Derived Algebraic Geometry Seminar: UPenn 2016

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Introduction

This contains notes livetexed for the Derived Algebraic Geometry Seminar currently being held at the University of Pennsylvania math department in the 2016-17 academic year. We are following Mauro Porta's plan to introduce derived algebraic geometry, and then to look at derived geometric objects with extra structure (initially we will be looking at the case of symplectic and Poisson structures).

This is a draft and errors should be expected.

Chapter 1

∞-category theory

Talk by Mauro Porta.

1.1 Why ∞-categories?

Our main reason for studying ∞-categories in this seminar is that derived schemes form an ∞-category. Some other applications of ∞-categories are the following.

1. Formal moduli problems over a field k of characteristic 0 are equivalent to dgLie_k, but this is an equivalence of ∞-categories. We can see explicitly why this equivalence is plausible. For F a formal moduli problem, $T_xF[1]$ is a dgLie algebra. Conversely, Maurer-Cartan elements on the RHS determine $F(k[\epsilon])$, i.e. infinitesimal formal moduli problems. Brackets then allow the complete recovery of F.

2. The ∞-category of rational homotopy types is equivalent to that of dgLie algebras over $\mathbb{Q}$, concentrated in positive degrees:
$$S_*^{\mathrm{rat}} \cong \mathrm{dgLie}_{\mathbb{Q}}^{\geq 1}$$
This statement is related to item 1: Lurie gives a nice proof using formal moduli problems, see [11].

3. To $X \in \mathrm{Sch}_k$, we associate its derived category of quasi-coherent sheaves, $D(X) = D(QCoh(X))$. It's a powerful invariant of X, especially when X is not smooth. For example, it contains the cotangent complex and dualizing complex, $\mathbb{L}_X, \omega_X \in D(X)$, which are not necessarily bounded if X is not smooth.

   The problem is that we cannot reconstruct $D(X)$, the derived category in the classical sense, by patching: $D(X) \not\cong \lim_{\{U\}\text{ Zariski cover}} D(U)$. For example, take $X = \mathbb{P}^1_k$, and its standard cover by 2 open affines U_0, U_1. We show that the functor:
$$D(\mathbb{P}^1) \to D(U_0) \times_{D(U_{01})} D(U_1)$$
   is not faithful, by exhibiting a morphism in $D(\mathbb{P}^1)$ which gets mapped to 0. Start from the observation that morphisms from the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ are the same as sections of the target sheaf, which implies:
$$\mathbb{R}(\mathrm{Hom})\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)[1]\big) \cong \mathbb{R}\Gamma\big(\mathcal{O}_{\mathbb{P}^1}(-2)[1]\big).$$
   This complex has nontrivial cohomology in degree 0:
$$H^0\mathbb{R}\Gamma\big(\mathcal{O}_{\mathbb{P}^1}(-2)[1]\big) \cong H^1\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big) \cong k. \tag{1.1.1}$$

However, when passing to the affine patches, $D(U_i) \simeq D(k[T] - Mod)$, and the complexes corresponding to the restrictions of $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-2)[1]$ are the following.

$$0 \longrightarrow 0 \longrightarrow k[T] \longrightarrow 0$$

$$0 \longrightarrow k[T] \longrightarrow 0 \longrightarrow 0$$

As such, there are no non-zero morphisms between the restrictions. Equivalently, when restricting to affine opens, the first cohomology in equation 1.1.1 is 0.

On the other hand, we will see that the ∞-derived category of X (which we temporarily denote by $L_{\mathrm{qcoh}}(X)$) can be patched using the homotopy fiber product:

$$L_{qcoh}(\mathbb{P}^1_k) \simeq L_{qcoh}(U_0) \times_{L_{qcoh}(U_{01})} L_{qcoh}(U_1).$$

4. Let $\mathcal{M}_{ell}$ be the moduli stack of elliptic curves, i.e. the functor F sending $\mathrm{Spec}(A)$ to the classes of elliptic curves over $\mathrm{Spec}(A)$. It is not a sheaf, because two elliptic curves can become isomorphic after a base extension. The problem here is that we were trying to take $F : \mathcal{A}ff^{\mathrm{op}} \to \mathcal{S}et$, and we can't talk about isomorphisms in $\mathcal{S}et$. Classically one solves this problem by replacing sets by groupoids, which are equivalent to 1-homotopy types.

$$\begin{tikzcd} & \mathcal{G}pd \cong \mathcal{S}^{\leq 1} \\ \mathcal{A}ff^{\mathrm{op}} \arrow[ur, "\text{stacks}"] \arrow[r, "\text{naive moduli problems}"'] & \mathcal{S}et \cong \mathcal{S}^{\leq 0} \arrow[u, hook] \end{tikzcd}$$

We can define higher stacks by extending the tower to higher homotopy types, and ultimately to the category of spaces.

$$\begin{tikzcd} & \mathcal{S} \\ & \vdots \arrow[u, hook] \\ & \mathcal{S}^{\leq 1} \arrow[u, hook] \\ \mathcal{A}ff^{\mathrm{op}} \arrow[uuur, "\text{higher stacks}"] \arrow[ur, "\text{stacks}"] \arrow[r, "\text{naive moduli problems}"'] & \mathcal{S}et \cong \mathcal{S}^{\leq 0} \arrow[u, hook] \end{tikzcd}$$

In later talks, we'll see that the perfect complexes $\mathcal{P}erf$ form an ∞-stack which doesn't factor through finite homotopy types.

1.2 Three ways of working with ∞-categories

To be attempted in order of desperation:

1. Reason model-independently to get a clean proof. The trick is that there are key statements (not proven model independently; some are proven by Lurie and can be found in [9]) which behave like a "non-minimal set of axioms". One should learn a roadmap to [9], in order to know where to find these statements. (A good start for this roadmap is reading [6].)

2. Internal rectification. **Rectification** is when something is defined up to homotopy, and we try to reduce the necessary homotopies. Suppose we have a 1-category $\mathcal{M}$, and consider $\mathrm{Fun}(\Delta^2, \mathcal{M})$ and $\mathrm{Fun}(\Delta^2, \infty(\mathcal{M}))$. The first is defined by specifying 3 objects and 3 morphisms, while the second also requires the specification of a 2-morphism. In fact, these are the same as topological spaces due to [9] 4.2.4.4. In this case, once the 1-morphisms are specified, the homotopy is defined up to a contractible space of choices, therefore forgetting it gives an equivalence.

   Internal rectification is where we do rectification, while working in the setting of ∞-categories. Example: an ∞-category with products, see it as a symmetric monoidal category with products. $Mon_{E_1}(\mathcal{C}) \simeq Fun^{\times}(\Delta^{\mathrm{op}}, \mathcal{C}) \to Fun(\Delta^{\mathrm{op}}, \mathcal{C})$. The reference is [12], 4.1.2.6.

3. Try a "real rectification" result, i.e. work with a model-categorical presentation. For example, take $\mathcal{S}$, the ∞-category of spaces, this is the Dwyer–Kan localization of the simplicial model category $sSet_{Kan}$. Suppose we wish to compute the limit (see [6] §2.5) of the functor $N(F)$, where F maps $\cdot \to \cdot \leftarrow \cdot$ to $\{x\} \to X \leftarrow \{y\}$ and N denotes the nerve functor, which takes categories to their associated ∞ categories. As ∞-categorical limits correspond to homotopy limits in the model category (by [9], Theorem 4.2.4.1), $Path_X(x, y)$ is the ∞-limit of this diagram. See [25] for more on model categories and their links to ∞-categories.

In what follows we give examples where we can get by with procedure 1.

Definition 1.2.1. An ∞-category is a simplicial set $\mathcal{C}$ such that all inner horns have fillers. In other words, for all $0 < i < n$, the dotted arrow in the following diagram exists.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Note that this achieves what we want: inner horn fillings act as composition of morphisms, but this composition is not unique. "Higher Topos Theory [9] is the book where all of category theory is carried out without ever talking about composition." A few problems arise from here:

1. How do we define Yoneda? A morphism $X \to Y$ is supposed to determine a morphism $h_X \to h_Y$ by composition, which is not well-defined.

2. Let $\mathcal{C}$ be an ∞-category. We want $f : x \to y$ in $\mathcal{C}$ to determine a functor $f_* : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ between over-categories, where, morally speaking, $g : Z \to X$ is sent to the composition $f \circ g$. Again, this composition is not well-defined.

