Chapter 1

∞-category theory

1.1 Motivations

Exercise 1.1.1. We fix a base field k. Let $X = \mathbb{P}^1_k$ and let U_0 and U_1 be the standard open affine cover of $\mathbb{P}^1_k$. For any k-algebra A, we have:

$$U_0(A) \coloneqq \{[x_0 : x_1] \in \mathbb{P}^1_k(A) \mid x_0 \neq 0\}, \qquad U_1(A) \coloneqq \{[x_0 : x_1] \in \mathbb{P}^1_k(A) \mid x_1 \neq 0\}.$$

Let $U_{01} = U_0 \cap U_1$ be their intersection. Show that the canonical functor

$$\mathrm{h}(\mathcal{D}(\mathbb{P}^1_k)) \to \mathrm{h}(\mathcal{D}(U_0)) \times_{\mathrm{h}(\mathcal{D}(U_{01}))} \mathrm{h}(\mathcal{D}(U_1))$$

is essentially surjective but not fully faithful.

Exercise 1.1.2. Let $\mathcal{C}$ be a triangulated category where countable products and countable direct sums exist. Show that if there exists a functor Tr from the category of arrows $\mathcal{C}^{\Delta^1}$ to the category of exact triangles in $\mathcal{C}$, then every triangle in $\mathcal{C}$ is split. (See [4, Proposition II.1.2.13].)

1.2 Reminders on simplicial sets

Exercise 1.2.1. Show that the nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful and its essential image is spanned by those simplicial sets K satisfying the following lifting condition: for every $n \geq 2$ and for every $0 < i < n$ every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a unique solution.

Solution. The nerve of a category $\mathcal{C}$ is:

$$(N\mathcal{C})_n = \{(f_1, \ldots, f_n) | \text{ composable morphisms}\}.$$

The face maps are:

$$d_j(f_1, \ldots, f_n) = \begin{cases} (f_1, \ldots, f_{n-1}), & j = 0 \\ (f_1, \ldots, f_j \circ f_{j-1}, \ldots, f_n), & 0 < j < n \\ (f_2, \ldots, f_n), & j = n. \end{cases}$$

The degeneracy s_j is obtained by inserting an identity map in the j^{th} slot.

A functor $F : \mathcal{C} \to \mathcal{D}$, induces a simplicial map:

$$\begin{aligned} N(F)_n : (N\mathcal{C})_n &\to (N\mathcal{D})_n \\ (f_1, \dots, f_n) &\mapsto \big(F(f_1), \dots, F(f_n)\big). \end{aligned}$$

If two functors F, F' induce simplicial maps $N(F) = N(F')$ which agree, then $F(f) = F'(f)$ for every morphism f. Hence N is faithful. Given a simplicial map $G : N\mathcal{C} \to N\mathcal{D}$, we define a functor $F : \mathcal{C} \to \mathcal{D}$ to be G_0 on objects and G_1 on morphisms. We show that F respects composition. Let f_1, f_2 be two composable morphisms in $\mathcal{C}$ and denote by x the 2-simplex (f_1, f_2). Then:

$$F(f_2 \circ f_1) = G(d_1 x) = d_1 G(x) = F(f_2) \circ F(f_1).$$

This proves that N is also full.

We move on to the essential image. Let K be the nerve of a category. The data of a map $\Lambda^n_i \to K$ is the same as the data of maps $y_j : \Delta^{n-1} \to K$ for $j \neq i$, which are compatible along their faces. By Yoneda, this is the same as simplices $\{y_j \in K_{n-1}\}_{j \neq i}$ compatible along faces. Given this data, we define the horn filler $x \in K_n$ by:

$$x = \left((d_0)^{n-2} y_{n-1}, (d_0)^{n-3} d_n y_{n-1}, \dots, d_0 (d_n)^{n-3} y_{n-1}, (d_n)^{n-2} y_0\right).$$

The simplicial identities ensure that $d_j x = y_j$ for $j \neq i$. [1] Using the compatibility of the y_j along faces, x is the unique solution to the lifting problem.

Conversely, given a K which has unique solutions to all lifting problems of inner horns, we define a category $\mathcal{C}$ such that $K \cong \mathcal{C}$. Let K_0 be the objects of $\mathcal{C}$, and for $X, Y \in K_0$, define:

$$\mathrm{Hom}(X, Y) := \{f \in K_1 | d_1 f = X, d_0 f = Y\}.$$

Given $f_1 : X \to Y$ and $f_2 : Y \to Z$, define a lifting problem by mapping the 1-simplices $0 \to 1$ and $1 \to 2$ in Λ^2_1 to f_1 and f_2, respectively. We define $f_2 \circ f_1$ to be d_1 of the unique lift. Associativity of this composition follows from the unique filling of the horn Λ^3_1; we don't give the details here. ☐

Exercise 1.2.2. Let S, S' be sets, seen as discrete simplicial set. Show that any morphism $f \colon S \to S'$ is a Kan fibration, and that f is a trivial Kan fibration if and only if f is a bijection.

