# Linearity and Superposition

Mauro Mongiardo[1]

[1] Department of Engineering, University of Perugia, Perugia, Italy.

# Table of contents

# Physical Constants

| Name | Symbol | Value | Unit |
|---|---|---|---|
| Number $\pi$ | $\pi$ | 3.14159265358979323846 | |
| Number e | e | 2.71828182845904523536 | |
| Euler's constant | $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n} 1/k - \ln(n)\right) = 0.5772156649$ | | |
| Elementary charge | $e$ | $1.60217733 \cdot 10^{-19}$ | C |
| Gravitational constant | $G, \kappa$ | $6.67259 \cdot 10^{-11}$ | $m^3 kg^{-1} s^{-2}$ |
| Speed of light in vacuum | $c$ | $2.99792458 \cdot 10^{8}$ | m/s (def) |
| Permittivity of the vacuum | $\varepsilon_0$ | $8.854187 \cdot 10^{-12}$ | F/m |
| Permeability of the vacuum | $\mu_0$ | $4\pi \cdot 10^{-7}$ | H/m |
| Electron mass | $m_{\mathrm{e}}$ | $9.1093897 \cdot 10^{-31}$ | kg |
| Proton mass | $m_{\mathrm{p}}$ | $1.6726231 \cdot 10^{-27}$ | kg |
| Neutron mass | $m_{\mathrm{n}}$ | $1.674954 \cdot 10^{-27}$ | kg |
| Diameter of the Sun | $D_{\odot}$ | $1392 \cdot 10^{6}$ | m |
| Mass of the Sun | $M_{\odot}$ | $1.989 \cdot 10^{30}$ | kg |
| Rotational period of the Sun | $T_{\odot}$ | 25.38 | days |
| Radius of Earth | $R_{\mathrm{A}}$ | $6.378 \cdot 10^{6}$ | m |
| Mass of Earth | $M_{\mathrm{A}}$ | $5.976 \cdot 10^{24}$ | kg |
| Rotational period of Earth | $T_{\mathrm{A}}$ | 23.96 | hours |
| Earth orbital period | Tropical year | 365.24219879 | days |
| Astronomical unit | AU | $1.4959787066 \cdot 10^{11}$ | m |
| Light year | lj | $9.4605 \cdot 10^{15}$ | m |

# Coulomb's law

# Coulomb's law

Two charged points with charges of the same sign are repulsing, while points with charges of opposite signs are attracting with force proportional to quantities of their charges and inverse proportional to square of distance between them:

$$F = k\frac{Q_1\,Q_2}{r^2}. \tag{1}$$

with

$$k = \frac{1}{4\pi\epsilon_0} = \frac{c^2\mu_0}{4\pi}. \tag{2}$$

# Coulomb's law

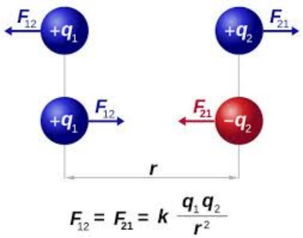


**Figure 1:** Figure illustrating the Force between two electrical charges

Force $F$ defined by the relationship (1) is strong.

However, in everyday life it does not reveal itself. This is due to the *screening*.

The numbers of positive and negative charges in nature are exactly balanced. Atoms and molecules, which constitute all observable matter around us, have the same amount of positive and negative charges.

Therefore they are electrically neutral in whole.

# Currents

# Currents

Electric current arises as a result of motion of charged points.

This occurs in metallic conductor, which usually have lengthy form (form of wire). Current in such conductor is determined by the *amount of charge passing through it within the unit of time*.

Let's consider straight conducting rod of the length $l$.

Current in it leads to misbalance of charges in its ends. Charges of definite sign move to one end of the rod, while lack of these charges in the other end of the rod is detected as the charge of opposite sign.

Then Coulomb force arises that tends to recover balance of charges in electrically neutral rod.

This means that in such rod current could not flow in constant direction during long time.

# Circular ring

Another situation when we consider a conductor of the form of ring or circuit.

Here current does not break the balance of charges.

Direct current can flow in it during unlimitedly long time.

Circular conductor itself thereby remains electrically neutral and no Coulomb forces arise.

In spite of absence of Coulomb forces, in experiments the interaction of two circular conductors with currents has been detected.

