

Derived Algebraic Geometry Seminar: UPenn 2017

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Introduction

This contains notes for the Derived Algebraic Geometry Seminar currently being held at the University of Pennsylvania math department in the 2016-17 academic year. Having introduced the machinery of Derived Algebraic Geometry the previous semester, we investigate its applications to producing Virtual Fundamental Cycles. Initially we will focus on moduli spaces of stable maps, with various boundary conditions, and how VFCs for these can be used to construct Gromov-Witten invariants and Floer-type theories.

This is a draft and errors should be expected.

Chapter 1

Stable Maps and Gromov-Witten Invariants

1.1 The Counting Problem

Basic idea of enumerative geometry, as explained in [4] 3.1: set up a moduli space M for the objects, e.g. curves, one wants to count: $\mathcal{M}_{g,n}(X, \beta)$, equipped with (flat) evaluation maps $\nu_i : \mathcal{M}_{g,n}(X, \beta) \rightarrow X$, given by $(C, p_1, \dots, p_n, \mu) \mapsto \mu(p_i)$. Each constraint $\nu_i \in \Gamma_i$, where $\Gamma_i \in H_*(X, \mathbb{Z})$, gives a subscheme, of $\mathcal{M}_{g,n}(X, \beta)$. We take the intersection of all these:

$$\bigcap_{i=1}^m \nu_i^* \Gamma_i.^1 \quad (1.1.1)$$

If the intersections are transverse and the result has dimension 0, can count the number of points. We would like to set up Γ_i such that:

$$\sum_{i=1}^m \text{codim } \Gamma_i = \dim \mathcal{M}_{g,n}(X, \beta).$$

Thus the enumerative problem is reduced to intersection theory in M . In order to do intersection theory successfully, M needs to be compact (proper), and we need to understand its Chow ring, where the subschemes live.

A first modification: in order to drop the transversality assumption on Γ_i , we replace them with the Poincaré dual cohomology classes γ_i , and take cup products then 1.1.1 is replaced by a first naive definition of the **Gromov-Witten invariants**:

$$I_{g,n,\beta} := \int_{[\mathcal{M}_{g,n}(X,\beta)]} \bigwedge_i \nu_i^* \gamma_i. \quad (1.1.2)$$

If $\mathcal{M}_{g,n}(X, \beta)$ is smooth and proper, then $[\mathcal{M}_{g,n}(X, \beta)]$ is the fundamental class, against which it makes sense to evaluate cohomology classes. $I_{g,n,\beta}$ is defined to be 0 unless $\sum_i \deg \gamma_i = \dim \mathcal{M}_{g,n}(X, \beta)$.

1.2 Axiomatic Definition of GW

The axiomatic approach of Kontsevich and Manin in [5] is as follows. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification by stable curves of the moduli stack of genus g curves with n marked points. We take this as a well-understood object and explain the rest.

¹This pullback is an umkehr map and we need some assumptions; is properness of μ_i enough?

Definition 1.2.1 (2.2 in [5]). A **system of Gromov-Witten classes for X** is a family of linear maps:

$$I_{g,n,\beta}^X : H^*(X, \mathbb{Q})^{\otimes n} \rightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$$

defined for $n + 2g - 3 \geq 0$, and satisfying the following axioms.

1. **Effectivity:** $I_{g,n,\beta} = 0$ for β non-effective, i.e. not in the dual of the Kähler cone.
2. **S_n -covariance:** equivariant with respect to the obvious S_n action on the domain and target.
3. **Grading:** $\deg I_{g,n,\beta} = -2 \int_{\beta} c_1(X) + (2 - 2g) \dim X$. More precisely, this means that we set $|\gamma| = i$ for $\gamma \in H^i(X, \mathbb{Q})$ and we require that:

$$|I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_m)| = \sum_{j=1}^m |\gamma_j| - 2 \int_{\beta} c_1(X) + (2g - 2) \dim X.$$

Some comments on the grading axiom:

- Following the convention in [5], we use the real, not complex, dimension.
- Informally we think of $I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_m)$ as obtained by pushing forward via the natural map:

$$\mathcal{M}_{g,n}(X, \beta) \rightarrow \mathcal{M}_{g,n}.$$

As a result, its degree is an expectation for $\dim \mathcal{M}_{g,n} - \dim \mathcal{M}_{g,n}(X, \beta)$. We know that $\dim \mathcal{M}_{g,n} = 2(3g - 3 + n)$. By deformation theory we also compute $\text{vdim } \mathcal{M}_{g,n}(X, \beta)$, called the **virtual dimension**, the expected dimension whenever first-order deformations are unobstructed.

