A} \tag{30}$$

Expanded version of (27)

The expanded version of (27), in rectangular coordinates, is:

$$\begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = - \begin{pmatrix} \partial_0 & 0 & \partial_z & \partial_x - i\,\partial_y \\ 0 & \partial_0 & i\,\partial_y + \partial_x & -\partial_z \\ -\partial_z & -\partial_x + i\,\partial_y & -\partial_0 & 0 \\ -i\,\partial_y - \partial_x & \partial_z & 0 & -\partial_0 \end{pmatrix} \begin{pmatrix} vA_z \\ v\,(i\,A_y + A_x) \\ \phi \\ 0 \end{pmatrix}$$

It is thus noted that we can write a single equation containing the sources the field and the potential as

$$-\Box^2 \bar{A} = -\partial\!\!\!/^2 \bar{A} = \partial\!\!\!/ \bar{F} = -\eta \bar{J} \tag{31}$$

Frequency–domain

Time–domain Maxwell's equations

Time–domain Maxwell's equations are commonly expressed as:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{32}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{33}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{34}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{35}$$

Maxwell's equations: frequency domain

In the following, we make use of equivalence theorems which introduce magnetic current density, $\mathbf{M}(\mathbf{r})$, and magnetic charge distributions, $\rho_m(\mathbf{r})$. These quantities, although not physically present, help in the solution of several boundary value problems. When considering also magnetic currents and charges, the frequency–domain Maxwell's equations become

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mathbf{B}(\mathbf{r}) - \mathbf{M}(\mathbf{r}) \,, \tag{36a}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \,, \tag{36b}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_e(\mathbf{r}) \,, \tag{36c}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = -\rho_m(\mathbf{r}) \,. \tag{36d}$$

Frequency–domain Maxwell's Equation in compact form

In the following we initially consider only electric sources and an homogeneous region of space of constant permittivity and permeability, so that we can use $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$.

We multiply both sides of (36a) times i and applying ($\nabla \wedge \mathbf{E} = i\,\nabla \times \mathbf{E}$) obtaining:

$$\nabla \wedge \mathbf{E} = -j\omega\mu i\,\mathbf{H}\,. \tag{37}$$

By summing together (37) and the divergence equation one gets

$$\nabla\,\mathbf{E} = -j\omega\mu\, i\,\mathbf{H} + \frac{\rho}{\epsilon}\,. \tag{38}$$

One may rewrite the above eq. as

$$\nabla\,\mathbf{E} = -j\omega\mu\;\hat{\mathbf{H}} + \frac{\rho}{\epsilon}\,. \tag{39}$$

Frequency–domain Maxwell's Equation in compact form

We can now consider (36b) multiply by $i\,i = -1$ and summing with (35) obtaining

$$\nabla\,\hat{\mathbf{H}} = -j\omega\epsilon\,\mathbf{E} - \mathbf{J}\,. \tag{40}$$

Equations (39) and (40) can be expressed as

$$\begin{aligned} \nabla\,\mathbf{E} &= -j\,k\,\eta\,\hat{\mathbf{H}} + \frac{\rho}{\epsilon} && (41)\\ \nabla\eta\,\hat{\mathbf{H}} &= -j\,k\,\mathbf{E} - \eta\,\mathbf{J} && (42) \end{aligned}$$

Frequency–domain Maxwell's Equation: GA

By summing together (41) and (42) and using (1), the well known results that allows to express the four Maxwells' equation as a single one in the frequency–domain is recovered:

$$(\nabla + j\,k\,)\,\mathcal{F} = \frac{\rho}{\epsilon} - \eta\,\mathbf{J} \tag{43}$$

or, synthetically,

$$\underline{\partial}^{+}\mathcal{F} = \mathcal{J} \tag{44}$$

Frequency–domain potentials: GA approach

Frequency–domain potentials: GA approach

It is assumed that the field $\mathcal{F}$ can be recovered from a scalar potential ϕ and a vector potential $\mathbf{A}$, using the following expression:

$$\mathcal{F} = (\nabla - j\,k\,)\,(v\,\mathbf{A} - \phi)\ . \tag{45}$$

When (45) is inserted into (43) we recover the following equation:

$$\left(\nabla^2 + k^2\right)(v\,\mathbf{A} - \phi) = \frac{\rho}{\epsilon} - \eta\,\mathbf{J}\,. \tag{46}$$

and, by separating the scalar and the vector part, we recover respectively the scalar and vector wave equations

$$\left(\nabla^2 + k^2\right)\phi \;=\; -\frac{\rho}{\epsilon} \tag{47}$$

$$\left(\nabla^2 + k^2\right)\mathbf{A} \;=\; -\mu\mathbf{J}\,. \tag{48}$$

The grade structure

Finally, by equating (45) and (1) we get:

$$\begin{aligned}
\mathcal{F} &= \mathbf{E} + \eta\,\hat{\mathbf{H}} && (49)\\
&= (\nabla - j\,k\,)\,(v\,\mathbf{A} - \phi) && (50)\\
&= j\,k\,\phi + v\nabla\cdot\mathbf{A} - \nabla\wedge\phi - j\,k\,v\,\mathbf{A} + v\,\nabla\wedge\mathbf{A}\,. && (51)
\end{aligned}$$