To the rescue comes Corollary 2.4.7.12 in [9].

Theorem 1.2.2. *Let $f : \mathcal{C} \to \mathcal{D}$ be an ∞-functor between ∞-categories. Then the projection*

$$\mathcal{P} : Fun(\Delta^1, \mathcal{D}) \times_{Fun(\{1\}, \mathcal{D})} \mathcal{C} \to Fun(\{0\}, \mathcal{D})$$

is a ***cartesian fibration****. Moreover, a morphism in the source is $\mathcal{P}$-**cartesian** iff its image in $\mathcal{C}$ is an* ***equivalence****.*

Note that the ∞-functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are nothing but the internal Hom in $s\mathcal{S}et$.

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})_n = s\mathcal{S}et(\mathcal{C} \times \Delta^n, \mathcal{D})$$

It's standard to prove that, if $\mathcal{C}, \mathcal{D}$ are ∞-categories, then so is $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$.

We will spend much of section 1.3 defining the terms in bold in Theorem 1.2.2. In Example 1.3.5, we will use Theorem 1.2.2 to obtain the desired pushforward map between overcategories.

1.3 Equivalences and Cartesian fibrations

Definition 1.3.1. $g : x \to y$ in $\mathcal{C}$ is an **equivalence** if any of the following equivalent conditions hold.

(a) The map $g' : \Lambda^2_0 \to \mathcal{C}$ given by $\{1 \leftarrow 0 \to 2\} \mapsto \{y \xleftarrow{g} x \xrightarrow{1_x} x\}$ admits an extension:

$$\begin{tikzcd} \Lambda^2_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

Morally speaking, the restriction of the dotted arrow to the face 12 of Δ^2 is the right inverse of g. Moreover, the map $g'' : \Lambda^2_2 \to \mathcal{C}$ given by $\{0 \leftarrow 2 \to 1\} \mapsto \{y \xrightarrow{1_y} y \xleftarrow{g} x\}$ admits an extension:

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

Morally speaking, the restriction of the dotted arrow to the face 01 of Δ^2 is the left inverse of g.

(b) The same as variant a, but with higher homotopies included. Formally, we introduce the Kan complex S^∞, defined as 0-coskeleton of the discrete simplicial set with 2 vertices. (For more details see the exercises [16].) We say that g is equivalence if there is a lift in the following diagram.

$$\begin{tikzcd} \Delta^1 \arrow[r, "g"] \arrow[d, hook] & \mathcal{C} \\ S^\infty \arrow[ur, dotted] & \end{tikzcd}$$

(c) We say that g is an equivalence if its image in the homotopy category $h(\mathcal{C})$ is an isomorphism. [1]

In the definition, going from version b to version a of is a rectification result, in the sense of procedure 2 described above.

Next, we recall the notions of cartesian morphism and cartesian fibration in the context of 1-categories.

Definition 1.3.2. Let $\mathcal{P} : C \to D$ be a functor between 1-categories. If $x \in Ob(C)$ and $f \in \mathrm{Hom}(x, y)$, we use the notation $\bar{x} := \mathcal{P}(x)$, $\bar{f} = \mathcal{P}(f)$. In the following diagram, the first 2 rows are in C, while the third one is in D. However, we would like to think about the "square" as a pullback square.

$$\begin{tikzcd} z \arrow[ddd, dashed, "\mathcal{P}"] & & \\ & x \arrow[r, "f"] \arrow[d, dashed, "\mathcal{P}"] & y \arrow[d, dashed, "\mathcal{P}"] \\ & \bar{x} \arrow[r, "\bar{f}"] & \bar{y} \\ \bar{z} & & \end{tikzcd}$$

We say that f is a $\mathcal{P}$-**cartesian morphism** if the data of a morphism $z \to y$ in C and a morphism $\bar{z} \to \bar{x}$ in D uniquely determine a morphism $z \to x$ in C, such that the "diagram" commutes.

We say that $\mathcal{P}$ is a **cartesian fibration** if for all $y \in C$ and all $\bar{x} \xrightarrow{\bar{f}} \bar{y}$ morphism in D, $\exists f : x \to y \in \mathcal{C}$ such that $\mathcal{P}(f) = \bar{f}$ and f is $\mathcal{P}$-cartesian.

The analogous definitions for ∞-categories are the following.

[1] Recall that this is a 1-category with objects $\mathrm{Ob}(\mathcal{C})$ and morphisms $\mathrm{Hom}(x, y) = \pi_0(\mathcal{C}(x, y))$.

Definition 1.3.3. Let $\mathcal{P} : \mathcal{C} \to \mathcal{D}$ be an ∞-functor. A 1-morphism in $\mathcal{C}$, which is the same as an edge $f : \Delta^1 \to \mathcal{C}$, is $\mathcal{P}$**-cartesian** if for all $n \geq 2$, the following outer horn has a filler.

$$\begin{tikzcd}
\Delta^1 = \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\
\Lambda^n_n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\mathcal{P}"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

Morally speaking, when $n = 2$, this says that for any edge $g : z \to f(1)$ and edge $\bar{h} : \bar{z} \to \overline{f(0)}$, there exist an edge $h : z \to f(0)$ and a homotopy $g \simeq f \circ h$, such that $\mathcal{P}(h) = \bar{h}$.

We say that $\mathcal{P}$ is a **cartesian fibration** if for every edge $a : \bar{x} \to \bar{y}$ of $\mathcal{D}$, and every object y of $\mathcal{C}$ such that $\mathcal{P}(y) = \bar{y}$, there exists a $\mathcal{P}$-cartesian edge $f : x \to y$ such that $\mathcal{P}(f) = a$.

Recall that, in the study of fibered 1-categories, one proves that cartesian fibrations with base D are the same as lax 2-functors from D to the 2-category of 1-categories. (This is known as the "Grothendieck construction", see for example, Proposition I.3.26 in [4].) Explicitly, given a cartesian fibration $\mathcal{P} : C \to D$, the corresponding lax 2-functor maps an object $d \in D$ to the fiber $\mathcal{P}^{-1}(d)$. Theorem 3.2.0.1, the main theorem of Chapter 3 in [9], is the analog of this result for the setting of ∞-categories.

Theorem 1.3.4. *For any ∞-category $\mathcal{C}$, there is an equivalence of ∞-categories:*

$$\mathit{CartFib}/\mathcal{C} \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty). \tag{1.3.1}$$

Example 1.3.5. Recall that we started out by trying to construct an ∞-functor $f_* : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ between overcategories, given an 1-morphism $f : x \to y$ in $\mathcal{C}$. Taking $F : \mathcal{C} \to \mathcal{C}$ as the identity, Theorem 1.2.2 gives a Cartesian fibration over $\mathcal{C}$:

$$\big\{(f : x \to y, a) | \{f : x \to y\} \in \mathcal{C}, F(a) \cong y\big\} \to \mathcal{C},$$

where a pair $(f : x \to y, a)$ maps to x. We recognize the fiber over x as the undercategory $\mathcal{C}_{x/}$:

$$\mathrm{Hom}_{sSet}(\Delta^n, \mathcal{C}_{x/}) = \big\{\alpha : \Delta^{n+1} \to \mathcal{C} | \alpha_{\Delta^{[0,\ldots,n}} = x\big\}.$$

Theorem 1.3.4 then produces an ∞-functor:

$$\mathcal{C}^{\mathrm{op}} \to \mathcal{C}at_\infty$$
$$x \mapsto \mathcal{C}_{x/}$$
$$f : x \to y \mapsto f^* : \mathcal{C}_{y/} \to \mathcal{C}_{x/}.$$

We have obtained a pullback map on undercategories. To obtain the pushforward on overcategories, start with $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ as the contravariant identity functor instead.(Todo: we probably want co-cartesian fibrations actually)

Next, we discuss a simpler example. Let $\mathcal{C}$ be an ∞-category, and let $x \in \mathcal{C}$ be an initial object. We want to construct a functor $\mathcal{C} \to \mathcal{C}_{x/}$. Note that this is silly in 1-category theory, since there's a unique morphism $x \to y$. To aid us in the context of ∞-categories, we start by giving a good definition.

Definition 1.3.6. $x \in \mathcal{C}$ **initial** if $\forall y \in \mathcal{C}$, $Map_{\mathcal{C}}(x, y)$ is contractible.

The key result, proved, for example, in [6], is the following.