The tangent space to $\mathcal{M}_{g,n}(X, \beta)$ at a point $(C, p_1, \dots, p_n, \mu)$ is:

$$H^1(C, T_C(-p_1 - \dots - p_n)) \oplus H^0(C, \mu^* T_X).$$

By Serre duality this is:

$$H^0(C, \Omega_C^{\otimes 2}(p_1 + \dots + p_n))^{\vee} \oplus H^0(C, \mu^* T_X).$$

Approximating the dimensions with the Euler characteristic, we get via Riemann-Roch:

$$\text{vdim } \mathcal{M}_{g,n}(X, \beta) = 2(\dim X - 3)(1 - g) + 2 \int_{\beta} c_1(T_X) + 2n. \quad (1.2.1)$$

Subtracting these we get what the grading axiom requires:

$$\dim \mathcal{M}_{g,n} - \dim \mathcal{M}_{g,n}(X, \beta) = 2 \int_{\beta} c_1(X) - (2 - 2g) \dim X.$$

- Assume that $I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_m)$ is of **codimension zero**, i.e. that:

$$\sum_{j=1}^n |\gamma_j| = 2 \int_{\beta} c_1(X) - (2 - 2g) \dim X. \quad (1.2.2)$$

Then $|I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_m)| = \dim \overline{\mathcal{M}}_{g,n}$. We can integrate this against the fundamental class of $\overline{\mathcal{M}}_{g,n}$, which is a proper smooth Deligne-Mumford stack. (Todo: reference?) We obtain a finite number, which we take as the result of the curve count.

4. **Fundamental class.** We introduce some more terminology. Call a class **basic** if it has the smallest n which makes sense, namely:

$$I_{0,3,\beta}^X(\gamma_1, \gamma_2, \gamma_3) \quad I_{1,1,\beta}^X(\gamma_1) \quad I_{g,0,\beta}^X \text{ for } g \geq 2.$$

Let $\pi : \overline{\mathcal{M}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n-1}$ be the projection that forgets the last marked point. Let $e_X^0 \in H^0(X, \mathbb{Q})$ be the identity of the cohomology ring. Unless the class on the LHS is basic, we require that:

$$I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_{n-1}, e_X^0) = \pi^* I_{g,n-1,\beta}^X(\gamma_1, \dots, \gamma_{n-1}).$$

In addition, we set:

$$I_{0,3,\beta}^X(\gamma_1, \gamma_2, e_X^0) = \begin{cases} \int_X \gamma_1 \wedge \gamma_2, & \text{if } \beta = 0, \\ 0, & \text{if } \beta \neq 0. \end{cases}$$

5. **Divisor.** In the case $|\gamma_n| = 2$, i.e. γ_n is the Poincaré dual class of a divisor, and if the LHS is a non-basic class, we require:

$$\pi_{n*} I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_n) = \int_{\beta} \gamma_n I_{g,n-1,\beta}^X(\gamma_1, \dots, \gamma_{n-1}).$$

6. **Splitting.** This axiom and the next are very important: they postulate a manageable structure of the boundary of the compactification $\overline{\mathcal{M}}_{g,n}(X, \beta)$, compatible with that of the boundary of $\overline{\mathcal{M}}_{g,n}$. One way to get boundary maps is to let the curves have 2 irreducible components, with genera g_1, g_2 and marked points $n_1 + 1, n_2 + 1$ such that $g = g_1 + g_2$, $n = n_1 + n_2$. The extra marked point on each irreducible component is where we glue them; they become one singular point in the resulting reducible curve. For S some partition of the n marked points into 2 sets of cardinality n_1 and n_2 , we let $\phi_S : \overline{\mathcal{M}}_{g_1, n_1+1} \times \overline{\mathcal{M}}_{g_2, n_2+1} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the gluing map. Choose a basis $\{\Delta_a\}$ of $H^*(X, \mathbb{Q})$ and define $g_{ab} = \int_V \Delta_a \wedge \Delta_b$; let $(g^{ab}) = (g_{ab})^{-1}$ denote the entries of the inverse matrix. Then:

$$\phi_S^* I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_n) = (-1)^S \sum_{\beta_1 + \beta_2 = \beta} \sum_{a,b} I_{g_1, n_1+1, \beta_1}^X(\otimes_{j \in S_1} \gamma_j \otimes \Delta_a) g^{ab} \otimes I_{g_2, n_2+1, \beta_2}^X(\Delta_b \otimes \otimes_{j \in S_2} \gamma_j).$$

Roughly speaking, we need to introduce $\sum_{a,b} (\Delta_a \otimes \Delta_b)$ to account for the position of the extra marked points. Integrating over these produces a factor $g_{a,b}$ that wasn't there on the LHS, so we need to multiply by g^{ab} to compensate for it.

