

Paraxial meridional ray tracing equations from the unified reflection-refraction law via geometric algebra

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Abstract. We derive the paraxial meridional ray tracing equations from the unified reflection-refraction law using geometric algebra. This unified law states that the normal vector to the interface is a rotation of the incident ray or of the refracted ray or of the reflected ray by an angle equal to the angle of incidence or of refraction. We obtain the finite meridional ray tracing equations by simply equating the arguments of the exponential rotation operators. We then derive the paraxial limits of these equations with the help of sign function identities. We show that by embedding the sign functions in the ray tracing equations, we explicitly declare our chosen sign conventions in symbols and not in prose.

1 Introduction

In paraxial optics, it is customary to declare beforehand the adopted set of sign conventions, as done for example in Nussbaum and Philips[?]. But there are as many sign conventions as there matrix optics authors. And debate ensues[?]-[?]. To get a taste of this controversy, let us quote Welford's 1974 critique of Conrady's convention[?][?]:

I have avoided the issue [of sign conventions] by simply using the universally accepted conventions of coordinate geometry, together with vectors and direction cosines, as a consistent system which agrees with what is done in other branches of physics. This conflicts with what has been taught at Imperial College on one small point: the sign of the paraxial convergence angle, u , has always been taken according to Conrady's convention but after due discussing with my colleagues I decided to reverse it so as to agree with the convention for direction cosines; the inconsistency could, we felt, no longer be justified.

To resolve this sign problem, we propose the use of sign functions that take values of ± 1 , such as the three axial direction functions for the vector \mathbf{v} :

$$c_{vx} = \frac{\mathbf{v} \cdot \mathbf{e}_1}{|\mathbf{v} \cdot \mathbf{e}_1|}, \quad (1)$$

$$c_{vy} = \frac{\mathbf{v} \cdot \mathbf{e}_2}{|\mathbf{v} \cdot \mathbf{e}_2|}, \quad (2)$$

$$c_{vz} = \frac{\mathbf{v} \cdot \mathbf{e}_3}{|\mathbf{v} \cdot \mathbf{e}_3|}. \quad (3)$$

These functions correspond to the signs of the direction cosines of a vector \mathbf{v} . So by using these sign functions, we explicitly adopt Welford's Cartesian sign conventions.

Another example of a sign function is the concavity function[?]:

$$c_{\sigma\eta} = \frac{\boldsymbol{\sigma} \cdot \boldsymbol{\eta}}{|\boldsymbol{\sigma} \cdot \boldsymbol{\eta}|}, \quad (4)$$

where $\boldsymbol{\sigma}$ is the incident ray and $\boldsymbol{\eta}$ is the normal vector to the interface. If the interface is concave, then $c_{\sigma\eta} = 1$; if convex, $c_{\sigma\eta} = -1$. We used the concavity function before when we wrote down the unified reflection-refraction law in exponential form[?]:

$$c_{\sigma\eta}\boldsymbol{\eta} = \boldsymbol{\sigma}e^{ic_{\sigma\eta}\beta\mathbf{e}_{\sigma\times\eta}} = \boldsymbol{\sigma}'e^{ic_{\sigma\eta}\beta'\mathbf{e}_{\sigma\times\eta}} = -\boldsymbol{\sigma}''e^{-ic_{\sigma\eta}\beta''\mathbf{e}_{\sigma\times\eta}}, \quad (5)$$

where $\boldsymbol{\sigma}'$ is the refracted ray and $\boldsymbol{\sigma}''$ is the reflected ray. From this unified law we derived the ray tracing equations for finite and paraxial skew rays in spherical coordinates and for finite meridional rays in polar coordinates. The paraxial meridional rays we deemed then to require a separate treatment; we set it aside for a future work.

In this paper, we shall continue our work. We shall start with a short review of geometric algebra and then proceed to geometric optics. We shall summarize the equations for finite skew and finite meridional ray tracing, and then use these to derive those for paraxial meridional rays in polar coordinates. We shall see how the use of sign functions makes the discussion of sign conventions unnecessary.

2 Geometric Algebra

In Clifford (geometric) algebra $\mathcal{Cl}_{3,0}$ the product of two vectors \mathbf{a} and \mathbf{b} is given by the Pauli identity[?]:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}), \quad (6)$$

where i is the unit imaginary scalar. Note that the geometric product \mathbf{ab} is an associative product, unlike the dot product $\mathbf{a} \cdot \mathbf{b}$ and the cross product $\mathbf{a} \times \mathbf{b}$.

