Electromagnetic energy-momentum equation without tensors: a geometric algebra approach

Quirino M. Sugon Jr.* and Daniel J. McNamara

Ateneo de Manila University, Department of Physics, Loyola Heights, Quezon City, Philippines 1108

*Also at Manila Observatory, Upper Atmosphere Division, Ateneo de Manila University Campus

e-mail: qsugon@observatory.ph

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Abstract. In this paper, we define energy-momentum density as a product of the complex vector electromagnetic field and its complex conjugate. We derive an equation for the spacetime derivative of the energy-momentum density. We show that the scalar and vector parts of this equation are the differential conservation laws for energy and momentum, and the imaginary vector part is a relation for the curl of the Poynting vector. We can show that the spacetime derivative of this energy-momentum equation is a wave equation. Our formalism is Dirac-Pauli-Hestenes algebra in the framework of Clifford (Geometric) algebra $\mathcal{C}l_{4,0}$.

1 Introduction

The conservation laws for electromagnetic energy and momentum are given in Simmons and Guttmann[1] as

$$\begin{aligned} -\mathbf{E}\cdot\mathbf{J} &= \frac{\partial U}{\partial t} + \nabla\cdot\mathbf{S}, & (1)\\ \rho\mathbf{E}+\mathbf{j}\times\mathbf{B} &= -\nabla U - \frac{1}{c^2}\frac{\partial \mathbf{S}}{\partial t} + \epsilon_0(\nabla\cdot\mathbf{E})\mathbf{E} + (\mathbf{E}\cdot\nabla)\mathbf{E}) \\ &\quad +\mu_0(\nabla\cdot\mathbf{H})\mathbf{H} + (\mathbf{H}\cdot\nabla)\mathbf{H}). & (2)\end{aligned}$$

where U is the energy density and $\mathbf{S}$ is the Poynting vector. Our aim is to unify these two equations.

The standard way to unify Eqs. (1) and (2) is through tensors as given in Jackson:[2]

$$\partial_\alpha \Theta^{\alpha\beta} = -\frac{1}{c}F^{\beta\lambda}J_\lambda, \tag{3}$$

where ∂_α is the spacetime derivative operator, $\Theta^{\alpha\beta}$ is the symmetric stress tensor, $F_{\beta\lambda}$ is the electromagnetic field tensor, and J_λ is the four-current density.

Yet, it is not obvious that the tensor equation in Eq. (3) is equivalent to the vector equations in Eqs. (1) and (2). To prove this equivalence is straightforward but tedious. What we need is a formulation that would enable us to extract the two conservation laws in a single step.

To answer this problem, we propose geometric algebra.[3, 4] This algebra combines the imaginary number i with the dot and cross products of vectors in a single associative vector product:[5]

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b}), \tag{4}$$

which looks like the Pauli identity in Quantum Mechanics[6] but without the matrices.

One use of the Pauli identity is in the unification of the four Maxwell's equations in Electrodynamics:[7, 8]

$$(\frac{1}{c}\partial_t + \nabla)(\mathbf{E}+i\mathbf{B}) = \rho - \frac{\mathbf{j}}{c}. \tag{5}$$

Separating the scalar, vector, imaginary vector, and imaginary scalar parts of Eq. (5) yields Gauss's, Ampere's, Faraday's, and magnetic flux continuity laws. (For comparison, tensor calculus[9] and differential forms[10] can only reduce Maxwell's equations into two.)

So we ask: is there an energy-momentum counterpart to the Maxwell's equation?

This question was answered by Vold[11] by left-multiplying the Maxwell's equation by the reverse (†) or conjugate of the electromagnetic field $F = \mathbf{E}+i\mathbf{B}$, and adding the resulting equation with its reverse:

$$F^\dagger(\dot{\partial}_t + \dot{\nabla})\dot{F} + \dot{F}^\dagger(\dot{\partial}_t+\dot{\nabla})F = F^\dagger S + S^\dagger F, \tag{6}$$

where $S = \rho - \mathbf{j}/c$ and the overdots determine the direction of differentiation. The reversion operator, just like the Hermitian conjugation operator in Quantum Mechanics[12], changes the order of factors and operators–even the direction of differentiation.