7. **Genus reduction.** Let $\psi : \overline{\mathcal{M}}_{g-1, n+2} \rightarrow \overline{\mathcal{M}}_{g,n}$ be the map which glues together the last 2 marked points. Then:

$$\psi^* I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_n) = \sum_{a,b} I_{g-1, n+2, \beta}^X(\gamma_1, \dots, \gamma_n, \Delta_a, \Delta_b) g^{ab}.$$

The splitting and genus reduction axioms motivate the choice of stable maps compactification, see ??.

8. **Motivic axiom.** The maps $I_{g,n,\beta}^X$ are induced by correspondences in the Chow rings:

$$C_{g,n,\beta}^X \in C^*(X^n \times \overline{\mathcal{M}}_{g,n}).$$

Namely, consider the two projection maps:

$$\begin{array}{ccc} & X^n \times \overline{\mathcal{M}}_{g,n} & \\ p \swarrow & & \searrow q \\ X^n & & \overline{\mathcal{M}}_{g,n}. \end{array}$$

We require that:

$$I_{g,n,\beta}^X(\gamma_1, \dots, \gamma_n) = q_* (C_{g,n,\beta}^X \wedge p^*(\gamma_1 \otimes \dots \otimes \gamma_n)).$$

This axiom is motivated as follows in [5], 2.3.8. Suppose we construct a good compactification $\overline{\mathcal{M}}_{0,n}(X, \beta)$, together with a virtual fundamental class $[\overline{\mathcal{M}}_{0,n}(X, \beta)]$. Consider then the map:

$$\begin{aligned} \alpha : \overline{\mathcal{M}}_{0,n}(X, \beta) &\rightarrow X^n \times \overline{\mathcal{M}}_{0,n} \\ (C, x_1, \dots, x_n, f) &\mapsto (f(x_1), \dots, f(x_n), (\bar{C}, x_1, \dots, x_n)). \end{aligned}$$

We would like \bar{C} to be C , but we may need to contract certain components to get a stable curve from a stable map. Compare definitions 1.3.2 and ???. Ignoring this for now, we set $C_{\bar{C}}^X g, n, \beta) = \alpha_*([\overline{\mathcal{M}}_{0,n}(X, \beta)])$. This means, roughly speaking, we're integrating over $\overline{\mathcal{M}}_{0,n}(X, \beta)$, like the naive definition 1.1.2 suggests.

We are mostly interested in codimension zero invariants, which informally are those where we imposed enough constraints to get a finite number of curves. For example, if we want to count degree d rational curves in \mathbb{P}^2 , the relevant codimension zero condition says:

$$\sum_{i=1}^n |\gamma_i| = 2 \int_{d[H]} c_1(\mathbb{P}^2) - 2 \dim \mathbb{P}^2 = 6d - 4.$$

For example, we could ask that the curves pass through n given points in \mathbb{P}^2 , then $|\gamma_i| = 4$, so we obtain $4n = 6d - 4$. If the computation were done right, this would be $12d - 4$, so that we get $n = 3d - 1$. So the relevant thing to count are degree d rational curves passing through $3d - 1$ points. (Todo: fix this)

1.3 Stable Map Compactification

To give a naive compactification of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, we could just look at the space $W(r, d)$ of $r + 1$ -tuples of degree d polynomials in 2 variables, up to scaling, and take the subset of tuples which don't vanish simultaneously. We get a subset of a projective space:

$$W(r, d) \subset \mathbb{P} \left(\bigoplus_{i=0}^r H^0(\mathbb{P}^1, \mathcal{O}(d)) \right).$$

We need to quotient by $\text{Aut}(\mathbb{P}^1)$ to identify maps that differ by a reparametrization; ignoring this for the moment, one hopes to take the closure of $W(r, d)$ in $\mathbb{P} \left(\bigoplus_{i=0}^r H^0(\mathbb{P}^1, \mathcal{O}(d)) \right)$ to obtain a compactification. However, for $g \neq 0$ and $X \neq \mathbb{P}^r$, this doesn't work and we need a less ad-hoc approach.

The choice of compactification matters; different choice leads to different numbers. That's because the numbers now count things in the boundary as well.