The exponential function in geometric algebra is also well-defined:

$$e^{i\theta} = \cos |\theta| + i \frac{\theta}{|\theta|} \sin |\theta|, \quad (7)$$

which is the generalization of Euler's theorem in complex analysis. If \mathbf{a} is a vector perpendicular to θ , then we can show that

$$\mathbf{a}e^{i\theta} = \mathbf{a} \cos |\theta| - \mathbf{a} \times \frac{\theta}{|\theta|} \sin |\theta|. \quad (8)$$

Equation (??) states that $\mathbf{a}e^{i\theta}$ is the vector \mathbf{a} rotated counterclockwise about the vector θ by an angle $|\theta|$. [?]

3 Geometric Optics

3.1 Finite Skew Rays

Finite skew rays[?] are the most general type of rays. The ray tracing equations for these rays are expressed in vector form .

From its initial position \mathbf{r}_0 , a light particle travels by a distance s in the direction of the unit vector σ . The final position \mathbf{r} of the light particle is

$$\mathbf{r} = \mathbf{r}_0 + s\sigma. \quad (9)$$

If at the position \mathbf{r} , the outward normal unit vector to the interface is η and the interface is spherical of radius R centered at C , then

$$\mathbf{r} = \mathbf{r}_0 + s\sigma = \mathbf{C} + R\eta. \quad (10)$$

After the light particle strikes the interface, the particle is either be refracted or reflected. The directions σ' and σ'' of the refracted and reflected vectors may be expressed in terms of the directions σ of the incident ray and (outward) normal vector η to the interface:

$$\sigma' = \sigma e^{i(\beta-\beta')\mathbf{e}_{\sigma \times \eta}}, \quad (11)$$

$$\sigma'' = \sigma e^{2ic_{\sigma\eta}\beta\mathbf{e}_{\sigma \times \eta}}, \quad (12)$$

where β and β' are the angles of incidence and refraction,

$$\beta = \sin^{-1} |\sigma \times \eta|, \quad (13)$$

$$\beta' = \sin^{-1} |\sigma \times \eta|, \quad (14)$$

and $\mathbf{e}_{\sigma \times \eta}$ is the rotational axis direction,

$$\mathbf{e}_{\sigma \times \eta} = \frac{\sigma \times \eta}{|\sigma \times \eta|}. \quad (15)$$

We may also combine the laws of refraction and reflection in Eqs. (??) and (??) into one, as given in Eq. (??). This unified law expresses the normal vector in terms of the rotations of the incident, refracted, and reflected rays about the vector $\mathbf{e}_{\sigma \times \eta}$ by angles β , β' , and $\beta'' = \beta$, respectively. The rotations are counterclockwise or clockwise depending on the sign value of the concavity function $c_{\sigma\eta}$. (Figure 1)

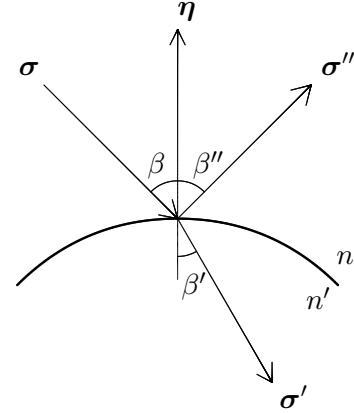


Figure 1: The incident ray σ , refracted ray σ' , and reflected ray σ'' . The rays make an angle of β , β' , and β'' with respect to the normal vector η , respectively. The interface is convex, $c_{\sigma\eta} = -1$.

3.2 Finite Meridional Rays

Finite meridional rays [?] are rays that lie on the same plane. Here, we choose this plane to be the zx -plane, with the z -axis along \mathbf{e}_3 as the optical axis.