This interaction has other nature, it is not due to electrical, but due to magnetic forces.

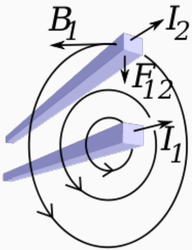


**Figure 2:** Force between two filaments with currents

The magnitude of magnetic forces depends essentially on the shape and mutual arrangement of circular conductors.

In order to reveal quantitative characteristics for magnetic forces it is convenient to simplify the geometry of conductors. To this end, they are deformed so that each possesses straight rod-shaped part of sufficiently large length $l$.

These rod–shaped parts are arranged parallel to each other with the distance $r$ between them.

In the limit, when $l$ is much larger than $r$, this configuration of conductors can be treated as a pair of infinitely long parallel conductors.

# Ampere Law

# Amper Law

In experiments it was found that such conductors do interact according to the following law.

Force of interaction of two infinite parallel conductors with currents per unit length of them is proportional to the values of currents in them and inverse proportional to the distance between them:

$$\frac{F}{l} = 2k_A \frac{I_1\, I_2}{r}. \tag{3}$$

with

$$k_A = \frac{\mu_0}{4\pi} \tag{4}$$

Two co-directed currents attract each other, while opposite directed currents repulse each other.

What is $\mu_0$?

In SI measure unit of current 1 A (one *ampere*) is a fundamental unit.

It is determined such that we have

$$\frac{F}{l} = \frac{2\,\mu_0}{4\pi}\frac{I_1\,I_2}{r}. \tag{5}$$

Here $\pi = 3.14\ldots$ is exact (though it is irrational) mathematical constant with no measure unit.

Constant $\mu_0$ is called *magnetic permeability* of vacuum. It has the (conventional) measure unit reported in Table 3

In contrast to constant $c$, the speed of light, $\mu_0$ is an exact constant.

Its value should not be determined experimentally. One could choose it to be equal to unity, but the above value for this constant was chosen by convention when SI system was established.

Due to this value of constant $\mu_0$ current of 1 *ampere* appears to be in that range of currents, that really appear in industrial and household devices.

Coefficient $4\pi$ in denominator (5) is used in order to simplify some other formulas, which are more often used for engineering calculations in electric technology.

Being basic unit in SI, unit of current *ampere* is used for defining unit of charge of 1 *coulomb* as $1\,\mathrm{C}$ : $1C = 1\,\mathrm{A}\cdot 1\,\mathrm{s}$.

Then the coefficient of proportionality in Coulomb law (1) appears to be not equal to unity.

In SI Coulomb law is written as

$$F = \frac{1}{4\pi\epsilon_0}\frac{Q_1 Q_2}{r^2}. \tag{6}$$

The constant $\epsilon_0$ is called dielectric permittivity of vacuum.

In contrast to constant $\mu_0$ this is a physical constant determined experimentally:

$$\epsilon_0 \approx 8.85 \cdot 10^{-12}. \tag{7}$$

## Relation between speed of light and medium parameters

Constants $\mu_0$, $\epsilon_0$, and $c$ are related to each other by the following equality:

$$c = \frac{1}{\sqrt{\epsilon_0\,\mu_0}} \approx 2.998 \cdot 10^8 \;\; m/sec. \tag{8}$$

The SI better suits for engineering calculations and is nowadays widely used.

Comparing Coulomb law and Ampere law we see that electrical and magnetic forces reveal themselves in quite different way.

However, they have common origin: they both are due to electric charges.

Below we shall see that their relation is much more close. Therefore theories of electricity and magnetism are usually united into one theory of electromagnetic phenomena.

Theory of electromagnetism is a theory with one measurable constant: this is the light velocity $c$.

Classical mechanics (without Newton's theory of gravitation) has no measurable constants.

Newton's theory of gravitation has the constant $G$ given in Table 3.

# Universal law of gravitation

# Universal law of gravitation

Two point masses attract each other with the force proportional to their masses and inverse proportional to the square of distance between them.

Universal law of gravitation is given by the formula

$$F = G\,\frac{M_1\,M_2}{r^2} \tag{9}$$

in SI system.

According to modern notion of nature classical mechanics and Newton's theory of gravitation are approximate theories.

Currently they are replaced by special theory of relativity and general theory of relativity.

Historically relativity has appeared as a result of development of the theory of electromagnetism.