Proposition 1.3.7. *If $\mathcal{C}$ is an ∞ category, then $x \in \mathcal{C}$ is initial iff the canonical projection $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial Kan fibration.*

To solve our problem, note that $\mathcal{C}$ is cofibrant in the Kan model structure, so there exists a lift in the diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \mathcal{C}. \end{tikzcd}$$

In the exercises, we also encounter the following problem. Suppose $\mathcal{C}$ has pushouts and a zero object. Construct an ∞-functor $\mathcal{C} \to \mathcal{C}$ sending x to the pushout of 0 and 0 over x. (Todo: write this up, either here or in the exercises)

Chapter 2

Derived Affines

Talk by Benedict Morrissey.

2.1 3 perspectives on derived affines

First recall the notion of affines in classical AG: $\mathrm{Aff}_k^{Cl} \simeq (\mathcal{C}Ring)^{\mathrm{op}}$. We get schemes by gluing these together. There's also the functor of points viewpoint: $X \in \mathrm{Aff}_k^{Cl}$ defines a sheaf by sending $\mathrm{Spec}\, R \mapsto \mathrm{Hom}(\mathrm{Spec}\, R, X)$. The schemes are then precisely the sheaves in the Zariski topology. Already in classical AG, there exist constructions which move us out of this category: both Serre's intersection theorem and Illusie's notion of the cotangent complex use derived functors. So by introducing DAG, we will understand better these structures in classical AG.

We will talk about 3 approaches to derived affines — all of these consist of embedding the classical category $CRing_k$ into a larger category in which we have a derived tensor product. In this section we assume that we are working over a ring k of characteristic zero[1].

1. Simplicial commutative rings;
2. Commutative differential graded algebras. (CDGA's)
3. Lawvere theory;

Remark 2.1.1. Classically gluing is easy. For example, fiber products are computed by reducing to the affine case, where it's just the tensor product of rings. In DAG, the derived tensor product is only defined up to quasi-isomorphism, so gluing can only be defined in a category which allows homotopy, such as an ∞-category. For today's talk we mostly use the model category description; an application of Dwyer-Kan localization produces an ∞-category.

2.1.1 Simplicial Commutative Rings

For approach 1, recall that the simplicial category Δ is:

$$Ob(\Delta) = \{n \in \mathbb{N} \cup \{0\}\}$$

where morphisms are compositions of face maps $\delta_i^n : [n-1] \to [n]$ for $0 \leq i < n$ and degeneracy maps $s_i^n : [n+1] \to [n]$, subject to the simplicial identities as can be found in e.g. [5].

[1] We note here that we neglect to mention the important generalization of rings given by E-∞ ring spectra, as described in chapter 7 of [12]. In the case where we are working over k a $\mathbb{Q}$-algebra this infinity category is equivalent to those described in this section as shown in [**?**] proposition 4.1.11. When we remove the characteristic zero assumption the statements about the Model structure on CDGA's no longer hold. One can still use simplicial commutative rings or E-∞ algebras, though these give different ∞ categories.

Definition 2.1.2. The **category of simplicial commutative rings** is the category of contravariant functors:

$$SCR_k = \mathrm{Hom}(\Delta^{\mathrm{op}}, CRing_k).$$

Remark 2.1.3. There's a model category structure on this: fibrations are Kan fibrations on the underlying simplicial sets, i.e. morphisms $f : A \to B$ of simplicial commutative rings, such that all horns have fillers:

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

Weak equivalences are weak homotopy equivalences on the underlying simplicial sets. Cofibrations are then determined from the axioms of a model category; note that they are *not* the same as cofibrations of the underlying simplicial sets.

Remark 2.1.4. We're using transfer to put the model structure on SCR_k. To explain what that means, under suitable conditions, there's a general procedure for defining a model structure on a category $\mathcal{B}$, given a model category $\mathcal{A}$ and an adjoint functor pair:

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}.$$

The procedure forces the adjoint functor pair to be a Quillen adjunction. In our case, we use the free-forgetful adjunction:

$$s\mathcal{S}et \underset{U}{\overset{F}{\rightleftarrows}} SCR_k$$

to transfer the Kan model structure to SCR_k. The key point which allows this to work is that all objects are fibrant. Cofibrations are more difficult to characterize, but the cofibrant objects are precisely the quasi-free ones. (That is, the ones isomorphic to a free object.)

2.1.2 CDGA's

Next, we introduce CDGA's and the Dold-Kan equivalence — which shows that this category is the same as that of simplicial commutative rings under our assumption that we are working over a characteristic zero field. Recall that we have a Quillen equivalence:

$$s\mathcal{V}ect \rightleftarrows dg - \mathcal{V}ect^{\leq 0}$$

between simplicial vector spaces and differential graded vector spaces, concentrated in nonpositive degrees. We want to talk about commutative monoids in these categories, $sc\mathcal{A}lg_k$ and $cdg - \mathcal{A}lg^{\leq 0}$, respectively. The model structure on $cdg - \mathcal{A}lg^{\leq 0}$ can also be obtained by transfer from the free-forgetful adjunction; we obtain that the weak equivalences are quasi-isomorphisms, and the fibrations are degree-wise surjections.

Theorem 2.1.5 (Symmetric monoidal Dold-Kan (A proof can be found in [19]))**.** *There is a Quillen equivalence:*[2]

$$sc\mathcal{A}lg_k \underset{\Gamma}{\overset{N}{\rightleftarrows}} cdg - \mathcal{A}lg_k^{\leq 0}.$$

Moreover, if the simplicial commutative algebra A_* corresponds to the commutative dg-algebra $B_\bullet$, then $\pi_i(A_*) \cong H^i(B_\bullet)$.

[2] Note that, in general, a Quillen equivalence is not an equivalence of categories. It does, however, induce an equivalence of Dwyer-Kan localizations (and hence also of homotopy categories).

Remark 2.1.6. We describe N: $A_* \in sc\mathcal{A}lg_k$ maps in the first stage to $\tilde{A}_\bullet$, where $\tilde{A}_{-n} = A_n$, and the differential is the alternating sum of the face maps. $N(A_*)$ is then the quotient of $\tilde{A}_\bullet$ by the images of the degeneracy maps. For C a CDGA we can describe $(\Gamma C)_n := Hom_{ch^-}(N(\Delta^n), C)$ where we are in fact using the above definition of N to give a functor from simplicial abelian groups to ch^- — the category of non positively graded chain complexes, and Δ^n is the simplicial abelian group freely generated by an n-simplex.

We have a similar result for the category of simplicial modules for a given simplicial ring, and the category of dg-modules for its image in CDGA's. Note that the categories of simplicial modules and of non-positively graded modules for a given CDGA both have model structures.

Theorem 2.1.7 ([19])**.** *If A is a simplicial ring the categories of simplicial A-modules and of negatively graded $N(A)$-modules are Quillen equivalent.*

If A is a CDGA, the categories of negatively graded A-modules and of simplicial $\Gamma(A)$-modules are equivalent.

We now define the truncation functor of a CDGA. We can use the above Quillen equivalence to also define truncation functors on the category of Simplicial Commutative Rings.

Let $CDGA_k^{\leq n} \hookrightarrow CDGA_k$ denote the subcategory of $CDGA_k$ consisting of objects A such that $H^i(A) = 0$ for all $i > n$. Note that $CDGA_k^{\leq 0} \cong CRing_k$. The inclusion has a right adjoing $\tau^{\leq n}$: $CDGA_k \to CDGA^{\leq n}$. For $A = (A_n)$,

$$(\tau^{\leq n}(A))_m = \begin{cases} A_m & 0 > m > -n \\ A/im(d^{m+1}) & m = n \\ 0 & m < n. \end{cases}$$

2.1.3 Lawvere Theories

We move on to approach 3 to derived affines, the Lawvere Theory description. This is important because it's the only one of the 3 procedures which carries through in the analytic setting (see e.g. [18]).

The idea of Lawvere theory is to describe all objects with some type of algebraic structure as functors between the free objects and the category Set, or in the ∞-categorical case $\mathcal{S}$ — the infinity category of (topological) spaces.