Yet, it is still not obvious that the scalar and vector parts of Eq. (6) are indeed Eqs. (1) and (2). Showing this result is not trivial, because Eq. (6) must first be converted into a form involving the spacetime derivative of the electromagnetic energy-momentum density[13]

$$\frac{1}{2}FF^\dagger = (\mathbf{E}+i\mathbf{B})(\mathbf{E}-i\mathbf{B}) = \frac{1}{2}(|\mathbf{E}|^2+|\mathbf{B}|^2) + \mathbf{E}\times\mathbf{B}. \tag{7}$$

To perform this differentiation properly, we shall use a theorem in Jancewicz[14] for the spatial derivative of a product of two vectors:

$$\nabla(\mathbf{ab}) = (\nabla\mathbf{a})\mathbf{b} - \mathbf{a}(\nabla\mathbf{b}) + 2(\mathbf{a}\cdot\nabla)\mathbf{b}. \tag{8}$$

(In Jancewicz, $\mathbf{b}$ is replaced by $\hat{\mathbf{b}} = i\mathbf{b}$; we only factored out the i.)

We shall divide this paper into four sections. The first is Introduction. In the second section, we shall review geometric algebra and calculus within the framework of Hestenes's spacetime algebra in spacetime split form via Clifford (Dirac) algebra $\mathcal{C}l_{4,0}$.[15, 16] In the third section, we shall revisit Maxwell's equation and use it to derive the Energy-Momentum equation. We shall show that the scalar and vector parts of the latter are the two conservations laws, while the imaginary vector part is a relation for the curl of the Poynting vector. The fourth section is Conclusions.

2 Clifford (Geometric) Analysis

2.1 Geometric Algebra

The Clifford (Dirac) algebra $\mathcal{C}l_{4,0}$ is generated by four vectors $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ that satisfies the orthonormality relation

$$\mathbf{e}_\mu \mathbf{e}_\nu + \mathbf{e}_\nu \mathbf{e}_\mu = 2\delta_{\mu\nu}, \tag{9}$$

for $\mu, \nu = 0, 1, 2, 3$. That is, $\mathbf{e}_\mu^2 = 1$ and $\mathbf{e}_\mu \mathbf{e}_\nu = -\mathbf{e}_\nu \mathbf{e}_\mu$. We shall refer to $\mathbf{e}_0$ as the unit temporal vector and to $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ as the three unit spatial vectors.

One important subalgebra of $\mathcal{C}l_{4,0}$ is the Pauli algebra $\mathcal{C}l_{3,0}$. If $\mathbf{a}$ and $\mathbf{b}$ be two vectors spanned by the three unit spatial vectors in $\mathcal{C}l_{3,0}$, then by the orthonormality axiom in Eq. (9), we can show that $\mathbf{a}$ and $\mathbf{b}$ satisfy the Pauli identity:

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + i(\mathbf{a} \times \mathbf{b}), \tag{10}$$

where the imaginary scalar $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. In words, the Pauli identity states that the geometric product $\mathbf{ab}$ of two vectors is equal to the sum of their dot product $\mathbf{a}\cdot\mathbf{b}$ and their imaginary cross product $i(\mathbf{a} \times \mathbf{b})$.

In general, every element $\hat{A}$ in $\mathcal{C}l_{3,0}$ is expressible as a cliffor or a linear combination of a scalar, a vector, an imaginary vector, and an imaginary scalar:

$$\hat{A} = A_0 + \mathbf{A}_1 + i\mathbf{A}_2 + iA_3. \tag{11}$$

The spatial inverse or the automorphic grade involution[17] of $\hat{A}$ is defined in terms of the unit temporal vector $\mathbf{e}_0 \equiv \,^\circ$:

$$\hat{A}^\circ = \,^\circ\hat{A}^\dagger. \tag{12}$$

So by the orthonormality axiom in Eq. (9), we have

$$\hat{A}^\dagger = A_0 - \mathbf{A}_1 + i\mathbf{A}_2 - iA_3. \tag{13}$$

Notice that the spatial inversion operator changes the sign only of vectors and imaginary scalars. This differs from the reversion operator which changes sign only of imaginary vectors and imaginary scalars.[18]

