Clifford Algebra

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Clifford Algebra Definition

Clifford basis

Let us first introduce Cl_n in the following way. We consider an orthonormal basis $e_1, e_2, \dots, e_n$ such that for $j = 1, \dots, n$

$$e_j^2 = 1 \tag{1}$$

and for $i \neq j$

$$e_i\, e_j + e_j\, e_i = 0 \tag{2}$$

anti–commutativity.

This basis is a Clifford basis of order n.

This is all we need for defining the Clifford algebra!

Three–dimensional case

For the three-dimensional case let us introduce the correspondence (for $i = 1, 2, 3$)

$$e_i = \sigma_i \tag{3}$$

with σ_i denoting the Pauli matrices.

The Pauli matrices are a basis for the Clifford algebra in 3D.

Therefore if you know how to operate with the Pauli matrices, you know the Clifford algebra of dimension 3!

A more general definition $Cl(n, m)$

A more general definition is obtained by considering, in addition to the n elements of the basis $e_1, e_2, \ldots, e_n$ other m elements which squares to -1

$$e_j^2 = -1 \tag{4}$$

for $j = n + 1, \ldots, n + m$.

An even more general definition can include also elements that square to zero (but we will not use it).

Formal definition (*)

Let $\{e_1, e_2, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, e_{p+q+1}, \ldots, e_n\}$, with $n = p + q + r$, $e_k^2 = \varepsilon_k$, $\varepsilon_k = +1$ for $k = 1, \ldots, p$, $\varepsilon_k = -1$ for $k = p + 1, \ldots, p + q$, $\varepsilon_k = 0$ for $k = p + q + 1, \ldots, n$, be an *orthonormal base* of the inner product vector space $\mathbb{R}^{p,q,r}$ with a geometric product according to the multiplication rules

$$e_k e_l + e_l e_k = 2\varepsilon_k \delta_{k,l}, \qquad k, l = 1, \ldots n, \tag{5}$$

where $\delta_{k,l}$ is the Kronecker symbol with $\delta_{k,l} = 1$ for $k = l$, and $\delta_{k,l} = 0$ for $k \neq l$.

Clifford's GA in two dimensions $Cl(2,0)$

Clifford's GA in two dimensions $Cl(2,0)$

A Euclidean plane is spanned by e_1, e_2 with

$$e_1 \cdot e_1 = e_2 \cdot e_2 = 1, \quad e_1 \cdot e_2 = 0. \tag{6}$$

$\{e_1, e_2\}$ is an *orthonormal* vector basis.

Under Clifford's *associative* geometric product we set

$$\begin{aligned} e_1^2 &= e_1 e_1 := e_1 \cdot e_1 = 1, \\ e_2^2 &= e_2 e_2 := e_2 \cdot e_2 = 1, \end{aligned} \tag{7}$$

$$\text{and} \quad (e_1 + e_2)(e_1 + e_2) = e_1^2 + e_2^2 + e_1 e_2 + e_2 e_1 \tag{8}$$

$$= 2 + e_1 e_2 + e_2 e_1 := (e_1 + e_2) \cdot (e_1 + e_2) = 2.$$

and

$$e_1 e_2 + e_2 e_1 = 0 \;\; \Leftrightarrow \;\; e_1 e_2 = -e_2 e_1, \tag{9}$$

The geometric product of orthogonal vectors forms a new entity, called unit bivector $e_{12} = e_1 e_2$ by Grassmann, and is anti-symmetric. General bivectors in $Cl(2,0)$ are e.g. βe_{12}. For orthogonal vectors the geometric product equals Grassmann's anti-symmetric outer product (exterior product, symbol $\wedge$)

$$\begin{aligned} e_{12} &= e_1 e_2 = e_1 \wedge e_2 \\ &= -e_2 \wedge e_1 = -e_2 e_1 = -e_{21}. \end{aligned} \tag{10}$$

Using associativity, we can compute the products

$$e_1 e_{12} = e_1 e_1 e_2 = e_1^2 e_2 = e_2, \quad e_2 e_{12} = -e_2 e_{21} = -e_1, \tag{11}$$

which represent a mathematically *positive* (anti-clockwise) 90° *rotation*.