Example 2.1.8. Let Ab be the category of abelian groups, and let FAb be the category of free abelian groups. There is an equivalence of categories

$$Ab \cong \mathrm{Fun}^{\times}(FAb^{\mathrm{op}}, Set).$$

To an abelian group G, we associate the functor $Hom(-, G)$. Note that we have an equality $Hom(\mathbb{Z}, G) =_{Sets} G$ (considering $\mathbb{Z}$ as an additive group), by mapping a homomorphism $f : \mathbb{Z} \to G$ to $f(1)$. Furthermore $Hom(\mathbb{Z} \times \mathbb{Z}, G) = G \times G$. Let $f \in Hom(\mathbb{Z} \times \mathbb{Z}, G) = G \times G$ map $(1, 0)$ and $(0, 1)$ to a and b respectively. Precomposing with the map $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ given by $1 \mapsto (1, 1)$ gives the map $Z \to G$ which maps $1 \mapsto ab$.

For a functor F, we can associate an abelian group as follows. There's a map $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, which sends $1 \mapsto 1 \times 1$. Since F preserves products, we have a map $F(\mathbb{Z}) \times F(\mathbb{Z}) \cong F(\mathbb{Z} \times \mathbb{Z}) \to F(\mathbb{Z})$. This endows $F(\mathbb{Z})$ with the structure of an abelian (by definition of the multiplication) group.

To a morphism of groups $G \to H$ we gain a natural transformation of functors $Hom(-, G) \to Hom(-, H)$ by composing an element of $Hom(-, G)$ with this morphism. A natural transformation of functors $F_1 \to F_2$ gives a map $F_1(Z) \to F_2(Z)$ such that the diagram

$$\begin{array}{ccc} F_1(\mathbb{Z} \times \mathbb{Z}) & \longrightarrow & F_1(\mathbb{Z}) \\ \downarrow & & \downarrow \\ F_2(\mathbb{Z} \times \mathbb{Z}) & \longrightarrow & F_2(\mathbb{Z}) \end{array}$$

commutes. This shows that the map $F_1(Z) \to F_2(Z)$ is a group homomorphism.

We denote by T_{disc} the opposite category of free commutative rings over k. Free commutative rings (over k) are the rings $k[x_1, \dots, x_n]$. Hence T_{disc} is the subcategory of the category of affine schemes with objects the planes $\{\mathbb{A}^n\}$. We denote by $CRing$ the category of commutative rings.

Proposition 2.1.9. *There is an equivalence of categories:*

$$\mathrm{Fun}^{\times}(T_{disc}, Set) \cong CRing.$$

On objects we map a functor to it's value on the group ring $\mathbb{A}^1$, $F \mapsto F(\mathbb{A}^1)$. The inverse map is given by taking a ring R to the functor $Hom_{CRing}(-, R)$. The (commutative) addition and multiplication on $\mathbb{A}^1$ ($\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ given on points by $(x, y) \mapsto x + y$ and $(x, y) \mapsto xy$ respectively) give $F(\mathbb{A}^1)$ the structure of a (commutative) ring.

Now pass to

$$SCR_k \cong s\,\mathrm{Fun}^{\times}(T_{disc}, Set) \cong \mathrm{Fun}^{\times}(T_{disc}, sSet) \cong \mathrm{Fun}^{\times}(T_{disc}, S),$$

where S is the infinity category of spaces. The last step is a very hard rectification theorem, proved by Lurie-Bergner.[3]

2.2 Derived Affines as Ringed Spaces

Finally, we take the viewpoint of seeing a scheme as a locally ringed space. For $A \in cdg - \mathcal{A}lg_k$, we look at the truncation $\mathrm{Spec}\, H^0(A)$, which is an affine scheme in the classical sense. We can regard A as a sheaf of cdg-algebras on the truncation, as long as we can understand how localization works for cdg-algebras. We claim that it suffices to localize the commutative algebra A_0. Indeed, we have the multiplication map:

$$\mu : A_0 \times A_i \to A_i,$$

so given a multiplicative subset $S \subset A_0$, we define the localization $S^{-1}A_i$ as $\mu(S^{-1}A_0 \times A_i)$. If this makes sense, we get a sheaf $\mathcal{O}_A$ of cdg-algebras.

We would like to define derived affines as pairs $\big(\mathrm{Spec}\, H^0(A), \mathcal{O}_A\big)$. There is a subtlety: a priori this only gives a 2-category, and we need ∞-categories. The key to resolving this is to define the notion of a sheaf valued in an $(\infty, 1)$-category.

2.3 Our favorite classes of morphisms

Definition 2.3.1. Given $f : A \to B$ in SCR_k, we get maps:

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$$

of graded modules. We say that f is **strong** if this is an isomorphism of graded modules.

Definition 2.3.2. We define $f : A \to B$ to be **étale** (resp. **smooth**, **Zariski open immersion**, **flat**) if f is strong and $\pi_0(A) \to \pi_0(B)$ is étale (resp. smooth, Zariski open immersion, flat) in the classical sense.

Remark 2.3.3. The strength condition on f is quite restrictive: for example, a strong map from a non-derived domain must have a non-derived target.

Definition 2.3.4. Let $X = \mathrm{Spec}(A)$ be a derived affine over k. Then the **small étale site** of X is:

$$X_{\text{ét}} = \{\text{étale maps } \mathrm{Spec}(B) \to \mathrm{Spec}(A)\}.$$

[3] HTT Propositions 5.5.9.2

In order to obtain the small étale site in the sense of classical AG, one needs to pass to the truncated version of the étale maps: $\pi_0(f) : \mathrm{Spec}(\pi_0(B)) \to \mathrm{Spec}(\pi_0(A))$. Then one can prove there's an equivalence of ∞-categories between the derived and classical étale sites. In particular, this shows that $X_{\text{ét}}$ is a 1-category. This is one of the ingredients in the proof of the easy version of Lurie representability. Moreover, the same holds for the small smooth site and the small Zariski site.

After introducing the cotangent complex $\mathbb{L}_f$ of a morphism f, we will see that f is étale iff $\pi_0(f)$ is of finite presentation and $\mathbb{L}_f \simeq 0$.

Definition 2.3.5. $f : A \to B$ is **of finite presentation** if the functor $\mathrm{Map}_A(B, -) : sc\mathcal{R}ing_k \to \mathcal{S}$ commutes with filtered colimits.

Unlike in the underived case, being of finite presentation is very strong, because it has a hidden regularity condition. In particular, we have the proposition due to Lurie:

Proposition 2.3.6. *$f : A \to B$ is of finite presentation in the derived sense iff $\pi_0(f)$ is of finite presentation in the classical sense (also called to order 0) and the cotangent complex $\mathbb{L}_f$ is perfect.*

Example 2.3.7. Let $X = \mathbb{A}^3$, and Y a closed subscheme of X which is not a local complete intersection. Then the inclusion $\iota : Y \to X$ is not of finite presentation in the derived sense. Indeed, by a conjecture of Quillen, which is now a theorem of Abramov, for maps between classical schemes, the cotangent complex is either concentrated in degrees 0 and -1, or it's unbounded. Since Y is not lci, the first case is ruled out, and $\mathbb{L}_\iota$ is unbounded.

Chapter 3

Stable ∞-categories

Talk by Michael Gerapetritis.

3.1 Motivation

In the 1-categorical setting, if $\mathcal{C}$ is a category, we may require that $\mathcal{C}(A, B)$ be a set. To get particularly well-behaved categories, namely the additive categories, we require that $\mathcal{C}(A, B)$ is actually an abelian group.

We try to replicate this in the ∞-category setting. Let $\mathcal{C}$ be an ∞-category, then $\mathcal{C}(X, Y)$ is a space. We want to discover what is the good extra structure to have on this space; we will call the corresponding ∞-categories stable.

3.2 Stable ∞-categories and triangulated 1-categories

Definition 3.2.1. An ∞-category $\mathcal{C}$ is **stable** if:

- $\mathcal{C}$ is pointed, i.e. it has a zero object;
- every morphism $f : X \to Y$ admits fibers and cofibers;
- a triangle is a fiber iff it is a cofiber.

Recall that a **triangle** in $\mathcal{C}$ is a map of simplicial sets $\Delta^1 \times \Delta^1 \to \mathcal{C}$, i.e. a homotopy commutative diagram with the zero object in the bottom-left corner:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

The triangle is a **fiber** if it is a pullback square, and a **cofiber** if it is a pushout square. We say that $f : X \to Y$ admits a fiber (resp. cofiber) when $\exists W$ (resp Z) such that:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

is a pullback square (or, respectively:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a pushout square).