2.2 Geometric Calculus

2.2.1 Time Differentiation

The time derivative of the product of two vector functions $\mathbf{a}$ and $\mathbf{b}$ is

$$\frac{\partial}{\partial t}(\mathbf{ab}) = \frac{\partial \mathbf{a}}{\partial t}\mathbf{b} + \mathbf{a}\frac{\partial \mathbf{b}}{\partial t}. \tag{14}$$

The scalar and imaginary vector part of this equation are

$$\frac{\partial}{\partial t}(\mathbf{a} \cdot \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial t} \cdot \mathbf{b} + \mathbf{a} \cdot \frac{\partial \mathbf{b}}{\partial t}, \tag{15}$$

$$\frac{\partial}{\partial t}(\mathbf{a} \times \mathbf{b}) = \frac{\partial \mathbf{a}}{\partial t} \times \mathbf{b} + \mathbf{a} \times \frac{\partial \mathbf{b}}{\partial t}, \tag{16}$$

which are the familiar identities in vector calculus.

2.2.2 Space Differentiation

The spatial derivative operator ∇ is defined as

$$\nabla = \frac{\partial}{\partial \mathbf{r}} = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3}, \tag{17}$$

where we used the identity $\mathbf{e}_k^{-1} = \mathbf{e}_k$. Because ∇ is a vector operator, we may use the Pauli identity in Eq. (10) to write

$$\nabla\mathbf{a} = \nabla \cdot \mathbf{a} + i(\nabla \times \mathbf{a}), \tag{18}$$

$$\mathbf{a}\nabla = \mathbf{a} \cdot \nabla + i(\mathbf{a} \times \nabla). \tag{19}$$

Notice that $\nabla\mathbf{a}$ is a function, while $\mathbf{a}\nabla$ is an operator.

For the spatial derivative of the geometric product of two vector functions $\mathbf{a}$ and $\mathbf{b}$, we use overdot notation[19]:

$$\nabla(\mathbf{ab}) = (\nabla\mathbf{a})\mathbf{b} + \dot{\nabla}\mathbf{a}\dot{\mathbf{b}}, \tag{20}$$

where

$$\dot{\nabla}\mathbf{a}\dot{\mathbf{b}} = \mathbf{e}_1\mathbf{a}\frac{\partial \mathbf{b}}{\partial x_1} + \mathbf{e}_2\mathbf{a}\frac{\partial \mathbf{b}}{\partial x_2} + \mathbf{e}_3\mathbf{a}\frac{\partial \mathbf{b}}{\partial x_3}. \tag{21}$$

If we employ the orthonormality axiom in Eq. (9), Eq. (21) becomes

$$\begin{aligned}\dot{\nabla}\mathbf{a}\dot{\mathbf{b}} &= (a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3)\mathbf{e}_1\frac{\partial \mathbf{b}}{\partial x_1} \\ &\quad + (-a_1\mathbf{e}_1 + a_2\mathbf{e}_2 - a_3\mathbf{e}_3)\mathbf{e}_2\frac{\partial \mathbf{b}}{\partial x_3} \\ &\quad + (-a_1\mathbf{e}_1 - a_2\mathbf{e}_2 + a_3\mathbf{e}_3)\mathbf{e}_3\frac{\partial \mathbf{b}}{\partial x_3}.\end{aligned} \tag{22}$$

Adding and subtracting $(\mathbf{a} \cdot \nabla)\mathbf{b}$ and rearranging the terms, we get

$$\dot{\nabla}\mathbf{a}\dot{\mathbf{b}} = -\mathbf{a}\nabla\mathbf{b} + 2(\mathbf{a} \cdot \nabla)\mathbf{b}. \tag{23}$$

Hence,

$$\nabla(\mathbf{ab}) = (\nabla\mathbf{a})\mathbf{b} - \mathbf{a}(\nabla\mathbf{b}) + 2(\mathbf{a} \cdot \nabla)\mathbf{b}, \tag{24}$$

which is Jancewicz's theorem in Eq. (8).

