CLIFFORD ALGEBRA DERIVATION of the CHARACTERISTIC HYPERSURFACES of MAXWELL'S EQUATIONS

WILLIAM M. PEZZAGLIA JR.
Department of Physics and Astronomy
San Francisco State University
1600 Holloway Avenue
San Francisco, California 94132
U.S.A.

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Abstract. An alternative, pedagogically simpler derivation of the allowed physical wave fronts of a propagating electromagnetic signal is presented using geometric algebra. Maxwell's equations can be expressed in a single multivector equation using 3D Clifford algebra (isomorphic to Pauli algebra spinorial formulation of electromagnetism). Subsequently one can more easily solve for the time evolution of both the electric and magnetic field simultaneously in terms of the fields evaluated only on a 3D hypersurface. The form of the special "characteristic" surfaces for which the time derivative of the fields can be singular are quickly deduced with little effort. **Key words:**

characteristics – multivector – clifford – electromagnetism

1. Introduction

Maxwell's equations (in a vacuum) are a set of (coupled) hyperbolic first-order differential equations. As such, there are "characteristic" hypersurfaces over which the first derivatives of the fields can be discontinuous[1]. Specifically these are unique three-dimensional surfaces embedded in the four-dimensional continuum. They correspond physically to the allowable physical wave fronts of a propagating electromagnetic disturbance of the field.

A derivation is provided by Adler[2] (based upon the earlier work of Fock[3]) which shows the characteristics to be light-cones in Minkowski space; equivalently three-dimensional spheres expanding at the speed of light. As the goal is to describe a three-surface, the problem is most naturally formulated in standard three-dimensional Gibbs vectors. Hence the standard derivation consists of a rather lengthy manipulation of Maxwell's equations in vector form. To arrive at the desired result requires many convoluted steps which can only be motivated by considerable experience in vector identities.

What is presented in this paper is an alternative derivation which is pedagogically simpler. Maxwell's four equations are expressed in a single multivector equation using three-dimensional Clifford Algebra (isomorphic to the "spinorial formulation" of electromagnetism). The notational economy of "four equations in one" cuts the number of steps by a corresponding factor of four. Further the associativity of the Clifford product (replacing non-associative Gibbs cross product) and its duplicity (decomposes into inner and outer portions) simplifies the vector identities to transparency. The derivation of equation for characteristics requires only two straight-forward steps.

The following section reviews the geometric algebra notation used in the derivation. In section 3 the multivector formulation of Maxwell's equations is reviewed. Section 4 contains the new Clifford algebra derivation of the characteristics for electromagnetic signals.

2. Algebraic Notation

For the three-dimensional geometry the "Clifford" algebra[4,5,6] is isomorphic to the familiar Pauli matrix algebra $\mathbf{C}(2)$ [2-by-2 complex matrices]. However, in contrast to standard view, the intrinsic 8 degrees of freedom are given concrete geometric interpretation.

2.1. The Clifford Group

The geometric interpretation of the multiplication rule,

$$\{\sigma_j, \sigma_k\} = 2\delta_{jk}, \quad (j, k = 1, 2, 3), \tag{1}$$

is that perpendicular basis vectors anticommute. The basis trivector: $i = \sigma_1\sigma_2\sigma_3$ is associated with the unit volume (pseudoscalar). As it commutes with all elements and has negative signature, Hestenes[4,6] declares it a geometric definition of the usual abstract i. Bivectors (pseudovectors) are direct products of 2 basis vectors, and geometrically associated with planes. Multiplication of a vector by i yields the (Hodge) dual plane and visa versa, e.g. $\sigma_1\sigma_2 = i\sigma_3$. The unit scalar $\{1\}$, 3 basis vectors $\{\sigma_j\}$, 3 basis bivectors $\{i\sigma_j\}$ and trivector $\{i\}$ make up the 8 element Clifford group of 3D orthogonal space.