Example 1.3.1. In the stable maps compactification that we introduce shortly, which produces Gromov-Witten invariants, we keep the domain curves well-behaved: they acquire nodal singularities, but no non-reduced structure. However, the maps themselves can be highly non-injective. A different choice is the Donaldson-Thomas compactification via Hilbert schemes: here we work with ideal sheaves, which always represent embeddings, however the domain curve can now be non-reduced or have singularities worse than nodal. Section 3 $\frac{1}{2}$ of [8] illustrates the differences with the following example. We work locally and consider the family of conics:

$$C_t = \{x^2 + ty = 0\} \subset \mathbb{C}^2,$$

which becomes singular as $t \rightarrow 0$. In the DT compactification, we take the limit in the defining equation, and get $x^2 = 0$, which is a thickened y -axis. In the stable map compactification, we parametrize the conics:

$$C_t \longleftrightarrow \xi \mapsto (-\sqrt{t}\xi, \xi^2).$$

This is a parametrization modulo automorphisms of the curve, namely $\xi \leftrightarrow -\xi$. Now as $t \rightarrow 0$, the limiting map is $\xi \mapsto (0, \xi^2)$, which is a double cover of the y -axis. You can't see from this example, but the different choices of compactification actually give different answers for the counting problem.

With that in mind, let's finally define stable maps. For reference and comparison we include the definition of stable curves:

Definition 1.3.2. (Todo: write this up)

Think about graphs of curves, such that each “twig” has no infinitesimal automorphisms. This means that twigs of genus g must have at least $3 - 2g$ special points, which means either marked points or singular ones.

(Todo: figure out an easy way to include the pictures of graphs)

Definition 1.3.3 (2.4.1 in [5]). A **stable map** to X is a structure (C, x_1, \dots, x_n, f) where:

- (C, x_1, \dots, x_n) is a connected reduced curve with n pairwise distinct marked non-singular points, and at worst additional singular double points.
- $f : C \rightarrow X$ is a map with no non-trivial infinitesimal automorphisms. This means that every irreducible component of C of genus g which is contracted to a point (of degree 0) must have at least $3 - 2g$ special points.

Remark 1.3.4. Note that, in the definition of stable maps (C, x_1, \dots, x_n, f) , the underlying curve (C, x_1, \dots, x_n) need not be stable. Therefore the forgetful map $\overline{\mathcal{M}}_{g,n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{g,n}$ must contract components of (C, x_1, \dots, x_n) which have infinitesimal automorphisms.

In his talk notes, Mauro provides the following construction of the moduli stacks of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Start from $\overline{\mathcal{M}}_{g,n}$, which are fine moduli spaces of curves, and therefore admit a universal family $\mathcal{C}_{g,n}$. Then define:

$$\overline{\mathcal{M}}_{g,n}(X) = \text{Map}_{\mathbf{St}/\overline{\mathcal{M}}_{g,n}}(\mathcal{C}_{g,n}, X \times \overline{\mathcal{M}}_{g,n}). \quad (1.3.1)$$

To obtain $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we must take maps α with the additional constraint that $\alpha_*[\mathcal{C}_{g,n}] = [\beta] \times [\overline{\mathcal{M}}_{g,n}]$.

(Todo: figure out the actual condition)

Remark 1.3.5. When we introduce a derived structure on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we follow the same approach, but take maps in \mathbf{dSt} instead of \mathbf{St} .

Theorem 1.3.6 (3.14 in [2]). $\overline{\mathcal{M}}_{g,n}(X, \beta)$ are proper, algebraic Deligne-Mumford stacks.

(Todo: we should say something about the proof, but the paper is very technical)

Definition 1.3.7. A smooth projective scheme X is **convex** if for every $f : \mathbb{P}^1 \rightarrow X$, $H^1(\mathbb{P}^1, f^*T_X) = 0$.

²

For example, \mathbb{P}^r is convex for every r . This notion is relevant due to:

Proposition 1.3.8. If X is convex, then $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is a smooth, proper Deligne-Mumford stack. ³

(Todo: what's a reference for this? [5] say it's an expectation in 2.4.2, but Mauro's notes imply that it's proved.)

Thus, in the situation of convex X , $[\mathcal{M}_{g,n}(X, \beta)]$ can be taken to be the fundamental class. Otherwise we will need to build a virtual fundamental class.

One of the most important properties of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the recursive structure of the boundary; this leads to a proof of the splitting and genus lowering axioms. We first do the case $g = 0$, which is formula 2.7.3.1 in [4].

Choose a partition $S_1 \cup S_2$ of the marked points, and classes β_1, β_2 such that $\beta_1 + \beta_2 = \beta$. Let $D(S_1, S_2; \beta_1, \beta_2) \subset \overline{\mathcal{M}}_{0,n}(X, \beta)$ be the boundary divisor consisting of curves of genus 0 with 2 irreducible components, with marked points S_i and mapping to β_i respectively.

