Title

Author
(Dated: Today)

PACS numbers:

I. GEOMETRIC ALGEBRA

A. Scalars, Vectors, Bivectors, and Trivectors

In Clifford (Geometric) Algebra $\mathcal{C}l_{3,0}$, also known as the Pauli Algebra, the product of the three unit vectors $\mathbf{e}_1$,$\mathbf{e}_2$, and $\mathbf{e}_3$ satisfies the orthonormality relation [**? ? ?**]

$$\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = 2\delta_{jk}, \tag{1}$$

where δ_{jk} is the Kronecker delta function. In other words, the square of the length of the vectors is equal to one and the product of two perpendicular vectors anticommute.

Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors spanned by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$. We can show that their product satisfies the Pauli identity[**? ?**]

$$\mathbf{ab} = \mathbf{a}\cdot\mathbf{b} + i(\mathbf{a}\times\mathbf{b}), \tag{2}$$

where $i = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is the unit trivector which behaves like an imaginary scalar that transforms vectors to bivectors. The Pauli identity states that the geometric product of two vectors is equal to the sum of their scalar dot product and their imaginary cross product.

B. Exponential Function and Rotations

Let $i\mathbf{e}_3\theta$ be the product of a bivector $i\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2$ with the scalar θ. Since the square of $i\mathbf{e}_3\theta$ is negative, then the exponential of $i\mathbf{e}_3\theta$ is given by Euler's theorem

$$e^{i\mathbf{e}_3\theta} = \cos\theta + i\mathbf{e}_3\sin\theta. \tag{3}$$

From this we can see that

$$\cos\theta = \frac{1}{2}(e^{i\mathbf{e}_3\theta} + e^{-i\mathbf{e}_3\theta}), \tag{4a}$$
$$\sin\theta = \frac{1}{2i\mathbf{e}_3}(e^{i\mathbf{e}_3\theta} - e^{-i\mathbf{e}_3\theta}), \tag{4b}$$

which are the known exponential definitions of cosine and sine functions.

Multiplying Eq. (3) by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, we obtain

$$\mathbf{e}_1e^{i\mathbf{e}_3\theta} = \mathbf{e}_1\cos\theta + \mathbf{e}_2\sin\theta = e^{-i\mathbf{e}_3\theta}\mathbf{e}_1, \tag{5a}$$
$$\mathbf{e}_2e^{i\mathbf{e}_3\theta} = \mathbf{e}_2\cos\theta - \mathbf{e}_1\sin\theta = e^{-i\mathbf{e}_3\theta}\mathbf{e}_2, \tag{5b}$$
$$\mathbf{e}_3e^{i\mathbf{e}_3\theta} = \mathbf{e}_3\cos\theta + \mathbf{e}_3i\mathbf{e}_3\sin\theta = e^{i\mathbf{e}_3\theta}\mathbf{e}_3. \tag{5c}$$

Notice that $\mathbf{e}_1e^{i\mathbf{e}_3\theta}$ is a rotation of $\mathbf{e}_1$ counterclockwise about $\mathbf{e}_3$ by an angle θ, while $\mathbf{e}_2e^{i\mathbf{e}_3\theta}$ is a rotation of $\mathbf{e}_2$ counterclockwise about the same direction and the same angle. Notice, too, that the argument of the exponential changes sign when $\mathbf{e}_1$ or $\mathbf{e}_2$ trades places with the exponential, while $\mathbf{e}_3$ commutes with the exponential.

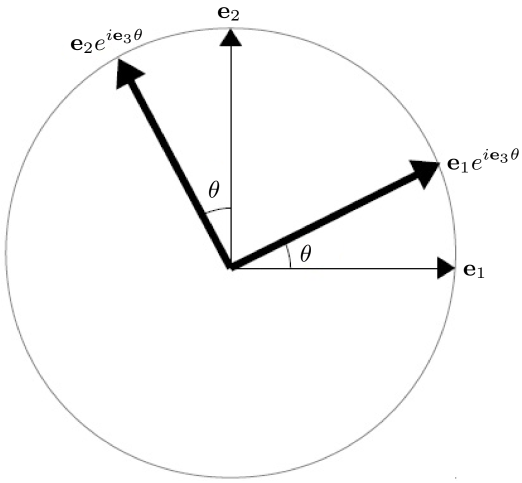

FIG. 1: Rotation of $\mathbf{e}_1$ and $\mathbf{e}_2$ about $\mathbf{e}_3$ counterclockwise by an angle θ

A vector $\mathbf{a}$ in 2D can be expressed in both rectangular and polar forms:

$$\mathbf{a} = a_x\mathbf{e}_1 + a_y\mathbf{e}_2 = a\mathbf{e}_1e^{i\mathbf{e}_3\theta}. \tag{6}$$

Expanding the exponential using Eq. (5a) and separating the $\mathbf{e}_1$ and $\mathbf{e}_2$ components, we arrive at the standard transformation equations for polar to rectangular coordinates:

$$x = a\cos\theta, \tag{7a}$$
$$y = a\sin\theta. \tag{7b}$$

We may also factor out $\mathbf{e}_1$ in Eq. (6) either to the left or to the right to get

$$\mathbf{a} = \mathbf{e}_1\hat{a} = \mathbf{e}_1(x + i\mathbf{e}_3y) = \mathbf{e}_1ae^{i\mathbf{e}_3\theta}, \tag{8a}$$
$$\mathbf{a} = \hat{a}^*\mathbf{e}_1 = (x - i\mathbf{e}_3y)\mathbf{e}_1 = ae^{-i\mathbf{e}_3\theta}\mathbf{e}_1. \tag{8b}$$

Factoring out $\mathbf{e}_1$ yields the definition of the complex number $\hat{a}$ and that of its complex conjugate $\hat{a}^*$:

$$\hat{a} = a_x + i\mathbf{e}_3a_y = ae^{i\mathbf{e}_3\theta}, \tag{9a}$$
$$\hat{a}^* = a_x - i\mathbf{e}_3a_y = ae^{-i\mathbf{e}_3\theta}. \tag{9b}$$

In general, we have the following relations:

$$\mathbf{e}_1\hat{a} = \hat{a}^*\mathbf{e}_1, \tag{10a}$$

$$\mathbf{e}_2\hat{a} = \hat{a}^*\mathbf{e}_2, \tag{10b}$$

$$\mathbf{e}_3\hat{a} = \hat{a}\mathbf{e}_3. \tag{10c}$$

That is, $\mathbf{e}_1$ and $\mathbf{e}_2$ both changes the complex number $\hat{a}$ to its conjugate $\hat{a}^*$ after commutation, while $\mathbf{e}_3$ simply commutes with $\hat{a}$ [**? ? ? ?**].