2.2. Vector Algebra

The direct "Clifford" product of two vectors can be decomposed into Grassmann symmetric "inner" (dot) and antisymmetric "exterior" (wedge) products respectively,

$$\mathbf{AB} = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \wedge \mathbf{B}, \tag{2}$$

where,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} = \frac{1}{2}\{\mathbf{A}, \mathbf{B}\} = A_j B_k \delta^{jk}, \tag{3a}$$

$$\mathbf{A} \wedge \mathbf{B} = -\mathbf{B} \wedge \mathbf{A} = \frac{1}{2}[\mathbf{A}, \mathbf{B}] = i\mathbf{A} \times \mathbf{B}. \tag{3b}$$

The symmetric part is the usual Gibbs dot product, and the wedge is a bivector, which in 3D is dual to the Gibbs cross product. So eq. (2) is a *multivector*, i.e. a conglomorate of scalar plus bivector.

2.3. Multivector Algebra

A general *multivector* or *cliffor*[5] is an aggregate sum of the four ranks of geometry with 8 degrees of freedom encoded,

$$M = s + \mathbf{E} + i\mathbf{H} + ip, \tag{4}$$

where s and p are real scalars, and $\mathbf{E}$ and $\mathbf{H}$ are real vectors. Under a parity inversion the odd ranked geometries will invert (vector $\mathbf{E}$, trivector ip), while the even (scalar s, bivector $i\mathbf{H}$) will not. Hence in 3D we associate the alternate terms of *pseudovector* for bivector, and *pseudoscalar* for the trivector.

The geometric product of general multivectors is completely associative, $(AB)C = A(BC)$. The product of a vector with a bivector (or trivector) can also be decomposed into symmetric and antisymmetric portions. We write down a summary of products and their Gibbs vector equivalents,

$$\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C} = i[\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})], \tag{5a}$$

$$\mathbf{A} \cdot (\mathbf{B} \wedge \mathbf{C}) = -\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{B})\mathbf{C} - (\mathbf{A} \cdot \mathbf{C})\mathbf{B}, \tag{5b}$$

$$\mathbf{A} \cdot (\mathbf{B} \wedge \mathbf{C} \wedge \mathbf{D}) = (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \wedge \mathbf{D} - (\mathbf{A} \cdot \mathbf{C})\mathbf{B} \wedge \mathbf{D} + (\mathbf{A} \cdot \mathbf{D})\mathbf{B} \wedge \mathbf{C}. \tag{5c}$$

3. Multivector Electromagnetism

Maxwell's four equations can be expressed in a single multivector form using the geometric algebra. It is this notational economy that will allow us a simple derivation of the characteristics. Here we review the Clifford algebra formulation of electrodynamics.

3.1. Vector Calculus

The application of a gradient operator[4] on a vector field yields a scalar plus bivector (pseudovector) field,

$$\nabla \mathbf{E} = \nabla \cdot \mathbf{E} + \nabla \wedge \mathbf{E} = \nabla \cdot \mathbf{E} + i(\nabla \times \mathbf{E}), \tag{6}$$

Note the interpretation differs from standard in that $(\nabla \times \mathbf{E})$ is a vector (not a pseudovector!). Multipled by i, it becomes a bivector/pseudovector.

3.2. Multivector Field

We define the electromagnetic field multivector[5,6,7] to be the aggregate sum of "vector" electric plus "bivector" (pseudovector) magnetic fields, $F = \mathbf{E} + i\mathbf{H}$. The gradient of the field has all four ranks of geometry represented,

$$\nabla F = (\nabla \cdot \mathbf{E}) - (\nabla \times \mathbf{H}) + i(\nabla \times \mathbf{E}) + i(\nabla \cdot \mathbf{H}), \tag{7}$$

in the order scalar, vector, bivector (pseudovector) and trivector (pseudoscalar) respectively (parenthesis included for brevity).

3.3. Maxwell's Equations

One can encode all four Maxwell's equations in the single multivector equation[5,6],

$$c\nabla F + \partial_t F = c\rho - \mathbf{J}, \tag{8a}$$

(assuming Heaviside-Lorentz units where c is the speed of light), which in the sourceless case becomes,

$$c\nabla F = -\partial_t F. \tag{8b}$$

Each of the 4 distinct geometric parts of eq. (8a) yields one of the standard Maxwell equations. Specifically, the scalar part of eq. (8a) yields Gauss's law, the vector portion is Ampere's law, the bivector is Faraday's law and the trivector is the magnetic monopole equation.

4. The Characteristic Solution

The goal is to obtain an equation for the time evolution of the multivector field F in terms of the field evaluated only on the the hypersurface. The presentation here will parallel that of Adler[2] (which is based on the earlier work of Fock[3]), except in the economical multivector formulation.