²We may want to restrict f to be stable, but we haven't defined this yet, so we'll ignore it for now.

³Here we are using the compactification by stable maps; this is defined in ??.

Lemma 1.3.9. *The boundary divisors are given by:*

$$D(S_1, S_2; \beta_1, \beta_2) = \mathcal{M}_{0, S_1 \cup \{x\}}(X, \beta_1) \otimes_X \mathcal{M}_{0, S_2 \cup \{x\}}(X, \beta_2).$$

Inducting on this formula, we obtain the structure of the lower dimensional strata as well; we don't write this down though.

Remark 1.3.10. The straight up generalization for curves of any genus would be:

$$\coprod_{g_1 + g_2 = g} \mathcal{M}_{g_1, S_1 \cup \{x\}}(X, \beta_1) \otimes_X \mathcal{M}_{g_2, S_2 \cup \{x\}}(X, \beta_2).$$

where $g_1 + g_2 = g$, and $[\beta_1] + [\beta_2] = [\beta]$. I haven't computed the dimensions, though, to see for what values of g_1, g_2 we get codimension 1 strata. Moreover, we have extra contributions from cycles of lower genus curves. (Todo: finish this)

To illustrate the need for virtual fundamental classes, we look at an example where $\overline{\mathcal{M}}_{g,n}(X, \beta)$ contains strata of higher dimension than vdim ; in this case, taking the straight up fundamental class would break the grading dimension of Kontsevich-Manin. The following example is worked out in full detail Section 4 of [7].

Example 1.3.11. We compute the dimension and virtual dimension of $\overline{\mathcal{M}}_{0,0}(X, 3\pi^*H)$, where $X = \text{Bl}_p \mathbb{P}^2$, $\pi : X \rightarrow \mathbb{P}^2$ is the blowup map, and $[H] \in H_2(\mathbb{P}^2, \mathbb{Z})$ is the hyperplane class. Using equation 1.2.1, we have:

$$\text{vdim } \overline{\mathcal{M}}_{0,0}(X, 3\pi^*H) = \int_{3\pi^*H} c_1(T_X) - 1 = 8.$$

One could look, for example, at rational curves of degree 3 in \mathbb{P}^2 which avoid p , i.e. $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^2, 3H)$. This is a stratum in $\overline{\mathcal{M}}_{0,0}(X, 3\pi^*H)$ of the correct dimension 8 (the space of cubics in \mathbb{P}^2 is 9-dimensional, and we subtract 1 for reparametrizations of the domain \mathbb{P}^1 .) More strata are given by rational cubics in \mathbb{P}^2 which pass through p with multiplicity k , and therefore lift to a curve in X of class $3\pi^*H - rE$, where $E \subset X$ is the exceptional divisor. To obtain a stable map in the appropriate class $3\pi^*H$, we add r components isomorphic to \mathbb{P}^1 which map to E . The dimension of this stratum is:

$$\dim \overline{\mathcal{M}}_{0,0}(X, 3\pi^*H - rE) + \dim \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1, r) = (8 - r) + (2r - 2) = 6 + r.$$

The farthest we can go while keeping $[\beta]$ effective (that is, $\beta \cdot K_X \leq 0$) is $r = 3$. This gives a stratum (supposedly a boundary stratum!) of dimension $9 > 8$.

Chapter 2

Obstruction Theories and Virtual Fundamental Classes

Talk by Benedict Morrissey.

Given a stack X , our objective is to construct a virtual fundamental class $[X]$ for it, motivated by the discussion in 1. We will see two ways in which a derived enhancement of X helps achieve this. We would like $[X]$ to come from an algebraic cycle, i.e. an element of the Chow group. In this case, given $f : X \rightarrow Y$ proper, there is a well-defined pushforward $f_*[X] \rightarrow [Y]$, which induces a pushforward $f_*[X]^H \rightarrow [Y]^H$ on the images $[X]^H, [Y]^H$ of the VFCs in any Weil cohomology theory H .

However, derived Chow groups have yet to be defined, so we start with a piecemeal approach, by defining a class in G-theory only.

2.1 Construction from G-Theory

Definition 2.1.1. The **G-theory** $G_0(X)$ of a classical stack X is defined as the K-theory of the category of coherent sheaves on X :¹

$$G_0(X) := K_0(\mathrm{Coh}(X)).$$

If \tilde{X} is a derived stack, we set $G_0(\tilde{X}) = K_0((\mathrm{Coh}\tilde{X})^\heartsuit)$.