Considering the scalar, vector and bivectors parts one gets

$$\begin{aligned}
0 &= j\,k\,\phi + v\,\nabla\cdot\mathbf{A} && (52)\\
\mathbf{E} &= -\nabla\wedge\phi - j\,k\,v\,\mathbf{A} && (53)\\
\eta\,\hat{\mathbf{H}} &= v\,\nabla\wedge\mathbf{A} && (54)
\end{aligned}$$

Potentials: GA

To summarize:

$$\left(\nabla^2 + k^2\right)\phi = -\frac{\rho}{\epsilon} \tag{55}$$

$$\left(\nabla^2 + k^2\right)\mathbf{A} = -\mu\mathbf{J}\,. \tag{56}$$

and

$$\phi = -\frac{v\,\nabla\cdot\mathbf{A}}{j\,k} \tag{57}$$

$$\mathbf{E} = -j\,k\,v\,\mathbf{A} - \nabla\phi \tag{58}$$

$$\hat{\mathbf{H}} = \frac{1}{\mu}\nabla\wedge\mathbf{A} \tag{59}$$

Note that the Lorenz condition is not imposed, but it is derived directly from the assumption in (45).

Potentials in spinor form

Potentials in spinor form

We have seen that in frequency domain the potentials are given by:

$$\begin{aligned} \phi &= -\frac{v\,\nabla\cdot\mathbf{A}}{j\,k} && (60)\\ \mathbf{E} &= -j\,k\,v\,\mathbf{A} - \nabla\phi && (61)\\ i\,\mathbf{H} &= \frac{1}{\mu}\nabla\wedge\mathbf{A}\,. && (62) \end{aligned}$$

It is convenient to express (62) in a different form noting that

$$\nabla\wedge\mathbf{A} = \nabla\mathbf{A} - \nabla\cdot\mathbf{A} = \nabla\,\mathbf{A} + j\,\frac{k}{v}\phi \tag{63}$$

and therefore

$$i\,\eta\,\mathbf{H} = v\,\nabla\mathbf{A} + j\,k\,\phi\,. \tag{64}$$

Symmetric form

So far we have therefore expressed the field in terms of the potentials, taking into account Lorenz condition, as

$$\begin{aligned}\mathbf{E} &= v\left(-j\,k\,\mathbf{A}-\frac{1}{v}\nabla\phi\right)\\ i\,\eta\,\mathbf{H} &= v\left[\nabla\mathbf{A}+(-j\,k)\left(-\frac{\phi}{v}\right)\right]\end{aligned}\tag{65}$$

Is apparent that (65) exhibit a symmetry suitable for expressing it by means of Dirac matrices. Let us introduce the following notation:

$$\begin{aligned}\mathbf{e} &= \begin{pmatrix} E_z \\ i\,E_y+E_x \end{pmatrix}\\ \mathbf{h} &= \begin{pmatrix} \eta\,i\,H_z \\ \eta\,(i\,H_x-H_y) \end{pmatrix}\end{aligned}\tag{66}$$

Potentials as quadrivectors

By using (66) we have

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix} = v \begin{pmatrix} -j\,k\,\sigma_0 & \tilde{\nabla} \\ \tilde{\nabla} & -j\,k\,\sigma_0 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{\phi} \end{pmatrix}. \tag{67}$$

with the position

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} A_z \\ A_x + i\,A_y \end{pmatrix} \\ \tilde{\phi} &= \begin{pmatrix} -\frac{\phi}{v} \\ 0 \end{pmatrix}. \end{aligned} \tag{68}$$

Expanded version

The expanded version of (67), in rectangular coordinates, is:

$$\begin{pmatrix} E_z \\ i\,E_y + E_x \\ \eta\, i\, H_z \\ \eta\,(i\,H_x - H_y) \end{pmatrix} = v \begin{pmatrix} -jk & 0 & \partial_z & \partial_x - i\,\partial_y \\ 0 & -jk & i\,\partial_y + \partial_x & -\partial_z \\ \partial_z & \partial_x - i\,\partial_y & -jk & 0 \\ i\,\partial_y + \partial_x & -\partial_z & 0 & -jk \end{pmatrix} \begin{pmatrix} A_z \\ i\,A_y + A_x \\ -\frac{\phi}{v} \\ 0 \end{pmatrix} \tag{69}$$

Magnetic sources

Sometimes it is convenient to consider also magnetic sources and, in a piecewise constant medium, we obtain the following local form of Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon}\,, \qquad \text{grade 0} \tag{70a}$$

$$\nabla \cdot (i\eta\mathbf{H}) = -\frac{1}{v}\,\partial_t \mathbf{E} - \eta\,\mathbf{J}\,, \qquad \text{grade 1} \tag{70b}$$

$$\nabla \wedge \mathbf{E} = -\frac{1}{v}\,\partial_t\,(i\eta\mathbf{H}) - i\,\mathbf{M}, \qquad \text{grade 2} \tag{70c}$$

$$\nabla \wedge (i\eta\mathbf{H}) = -i\,v\,\rho_m\,, \qquad \text{grade 3} \tag{70d}$$