The opposite order gives

$$e_{12}e_1 = -e_{21}e_1 = -e_2, \quad e_{12}e_2 = e_1, \tag{12}$$

which represents a mathematically *negative* (clockwise) 90° *rotation*.

The bivector e_{12} acts like a *rotation operator*, and we observe the general anti–commutation property

$$ae_{12} = -e_{12}a, \quad \forall a = a_1e_1 + a_2e_2 \in \mathbb{R}^2, \; a_1, a_2 \in \mathbb{R}. \tag{13}$$

The square of the unit bivector is -1,

$$e_{12}^2 = e_1e_2e_{12} = e_1(-e_1) = -1, \tag{14}$$

just like the imaginary unit j of complex numbers $\mathbb{C}$.

Multiplication table

Table 1 is the complete multiplication table of the Clifford algebra $Cl(2,0)$ with algebra basis elements $\{1, e_1, e_2, e_{12}\}$.

The even subalgebra spanned by $\{1, e_{12}\}$ (closed under geometric multiplication), consisting of even grade scalars (0-vectors) and bivectors (2-vectors), is isomorphic to $\mathbb{C}$.

Table 1: Multiplication table of plane Clifford algebra $Cl(2,0)$.

	1	e_1	e_2	e_{12}
1	1	e_1	e_2	e_{12}
e_1	e_1	1	e_{12}	e_2
e_2	e_2	$-e_{12}$	1	$-e_1$
e_{12}	e_{12}	$-e_2$	e_1	-1

The general geometric product of two vectors $a, b \in \mathbb{R}^2$

$$\begin{aligned} ab &= (a_1 e_1 + a_2 e_2)(b_1 e_1 + b_2 e_2) \\ &= a_1 b_1 + a_2 b_2 + (a_1 b_2 - a_2 b_1) e_{12} \\ &= \frac{1}{2}(ab + ba) + \frac{1}{2}(ab - ba) = a \cdot b + a \wedge b, \end{aligned} \tag{15}$$

has therefore a scalar *symmetric* inner product part

$$\begin{aligned} \frac{1}{2}(ab + ba) &= a \cdot b = a_1 b_1 + a_2 b_2 \\ &= |a||b| \cos \theta_{a,b}, \end{aligned} \tag{16}$$

and a bi-vector *skew-symmetric* outer product part

$$\frac{1}{2}(ab - ba) = a \wedge b = (a_1 b_2 - a_2 b_1) e_{12} = |a||b| e_{12} \sin \theta_{a,b}. \tag{17}$$

We observe that parallel vectors ($\theta_{a,b} = 0$) commute, $ab = a \cdot b = ba$, and orthogonal vectors ($\theta_{a,b} = 90^\circ$) anti-commute, $ab = a \wedge b = -ba$. The outer product part $a \wedge b$ represents the *oriented area* of the parallelogram spanned by the vectors a, b in the plane of $\mathbb{R}^2$, with oriented magnitude

$$\det(a, b) = |a||b| \sin \theta_{a,b} = (a \wedge b) e_{12}^{-1}, \tag{18}$$

where $e_{12}^{-1} = -e_{12}$, because $e_{12}^2 = -1$.

With the *Euler* formula we can rewrite the geometric product as

$$
\begin{aligned}
ab &= |a||b|(\cos\theta_{a,b} + e_{12}\sin\theta_{a,b}) \\
&= |a||b|e^{\theta_{a,b}e_{12}}\,,
\end{aligned}
\tag{19}
$$

again because $e_{12}^2 = -1$.

The geometric product of vectors is *invertible* for all vectors with non-zero square $a^2 \neq 0$

$$
\begin{aligned}
a^{-1} := a/a^2, \quad aa^{-1} = aa/a^2 = 1, \\
a^{-1}a = \frac{a}{a^2}a = a^2/a^2 = 1.
\end{aligned}
\tag{20}
$$

The inverse vector a/a^2 is a rescaled version (reflected at the unit circle) of the vector a.

This invertibility leads to significant simplifications and ease in computations.