Definition 2.1.2. A **derived enhancement** of a stack X is a derived stack \tilde{X} such that $t_0(\tilde{X}) = X$.

There is a natural inclusion, left-adjoint to the truncation, which we denote $j : X \rightarrow \tilde{X}$. Using the fact that pushforwards of coherent sheaves by proper maps are coherent, (**Todo: check if there are other conditions, and whether \tilde{X} derived changes anything**) we obtain $j_* : G_0(X) \rightarrow G_0(\tilde{X})$.

Proposition 2.1.3. *If X is quasi-compact, then $j_* : G_0(X) \rightarrow G_0(\tilde{X})$ is a bijection. In this case we define:*

$$[X]^{\mathrm{vir}} := j_*^{-1}[\mathcal{O}_{\tilde{X}}].$$

Proof. The identification actually works on the full spectrum of G -theory. We're using the theorem of the heart for K -theory. The identification is done as follows.

1. Theorem of the heart for K -theory. (Due to Quillen, and Batwick in the DG category setting.) If you have \mathcal{C} a stable ∞ -category, idempotent complete, with t -structure, and every object in the heart is bounded, then $K(\mathcal{C}) = K(\mathcal{C}^\heartsuit)$.

¹One can also define higher G-theory G_i , but we won't need this.

2. $\mathrm{Coh}(\tilde{X})^\heartsuit \simeq \mathrm{Coh}(X)^\heartsuit$, which follows from descent and the analogous result for derived affines, which was proved during the first semester, in the talk on Stable ∞ -categories. ²

□

Theorem 2.1.4. *For \tilde{X} quasi-compact,³ $\mathcal{O}_{\tilde{X}}$ is bounded. It follows that the following sum is finite:*

$$j_*^{-1}[\mathcal{O}_{\tilde{X}}] = \sum_{i=0}^{\infty} (-1)^i [H^i(\mathcal{O}_{\tilde{X}})],$$

so it defines an element in $G_0(X)$.

Remark 2.1.5. Note that the cohomology in Theorem 2.1.4 is just the cohomology of the complex, NOT sheaf cohomology. Moreover it wouldn't make sense to use K theory instead of G theory, because even if $\mathcal{O}_{\tilde{X}}$ is perfect, the kernels and cokernels of the various differentials don't need to be.

Proof. We start with a vague understanding of why the theorem may be true. The counterexample is $\mathrm{Spec}(\mathrm{Sym} k[2])$, where the cotangent complex is unbounded. But if it's in amplitude $[-1, 0]$, it's like an exterior algebra and it works.

We work locally, $\mathrm{Spec} B \rightarrow \mathrm{Spec} A$, $\mathrm{supp} \mathbb{L}_{B/A} \subset [-1, 0]$. B is a derived lci over A , so the cotangent complex is perfect, so there's a theorem that says that B is homotopically of finite type over A . These can be constructed by attaching finitely many cells:

$$A = B_0 \rightarrow B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow B_k = B.$$

Attaching the $i + 1^{\mathrm{th}}$ cells of B looks like:

$$\begin{array}{ccc} B_i & \longrightarrow & B_{i+1} \\ \uparrow & & \uparrow \\ \bigotimes A[\partial \Delta^{i+1}] & \longrightarrow & A[\Delta^{i+1}]. \end{array}$$

B_1 is obtained by attaching cells in degree 1. The map $B_1 \rightarrow B$ is an isomorphism on π_0 . (Todo: review this proof)

There's another proof by Lowrey and Schürg, in [6], which is more intuitive. Having a quasi-smooth structure allows one to describe the derived space locally as the derived zero locus of a section of a vector bundle. Then the derived intersection can be computed as a Koszul resolution, so $\mathcal{O}_{\tilde{X}}$ behaves like an exterior algebra, which means it's bounded. Here the quasi-compactness is used in order to reduce to finitely many local charts, which means that the bound on $\mathcal{O}_{\tilde{X}}$ is uniform. □

Remark 2.1.6. The idea behind the proof of Lowrey and Schürg is also that of **Kuranishi structures**. These are essentially a machinery for working with derived stacks which remembers the local description as zero locus, in order to avoid using the machinery of derived geometry. In DAG quasi-smoothness is an intrinsic property that one can check at the level of the cotangent complex, so that one doesn't need to remember the local descriptions, which are cumbersome and don't glue well.