# Concept of near action

# near and distant action

Let us consider pair of charged bodies, which are initially fixed, and let us do the following mental experiment with them. When we start moving second body apart from first one, the distance $r$ begins increasing and consequently force of Coulomb interaction will decrease.

In this situation we have a natural question: how soon after second body starts moving second body will feel change of Coulomb force of interaction?

There are two possible answers to this question:

- immediately;
- with some delay depending on the distance between bodies.

First answer is known as concept of *distant action*. Taking this concept we should take formula (1) as absolutely exact formula applicable for charges at rest and for moving charges as well.

# near action

Second answer is based on the concept of *near action*.

According to this concept, each interaction (and electric interaction among others) can be transmitted immediately only to the point of space infinitesimally close to initial one.

Transmission of any action to finite distance should be considered as a process of successive transmission from point to point. This process always leads to some finite velocity of transmission for any action.

In the framework of the concept of near action Coulomb law is treated as approximate law, which is exact only for the charges at rest that stayed at rest during sufficiently long time so that process of transmission of electric interaction has been terminated.

## Electromagnetic theory vs. Newton's theory of gravitation.

Theory of electromagnetism has measurable constant $c$ (light velocity), which is first pretender for the role of transmission velocity of electric and magnetic interactions.

For this reason electromagnetic theory is much more favorable as compared to Newton's theory of gravitation.

The value of light velocity is a very large quantity. If we settle an experiment of measuring Coulomb force at the distances of the order of $r \approx 10\ cm$, for the time of transmission of interaction we would get times of the order of $t \approx 3 \cdot 10^{-10}\ sec$.

# Experimental technique

Experimental technique of XIX-th century was unable to detect such a short interval of time.

Therefore the problem of choosing concept could not be solved experimentally.

In XIX-th century it was subject for contests. The only argument against the concept of distant action at that time, quite likely, was its straightness, its self-completeness, and hence its scarcity.

# Introduction of Field concept

In present time the concept of near action is commonly accepted.

Now we have the opportunity for testing it experimentally in the scope of electromagnetic phenomena.

According to the concept of near action, process of transmitting interaction to far distance exhibits an inertia.

Starting at one point, where moving charge is placed, for some time this process exist in hidden form with no influence to both charges.

In order to describe this stage of process we need to introduce new concept. This concept is a *field*.

# Field concept

# Field concept

A Field is a material entity able to fill the whole space and able to act upon other material bodies transmitting mutual interaction of them.

The number of fields definitely known to scientists is not large.

There are only four fundamental fields: *strong field, weak field, electromagnetic field*, and *gravitational field*.

Strong and weak fields are very short distance fields, they reveal themselves only in atomic nuclei, in collisions and decay of elementary particles, and in stellar objects of extremely high density, which are called neutron stars.

Strong and weak interactions and fields are not considered in this course.

There are various terms using the word field: *vector field, tensor field, spinor field, gauge field,* and others.

These are mathematical terms reflecting some definite properties of real physical fields.

Let's apply concept of near action to Coulomb law for two charged points. Coulomb force in the framework of this concept can be interpreted as follows: first charge produces electric field around itself, and this field acts upon other charge. Result of such action is detected as a force $F$ applied to second charge.

Force is vectorial quantity. Let's denote by **F** vector of force and take into account the direction of this vector determined by verbal statement of Coulomb law above. This yields

$$\mathbf{F} = k\, Q_1\, Q_2\, \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{10}$$

# Electric Field

Here $\mathbf{r}_1$ and $\mathbf{r}_2$ are radius-vectors of points, where charges $Q_1$ and $Q_2$ are placed.

Let's consider vector $\mathbf{E}$ determined as the ratio $\mathbf{E} = \mathbf{F}/Q_2$.

For this vector from formula (10) we derive

$$\mathbf{E} = k\,Q_1\,\frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3}. \tag{11}$$

# Determination of the field

Vector $\mathbf{E}$ depends upon the position of first charge and upon its value.

It depends also on the position of second charge, but it doesn't depend on the value of second charge.

One can take vector $\mathbf{E}$ for quantitative measure of electric field produced by first charge $Q_1$ at the point $\mathbf{r}_2$, where second charge is placed.