Projection and Rejection

For example, the *projection* of one vector $x \in \mathbb{R}^2$ onto another $a \in \mathbb{R}^2$ is

$$x_{\|} = |x\,|\cos\theta_{a,x}\frac{a}{|a|} = (x \cdot \frac{a}{|a|})\frac{a}{|a|} = (x \cdot a)\frac{a}{|a|^2} = (x \cdot a)a^{-1}. \tag{21}$$

The *rejection* (perpendicular part) is

$$\begin{aligned} x_{\perp} = x - x_{\|} &= xaa^{-1} - (x \cdot a)a^{-1} \\ &= (xa - x \cdot a)a^{-1} = (x \wedge a)a^{-1}. \end{aligned} \tag{22}$$

We can now use $x_{\|}, x_{\perp}$ to compute the reflection[1] of $x = x_{\|} + x_{\perp}$ at the line (hyperplane[2]) with normal vector a, which means to reverse $x_{\|} \to -x_{\|}$

$$
\begin{aligned}
x' &= -x_{\|} + x_{\perp} = -a^{-1} a\, x_{\|} + a^{-1} a\, x_{\perp} \\
&= -a^{-1} x_{\|} a - a^{-1} x_{\perp} a = -a^{-1}(x_{\|} + x_{\perp}) a = -a^{-1} x a.
\end{aligned} \tag{23}
$$

[1]Note that reflections at hyperplanes are nothing but the *Householder transformations* of matrix analysis.

[2]A hyperplane of a nD space is a $(n-1)$D subspace, thus a hyperplane of $\mathbb{R}^2$, $n = 2$, is a 1D $(2 - 1 = 1)$ subspace, i.e. a line. Every hyperplane is characterized by a vector normal to the hyperplane.

Rotor

The combination of two reflections at two lines (hyperplanes) with normals a, b

$$x'' = -b^{-1}x'b = b^{-1}a^{-1}xab = (ab)^{-1}xab = R^{-1}xR, \tag{24}$$

gives a rotation. The rotation angle is $\alpha = 2\theta_{a,b}$ and the *rotor*

$$R = e^{\theta_{a,b}e_{12}} = e^{\frac{1}{2}\alpha e_{12}}, \tag{25}$$

where the lengths $|a||b|$ of ab cancel against $|a|^{-1}|b|^{-1}$ in $(ab)^{-1}$. The rotor R gives the *spinor* form of rotations, fully replacing rotation matrices, and introducing the same elegance to *real* rotations in $\mathbb{R}^2$, like in the complex plane.

Reflections and Rotations

In 2D, the product of three reflections, i.e. of a rotation and a reflection, leads to another reflection.

In 2D the product of an *odd* number of reflections always results in a *reflection*.

That the product of an *even* number of reflections leads to a *rotation* is true in general dimensions.

These transformations are in Clifford algebra simply described by the products of the vectors normal to the lines (hyperplanes) of reflection and called versors.

Geometric algebra in 3D

Geometric algebra of 3D Euclidean space

The Clifford algebra $Cl(\mathbb{R}^3) = Cl(3,0)$ is probably the most thoroughly studied and applied GA.

In physics it is also known as *Pauli algebra*, since Pauli's spin matrices provide a 2×2 matrix representation. This shows how GA unifies *classical* and *quantum* mechanics and electromagnetism.

Given an orthonormal vector basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$, the eight-dimensional ($2^3 = 8$) Clifford algebra $Cl(\mathbb{R}^3) = Cl(3,0)$ has a basis of one scalar, three vectors, three bivectors and one trivector

$$\{1, e_1, e_2, e_3, e_{23}, e_{31}, e_{12}, e_{123}\}, \tag{26}$$

where as before $e_{23} = e_2 e_3$, $e_{123} = e_1 e_2 e_3$, etc. All basis bivectors square to -1, and the product of two basis bivectors gives the third

$$e_{23} e_{31} = e_{21} = -e_{12}, \text{ etc.} \tag{27}$$

Quaternions

Therefore the even subalgebra $Cl^+(3,0)$ with basis[3] $\{1, -e_{23}, -e_{31}, -e_{12}\}$ is indeed found to be isomorphic to quaternions $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$.

This isomorphism is not incidental. As we have learned already for $Cl(2,0)$, also in $Cl(3,0)$, the even subalgebra is the algebra of rotors (rotation operators) or spinors, and describes rotations in the same efficient way as do quaternions.

We therefore gain a *real geometric* interpretation of quaternions, as the oriented bi-vector side faces of a unit cube, with edge vectors $\{e_1, e_2, e_3\}$.

[3] The minus signs are only chosen, to make the product of two bivectors identical to the third, and not minus the third.