The VFC in ordinary cohomology is defined by Konsevich to be:

$$[X]^{\mathrm{vir}} = \mathrm{Ch}([X]_G^{\mathrm{vir}}) \mathrm{Td}(j^* \mathbb{T}_{\tilde{X}}). \quad (2.1.1)$$

Conjecture 2.1.7. *Definition 2.1.1 agrees with the construction of Behrend-Fantechi, ??.* (Todo: ref this)

The conjecture has been verified for schemes (not stacks) by Ciocan-Fontanine and Kapranov, in [3], using the additional assumption (which is made in Behrend Fantechi anyway) that the cotangent complex admits a global resolution by vector bundles.

²Throughout when we write Coh we mean Coh^b .

³Note that we don't need to assume that X is quasi-compact.

2.2 Obstruction Theories

We introduce the alternative construction of VFCs, following [1]. In the words of Mauro, we want to use this as a black box which achieves:

$$\text{Obstruction Theory} \implies \text{VFC}.$$

Throughout we will use X, Y for underived stacks, and \tilde{X}, \tilde{Y} for their derived enhancements.

Definition 2.2.1. An **obstruction theory** for X is a morphism $\phi : E \rightarrow \mathbb{L}_X$ in $D(\text{Coh}(X))$, such that:

$$\begin{aligned} h^0(\phi) : H^0(E) &\rightarrow H^0(\mathbb{L}_X) \text{ is an isomorphism,} \\ h^{-1}(\phi) : H^{-1}(E) &\rightarrow H^{-1}(\mathbb{L}_X) \text{ is surjective,} \\ H^i(E) &= 0 \text{ for } i \neq -1, 0. \end{aligned}$$

Definition 2.2.2. A **perfect obstruction theory** is an obstruction theory such that E is in perfect amplitude $[-1, 0]$, which means that locally E is isomorphic to a 2-term complex of vector bundles $[E^{-1} \rightarrow E^0]$.

The link to derived geometry is as follows.

Proposition 2.2.3. *Given a derived enhancement $j : X \rightarrow \tilde{X}$, with \tilde{X} a quasi-smooth DM stack, there is a perfect obstruction theory:*

$$j^* \mathbb{L}_{\tilde{X}} \rightarrow \mathbb{L}_X.$$

Proof. By descent we reduce this to the case of affines, and we need only consider $A \rightarrow t_0(A)$. We have the fiber sequence:

$$j \mathbb{L}_A \rightarrow \mathbb{L}_{\pi_0(A)} \rightarrow \mathbb{L}_{\pi_0(A)/A}.$$

Due to the connectivity estimates, which we introduced last semester in the talk about the cotangent complex, $\mathbb{L}_{\pi_0(A)/A}$ is 2-connective. Indeed, the fiber of $A \rightarrow \pi_0(A)$ is 1-connective, so the cofiber, which is the shift of the fiber by 1, is 2-connective.⁴ \square

Throughout the rest of the talk, the goal is to describe how to construct a VFC, starting with an obstruction theory. In the smooth case, if you take the G -construction we did earlier, you'd get the same answer.

We also want to describe functoriality properties for the VFC, and to that effect we introduce compatibility data between obstruction theories. During the check that Kontsevich-Manin axioms are satisfied, we will need to use functoriality a lot. The following is Definition 5.8 in [1].

Definition 2.2.4. Let $u : X' \rightarrow X$ be a morphism. A **compatibility datum between obstruction theories** E for X and F for X' is a choice of embeddings $f : X \rightarrow Y$, $g : X' \rightarrow Y'$ into smooth stacks, such that the following diagrams commute:

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ g \downarrow & & \downarrow f \\ Y' & \xrightarrow{v} & Y, \end{array}$$

$$\begin{array}{ccccc} u^* E & \xrightarrow{\phi} & F & \xrightarrow{\psi} & g^* \mathbb{L}_{Y'/Y} \\ \downarrow & & \downarrow & & \downarrow \\ u^* \mathbb{L}_X & \longrightarrow & \mathbb{L}_{X'} & \longrightarrow & \mathbb{L}_{X'/X}. \end{array}$$

Moreover, we require the two rows to be fibration sequences in $D(\text{Coh}(X'))$.

⁴In order to relate the fiber and cofiber of the morphism $A \rightarrow \pi_0(A)$, we use the fact that we are working in the stable ∞ -category $\pi_0(A)\text{-Mod}$, and not in $\pi_0(A) - \text{Alg}$.