Solution. Since S and S' are sets, all k-simplices are of the form $s^k x$, for x a 0-simplex. Given a lifting problem:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S' \end{tikzcd}$$

all k-simplices of Λ^n_i, for $k > 0$, must map to degenerate k-simplices in S. Hence Λ^n_i maps to a point $s \in S$. Similarly, Δ^n maps to $f(s)$. The constant map from Δ^n to s is then the unique solution to the lifting problem. It follows that f is a Kan fibration, and moreover that all sets S are ∞-groupoids.

By definition, f is a weak equivalence if it induces a weak equivalence on geometric realizations. $|S|$ and $|S'|$ are discrete topological spaces, therefore $|f|$ is a weak equivalence iff it is a bijection. ☐

Exercise 1.2.3. Let G and H be simplicial groups and let $f \colon G \to H$ be a surjective group homomorphism. Show that f is a Kan fibration.

Solution. There is an algorithm for constructing fillers on nLab. [2] We don't have any intuition for it, so we should work on building that.

The algorithm produces unique fillers for all horns, so in particular simplicial groups are ∞-groupoids. ☐

[1] Note that it's essential that both y_0 and y_{n-1} are available to use in the definition of x, i.e. that Λ^n_i is an inner horn.
[2] `https://ncatlab.org/nlab/show/simplicial+group`

Exercise 1.2.4. Let $\partial\Delta^2$ be the smallest full subsimplicial set of Δ^2 spanned by its non-degenerate edges $\Delta^1 \to \Delta^2$. Show that $\partial\Delta^2$ fits into a coequalizer diagram

$$(\Delta^0)^{\amalg 6} \rightrightarrows (\Delta^1)^{\amalg 3} \to \partial\Delta^2.$$

(Hint: Have a look at [2, Theorem III.3.1].)

Exercise 1.2.5. Let S be a set, seen as a discrete simplicial set. Show that $\mathrm{cosk}_n(S)$ satisfies the following property: for every $m \geq n$ and every $0 \leq i \leq m$ the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathrm{cosk}_n(S) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. In particular, deduce that $\mathrm{cosk}_0(S)$ is a Kan complex.

1.3 ∞-categories

Exercise 1.3.1. Show that every Kan complexes and 1-categories are ∞-categories (quasicategories).

Solution. Kan complexes have fillers for all horns. 1-categories have unique fillers for all inner horns. In particular, both have fillers for all inner horns, which is the definition of ∞-categories. ☐

Exercise 1.3.2. A morphism $f\colon X \to Y$ in an ∞-category $\mathcal{C}$ is said to be an equivalence if its image in $\mathrm{h}(\mathcal{C})$ is an isomorphism. Define $S^\infty := \mathrm{cosk}_0(\{0,1\})$ and let $j\colon \Delta^1 \to S^\infty$ be the map classified by

$$\mathrm{sk}_0(\Delta^1) = \{0,1\} \xrightarrow{\mathrm{id}} \{0,1\}.$$

To give a morphism $f\colon X \to Y$ in an ∞-category $\mathcal{C}$ it is equivalent to specify a morphism of simplicial sets $e_f\colon \Delta^1 \to \mathcal{C}$. Show that f is an equivalence if and only if the lifting problem

$$\begin{tikzcd} \Delta^1 \arrow[r, "e_f"] \arrow[d, hook, "j"] & \mathcal{C} \\ S^\infty \arrow[ur, dashed] & \end{tikzcd}$$

has at least one solution. Next, show that any two such solution are homotopic. (Hint: have a look at Exercises 1.2.5 and 1.4.1.)

Exercise 1.3.3. In [3] a functor of ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$ is said to be a *categorical equivalence* if and only if the induced functor $\mathfrak{C}[f]\colon \mathfrak{C}[\mathcal{C}] \to \mathfrak{C}[\mathcal{D}]$ is an equivalence of simplicial categories. Show that f is a categorical equivalence if and only if it is fully faithful and essentially surjective.

Exercise 1.3.4. Let E denote the walking isomorphism (i.e. the 1-category with two objects and an isomorphism between them). Recall the definition of S^∞ from the previous exercise. Show that there is a canonical map $E \to S^\infty$ and that this is a categorical equivalence. In particular, for every ∞-category $\mathcal{C}$, the functor

$$\mathrm{Fun}(S^\infty, \mathcal{C}) \to \mathrm{Fun}(E, \mathcal{C})$$

is a categorical equivalence. (This is a very simple example of what an "internal rectification theorem" looks like.)