To verify the correctness of the product rule in Eq. (24), we separate its vector and imaginary scalar parts:

$$\begin{aligned}\nabla(\mathbf{a} \cdot \mathbf{b}) &- \nabla \times (\mathbf{a} \times \mathbf{b}) \\ &= (\nabla \cdot \mathbf{a})\mathbf{b} - (\nabla \times \mathbf{a}) \times \mathbf{b} - \mathbf{a}(\nabla \cdot \mathbf{b}) \\ &\quad + \mathbf{a} \times (\nabla \times \mathbf{b}) + 2(\mathbf{a} \cdot \nabla)\mathbf{b},\end{aligned} \tag{25}$$

$$i\nabla \cdot (\mathbf{a} \times \mathbf{b}) = i(\nabla \times \mathbf{a}) \cdot \mathbf{b} - i\mathbf{a} \cdot (\nabla \times \mathbf{b}). \tag{26}$$

The second equation is a familiar identity, while the first can be verified through two other known identities:[20]

$$\begin{aligned}\nabla(\mathbf{a} \cdot \mathbf{b}) &= (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) \\ &\quad + \mathbf{b} \times (\nabla \times \mathbf{a}),\end{aligned} \tag{27}$$

$$\begin{aligned}\nabla \times (\mathbf{a} \times \mathbf{b}) &= \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} \\ &\quad - (\mathbf{a} \cdot \nabla)\mathbf{b}.\end{aligned} \tag{28}$$

2.2.3 Spacetime Differentiation

An event $\hat{r}^\circ$ is defined by its position $\mathbf{r}$ and time t:

$$\hat{r}^\circ = (ct + \mathbf{r})^\circ = \,^\circ(ct - \mathbf{r}) = \,^\circ\hat{r}^\dagger, \tag{29}$$

where c is the speed of light. The square of the event is the Minkowski metric:

$$\hat{r}^\circ\hat{r}^\circ = \hat{r}^{\circ\circ}\hat{r}^\dagger = \hat{r}\hat{r}^\dagger = c^2t^2 - |\mathbf{r}|^2. \tag{30}$$

(In Hestenes's spacetime algebra, the event $x \equiv \hat{r}^\circ$, so that $x\gamma_0 = \hat{r}^{\circ\circ} = \hat{r} = ct + \mathbf{r}$ and $x^2 = \hat{r}^\circ \hat{r}^\circ$.)

Corresponding to the event $\hat{r}^\circ$ is the event derivative operator

$$\frac{\partial}{\partial \hat{r}^\circ} = {}^\circ(\frac{1}{c}\frac{\partial}{\partial t} + \nabla) = {}^\circ\frac{\partial}{\partial \hat{r}}, \tag{31}$$

where we used the identities $\mathbf{e}_0^{-1} = \mathbf{e}_0$ and $(\mathbf{e}_k\mathbf{e}_0)^{-1} = \mathbf{e}_0\mathbf{e}_k$. We can show that the square of event derivative operator is the d'Alembertian operator for wave equations.[21]

3 Classical Electrodynamics

3.1 Maxwell's Equation

In a linear and isotropic medium characterized by the speed of light $c = 1/\sqrt{\mu\epsilon}$ and the radiation resistance $\zeta = \sqrt{\mu/\epsilon}$, the Maxwell's equation may be written as

$$\frac{\partial \hat{E}}{\partial \hat{r}^\circ} = {}^\circ\frac{\partial \hat{E}}{\partial \hat{r}} = \zeta \hat{j}^\circ, \tag{32}$$

where

$$\begin{aligned} \hat{E} &= \mathbf{E} + i\zeta\mathbf{H}, & (33)\\ \hat{j} &= (\rho c + \mathbf{j}). & (34) \end{aligned}$$