Behrend and Fantechi prove:

Proposition 2.2.5. *Given compatibility data between obstruction theories E for X and F for X' , it follows that $u^*[X]^{\text{vir},E} = [X']^{\text{vir},F}$.*

For us obstruction theories come from derived enhancements \tilde{X}, \tilde{X}' . In this case, we obtain the functoriality of VFCs in a cleaner way, by giving a morphism between derived enhancements $w : \tilde{X}' \rightarrow \tilde{X}$, fitting in the commutative diagram:

$$\begin{array}{ccc}
 & \tilde{X}' & \xrightarrow{w} \tilde{X} \\
 j \nearrow & & \nwarrow i \\
 X' & \xrightarrow{u} X & \\
 g \downarrow & \tilde{g} & \downarrow f \\
 Y' & \xrightarrow{v} Y & \tilde{f} \nearrow
 \end{array}$$

Moreover, we require that the top and back square are homotopy pullbacks.

Remark 2.2.6. I was hoping that the top square would be enough. Unfortunately, we still need the choice of ambient spaces Y, Y' , as well as morphisms \tilde{g}, \tilde{f} , and the data for the homotopy commutativity of the back square. However Mauro says:

1. In the applications we care about (stable maps), the entire back square will be there naturally.
2. Working with the derived compatibility data is still easier, in practice, than with the fibration sequences in Definition 2.2.4.

Let us see why the derived compatibility data implies the diagram between fibration sequences in Definition 2.2.4. The assumption is that $E = i^*\mathbb{L}_{\tilde{X}}$ and $F = j^*\mathbb{L}_{\tilde{X}'}$. We first need the map:

$$\phi : u^*i^*\mathbb{L}_{\tilde{X}} \rightarrow j^*\mathbb{L}_{\tilde{X}'}$$

This is just given by w . More precisely, the commutativity of the top square gives the map on the left in the following diagram, and we define the top map as the composition:

$$\begin{array}{ccc}
 u^*i^*\mathbb{L}_{\tilde{X}} & \longrightarrow & j^*\mathbb{L}_{\tilde{X}'} \\
 \downarrow & \nearrow & \\
 j^*w^*\mathbb{L}_{\tilde{X}} & &
 \end{array}$$

To get ψ , which must be such that the row is a fiber sequence, we make use of the maps \tilde{g}, \tilde{f} . Question: how to identify $j^*\mathbb{L}_{\tilde{X}'/\tilde{X}}$ with $g^*\mathbb{L}_{Y'/Y}$? Since the back square is a pullback, we have a canonical identification $\mathbb{L}_{\tilde{X}'/\tilde{X}} \simeq \tilde{g}^*\mathbb{L}_{Y'/Y}$, and this gives:

$$j^*\mathbb{L}_{\tilde{X}'/\tilde{X}} \simeq j^*\tilde{g}^*\mathbb{L}_{Y'/Y} \simeq g^*\mathbb{L}_{Y'/Y}. \quad (2.2.1)$$

We take this composition to be ψ . Note that this chain of equivalences depends very much on the extra data of the homotopy commutative back square.

Remark 2.2.7. Throughout, we want Y', Y to be smooth, and \tilde{f}, \tilde{g} to be quasi-smooth. Therefore, if X, X' are not smooth, we cannot expect f, g to be just identity maps. In fact, the point that Behrend-Fantechi make is that Y and Y' should only be expected to exist locally.

Definition 2.2.8. A **local embedding** (U, M) of X is the data of $U \rightarrow X$ an étale map and $U \rightarrow M$ a local immersion, where M smooth affine k -scheme of finite type. Given a local embedding, the associated **normal bundle** is $N_{U|M} := \text{Spec}_M(\text{Sym}(I/I^2))$. Inside this we have the **normal cone** $C_{U/M} = \text{Spec}_M(\oplus_{n \geq 0} I^n/I^{n+1})$. The ring homomorphism $\text{Sym}(I/I^2) \rightarrow \oplus_{n \geq 0} I^n/I^{n+1}$ is surjective, so the map $C_{U/M} \rightarrow N_{U|M}$ is a closed embedding.

Given an obstruction theory $E \rightarrow \mathbb{L}_X$, if we can write $E = [F_{-1} \rightarrow F_0]$ globally, and define $F^1 := F_{-1}^*$, then we get the pullback diagram:

$$\begin{array}{ccc} C(F^\bullet) & \longrightarrow & F_1 \\ \downarrow & & \downarrow \\ C_X & \longrightarrow & N_X. \end{array}$$

Definition 2.2.9. Let $0 : X \rightarrow F_1$ be the zero section. The **virtual fundamental class** of X induced by the obstruction theory E is the intersection of $[C(F^\bullet)] \in \text{Chow}(F_1)$ with the zero section, i.e. $[X]^{\text{vir}, E} := o^! [C(F^\bullet)]$. (Todo: why is this shriek and not star?)

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