Exercise 1.3.5. Let $\mathcal{C}$ be an ∞-category. Let S_0 be a collection of *objects* in $\mathcal{C}$. Let $\mathcal{C}_0$ be the smallest full subsimplicial set of $\mathcal{C}$ containing S_0 (explicitly, an n-simplex $\sigma\colon \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}$ if and only if for every morphism $\Delta^0 \to \Delta^n$ the composition $\Delta^0 \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0.) Show that $\mathcal{C}_0$ is an ∞-category. Furthermore, show that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ of simplicial sets is a fully faithful functor of ∞-categories.

Exercise 1.3.6. Let $\mathcal{C}$ be an ∞-category. Let S_0 be a collection of *morphisms* in $\mathcal{C}$, and suppose that S_0 is closed under composition, in the sense that for every 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

is $\mathcal{C}$, if f and g belong to S_0 then so does h. Let $\mathcal{C}_0$ be the smallest full subsimplicial set of $\mathcal{C}$ containing S_0 (explicitly, an n-simplex $\sigma\colon \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}$ if and only if for every morphism $\Delta^1 \to \Delta^n$ the composition $\Delta^1 \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0). Show that $\mathcal{C}_0$ is an ∞-category.

Exercise 1.3.7. Let $\mathcal{C}$ be an ∞-category. Show that the collection of equivalences in $\mathcal{C}$ is closed under composition, in the sense of the previous exercise. Let $\mathcal{C}^{\simeq}$ be the ∞-subcategory of $\mathcal{C}$ spanned by equivalences in $\mathcal{C}$. Show that $\mathcal{C}^{\simeq}$ is a Kan complex.

1.4 Localization of ∞-categories

Exercise 1.4.1. Let $\mathcal{C}$ be an ∞-category (seen as a quasicategory). Let $\mathcal{C} \to \widetilde{\mathcal{C}}$ be a fibrant replacement for the Kan model structure on sSet. Show that $\widetilde{\mathcal{C}}$ enjoys the following universal property: for every ∞-category $\mathcal{D}$ the functor of ∞-categories

$$\mathrm{Fun}(\widetilde{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and its essential image is spanned by those morphisms $f\colon \mathcal{C} \to \mathcal{D}$ that send every morphism in $\mathcal{C}$ into an equivalence in $\mathcal{D}$. Thus, there is a categorical equivalence $\tilde{\mathcal{C}} \simeq \mathcal{C}[W^{-1}]$, where W denotes the collection of all arrows in $\mathcal{C}$. Deduce that if $\mathcal{C}$ is an ∞-category where every morphism is invertible, then $\mathcal{C}$ is categorically equivalent to a Kan complex.

Exercise 1.4.2. Let $\mathcal{C}$ be an ∞-category and let S be a (small) collection of arrows in $\mathcal{C}$. Show that $\mathrm{h}(\mathcal{C}[S^{-1}]) \in \mathrm{Cat}$ is canonically equivalent to the 1-categorical localization of $\mathrm{h}(\mathcal{C})$ at $\overline{S}$, the collection of morphism which is the image of S via the canonical functor $\mathcal{C} \to \mathrm{h}(\mathcal{C})$.

Exercise 1.4.3. Let $\mathcal{C}$ be an ∞-category with finite limits and let S be a (small) collection of arrows in $\mathcal{C}$. Suppose that $\mathcal{C}$ is stable under pullbacks. Then the ∞-categorical localization $\mathcal{C}[S^{-1}]$ has finite limits and the localization functor $L\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ commutes with them.

1.5 Limits and colimits

Exercise 1.5.1. Let $\mathcal{S}$ be the ∞-category of spaces and let X be an object in $\mathcal{S}$. Using [3, Theorem 4.2.4.1] show that the colimit of the diagram

$$* \longleftarrow X \longrightarrow *$$

can be canonically identified with $\Sigma(X)$.

Now fix two points $p, q\colon * \to X$. Show that the limit of the diagram

$$* \xrightarrow{\ p\ } X \xleftarrow{\ q\ } *$$

can be canonically identified with the path space $\mathrm{Path}_X(p, q)$.

Exercise 1.5.2. n-cofinality...