Remark 3.2.2. Note that the data of a triangle consists not only of homotopy commutative diagrams as above, but also of choices of homotopies between the branches. This is crucial, since it ensures that cones are functorial at the level of the homotopy category. This functoriality does not hold in a general triangulated category. (See Theorem 3.2.5 for the relation between stable ∞-categories and triangulated 1-categories.)

Example 3.2.3. Our two main examples are ∞-categories of spectra (see Section 3.5) and of modules over a CDGA or SCR (see Section 3.3).

Recall the data for a triangulated category.

Definition 3.2.4. A category $\mathcal{D}$ is triangulated if:

1. $\mathcal{D}$ is additive;
2. $\mathcal{D}$ admits a translation functor $T : \mathcal{D} \xrightarrow{\simeq} \mathcal{D}$;
3. $\mathcal{D}$ has a collection of distinguished triangles:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

This data is required to satisfy some axioms, but we won't go into details here.

Theorem 3.2.5. *If $\mathcal{C}$ is a stable ∞-category, then $h\mathcal{C}$ is triangulated.*

For a proof see [12]. We won't go over it, let's just say that translation is given by Σ, and distinguished triangles are precisely the images of fiber sequences (or equivalently, cofiber sequences), as resulting from the following diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Proposition 3.2.6. *$\mathcal{C}$ is stable iff the following hold:*

1. *$\mathcal{C}$ admits finite limits and colimits;*
2. *any square is a pushout iff it is a pullback.*

Proof. Again, we don't give a full proof. Let's just see why products and coproducts must exist in a stable ∞-category. Note first that Σ is an equivalence of ∞-categories. Indeed, Σ is a left adjoint functor; moreover, the unit and counit of the adjunction become isomorphisms in the homotopy category, due to condition 2 in the definition of a triangulated category. Then we use the following diagram.

$$\begin{tikzcd} \Omega(X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & X \arrow[r] & X \oplus Y \end{tikzcd}$$

We have defined $X \oplus Y$ as the cofiber of $\Omega(X) \xrightarrow{0} Y$, which is postulated to exist in a stable ∞ -category. This turns the outer rectangle into a pushout square, and it follows that the square on the right is also a pushout square. Thus $X \oplus Y$ is the coproduct of X and Y. We reason dually to obtain products. $\square$

Definition 3.2.7. Let $\mathcal{C}, \mathcal{C}'$ be stable ∞-categories, and $F : \mathcal{C} \to \mathcal{C}'$ an ∞-functor which maps 0 objects to 0 objects. Equivalently, F maps triangles to triangles. If F maps fiber sequences to fiber sequences, we say that F is **exact**.

Lemma 3.2.8. *TFAE:*

1. *F is exact;*
2. *F is right-exact, i.e. commutes with finite colimits;*
3. *F is left-exact, i.e. commutes with finite limits.*

This is very useful: sometimes it's really easy to check that a functor is right or left exact, e.g. if it's a left or right adjoint, respectively.

3.3 Modules

For a useful example of the result in Lemma 3.2.8, we look at $\mathcal{C} = A - \mathcal{M}od$, where A is a CDGA or SCR over k. (By $A - \mathcal{M}od$ we mean the unbounded derived category.) The easiest way to see $A - \mathcal{M}od$ as an ∞-category is to put a model structure on chain complexes, say the projective one, and then take the underlying ∞-category. We claim that $A - \mathcal{M}od$ is a stable ∞-category. Using the theorem Mauro talked about in Lecture 1, limits and colimits exists in the ∞-category iff they exist in the model category. (Todo: reference theorem) It remains to prove the following.

Lemma 3.3.1. *A triangle in $A - \mathcal{M}od$ is a fiber iff it is a cofiber.*

Proof. We prove one direction; the other argument is dual to this one. Assume that $f : M^\bullet \to N^\bullet$ is the fiber of a map g. Take a cofibrant replacement of f, get $\tilde{M}, \tilde{N}$ cofibrant and a homotopy pullback square: (Todo: figure out how to do the cartesian symbol in tikz)

$$\begin{tikzcd} \tilde{M}^\bullet \arrow[r, hook, "\tilde{f}"] \arrow[d] & \tilde{N}^\bullet \arrow[d, "\tilde{g}"] \\ 0 \arrow[r] & P^\bullet. \end{tikzcd}$$

$\tilde{f}$ is cofibrant, so it's a degree-wise injection. Then g is a degreewise surjection, and it follows that the square is a strict pushout. (Todo: wait, how did this work again?) □

Now suppose we have $f : A \to B$ a morphism of $CDGA_k^{\leq 0}$. It induces the adjunction of model categories:

$$A - Mod \underset{f_*}{\overset{f^*}{\rightleftarrows}} B - Mod,$$

where f_* is the forgetful functor, and $f^*(M) = M \otimes_A B$. So this gives an adjunction of ∞-categories: [1]

$$A - \mathcal{M}od \underset{Rf_*}{\overset{Lf^*}{\rightleftarrows}} B - \mathcal{M}od.$$

Explicitly, Lf^* is constructed by first choosing a cofibrant replacement $\tilde{M}$ for M, and then taking $\tilde{M} \otimes_A B$. The answer doesn't depend on cofibrant replacement, up to coherent isomorphism. Then Lf^* is a left adjoint functor, so it follows from general nonsense that it's right exact. Lemma 3.2.8 then implies that Lf^* is also left exact and exact.

Remark 3.3.2. If f is not flat in the sense of Definition 2.3.2, then the exactness of Lf^* comes at the price of losing t-exactness. To explain what we mean, pick $M \in A - Mod$, such that $H^i(M) = 0$ unless $i = 0$. But then $Lf^*(M) = M \otimes_A^{\mathbb{L}} B$, and $H^{-i}(M \otimes_A^{\mathbb{L}} B) = \mathrm{Tor}_i^A(M, B)$, which is $\neq 0$ in general, because f is not flat. So even though M was homologically concentrated in degree 0, $Lf^*(M)$ may not be. In other words, the failure of a functor of (Grothendieck) abelian categories to preserve limits translates into a lack of t-exactness of the derived functor. In the following section we define t-structures and t-exactness for ∞-categories.

[1] Here we use L and R to indicate that the functors are derived. In later talks derived functors will be the default, and we will omit the symbols L and R.

3.4 t-structures

Definition 3.4.1. If $\mathcal{C}$ is a stable ∞-category, a **t-structure**[2] on $\mathcal{C}$ is the data of two full subcategories of $\mathcal{C}$, $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$, [3] such that:

1. $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y[-1]) = 0$ if $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 0}$. [4]
2. $X \in \mathcal{C}^{\leq 0}, X[1] \in \mathcal{C}^{\leq 0}$;
3. $\forall X$, $\exists$ fiber sequence $X' \to X \to X''$, where $X' \in \mathcal{C}^{\leq 0}$, $X'' \in \mathcal{C}^{\geq 1}$.

Remark 3.4.2. Condition 1 has the following intuitive meaning in the case $\mathcal{C} = A - \mathcal{M}od$. 0-morphisms in $\mathcal{C}$ are chain maps which preserve degree, while higher morphisms are homotopies which shift the degree to the left; morphisms that shift degree to the right are not allowed. Then, if $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 0}$, no nonzero morphisms should be allowed between X and $Y[-1]$:

$$\ldots \longrightarrow X_{-2} \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

$$\ldots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \ldots$$

Remark 3.4.3. X' and X'' are uniquely determined by X.

Theorem 3.4.4. *The inclusion $\mathcal{C}^{\leq 0} \to \mathcal{C}$ has a right adjoint, which we denote $\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}^{\leq 0}$. Similarly we get $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}^{\geq 0}$.*

Corollary 3.4.5. *For all $X \in \mathcal{C}$, the fiber sequence of 3 is just:*

$$\tau_{\leq 0} X \to X \to \tau_{\geq 1} X.$$

Proposition 3.4.6. *Denote by $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$, the **heart** or **core** of the t-structure. It is an abelian 1-category.*

Proposition 3.4.7. *Let $\mathcal{C}$ be stable. Then if:*

$$X \to Y \to Z$$

is a fiber sequence, then we have a long exact sequence of H^i, where $H^i(X) := \tau_{\geq i} \circ \tau_{\leq i}(X)$.