In words, the Maxwell's equation states that the event derivative $\partial/\partial\hat{r}^\circ$ of the electromagnetic field $\hat{E}$ is proportional to the event curent density $\hat{j}^\circ$.[22]

Factoring out the unit temporal vector to the right of Eq. (32), we get

$$\frac{\partial \hat{E}}{\partial \hat{r}} = \zeta \hat{j}^\dagger. \tag{35}$$

That is,

$$(\frac{1}{c}\frac{\partial}{\partial t} + \nabla)(\mathbf{E} + i\zeta\mathbf{H}) = \zeta(\rho c - \mathbf{j}). \tag{36}$$

Separating the scalar, vector, imaginary vector, and imaginary scalar parts of Eq. (36), we obtain Maxwell's equations:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \zeta\rho c, & (37)\\ \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \zeta\nabla\times\mathbf{H} &= \zeta\mathbf{j}, & (38)\\ i(\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t} + \nabla\times\mathbf{E}) &= 0, & (39)\\ i\zeta(\nabla\cdot\mathbf{H}) &= 0. & (40) \end{aligned}$$

3.2 Energy-Momentum Equation

Let us define the event momentum density as

$$\frac{\hat{S}^\circ}{c^2} = \frac{1}{c}(U + \frac{\mathbf{S}}{c})^\circ = -\frac{1}{2c}\epsilon\hat{E}\hat{E}^{\dagger\circ} = -\frac{1}{2c}\epsilon\hat{E}^\circ\hat{E}. \tag{41}$$

where

$$\begin{aligned} U &= \frac{1}{2}(\epsilon|\mathbf{E}|^2 + \mu|\mathbf{H}|^2), & (42)\\ \mathbf{S} &= \mathbf{E}\times\mathbf{H} & (43) \end{aligned}$$

are the energy density and the Poynting vector, respectively. The square of Eq. (41) is

$$\frac{\hat{S}^{\circ 2}}{c^4} = \frac{\hat{S}\hat{S}^\dagger}{c^4} = \frac{1}{c^2}(U^2 - \frac{|\mathbf{S}|^2}{c^2}). \tag{44}$$

If this quantity is zero, then $U = |\mathbf{S}|/c$, or equivalently, $\mathcal{E} = |\mathbf{p}|c$, which is the energy-momentum relation for light-like particles.

The event derivative $\partial/\partial\hat{r}^\circ$ of the energy-momentum density $\hat{S}^\circ/c^2$ is

$$\frac{1}{c^2}\frac{\partial \hat{S}^\circ}{\partial \hat{r}^\circ} = -\frac{1}{2c}\epsilon^\circ\frac{\partial}{\partial \hat{r}}(\hat{E}\hat{E}^\dagger)^\circ. \tag{45}$$

This leaves us with the problem of differentiating $\hat{E}\hat{E}^\dagger$.

By the chain rules in Eqs. (14) and (24), we can show that the temporal and spatial derivatives of $\hat{E}\hat{E}^\dagger$ are

$$\begin{aligned} \frac{\partial}{\partial t}(\hat{E}\hat{E}^\dagger) &= \frac{\partial E}{\partial t}\hat{E}^\dagger + E\frac{\partial E^\dagger}{\partial t}, & (46)\\ \nabla(\hat{E}\hat{E}^\dagger) &= (\nabla E)\hat{E}^\dagger - \hat{E}\nabla\hat{E}^\dagger + 2(\hat{E}\cdot\nabla)\hat{E}^\dagger, & (47) \end{aligned}$$

where

$$\hat{E}\cdot\nabla = \mathbf{E}\cdot\nabla + i\zeta\mathbf{H}\cdot\nabla. \tag{48}$$

Adding Eqs. (46) and (47) yields

$$\frac{\partial}{\partial \hat{r}}(\hat{E}\hat{E}^\dagger) = \frac{\partial \hat{E}}{\partial \hat{r}}\hat{E}^\dagger + \hat{E}\frac{\partial \hat{E}^\dagger}{\partial \hat{r}^\dagger} + 2(\hat{E}\cdot\nabla)\hat{E}^\dagger. \tag{49}$$