Let us define the incident, refracted, reflected, and normal vectors as vectors in zx -plane:

$$\sigma = \mathbf{e}_3 e^{i\mathbf{e}_2 \theta_\sigma} = \mathbf{e}_3 \cos \theta_\sigma + \mathbf{e}_1 \sin \theta_\sigma, \quad (16)$$

$$\sigma' = \mathbf{e}_3 e^{i\mathbf{e}_2 \theta_{\sigma'}} = \mathbf{e}_3 \cos \theta_{\sigma'} + \mathbf{e}_1 \sin \theta_{\sigma'}, \quad (17)$$

$$\sigma'' = \mathbf{e}_3 e^{i\mathbf{e}_2 \theta_{\sigma''}} = \mathbf{e}_3 \cos \theta_{\sigma''} + \mathbf{e}_1 \sin \theta_{\sigma''}, \quad (18)$$

$$\eta = \mathbf{e}_3 e^{i\mathbf{e}_2 \theta_\eta} = \mathbf{e}_3 \cos \theta_\eta + \mathbf{e}_1 \sin \theta_\eta, \quad (19)$$

where θ is a counterclockwise rotation angle measured from \mathbf{e}_3 .

If we also define the position vectors \mathbf{r} and \mathbf{r}_0 as vectors in the zx -plane,

$$\mathbf{r} = z\mathbf{e}_3 + x\mathbf{e}_1, \quad (20)$$

$$\mathbf{r}_0 = z_0\mathbf{e}_3 + x_0\mathbf{e}_1, \quad (21)$$

and the center

$$\mathbf{C} = z_C \mathbf{e}_3 \quad (22)$$

of the interface to be along the optical axis \mathbf{e}_3 , then Eq. (??) separates into

$$z = z_0 + s \cos \theta_\sigma = C + R \cos \theta_\eta, \quad (23)$$

$$x = x_0 + s \sin \theta_\sigma = R \sin \theta_\eta. \quad (24)$$

Equations (??) and (??) are the ray propagation equations for finite meridional rays. (Figure 2)

On the other hand, substituting the definitions in Eqs. (??) to (??) back to the unified reflection-refraction law in Eq. (??) and employing the identities

$$\mathbf{e}_{\sigma \times \eta} = c_{(\sigma \times \eta)y} \mathbf{e}_2, \quad (25)$$

$$c_{\sigma \eta} = e^{i \mathbf{e}_2 (c_{\sigma \eta} - 1) \pi / 2}, \quad (26)$$

$$-1 = e^{i \mathbf{e}_2 \pi} \quad (27)$$

we arrive at

$$\begin{aligned} \frac{\pi}{2}(c_{\sigma \eta} - 1) + \theta_\eta &= \theta_\sigma + c_{\sigma \eta} c_{(\sigma \times \eta)y} \beta, \\ &= \theta_{\sigma'} + c_{\sigma \eta} c_{(\sigma \times \eta)y} \beta', \\ &= \pi + \theta_{\sigma''} - c_{\sigma \eta} c_{(\sigma \times \eta)y} \beta. \end{aligned} \quad (28)$$

Equation (??) is the unified reflection-refraction law for finite meridional rays. We can show that this equation contains a restatement of the Bessel-Conrady refraction invariant, by replacing θ_σ by U and $c_{\sigma \eta} c_{(\sigma \times \eta)y} \beta$ by I . [?][?]

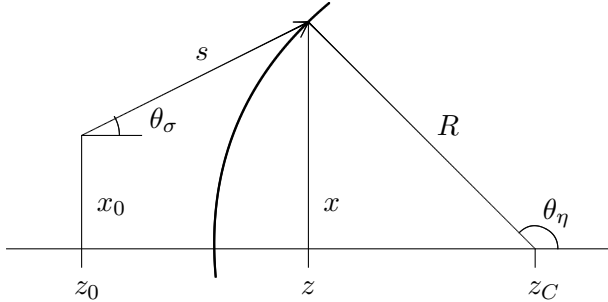


Figure 2: A ray travels a distance s until it intersects an interfaces of radius R .

3.3 Paraxial Meridional Rays

Paraxial meridional rays are rays that lie on the same plane (zx -plane) and make a small angle with respect to the optical axis \mathbf{e}_3 . Mathematically, we say that if \mathbf{v} is a paraxial meridional ray, then the polar angle θ_v of

\mathbf{v} may be expressed in terms of the small positive angle θ_{vz} that \mathbf{v} makes with the optical axis[?]:

$$\theta_v = \frac{\pi}{2}(1 - c_{vz}) + c_{vz} c_{vx} \theta_{vz}, \quad (29)$$

where θ_v is the polar angle of \mathbf{v} . We can easily verify that

$$\cos \theta_v = c_{vz}, \quad (30)$$

$$\sin \theta_v = c_{vx} \theta_{vz}. \quad (31)$$

Note that c_{vx} and c_{vz} are the axial direction functions defined in Eqs. (??) and (??).