Putting the last few results together, from $\mathcal{C}$ a presentable stable ∞-category with t-structure, the heart is Grothendieck abelian. Write $A = \mathcal{C}^{\heartsuit}$. Then we can form $\mathcal{D}(A)$, the ∞-derived category of A. The next theorem describes the relationship between $\mathcal{C}$ and $\mathcal{D}(A)$.

Theorem 3.4.8 (Lurie)**.** *$\mathcal{D}(A)$ has a universal property which produces an ∞-functor:*

$$\mathcal{D}(A) \to \mathcal{C}.$$

In general this is very far from being an equivalence.

Example 3.4.9. Let $A \in CDGA_k^{\leq 0}$. The theorem gives a map:

$$(A - \mathcal{M}od)^{\heartsuit} \to (H^0(A) - Mod)^{\heartsuit}. \tag{3.4.1}$$

This is one of the most important facts in DAG, because it reduces problems about the ∞-category of A-modules to problems in classical categories of modules, where one can work with generators and relations. The map in 3.4.1 is an equivalence iff $A \simeq H^0(A)$ are quasi-isomorphic.(Todo: figure out what's the precise relationship here)

[2] t stands for truncation

[3] Note that we use cohomological notation, while Lurie in [12] uses homological notation. Therefore gradings have opposite signs in this seminar and in [12].

[4] In a stable ∞-category, we sometimes use the shift notation $[n]$ to denote the $|n|$-fold iterated application of the Σ functor (if n is positive) or the Ω functor (if n is negative). This notation is justified by Proposition 3.5.4.

Definition 3.4.10. Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories with t-structures. Then an exact functor $F : \mathcal{C} \to \mathcal{D}$ is:

1. **left t-exact** if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$;
2. **right t-exact** if $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$;
3. **t-exact** if both.

Example 3.4.11. For $A, B \in CDGA_k^{\leq 0}$, $f : A \to B$, we have the adjunction:

$$A - \mathcal{M}od \underset{Rf_*}{\overset{Lf^*}{\rightleftarrows}} B - \mathcal{M}od.$$

Every object is fibrant, so we don't need to derive the functors. Rf_* is both left and right t-exact. Lf^* is not right t-exact, because of nontrivial Tor^i terms; see 3.3.2. However, Lf^* is right t-exact: morally speaking, Projective resolution only puts stuff in negative degrees. We give an ∞-categorical proof.

Pick $M \in A - Mod^{\geq 0}$. We want $Lf^*(M) \in B - Mod^{\leq 0}$. To check this is the same as checking that $\forall N \in B - Mod^{\geq 1}$, $\mathrm{Map}_{B-Mod}(Lf^*M, N) \cong 0$. But this is $Map_{A-Mod}(M, Rf_*N) \cong 0$, which follows since Rf_* was t-exact.

3.5 Spectra

Going back to the question left unanswered in Section 3.1, the extra structure we want on morphism spaces of stable ∞-categories is $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathrm{Sp}^{\leq 0}$.

Definition 3.5.1. **Spectra** are sequences $\{F_i\}$ of objects in $\mathcal{C}$ such that $F_n \simeq \Omega F_{n+1}$. Alternatively, we identify them with objects of the homotopy limit:

$$\ldots \overset{\Omega}{\to} \mathcal{C} \overset{\Omega}{\to} \mathcal{C} \overset{\Omega}{\to} \ldots$$

Remark 3.5.2. We must be careful with defining morphisms between spectra: we want squares to commute up to coherent homotopy. Moreover, it's hard to get a monoidal model structure on the category of spectra: this was done only in the 2000s, after Hovey introduced symmetric spectra. Lurie has a very categorical and very nice way of putting a monoidal structure at the level of the ∞-category directly. See the last chapter of [6], and also 4.8.2 of [12].

Theorem 3.5.3. *$Sp(\mathcal{C})$ is stable.*

This gives a canonical stabilization for every ∞-category. The proof of the theorem follows from the following characterization of stable ∞-categories, and the fact that $\Omega : Sp(\mathcal{C}) \to Sp(\mathcal{C})$ is an equivalence.

Proposition 3.5.4. *$\mathcal{C}$ is a pointed ∞-category. TFAE:*

1. *$\mathcal{C}$ is stable;*
2. *$\mathcal{C}$ admits colimits and $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence;*
3. *$\mathcal{C}$ admits limits and $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence;*

Chapter 4

The Cotangent Complex

Talk by Sukjoo Lee.

4.1 Motivation

We recall from classical AG: if $f : A \to B$ is a homomorphism between commutative rings and M is a B-module, an A-**derivation** of B into M is a map $d : B \to M$ such that:

- $d\big(f(a)\big) = 0$, for all $a \in A$;
- $d(bb') = db\, b' + b\, db'$ (Leibniz rule).

We denote by $Der_A(B, M)$ the set of all derivations of B into M. There is also an absolute version, where we take $f : 0 \to A$, and the first condition is automatic.

Definition 4.1.1. The **module of relative Kähler differentials** of B over A is a derivation $(\Omega^1_{B/A}, d_A)$ over A satisfying the universal property:

$$\begin{tikzcd} B \arrow[r, "d_A"] \arrow[dr, "d'"'] & \Omega^1_{B/A} \arrow[d, dotted, "\exists!"] \\ & M. \end{tikzcd}$$

Equivalently, $\mathrm{Hom}_{B-Mod}(\Omega^1_{B/A}, M) \simeq Der_A(B, M)$. (+ absolute version).

Proposition 4.1.2. *If $A \to B \to C$ is a sequence of maps of commutative rings, then the following sequence of C-modules is exact:*

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0. \tag{4.1.1}$$

One of the goals for this talk is to extend the sequence to the left. If $CRing$ was an Abelian category, we would attempt to derive the functor Ω^1; however, this is not the case. Instead, what we do is generalize the notion of Kähler differential to the ∞-categorical setting, and show that this gives an extension to the left of the sequence 4.1.1. Slogan: "∞-category theory allows us to do derived functors in a non-linear setting".

4.2 Generalization and definition

Note that generalizing the Leibniz rule to the ∞-category setting is hard, because we'd have to replace the equality with a homotopy. Instead, consider the following idea. For a ring homomorphism $\phi : A \to B$, we want a new homomorphism $\phi' : A \to B$ "sufficiently close" to ϕ. For example, take $I \subset B$ an ideal with $I^2 = 0$. Then "sufficiently close" means that $\phi' : A \to B$ is congruent to ϕ modulo I, i.e.:

$$\forall\, a \in A,\ \phi(a) - \phi'(a) \in I.$$

For a fixed ϕ, we have a bijective correspondence:

$$\left\{ \begin{array}{c} \phi' : A \to B \text{ such that} \\ \phi' \equiv \phi \bmod I \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} d : A \to I \text{ satisfying} \\ \text{the Leibniz rule} \end{array} \right\}. \tag{4.2.1}$$

If M an A-module, take $B := A \oplus M$ equipped with the ring structure such that $M^2 = 0$:

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 \dot{m}_2 + a_2 \dot{m}_1).$$

We fix $\phi : A \to B$ the natural inclusion of A into the coproduct (as abelian groups); with the given ring structure, ϕ is also a ring homomorphism. Take the ideal $I = M$. Then the correspondence 4.2.1 becomes:

$$\mathrm{Map}_{/A}(A, A \oplus M) \cong Der(A, M). \tag{4.2.2}$$

This is something we can generalize. We work with $A \in SCR_k$, and the category of $A - \mathcal{M}od$; all categories in the rest of the talk are ∞-categories. Take $M \in A - \mathcal{M}od$ and construct $A \oplus M$, whose underlying simplicial set is the coproduct, and whose ring structure is defined levelwise (see [22], 1.2.1.1 for details).

We adapt equation 4.2.2 to this setting, by defining:

$$\mathcal{D}er(A, M) = \mathrm{Map}_{/A}(A, A \oplus M) \in \mathcal{S}.$$

Moving from the absolute version of derivations to the relative one, for $f : A \to B$ in SCR_k and $M \in B - \mathcal{M}od$, we define:

$$\mathcal{D}er_A(B, M) = \mathrm{Map}_{A-Alg/B}(B, B \oplus M) \in \mathcal{S}.$$

We obtain functors $\mathcal{D}er(A, -) : A - Mod \to \mathcal{S}$ and $\mathcal{D}er_A(B, -) : B - Mod \to \mathcal{S}$. We claim that these functors are corepresentable, and call the corepresenting objects the **absolute cotangent complex** $\mathbb{L}_A$ and **relative cotangent complex** $\mathbb{L}_{B/A}$, respectively. Equivalently, this means:

$$\mathrm{Map}_{A-Mod}(\mathbb{L}_A, M) \cong \mathrm{Map}_{/A}(A, A \oplus M),$$
$$\mathrm{Map}_{B-Mod}(\mathbb{L}_{B/A}, M) \cong \mathrm{Map}_{A-Alg/B}(B, B \oplus M).$$

The proof of corepresentability relies on the following lemma.