If we substitute Eq. (49) back to Eq. (45) and use the Maxwell's equation in Eq. (32), we get

$$\frac{\partial \hat{S}^\circ}{\partial \hat{r}^\circ} = -\frac{1}{2}(\hat{j}^{\circ\circ}\hat{E} + {}^\circ\hat{E}\hat{j}^\circ) - \frac{1}{\zeta}(\hat{E}\cdot\nabla)\hat{E}, \tag{50}$$

after dividing by -2ζ and using the spatial inversion property of the temporal vector. Juxtaposing the two unit temporal vectors to cancel each other, and taking the spatial inverse of the resulting equation, we arrive at

$$\frac{\partial \hat{S}}{\partial \hat{r}} = -\frac{1}{2}(\hat{j}^\dagger\hat{E}^\dagger + \hat{E}\hat{j}) - \frac{1}{\zeta}(\hat{E}\cdot\nabla)\hat{E}^\dagger. \tag{51}$$

We shall refer to Eqs. (50) and (51) as the event-momentum and energy-momentum equations, respectively.

Expanding Eq. (51),

$$\begin{aligned} (\frac{1}{c}\frac{\partial}{\partial t} + \nabla)(cU + \mathbf{S}) &= -\frac{1}{2}(\rho c - \mathbf{j})(-\mathbf{E} + i\zeta\mathbf{H})\\ &\quad -\frac{1}{2}(\mathbf{E} + i\zeta\mathbf{H})(\rho c + \mathbf{j})\\ &\quad -\frac{1}{\zeta}\mathbf{E}\cdot\nabla(-\mathbf{E} + i\zeta\mathbf{H})\\ &\quad - i\mathbf{H}\cdot\nabla(-\mathbf{E} + i\zeta\mathbf{H}). & (52) \end{aligned}$$

and separating the scalar, vector, imaginary vector, and imaginary scalar parts, we get

$$\begin{aligned} \frac{\partial U}{\partial t} + \nabla\cdot\mathbf{S} &= -\mathbf{j}\cdot\mathbf{E}, & (53)\\ c\nabla U + \frac{1}{c}\frac{\partial \mathbf{S}}{\partial t} &= +\frac{1}{\zeta}(\mathbf{E}\cdot\nabla)\mathbf{E} + \zeta(\mathbf{H}\cdot\nabla)\mathbf{H}\\ &\quad -\zeta\mathbf{j}\times\mathbf{H}, & (54)\\ i(\nabla\times\mathbf{S}) &= i(-\frac{\rho}{\epsilon}\mathbf{H} - (\mathbf{E}\cdot\nabla)\mathbf{H} + (\mathbf{H}\cdot\nabla)\mathbf{E}), & (55)\\ 0 &= -\frac{i}{2}\zeta(-\mathbf{j}\cdot\mathbf{H} + \mathbf{H}\cdot\mathbf{j}), & (56) \end{aligned}$$

The first equation is the conservation of energy. The second is the conservation of momentum, which may be converted to that in Eq. (2) by adding Gauss's law and magnetic flux continuity law. The third is a relation for the curl of the Poynting vector $\mathbf{S}$, which follows from the product rule in Eq. (28). The fourth is insignificant.

4 Conclusions

In this paper, we used Hestenes spacetime algebra in spacetime split form to write down Maxwell's equation, which states that the event derivative oaf the electromagnetic field is proportional to the event current density. We showed that the four Maxwell's equations may be extracted from this equation by employing the spatial inversion of the temporal vector and Pauli expansion for the geometric product of vectors.

We defined the event momentum density in terms of the product of the electromagnetic field with its spatial inverse. We used Jancewicz's theorem for the differentiation of a product of two vectors together with the Maxwell's equation to derive the event momentum equation, whose scalar and vector parts are the conservation laws for energy and momentum, and whose imaginary vector part is a relation for the curl of the Poynting vector. We can show that the event derivative of the event momentum equation would yield its corresponding wave equation.

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