3.3.1 Propagation

Using the approximations in Eqs. (??) and (??), Eqs. (??) and (??) simplify to

$$z = z_0 + c_{\sigma z} s = C + c_{\eta z} R, \quad (32)$$

$$x = x_0 + c_{\sigma x} s \theta_{\sigma z} = c_{\eta x} R \theta_{\eta z}. \quad (33)$$

Equations (??) and (??) are the position-height relations for paraxial meridional ray propagation.

For the polar angle θ_σ , we know that it is conserved during propagation or translation:

$$\theta_\sigma = \theta_{\sigma 0}. \quad (34)$$

Using Eq. (??), Eq. (??) may be expanded as

$$\frac{\pi}{2}(1 - c_{\sigma z}) + c_{\sigma z} c_{\sigma x} \theta_{\sigma z} = \frac{\pi}{2}(1 - c_{\sigma 0 z}) + c_{\sigma 0 z} c_{\sigma 0 x} \theta_{\sigma 0 z}, \quad (35)$$

Because the light particle moves in the same direction of the z -axis during propagation, then

$$c_{\sigma z} = c_{\sigma 0 z} \quad (36)$$

so that Eq. (??) reduces to

$$c_{\sigma x} \theta_{\sigma z} = c_{\sigma 0 x} \theta_{\sigma 0 z}. \quad (37)$$

Thus, the inclination angle of a ray from the optical axis remains invariant under ray propagation.

Substituting Eq. (??) back to the propagation equation in Eq. (??), we obtain

$$x = x_0 + s c_{\sigma 0 x} \theta_{\sigma 0 z}. \quad (38)$$

Equation (??) expresses the height x of a ray from the optical axis as a function of the ray's initial height x_0 , distance travelled s , and direction angle $\theta_{\sigma 0 z}$. Notice that the magnitude $\theta_{\sigma 0 z}$ of the paraxial angle is divorced from its x -direction sign function $c_{\sigma 0 x}$.

3.3.2 Refraction

The refraction law may be extracted from the unified law in Eq. (??):

$$\theta_{\sigma'} = \theta_{\sigma} + c_{\sigma\eta}c_{(\sigma \times \eta)y}(\beta - \beta'). \quad (39)$$

We know that the angles of incidence and refraction are related by Descartes-Snell's law:

$$n' \sin \beta' = n \sin \beta, \quad (40)$$

where n and n' are the refractive indices of the medium containing the incident and refracted rays, respectively. In the paraxial limit, Eq. (??) becomes

$$n' \beta' = n \beta. \quad (41)$$

Hence,

$$\theta_{\sigma'} = \theta_{\sigma} + c_{\sigma\eta}c_{(\sigma \times \eta)y}(1 - \mu)\beta. \quad (42)$$

where

$$\mu = \frac{n}{n'}. \quad (43)$$

Equation (??) is the paraxial refraction law in terms of the polar angle θ_{σ} of the incident ray and the angle of incidence β .

Using the unified refraction-reflection law in Eq. (??), we solve for the angle of incidence β in terms of the polar angle θ_{η} of the normal vector:

$$c_{\sigma\eta}c_{(\sigma \times \eta)y}\beta = \frac{\pi}{2}(c_{\sigma\eta} - 1) + \theta_{\eta} - \theta_{\sigma}, \quad (44)$$

Substituting this back to Eq. (??) and rearranging the terms, we arrive at

$$\theta_{\sigma'} = \mu\theta_{\sigma} + (1 - \mu)\left(\frac{\pi}{2}(c_{\sigma\eta} - 1) + \theta_{\eta}\right). \quad (45)$$

In paraxial approximation, this is

$$\begin{aligned} \frac{\pi}{2}(1 - c_{\sigma'z}) + c_{\sigma'z}c_{\sigma'x}\theta_{\sigma'z} \\ = \mu(1 - c_{\sigma z})\frac{\pi}{2} + \mu c_{\sigma z}c_{\sigma x}\theta_{\sigma z} \\ + (1 - \mu)(c_{\sigma\eta} - c_{\eta z})\frac{\pi}{2} \\ + (1 - \mu)c_{\eta z}c_{\eta x}\theta_{\eta z}. \end{aligned} \quad (46)$$

Now, we can verify that the following sign identities hold:

$$c_{\sigma'z} = c_{\sigma z}, \quad (47)$$

$$c_{\sigma\eta} = c_{\sigma z}c_{\eta z}. \quad (48)$$

The first equation states that the relative directions of the incident and refracted rays with respect to the optical axis are the same; the second states that the concavity function is equal to the product of the relative directions

of the incident ray and normal vector with respect to the optical axis.