Reflections and rotations in 3D

In $Cl(3,0)$ a reflection at a plane (=hyperplane) is specified by the plane's normal vector $a \in \mathbb{R}^3$

$$x' = -a^{-1}xa, \tag{28}$$

the proof is identical to the one in (23) for $Cl(2,0)$.

The combination of two such reflections leads to a rotation by $\alpha = 2\theta_{a,b}$

$$\begin{aligned} x'' &= R^{-1}xR, \qquad (29)\\ R &= ab = |a||b|e^{\theta_{a,b}\mathbf{i}_{a,b}} = |a||b|e^{\frac{1}{2}\alpha\mathbf{i}_{a,b}}, \end{aligned}$$

where $\mathbf{i}_{a,b} = a \wedge b/(|a \wedge b|)$ specifies the oriented unit bivector of the plane spanned by $a, b \in \mathbb{R}^3$.

The unit trivector $i = e_{123}$ also squares to -1

$$\begin{aligned} i^2 &= e_1 e_2 e_3 e_1 e_2 e_3 = -e_1 e_2 e_1 e_3 e_2 e_3 \\ &= e_1 e_2 e_1 e_2 e_3 e_3 = (e_1 e_2)^2 (e_3)^2 = -1, \end{aligned} \tag{30}$$

where we only used that the permutation of two orthogonal vectors in the geometric product produces a minus sign. Hence $i^{-1} = -i$. We further find, that i commutes with every vector, e.g.

$$\begin{aligned} e_1 i &= e_1 e_1 e_2 e_3 = e_{23}, \\ i e_1 &= e_1 e_2 e_3 e_1 = -e_1 e_2 e_1 e_3 = e_1 e_1 e_2 e_3 = e_{23}, \end{aligned} \tag{31}$$

and the like for $e_2 i = i e_2$, $e_3 i = i e_3$.

i changes bivectors into orthogonal vectors

If i commutes with every vector, it also commutes with every bivector $a \wedge b = \frac{1}{2}(ab - ba)$.

Hence i commutes with every element of $Cl(3,0)$.

i changes bivectors into orthogonal vectors

$$e_{23} i = e_2 e_3 e_1 e_2 e_3 = e_1 e_{23}^2 = -e_1 \,, \quad \text{etc.} \tag{32}$$

Writing the basis in the simple product form (26), fully preserves the *geometric interpretation* in terms of scalars, vectors, bivectors and trivectors, and allows to *reduce* all products to elementary geometric products of basis vectors.

Multiplication table and subalgebras of $Cl(3,0)$

Table 2: Multiplication table of Clifford algebra $Cl(3,0)$ of Euclidean 3D space $\mathbb{R}^3$.

	1	e_1	e_2	e_3	e_{23}	e_{31}	e_{12}	e_{123}
1	1	e_1	e_2	e_3	e_{23}	e_{31}	e_{12}	e_{123}
e_1	e_1	1	e_{12}	$-e_{31}$	e_{123}	$-e_3$	e_2	e_{23}
e_2	e_2	$-e_{12}$	1	e_{23}	e_3	e_{123}	$-e_1$	e_{31}
e_3	e_3	e_{31}	$-e_{23}$	1	$-e_2$	e_1	e_{123}	e_{12}
e_{23}	e_{23}	e_{123}	$-e_3$	e_2	-1	$-e_{12}$	e_{31}	$-e_1$
e_{31}	e_{31}	e_3	e_{123}	$-e_1$	e_{12}	-1	$-e_{23}$	$-e_2$
e_{12}	e_{12}	$-e_2$	e_1	e_{123}	$-e_{31}$	e_{23}	-1	$-e_3$
e_{123}	e_{123}	e_{23}	e_{31}	e_{12}	$-e_1$	$-e_2$	$-e_3$	-1

For the full multiplication table of $Cl(3,0)$ we still need the geometric products of vectors and bivectors. By changing labels in Table 1 ($1 \leftrightarrow 3$ or $2 \leftrightarrow 3$), we get that

$$\begin{aligned} e_2 e_{23} &= -e_{23} e_2 = e_3, \\ e_3 e_{23} &= -e_{23} e_3 = -e_2 \qquad (33) \\ e_1 e_{31} &= -e_{31} e_1 = -e_3, \\ e_3 e_{31} &= -e_{31} e_3 = e_1, \qquad (34) \end{aligned}$$

which shows that in general a vector and a bivector, which includes the vector, anti-commute.