4.1. Hypersurface Description

The equation of a smooth three-dimensional hypersurface S embedded in the four-dimensional space-time manifold is parameterized,

$$\omega(x^0, x^1, x^2, x^3) = h(\mathbf{r}) - x^0 = 0, \tag{9}$$

where $\omega(x^0, x^1, x^2, x^3)$ is continuous in first order derivatives, $x^o = ct$ and $\partial\omega/\partial x^0$ must be nonzero [so that one can construct function $h(\mathbf{r})$]. The multivector field $\hat{F} = F(\mathbf{r}), \mathbf{r})$, on surface S is a function only of the space coordinates $\mathbf{r} = (x, y, z)$, where we have adopted the "hat" notation of Adler[2].

4.2. Fock Relations

Assuming the vector functions $\hat{\mathbf{E}}(\mathbf{r}) = \mathbf{E}(h(\mathbf{r}), \mathbf{r})$ and $\hat{\mathbf{H}}(\mathbf{r}) = \mathbf{H}(h(\mathbf{r}), \mathbf{r})$ have continuous first derivatives, the chain rule provides,

$$\partial_j \hat{F} = \partial_j F + \frac{1}{c}(\partial_t F \ \partial_j h). \tag{10a}$$

Contracting with the basis vectors σ_j yields the multivector form of the relations stated by Fock (i.e. contains equations 3.06-3.09 of ref[3]),

$$\nabla \hat{F} = \nabla F + \frac{1}{c} \nabla h(\partial_t F). \tag{10b}$$

Substituting the multivector form eq. (8b) of Maxwell's equations for ∇F,

$$\nabla \hat{F} = \frac{1}{c}(\nabla h - 1)(\partial_t F), \tag{10c}$$

gives a single multivector equation containing four identities derived by Adler (equations 4.22-4.25 of ref[2] respectively).

4.3. Characteristic Equation

The problem is now simply to invert eq. (10c) to obtain an equation for $\partial_t F$ in terms of only the field evaluated on the the hypersurface, i.e. in terms of $\hat{F}$ (and its space derivatives). The derivation is as trivial in the multivector formulation as it is tedious in standard vector analysis. We simply multiply eq. (10c) on the left by the multivector $(\nabla h + 1)$ to "scalarize" the right side,

$$(\nabla h + 1)\nabla \hat{F} = \frac{1}{c}(|\nabla h|^2 - 1)(\partial_t F). \tag{11}$$

In principle we are done; we need not look at each geometric component to deduce the nature of the characteristics.

If the scalar factor of $(|\nabla h|^2 - 1)$ which appears on the right side of eq. (11) is zero, then the time derivative of the field could be discontinuous across S. From this point on the discussion is the same as given in the references[2,3]. The equation that these "characteristic" hypersurfaces must obey is determined by the condition,

$$|\nabla h|^2 = 1, \tag{12a}$$

which according to the definition of the hypersurface is equivalent to ,

$$(\partial_t \omega)^2 - (\nabla \omega)^2 = 0. \tag{12b}$$

The propagation of an electromagnetic wave front must satisfy this equation. Some particular hypersurface solutions are of the form,

$$\omega(x^0, x^1, x^2, x^3) = |\mathbf{r} - \mathbf{r}_0| - c(t - t_0), \tag{13a}$$

$$\omega(x^0, x^1, x^2, x^3) = \mathbf{n} \cdot \mathbf{r} - ct, \tag{13b}$$

corresponding to a spherical wave front (expanding at speed of light about point $\mathbf{r}_0$), and a plane wave front (moving at the speed of light, in direction of unit vector $\mathbf{n}$) respectively.

5. Summary

Deriving the characteristics of Maxwell's equations via Clifford algebra formulation is embarrassingly trivial, essentially consisting of eqs. (10bc) and (11). The notation encodes four-equations-in-one such that, for example, Maxwell equations, the Fock relations, and Adler's identities are each represented by a single compact multivector statement [equations (8a), (10b) and (10c) respectively]. It remains to be seen if geometric algebra would provide similar clarity in the derivation of characteristics in other areas, n.b. electromagnetic waves in media, mechanical shock waves in solids or fluids, the Proca equation, relativistic quantum mechanics and gravitation.

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