Vector $\mathbf{E}$ can be determined by formula (11) or it can be measured experimentally. For this purpose one should place test charge $q$ to the point $\mathbf{r}_2$ and one should measure Coulomb force $\mathbf{F}$ acting upon this test charge.

Then vector $\mathbf{E}$ is determined by division of $\mathbf{F}$ by the value of test charge $q$:

$$\mathbf{E} = \mathbf{F}/q. \tag{12}$$

# Superposition principle

# vector of electric field

Now consider more complicated situation. Suppose that charges $Q_1, \ldots, Q_n$ are placed at the points $\mathbf{r}_1, \ldots, \mathbf{r}_n$. They produce electric field around them, and this field acts upon test charge $q$ placed at the point $\mathbf{r}$.

This action reveals as a force $\mathbf{F}$ applied to the charge $q$. Again we can define vector $\mathbf{E}$ of the form (12) and take it for the quantitative measure of electric field at the point $\mathbf{r}$.

This vector is called *vector of intensity of electric field* or simply *vector of electric field* at that point.

# Discretely distributed charges

Generally speaking, in this case one cannot be a priori sure that vector $\mathbf{E}$ does not depend on the quantity of test charge $q$. However, there is the following experimental fact.

Electric field $\mathbf{E}$ at the point $\mathbf{r}$ produced by a system of point charges $Q_1, \dots, Q_n$ is a vectorial sum of electric fields that would be produced at this point by each charge $Q_1, \dots, Q_n$ separately.

This is the *Superposition principle*.

Superposition principle combined with Coulomb law leads to the following formula for the intensity of electric field produced by a system of point charges at the point $\mathbf{r}$:

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n} Q_i \, \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3}. \tag{13}$$

# Continuously distributed charges

Using superposition principle, one can pass from point charges to continuously distributed charges.

Suppose that the number of point charges tends to infinity: $n \to \infty$. In such limit sum in formula (13) is replaced by integral over 3-dimensional space:

$$\mathbf{E}(\mathbf{r}) = \int \rho(\mathbf{r}') \, \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, d^3\mathbf{r}'. \tag{14}$$

Here $\rho(\mathbf{r}')$ is spatial density of charge at the point $\mathbf{r}'$. This value designates the amount of charge per unit volume.

In order to find force acting on test charge $q$ we should invert formula (12) As a result we obtain

$$\mathbf{F} = q \, \mathbf{E}(\mathbf{r}). \tag{15}$$

# testing Fields

Force acting on a charge $q$ in electric field is equal to the product of the quantity of this charge by the vector of intensity of field at the point, where charge is placed.

However, charge $q$ also produces electric field. Does it experience the action of its own field?

For point charges the answer to this question is negative.

This fact should be treated as a supplement to principle of superposition.

Total force acting on a system of distributed charges in electric field is determined by the following integral:

$$\mathbf{F} = \int \rho(\mathbf{r})\, \mathbf{E}(\mathbf{r})\, d^3\mathbf{r}. \tag{16}$$

# Electrostatics

Field $\mathbf{E}(\mathbf{r})$ in (16) is external field produced by external charges.

Field of charges with density $\rho(\mathbf{r})$ is not included into $\mathbf{E}(\mathbf{r})$.

Concluding this section, note that formulas (13) and (14) hold only for charges at rest, which stayed at rest for sufficiently long time so that process of interaction transmitting reached the point of observation $\mathbf{r}$.

Fields produced by such systems of charges are called *static fields*, while branch of theory of electromagnetism studying such fields is called *electrostatics*.

# Superposition principle: mathematical formulation

Superposition principle is a consequence of the linearity of the equations.

Consider a linear operator $\mathcal{L}$ that, with given sources $s_1$ produces a field $F_1$ as

$$F_1 = \mathcal{L}(s_1) \tag{17}$$

Next, consider other sources $s_2$ such that

$$F_2 = \mathcal{L}(s_2)\,. \tag{18}$$

# Superposition principle: mathematical formulation

When they are considered together for linearity we have

$$F_1 + F_2 = \mathcal{L}(s_1 + s_2)\,. \tag{19}$$

This is the superposition principle that apply for linear equations.

Fortunately, Maxwell equations describing the behavior of the Electromagnetic field are linear ones.

Therefore in the study of Electromagnetic fields we can always apply superposition.

As we have seen the field is, in general, a *vector* quantity. It is therefore appropriate to recall the main operations on vectors.