The products of a vector with its orthogonal bivector always gives the trivector i

$$\begin{aligned} e_1 e_{23} = e_{23} e_1 = i, \quad e_2 e_{31} = e_{31} e_2 = i, \\ e_3 e_{12} = e_{12} e_3 = i, \end{aligned} \tag{35}$$

which also shows that in general vectors and orthogonal bivectors necessarily commute.

Commutation relationships therefore clearly depend on both *orthogonality* properties and on the *grades* of the factors, which can frequently be exploited for computations even without the explicit use of coordinates.

The grade structure of $Cl(3,0)$ and duality

A general multivector in $Cl(3,0)$, can be represented as

$$\begin{aligned} M = m_0 + m_1 e_1 + m_2 e_2 + m_3 e_3 + m_{23} e_{23} + m_{31} e_{31} + m_{12} e_{12} \\ + m_{123} e_{123}, \quad m_0, \ldots, m_{123} \in \mathbb{R}. \end{aligned} \tag{36}$$

We have a scalar part $\langle M \rangle_0$ of grade 0, a vector part $\langle M \rangle_1$ of grade 1, a bivector part $\langle M \rangle_2$ of grade 2, and a trivector part $\langle M \rangle_3$ of grade 3

$$\begin{aligned} M = \langle M \rangle_0 + \langle M \rangle_1 + \langle M \rangle_2 + \langle M \rangle_3, \\ \langle M \rangle_0 = m_0, \quad \langle M \rangle_1 = m_1 e_1 + m_2 e_2 + m_3 e_3, \\ \langle M \rangle_2 = m_{23} e_{23} + m_{31} e_{31} + m_{12} e_{12}, \quad \langle M \rangle_3 = m_{123} e_{123}. \end{aligned} \tag{37}$$

The set of all grade k elements, $0 \leq k \leq 3$, is denoted $Cl^k(3,0)$.

The multiplication table of $Cl(3,0)$, Table 2, reveals that multiplication with i (or $i^{-1} = -i$) consistently changes an element of grade k, $0 \leq k \leq 3$, into an element of grade $3 - k$, i.e. scalars to trivectors (also called pseudoscalars) and vectors to bivectors, and vice versa.

The Telegrapher's equations: an example of $Cl(1,1)$

Telegrapher equations

Let us consider the voltage V and the current I along a transmission line in the x direction. The telegrapher's equations, for a lossless line, are

$$\begin{aligned} \partial_x V &= -L\,\partial_t I \\ \partial_x I &= -C\,\partial_t V \end{aligned} \tag{38}$$

where L and C are the inductances and capacitances per unit length. It is convenient to introduce the velocity v and the impedance η defined as

$$\begin{aligned} v &= \frac{1}{\sqrt{LC}} \\ \eta &= \sqrt{\frac{L}{C}} \end{aligned} \tag{39}$$

or, equivalently

$$\begin{aligned} L &= \frac{\eta}{v} \\ C &= \frac{1}{\eta v}\,. \end{aligned} \tag{40}$$

By substituting (40) into (41) we get

$$\begin{aligned} \partial_x V &= -\frac{1}{v}\,\partial_t\,(\eta\,I) \\ \partial_x\,(\eta\,I) &= -\frac{1}{v}\,\partial_t V \end{aligned} \tag{41}$$

It is also convenient to introduce the two variables x_0, x_1 as

$$\begin{aligned} x_0 &= v\,t \\ x_1 &= x \end{aligned} \tag{42}$$

so that

$$\begin{aligned} \frac{1}{v}\partial_t &= \frac{\partial}{\partial x_0} = \partial_0 \\ \partial_x &= \frac{\partial}{\partial x_1} = \partial_1\,. \end{aligned} \tag{43}$$

The equations in (41) can be written in matrix form as

$$\begin{pmatrix} \partial_0 & \partial_1 \\ \partial_1 & \partial_0 \end{pmatrix} \begin{pmatrix} V \\ \eta I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{44}$$

or, equivalently, by changing the sign in the second row, as

$$\begin{pmatrix} \partial_0 & \partial_1 \\ -\partial_1 & -\partial_0 \end{pmatrix} \begin{pmatrix} V \\ \eta I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{45}$$