Lemma 4.2.1.

$Der_A(B, -)$ and $Der(A, -)$ commute with limits and K-filtered colimits.[1] *(Todo: explain the footnote better)*

Then the result follows by Theorem 5.5.2.7 in [9], which we reproduce here.[2]

Theorem 4.2.2. *Let $\mathcal{C}$ be a presentable ∞-category and $F : \mathcal{C} \to \mathcal{S}$ a functor. Then F is corepresentable by an object of $\mathcal{C}$ if and only if F preserves K-filtered colimits and all small limits.*

[1] ω-filtered would mean that the representing guy can be given by presentation with gen and relation. Otherwise we just mean for everything strictly less than cardinality K, which could be continuum or more.

[2] Related to this is the Adjoint Functor Theorem 5.5.2.9, which is one of the most important theorems in ∞-category theory. It's also the reason we love presentable ∞-categories.

Remark 4.2.3. Note that preserving small limits is obviously necessary in order to be corepresentable, since Hom is a right adjoint, and thus preserves small limits.(Todo: think more about the small hypothesis) Let's explain this in more detail for 1-category theory. We have the diagram:

$$\begin{tikzcd} A - Mod \arrow[r, "\simeq"] & Ab(CRing_{/A}) \arrow[d, "\text{forget}"] & \\ & CRing_{/A} \arrow[r, "\mathrm{Hom}(A,-)"] & Set. \end{tikzcd}$$

The equivalence on the first line works by sending an A-module M to $A \oplus M$, and a ring B over A to the kernel of $B \to A$. (Todo: finish this)

For ∞-category theory, the relevant diagram is:

$$\begin{tikzcd} A - \mathcal{M}od \arrow[r, "\simeq"] & \mathcal{S}p(SCR_{/A}) \arrow[d, "\text{forget}"] & \\ & SCR_{/A} \arrow[r, "\mathrm{Map}_{/A}(A,-)"] & \mathcal{S}. \end{tikzcd}$$

The reference for this is [12], section 7.4.(Todo: wait, where exactly? can't find it)

In [22], Chapter 1, we find an explicit (although not useful in practice, according to Mauro) model for $\mathbb{L}_A$. Take a simplicial resolution $\tilde{A} \to A$, which is also a cofibrant replacement. Then we have:

$$\Omega^1_{\tilde{A}} \otimes^{\mathbb{L}}_{\tilde{A}} A \in A - Mod$$

is a model for $\mathbb{L}$, where the complex $\Omega^1_{\tilde{A}}$ is build by taking Kähler differentials degree-wise:

$$(\Omega^1_{\tilde{A}})_\bullet := \Omega^1_{\tilde{A}_\bullet}.$$

4.3 Examples and Properties

In this section we compute $\mathbb{L}_{k[x]/k}$ and $\mathbb{L}_{k/k[x]}$, where x is in degree -1, as opposed to 1, by our cohomological convention. (See remark 4.3.2 for what this implies; $k[x]$ is not what it seems.) In the process we go over some of the properties of cotangent complexes.

Lemma 4.3.1. *Let $A \in SCR_k$ and $M \in A - \mathcal{M}od$. The cotangent complex of $f : A \to \mathrm{Sym}_A M$ is:*

$$\mathbb{L}_{\mathrm{Sym}_A M/A} \cong M \otimes^{\mathbb{L}}_A \mathrm{Sym}_A M.$$

Proof. For all $\mathrm{Sym}_A M$-module N,

$$\begin{aligned} \mathrm{Map}_{\mathrm{Sym}_A M-\mathcal{M}od}(\mathbb{L}_{\mathrm{Sym}_A M/A}, N) &\cong \mathrm{Map}_{A-\mathcal{M}od}(\mathrm{Sym}, \mathrm{Sym} \oplus N) \cong \mathrm{Map}_{A-\mathcal{M}od}(M, f_* N) \\ &\cong \mathrm{Map}_{\mathrm{Sym}_A M-\mathcal{M}od}(M \otimes^{\mathbb{L}}_A \mathrm{Sym}_A M, N). \end{aligned}$$

Here the first equivalence is definitional, the second follows from the universal property of $\mathrm{Sym}_A M$, and the third is the adjunction 3.3. □

For our first example, note that $k[x] \cong \mathrm{Sym}_k(k[1])$. Then the answer is $k[1] \otimes^{\mathbb{L}}_k k[x]$, which is just $k[x]$ concentrated in degree -1.

Remark 4.3.2. Note that, since $k[1]$ is concentrated in degree -1, so is $\mathrm{Sym}_k(k[1])$; it does not have information in all nonnegative degrees, as the notation may mislead one into thinking. We just get a copy of k in degree 0 and one in degree -1, and this is what we call $k[x]$. This is because multiplication in the symmetric algebra is graded commutative:

$$xy = (-1)^{|x||y|} yx,$$

so in particular for x of degree 1 we get $x^2 = -x^2 = 0$. If we started with $k[2]$ instead, $\mathrm{Sym}_k(k[2])$ would be nontrivial in all negative even degrees and commutative in the classical sense. More generally, $\mathrm{Sym}(k[n])$ gives what we would classically call a symmetric algebra if n is even, or an alternating algebra if n is odd.

Some properties of cotangent complex:

Proposition 4.3.3. *1. For $A \to B \to C$ in SCR_k, there is a homotpy cofiber sequence in $C - Mod$:*

$$\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C \to \mathbb{L}_{C/A} \to \mathbb{L}_{C/B}.$$

2. Base change: given a homotopy pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ A' \arrow[r] & B', \end{tikzcd}$$

there is an equivalence $\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B B' \simeq \mathbb{L}_{B'/A'}$.

To compute $\mathbb{L}_{k/k[x]}$, use the cofiber sequence associated to the sequence of maps $k \to k[x] \to k$. We get the cofiber sequence in $k - \mathcal{M}od$:

$$\mathbb{L}_{k[x]/k} \otimes^{\mathbb{L}}_{k[x]} k \to \mathbb{L}_{k/k} \to \mathbb{L}_{k/k[x]}. \tag{4.3.1}$$

By our previous computation, the first term is:

$$\mathbb{L}_{k[x]/k} \otimes^{\mathbb{L}}_{k[x]} k \simeq k[1] \otimes^{\mathbb{L}}_k k[x] \otimes^{\mathbb{L}}_{k[x]} k \simeq k[1].$$

(Using associativity for derived tensor product.) The second term in 4.3.1 is 0, so the cofiber sequence is actually a suspension diagram.

$$\begin{tikzcd} k[1] \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathbb{L}_{k/k[x]} \end{tikzcd}$$

Then $\mathbb{L}_{k/k[x]} \simeq k[2]$.

Going back to Section 4.1, we complete the exact sequence 4.1.1 on the left. Using stability of $C - \mathcal{M}od$, the cofiber sequence gives a long exact sequence on homology (recall proposition 3.4.7; in particular, $H^i(X) = \tau^{\geq i} \circ \tau^{\leq i}(X)$).

$$\begin{tikzcd} \cdots \arrow[r] & H^{-1}(\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C) \arrow[r] & H^{-1}(\mathbb{L}_{C/A}) \arrow[r] & H^{-1}(\mathbb{L}_{C/B}) \arrow[dll] \\ & H^0(\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C) \arrow[r] & H^0(\mathbb{L}_{C/A}) \arrow[r] & H^0(\mathbb{L}_{C/B}) \arrow[dll] \\ & H^1(\mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C) \arrow[r] & H^1(\mathbb{L}_{C/A}) \arrow[r] & H^1(\mathbb{L}_{C/B}) \arrow[r] & \cdots \end{tikzcd}$$

We claim that:

1. *For underived rings,* $H^i(\mathbb{L}_{B/A}) = 0$ if $i > 0$;
2. *For underived rings,* $H^0(\mathbb{L}_{B/A}) \simeq \Omega^1_{B/A}$;
3. *In general,* $H^0(\mathbb{L}_{B/A}) \simeq \Omega^1_{\pi_0(B)/\pi_0(A)}$.