The spinor containing the excitations is therefore given by:

$$\begin{pmatrix} -\eta J_z - i\,v\,\rho_m \\ -\eta\,(i\,J_y + J_x) \\ \frac{\rho_e}{\epsilon} - i\,M_z \\ -i\,(i\,M_y + M_x) \end{pmatrix} = \begin{pmatrix} \mathbf{J}_e \\ \mathbf{J}_m \end{pmatrix} \tag{71}$$

with the vector $\mathbf{J}_e$ containing the first two rows and the vector $\mathbf{J}_m$ containing the third and fourth rows.

Potential for magnetic sources only

By application of superposition it is convenient to consider the case when only magnetic sources are present. In frequency domain we have

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ i\eta\nabla \cdot (\mathbf{H}) &= -iv\rho_m \\ \nabla \wedge \mathbf{E} &= -jk\,(i\eta\mathbf{H}) - i\,\mathbf{M} \\ i\eta\nabla\,(\mathbf{H}) &= -j\,k\,\mathbf{E} \end{aligned} \tag{72}$$

which can be condensed as

$$\begin{aligned} \nabla\mathbf{E} &= -jk\,(i\eta\mathbf{H}) - i\,\mathbf{M} \\ \nabla\,(i\eta\mathbf{H}) &= -j\,k\,\mathbf{E} - iv\rho_m \end{aligned} \tag{73}$$

or, by using the multivector form, yields:

$$(\nabla + j\,k)\,\mathcal{F} = -i\,(v\rho_m + \mathbf{M})\ . \tag{74}$$

Potentials for magnetic sources only

It is assumed that the field $\mathcal{F}$ can be recovered from a scalar potential ψ and a vector potential $\mathbf{F}$, using the following expression:

$$\mathcal{F} = i\,(\nabla - j\,k\,)\,(v\,\mathbf{F} + \psi)\ . \tag{75}$$

When (75) is inserted into (74) we recover the following equation:

$$\left(\nabla^2 + k^2\right)(v\,\mathbf{F} + \psi) = -\,(v\rho_m + \mathbf{M})\ . \tag{76}$$

and, by separating the scalar and the vector part, we recover, respectively, the scalar and vector wave equations:

$$\begin{aligned} \left(\nabla^2 + k^2\right)\psi &= -v\,\rho_m \\ \left(\nabla^2 + k^2\right)\mathbf{F} &= -\frac{1}{v}\,\mathbf{M}\,. \end{aligned} \tag{77}$$

Finally, we get:

$$\begin{aligned} \mathcal{F} &= \mathbf{E} + i\,\eta\,\mathbf{H} \\ &= i\,(\nabla - j\,k\,)\,(v\,\mathbf{F} + \psi) \\ &= i\,(-j\,k\,\psi + v\,\nabla\cdot\mathbf{F} + \nabla\psi - j\,k\,v\,\mathbf{F} + v\,\nabla\wedge\mathbf{F})\ . \end{aligned} \tag{78}$$

Grade separation

Considering the scalar, vector and bivectors parts one gets

$$\begin{aligned} 0 &= -j\,k\,\psi + v\,\nabla\cdot\mathbf{F} \\ \mathbf{E} &= i\,(v\,\nabla\wedge\mathbf{F}) \\ i\,\eta\,\mathbf{H} &= i\,(\nabla\psi - j\,k\,v\,\mathbf{F}) \end{aligned} \tag{79}$$

and therefore:

$$\begin{aligned} \psi &= \frac{v\,\nabla\cdot\mathbf{F}}{j\,k} \\ \mathbf{E} &= i\,(v\,\nabla\mathbf{F} - jk\psi) \\ i\eta\,\mathbf{H} &= i\,(\nabla\psi - j\,k\,v\,\mathbf{F})\ . \end{aligned} \tag{80}$$

The Lorenz condition (80) is derived directly from the assumption in (75).

Potentials for magnetic sources in spinor form

Potentials for magnetic sources in spinor form

It is apparent that (80) exhibit a symmetry suitable for expressing it by means of Dirac matrices. By using (66) we have

$$\begin{pmatrix} \mathbf{e} \\ \mathbf{h} \end{pmatrix} = i\, v \begin{pmatrix} j\, k\, \sigma_0 & \tilde{\nabla} \\ -\tilde{\nabla} & -j\, k\, \sigma_0 \end{pmatrix} \begin{pmatrix} -\frac{1}{v}\tilde{\psi} \\ \tilde{F} \end{pmatrix} . \tag{81}$$

with the position

$$\begin{aligned} \tilde{F} &= \begin{pmatrix} F_z \\ F_x + i\, F_y \end{pmatrix} \\ \tilde{\psi} &= \begin{pmatrix} \psi \\ 0 \end{pmatrix} . \end{aligned} \tag{82}$$