By making use of Pauli matrices we can therefore write the telegrapher equations in (45) as

$$(\sigma_3 \partial_0 + i\,\sigma_2 \partial_1)\,\psi = 0 \tag{46}$$

where we have introduced the quantity ψ defined as

$$\psi = \begin{pmatrix} V \\ \eta I \end{pmatrix} . \tag{47}$$

It is convenient, when the time variable is considered as in this case, to denote the elements of the Clifford basis starting from zero instead of one. We can now identify the elements of the basis $\{e_0, e_1\}$ as

$$\begin{aligned} e_0 &= \sigma_3 \\ e_1 &= i\,\sigma_2 \end{aligned} \tag{48}$$

by noting that

$$
\begin{aligned}
e_0^2 = \sigma_3\sigma_3 \quad &= \quad \sigma_0 \\
e_1^2 = -\sigma_2\sigma_2 \quad &= \quad -\sigma_0 \\
e_0 e_1 = \sigma_3\, i\, \sigma_2 \quad &= \quad -i\, \sigma_2\, \sigma_3 = -e_1 e_0 \, .
\end{aligned}
\tag{49}
$$

we have therefore realized an example of $Cl(1,1)$.

Therefore, the Clifford algebra $Cl(1,1)$ with the identification of the basis as in (48), is well suited to describe the telegrapher's equation.

The telegrapher's equation, in a geometric algebra form, is:

$$
(e_0\partial_0 + e_1\partial_1)\,\psi = 0 \tag{50}
$$

This is in a form similar to the Dirac equation.

Naturally, since e_0 squares to $1\sigma_0$ and e_1 squares to $-1\sigma_0$ and they anticommute we also have

$$\left(e_0\partial_0 + e_1\partial_1\right)\left(e_0\partial_0 + e_1\partial_1\right) = \left(\partial_0^2 - \partial_1^2\right)\sigma_0 \tag{51}$$

which provides the operator of the wave equation

$$\left(\partial_0^2 - \partial_1^2\right)\sigma_0\psi = 0\,. \tag{52}$$

With GA we have found the square root of the operator of the wave equation! This is not possible in conventional vector algebra.

Conventional procedure compared to GA

The conventional procedure to find the wave equation corresponding to (52) is the following. One starts from

$$\begin{aligned} \partial_0 V + \partial_1(\eta I) &= 0 &(53)\\ -\partial_1 V - \partial_0(\eta I) &= 0 &(54) \end{aligned}$$

perform a derivative of the first equation (53) w.r.t. ∂_0. Then perform a derivative of (54) w.r.t. ∂_1 and then substitute $\partial_0\partial_1(\eta I)$ so as to obtain the equation in V. Then repeat again the procedure for obtaining the other second order equation in I.

With (51) in just one passage we have obtained (52)!

Equivalent equations

As an alternative we could have considered (44) and write it in terms of Pauli matrices as:

$$(\sigma_0 \partial_0 + \sigma_1 \partial_1)\,\psi = 0\,. \tag{55}$$

Note, however, that the σ_0 matrix is not anti–commutative and therefore cannot be used to create the geometric algebra basis.

Nonetheless, it is still feasible to obtain the operator of the wave equation in the following way:

$$(\sigma_0 \partial_0 - \sigma_1 \partial_1)\,(\sigma_0 \partial_0 + \sigma_1 \partial_1) = \partial_0^2 - \partial_1^2\,. \tag{56}$$

Systematic way to generate Dirac–like equations

But there is one more systematic way to generate Dirac–like equation when σ_0 is present.

The expression appearing in (55) can be transformed without needing to change the sign at one equation, as we did before.

In fact, we can multiply (55) by one of the sigma not appearing in the equation (therefore either σ_2 or σ_3) and obtain a different equation composed exclusively by anti–commuting matrices.

An example

As an example, by pre mutltiplying (55) with σ_3 one obtains

$$\sigma_3 \left(\sigma_0 \partial_0 + \sigma_1 \partial_1\right) \psi = \left(\sigma_3 \partial_0 + i \sigma_2 \partial_1\right) \psi \,. \tag{57}$$

i.e. (46).