Employing the sign identities in Eqs. (??) and (??), Eq. (??) reduces to

$$c_{\sigma'x}\theta_{\sigma'z} = m' + \mu c_{\sigma x}\theta_{\sigma z} + (1 - \mu)c_{\sigma\eta}c_{\eta x}\theta_{\eta z}, \quad (49)$$

where

$$m' = (1 - \mu)(c_{\eta z} - c_{\sigma z}c_{\eta z} - c_{\sigma z} + 1)\frac{\pi}{2}. \quad (50)$$

Let us analyze the angular function m' in Eq. (??) by considering two cases:

$$m' = 0; \quad c_{\sigma z} = +1, \quad (51)$$

$$m' = \pi(1 - \mu)(1 + c_{\eta z}); \quad c_{\sigma z} = -1. \quad (52)$$

Because the direction $c_{\eta z}$ of the normal vector with respect to the \mathbf{e}_3 is arbitrary, then $m' \neq 0$ in general for backward propagating rays ($c_{\sigma z} = -1$). This is an inconvenient case. So we shall leave this for a future work and impose that

$$c_{\sigma z} = c_{\sigma'z} = +1 \quad (53)$$

for all our equations. Thus, Eq. (??) reduces further to

$$c_{\sigma'x}\theta_{\sigma'z} = \mu c_{\sigma x}\theta_{\sigma z} + (1 - \mu)c_{\eta z}c_{\eta x}\theta_{\eta z}. \quad (54)$$

Equation (??) is the refraction law for forward propagating paraxial rays.

Using the result in Eq. (??),

$$c_{\eta x}\theta_{\eta z} = \frac{x}{R}, \quad (55)$$

Eq. (??) becomes

$$c_{\sigma'x}\theta_{\sigma'z} = \mu c_{\sigma x}\theta_{\sigma z} + (1 - \mu)c_{\eta z}\frac{x}{R}. \quad (56)$$

Multiplying this by n' yields

$$n'c_{\sigma'x}\theta_{\sigma'z} = nc_{\sigma x}\theta_{\sigma z} - Px, \quad (57)$$

where

$$P = c_{\eta z}\frac{n - n'}{R} \quad (58)$$

is the power of the interface. Notice that the radius R of the interface is always positive; the z -axis direction function $c_{\eta z}$ of the normal vector $\boldsymbol{\eta}$ takes care of the sign traditionally possessed by R .

3.3.3 Reflection

From the unified refraction-reflection law in Eq. (??), we get

$$\theta_{\sigma''} = -\pi + \theta_{\sigma} + 2c_{\sigma\eta}c_{(\sigma \times \eta)y}\beta. \quad (59)$$

Using the result in Eq. (??), Eq. (??) becomes

$$\theta_{\sigma''} = (c_{\sigma\eta} - 2)\pi - \theta_{\sigma} + 2\theta_{\eta}. \quad (60)$$

In the paraxial limit, Eq. (??) reduces to

$$\frac{\pi}{2}(1 - c_{\sigma''z}) + c_{\sigma''z}c_{\sigma''x}\theta_{\sigma''z} \quad (61)$$

$$= \pi(c_{\sigma\eta} - c_{\eta z} - 1) + \frac{\pi}{2}(c_{\sigma z} - 1) - c_{\sigma z}c_{\sigma x}\theta_{\sigma z} + 2c_{\eta z}c_{\eta x}\theta_{\eta z}. \quad (62)$$

Because the z -direction of the reflected ray is opposite to that of the incident ray, then

$$c_{\sigma''z} = -c_{\sigma z}, \quad (63)$$

so that Eq. (??) becomes

$$c_{\sigma''x}\theta_{\sigma''z} = \pi(-c_{\eta z} + c_{\sigma z}c_{\eta z} + 2c_{\sigma z}) + c_{\sigma x}\theta_{\sigma z} + 2c_{\sigma\eta}c_{\eta x}\theta_{\eta z}, \quad (64)$$

after multiplying by $c_{\sigma z}$ and using the sign identity in Eq. (??).