Exercise 1.5.3. $\star$ Let K be a simplicial set and let $F\colon K^{\mathrm{op}}\mathcal{P}\mathrm{r}^{\mathrm{L}}$ be an ∞-functor. Let $\mathcal{C}$ be a presentable ∞-category and let $\Delta_{\mathcal{C}}\colon K^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the constant ∞-functor associated to F. Let $\varphi\colon \Delta_{\mathcal{C}} \to F$ be a natural transformation in $\mathrm{Fun}(K^{\mathrm{op}}, \mathcal{P}\mathrm{r}^{\mathrm{L}})$. We let

$$\Phi\colon \mathcal{C} \to \varprojlim F$$

be the induced functor. For every $x \in K$, the functor $\varphi_x\colon \mathcal{C} \to F(x)$ admits a right adjoint, which we denote $\psi_x\colon F(x) \to \mathcal{C}$. Show that there exists an ∞-functor

$$\overline{\Psi}\colon \varprojlim F \to \mathrm{Fun}(K, \mathcal{C})$$

which informally sends $Y = \{Y_x\}_{x\in K} \in \varprojlim F$ to the diagram $\overline{\Psi}(Y)\colon K \to \mathcal{C}$ given by

$$\overline{\Psi}(Y)(x) = \psi_x(Y_x).$$

Prove moreover that the composition

$$\varprojlim F \xrightarrow{\overline{\Psi}} \mathrm{Fun}(K, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

can be canonically identified with a right adjoint for Φ.

1.6 Left and right fibrations

Exercise 1.6.1. Let X be a connected Kan complex and let F be any other Kan complex. Let us further fix a point $x \in X$. Let $\mathrm{LF}_x(X; F)$ be the full subcategory of left fibrations $\mathrm{LF}(X)$ over X whose homotopy fiber at x is equivalent to F. Let $\mathrm{B}(\mathrm{hAut}(F))$ be the classifying space of the simplicial group of homotopy automorphisms of F. Show that there is a canonical equivalence of ∞-categories

$$\mathrm{LF}_x(X; F) \simeq \mathrm{Fun}(X, \mathrm{B}(\mathrm{hAut}(F))).$$

1.7 Cartesian and coCartesian fibrations

Exercise 1.7.1. Let $\mathcal{C}$ be an ∞-category and let $X \in \mathcal{C}$ be an object. Let $f\colon U \to X$ and $g\colon V \to X$ be two morphisms in $\mathcal{C}$. For every 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$ such that $d_0(\sigma) = f$ and $d_1(\sigma) = g$, show that there is a pullback square in $\mathcal{S}$:

$$\begin{tikzcd}
\mathrm{Path}_{\mathrm{Map}_{\mathcal{C}}(U,X)}(f, d_2(\sigma)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}_{/X}}(f, g) \arrow[d] \\
* \arrow[r, "d_2(\sigma)"] & \mathrm{Map}_{\mathcal{C}}(U, V).
\end{tikzcd}$$

(Hint: Use [3, Propositions 2.1.2.1 and 2.4.4.2].)

1.8 Adjunctions

Exercise 1.8.1. Let $\mathcal{C}$ be an ∞-category with a zero object 0. Suppose that for every object $X \in \mathcal{C}$ the span

$$0 \longleftarrow X \longrightarrow 0$$

has both a limit $\Omega(X)$ and a colimit $\Sigma(X)$. Construct in an explicit way ∞-functors $\Sigma, \Omega\colon \mathcal{C} \to \mathcal{C}$ informally given by $X \mapsto \Sigma(X)$ and $X \mapsto \Omega(X)$, respectively. Show that Σ and Ω are adjoint by explicitly constructing a fibration $\mathcal{D} \to \Delta^1$ which is both Cartesian and coCartesian.

Exercise 1.8.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an ∞-functor. Show that the following statements are equivalent:

1. F has a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$;
2. for every $Y \in \mathcal{D}$ there exists an object $X \in \mathcal{C}$ and a morphism $\varepsilon_X\colon F(X) \to Y$ such that for every other $X' \in \mathcal{C}$ the canonical composition

$$\mathrm{Map}_{\mathcal{C}}(X', X) \xrightarrow{f} \mathrm{Map}_{\mathcal{D}}(f(X'), f(X)) \xrightarrow{\varepsilon_{X*}} \mathrm{Map}_{\mathcal{D}}(f(X'), Y)$$

is a weak homotopy equivalence.

1.9 Stable ∞-categories

Exercise 1.9.1. Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full stable subcategory of $\mathcal{C}$. Let $S \coloneqq \{f\colon X \to Y \in \mathcal{C} \mid \mathrm{cofib}(f) \in \mathcal{D}\}$. Show that the ∞-categorical localization $\mathcal{C}[S^{-1}]$ is a stable ∞-category.

Exercise 1.9.2. It is shown in [1] that $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is a presentable ∞-category. Prove directly that cofibers in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ exist.

Chapter 2

Derived rings

Chapter 3

Derived stacks

Bibliography

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