Pre and post multiplication of (55) with σ_2 leads to other possible equations as

$$\begin{aligned} \sigma_2 \left(\sigma_0 \partial_0 + \sigma_1 \partial_1\right) &= \sigma_2 \partial_0 - i \sigma_3 \partial_1 \\ \left(\sigma_0 \partial_0 + \sigma_1 \partial_1\right) \sigma_2 &= \sigma_2 \partial_0 + i \sigma_3 \partial_1 \,. \end{aligned} \tag{58}$$

It is left as an exercise to perform the following computation:

$$\frac{1}{2}\left(\sigma_0 + i\,\sigma_2\right)\left(\sigma_2\partial_0 + i\,\sigma_3\partial_1\right)\left(\sigma_1 + \sigma_3\right) = i\,\sigma_1\partial_0 - \sigma_2\partial_1 \qquad (59)$$

and to recognize that the result is an anti–diagonal matrix, thus leading to two separated problems.

Similarly, if an expression is formed only by employing σ_0 and σ_3 will lead again to two separated problems.

The technique to obtain such diagonalization is called Weyl decomposition.

The Weyl decomposition

Let us introduce the quantities a, b defined as

$$\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \psi = \begin{pmatrix} V \\ \eta I \end{pmatrix} . \tag{60}$$

By using (60) in (45) we obtain the following two equations

$$\begin{aligned} \partial_0 \left(a+b\right) + \partial_1 \left(a-b\right) &= 0 & (61) \\ -\partial_1 \left(a+b\right) - \partial_0 \left(a-b\right) &= 0 \,. & (62) \end{aligned}$$

By summing and subtracting (61) and (62) we obtain two independent expressions as

$$\begin{aligned} \partial_0 a + \partial_1 a &= 0 \\ \partial_0 b - \partial_1 b &= 0 \end{aligned} \tag{63}$$

or, in matrix form,

$$\begin{pmatrix} \partial_0 + \partial_1 & 0 \\ 0 & \partial_0 - \partial_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0\,. \tag{64}$$

The a and b correspond to progressive and regressive waves, respectively.

What we have obtained here is of *considerable importance*.

If we consider our original problem (45) we have a *system of two coupled equations of the first order*.

By contrast, when considering (64) we have two *independent* first order equations.

Therefore, the traveling waves (both progressive and regressive) are the natural basis for having uncoupled equations!

Time–harmonic solution

We have seen that propagation along the transmission line can be described by the equations (64) here repeated for convenience:

$$\begin{pmatrix} \partial_0 + \partial_1 & 0 \\ 0 & \partial_0 - \partial_1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \,. \tag{65}$$

The a and b correspond to progressive and regressive waves, respectively. In order to find a solution it is advantageous to apply separation of variables and to consider a time–harmonic solution.

Separation of variables

The solution for the propagating wave a can be written in terms of two different functions a_0, a_1 as

$$a(x_0, x_1) = a_0(x_0)\, a_1(x_1)\,. \tag{66}$$

In addition we can assume a time–harmonic behavior for the a_0 part. In particular, it is typically chosen the following expansion

$$a_0(x_0) = A_0 e^{j\omega t} = A_0 e^{jkx_0} \tag{67}$$

with $k = \omega/v$ and $x_0 = vt$ defined in (42). It is noted that the derivative w.r.t. x_0 of a gives

$$\partial_0 a(x_0, x_1) = jk\, a_0(x_0) a_1(x_1)\,, \tag{68}$$

and therefore the equation becomes

$$jk\, a_0(x_0) a_1(x_1) + \partial_1 a_0(x_0) a_1(x_1) = 0\,. \tag{69}$$

Since $a_0(x_0)$ is present in all members can be factored out, obtaining the following equation

$$\partial_1 a_1(x_1) = -jk\, a_1(x_1) \tag{70}$$

with solution

$$a_1(x_1) = A_1 e^{-jkx_1}\,. \tag{71}$$

Therefore the waves a can be written as

$$a(x_0, x_1) = A_0 e^{jkx_0} A_1 e^{-jkx_1} = Ae^{jk(x_0 - x_1)} \tag{72}$$

with $A = A_0\, A_1$. This solution represent a progressive wave. By a similar procedure the solution for b can be obtained as

$$b(x_0, x_1) = Be^{jk(x_0 + x_1)}\,. \tag{73}$$