Let us analyze the angular function

$$m'' = \pi(-c_{\eta z} + c_{\sigma z}c_{\eta z} + 2c_{\sigma z}) \quad (65)$$

by considering two cases:

$$m'' = 2\pi; \quad c_{\sigma z} = +1, \quad (66)$$

$$m'' = -2\pi(c_{\eta z} + 1) = \{0, -4\pi\}; \quad c_{\sigma z} = -1, \quad (67)$$

since $c_{\eta z} = \pm 1$. Because $4\pi \equiv 2\pi \equiv 0$, then we may simply set $m'' = 0$, so that Eq. (??) simplifies further to

$$c_{\sigma''x}\theta_{\sigma''z} = c_{\sigma x}\theta_{\sigma z} + 2c_{\sigma\eta}c_{\eta x}\theta_{\eta z}. \quad (68)$$

Employing the identity in Eq. (??), Eq. (??) becomes

$$c_{\sigma''x}\theta_{\sigma''z} = c_{\sigma x}\theta_{\sigma z} - 2c_{\sigma\eta}\frac{x}{R}. \quad (69)$$

If we impose that the incident ray moves along \mathbf{e}_3 , then $c_{\sigma z} = +1$, so that Eq. (??) simplifies to

$$c_{\sigma''x}\theta_{\sigma''z} = c_{\sigma x}\theta_{\sigma z} - 2c_{\eta z}\frac{x}{R}. \quad (70)$$

Multiplying Eq. (??) by $n'' = n$,

$$n''c_{\sigma''x}\theta_{\sigma''z} = nc_{\sigma x}\theta_{\sigma z} - 2nc_{\eta z}\frac{x}{R}, \quad (71)$$

and comparing the result with the refraction law in Eq. (??), we see that we may express the mirror power P'' as

$$P'' = c_{\eta z}\frac{2n}{R}. \quad (72)$$

Notice that except for the factor of 2, the power P of a mirror Eq. (??) is similar to that of a lens in Eq. (??), which is what we expect.

4 Conclusions

In this paper, we used geometric algebra to derive from the unified reflection-refraction law in exponential form the paraxial meridional ray tracing equations in polar form, by equating the arguments of the exponentials and employing the properties of sign functions. The sign functions, such as the concavity and axial direction functions, make explicit the Cartesian sign convention used, though in symbols and not in words. We hope that these sign functions would be universally adopted to finally settle the never-ending debate on sign conventions.

References

- [1] Allen Nussbaum and Richard A. Phillips, *Contemporary Optics for Scientists and Engineers* (Prentice-Hall, Englewood Cliffs, NJ, 1976), p. 11–12.
- [2] B. D. Sukheeja, “Sign conventions in geometrical optics,” *Am. J. Phys.* **35**(7), 611–613 (1967).
- [3] B. D. Sukheeja, “Clarifications on sign conventions in geometrical optics,” *Am. J. Phys.* **37**(2), 230 (1969).
- [4] H. S. Sandhu and G. B. Friedmann, “Concerning sign conventions in geometrical optics,” *Am. J. Phys.* **37**(2), 229–230 (1969).
- [5] Fred Lemmerhirt, “Sign conventions in geometrical optics,” *Am. J. Phys.* **67**(5), 370 (1999).
- [6] Allen Nussbaum, “Sign conventions in geometrical optics,” *Am. J. Phys.* **68**(4), 305 (2000).
- [7] W. T. Welford, *Aberrations of the Symmetrical Optical System* (Academic, London, 1974), p. vi.
- [8] Alexander Eugen Conrady, *Applied Optics and Optical Design*, Part I. (Dover, New York, 1985).
- [9] Quirino M. Sugon Jr. and Daniel J. McNamara, “A geometric algebra reformulation of geometric optics,” *Am. J. Phys.* **72**(1), 92–97. See pp. 95.
- [10] Quirino M. Sugon Jr. and Daniel J. McNamara, “Ray tracing in spherical interfaces using geometric algebra,” in *Advances in Imaging and Electron Physics* **139**, 179–224. See pp. 192–195.
- [11] David Hestenes, “Oersted medal lecture 2002: Reforming the mathematical language of physics,” *Am. J. Phys.* **71**(2), 104–121 (2003). See p. 110.
- [12] See Ref. [?], p. 93.
- [13] See Ref. [?], pp. 188–196.
- [14] See Ref. [?], pp. 215–221.
- [15] See Ref. [?], p. 7.
- [16] See Ref. [?], pp. 219.
- [17] See Ref. [?], pp. 204.