An application of these facts is the desired extension to the left of the sequence 4.1.1. The facts are proved in [12], section 7.4.3. We will say more about claim 1, but first we need to talk about connectivity.

4.4 Connectivity

Definition 4.4.1. A space X is **n-connective** if $\pi_i(X,x) = 0$ for all $x \in X$ and $i < n$. We say X is **connective** if it's 0-connective, **connected** if it's 1-connective. $f : X \to Y$ is **n-connective** if fiber(f) is n-connective.

The following is in [12], 7.4.3.2, and it's VERY important.

Theorem 4.4.2 (Connectivity estimate)**.** *Assume $f : A \to B$ is a map in SCR_k and cofib(f) is n-connective. Then there exists a map:*

$$\mathcal{E}_f : B \otimes_A^{\mathbb{L}} Cofib(f) \to \mathbb{L}_{B/A}$$

in $B - Mod$, which is $2n$-connective.

Remark 4.4.3. The proof is not hard; the only difficulty is constructing the map, which we can do after we learn Postnikov towers. (Todo: reference once we have the postnikov notes)

Corollary 4.4.4. *The hypothesis of Theorem 4.4.2 implies $\mathbb{L}_{B/A}$ is n-connective.*

Proof. We look at the fiber sequence:

$$fib(\mathcal{E}_f) \to B \otimes_A^{\mathbb{L}} cofib(f) \to \mathbb{L}_{B/A},$$

and get a long exact sequence of homotopy groups. So it suffices to show that:

1. $B \otimes_A^{\mathbb{L}} Cofib(f)$ is n-connective;
2. $fib(\mathcal{E}_f)$ is $n-1$-connective.

2 is implied by Theorem 4.4.2; note that theorem is actually considerably stronger. Property 1 is proved in [17]. The proof there uses a spectral sequence due to Quillen: for $M, N \in A - Mod$, $A \in SCR_k$,

$$\mathrm{Tor}_p^{\pi_q(A)}(\pi_q M, \pi_q N) \Longrightarrow \pi_{p+q}(M \otimes_A^{\mathbb{L}} N).$$

□

Remark 4.4.5. In particular, cotangent complexes are 0-connective for commutative rings. This gives a proof of fact 1 at the end of the previous section.

Corollary 4.4.6. *For $A \in SCR_k$, $\mathbb{L}_A$ is 1-connective. Moreover, $f : A \to \pi_0(A)$ is 1-connective, so $\mathbb{L}_{\pi_0(A)/A}$ is 1-connective.*

The most important corollary:

Corollary 4.4.7. *$f : A \to B$ is an equivalence iff $\pi_0(f) : \pi_0(A) \to \pi_0(B)$ is and $\mathbb{L}_{B/A} \simeq 0$. One direction obvious, the other comes from the fact that $\mathbb{L}_{B/A}$ is n-connected for all n.*

Remark 4.4.8. Slogan: "DAG = classical AG + DDT". Lurie's representability theorm is a great example of the philosophy: it says that a derived stack is representable iff its truncation is representable and its cotangent complex is nice enough. We won't get to see this in the seminar, since we'll change course towards structured DAG instead.

Remark 4.4.9. Cotangent complexes we glue for free, which was not possible before ∞-categories. This allows to reduce many questions to the affine setting, where we may have to do actual computations if things go wrong.

We have one talk on Postnikov tower, and one on perfect complexes, then we leave the affine setting forever.

(Todo: look at last 2 exercises from stable ∞-category)

Chapter 5

Square Zero Extensions

Talk by Matei Ionita.

5.1 Square Zero Extensions

Recall that, given $A \in cdga_k^{\leq 0}$ and $M \in A - \mathcal{M}od$, we defined derivations from A into M as:

$$\mathcal{D}er_k(A, M) = \mathrm{Map}_{A-\mathcal{A}lg/k}(A, A \oplus M).$$

Alternatively, these are the same as sections of the projection map $A \oplus M \to A$. Morally speaking, we'd like to define square-zero extensions as homotopy fibers of derivations, i.e. $f : A^\eta \to A$ is a square-zero extension of A by M if there is a homotopy pullback square:

$$\begin{tikzcd} A^\eta \arrow[r, "f"] \arrow[d] & A \arrow[d, "d_\eta"] \\ 0 \arrow[r] & M[1]. \end{tikzcd}$$

The problem is that the above diagram doesn't make sense, because a derivation is not a morphism in $cdga_k^{\leq 0}$. In section 7.4.1 of [12], Lurie addresses this by using the category of tangent correspondences, which acts like a "tangent bundle" of the category $cdga_k^{\leq 0}$, with $A - \mathcal{M}od$ acting as the tangent space $T_A cdga_k^{\leq 0}$. In this new category the diagram makes sense. However, we don't introduce all this technology here, and instead translate Lurie's (more general) definition of square zero extensions into a more accessible version.

Definition 5.1.1. A map $\tilde{f} : \tilde{A} \to A$ is a **square-zero extension** of A by M if it's equivalent in the category $cdga_{/A}^{\leq 0}$ to a map $f : A^\eta \to A$ such that there is a homotopy pullback diagram in $cdga_k^{\leq 0}$:

$$\begin{tikzcd} A^\eta \arrow[r, "f"] \arrow[d] & A \arrow[d, "d_\eta"] \\ A \arrow[r, "d_0"] & A \oplus M[1]. \end{tikzcd}$$

Here d_0 is the zero derivation.

Remark 5.1.2. We explain why the shift by 1 is necessary in definition 5.1.1, by studying the split square-zero extension. We claim that, with the shift in place, the following diagram is a homotopy pullback.(Todo: replace with better explanation)

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \oplus M[1] \end{tikzcd}$$

To see this, extend the diagram by considering the map $0 \to A$, and the resulting pullback square in the category $A - \mathcal{M}od$:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & A \oplus M \arrow[r] \arrow[d] & A \arrow[d] \\ 0 \arrow[r] & A \arrow[r] & A \oplus M[1]. \end{tikzcd}$$

Indeed, the vertical map $A \oplus M \to A$ is surjective, hence a fibration in $A - \mathcal{M}od$, and then the naive pullback M is a homotopy pullback. Moreover, the outer square is also a homotopy pullback in $A - \mathcal{M}od$, because it's equivalent to:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & M[1]. \end{tikzcd}$$

It follows that the square on the right is a homotopy pullback in $A - \mathcal{M}od$. But all maps in this square are maps of A-algebras, so we claim that the square is actually a homotopy pullback in $A - \mathcal{A}lg$.

Remark 5.1.3. Definition 5.1.1 is easy and clean, but it is hard to see whether a given map satisfies it. For example, if $A \to B$ is a square zero extension of commutative rings by a B-module M, in the classical sense, the shift $M[1]$ makes us leave the classical category of modules. Moreover, it's hard to prove that the given map $A \to B$ comes from the fiber product structure of A.

We would like to construct a functor $\Phi : \mathcal{D}er(A, M) \to \mathrm{Fun}(\Delta^1, cdga_k^{\leq 0})$ whose essential image are the square-zero extensions. Morally speaking, Φ sends $d_\eta : A \to A \oplus M$ to its homotopy fiber. The rest of this section makes this construction precise.

Definition 5.1.4. The ∞-category $\mathcal{D}er_A$ of **derivations of A** has objects derivations $d : A \to M$ and spaces of morphisms $\mathcal{D}er_A(M_1, M_2) = A - \mathcal{M}od_{/A}(M_1, M_2)$. The ∞-category $\tilde{\mathcal{D}er}_A$ of **extended derivations of A** has objects consisting of homotopy pullback squares:

$$\begin{tikzcd} A^\eta \arrow[r, "f"] \arrow[d] & A \arrow[d, "d_\eta"] \\ A \arrow[r, "d_0"] & A \oplus M, \end{tikzcd}$$

and spaces of morphisms consisting of morphisms of squares.

Note that $\tilde{\mathcal{D}er}_A$ can be described as the full ∞-subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, cdga_{/A}^{\leq 0})$ whose objects are homotopy pullback squares and have prescribed restrictions: $F(\{0, 0\}) = A$ and $F(\{1\} \times \Delta_1) = d_0 : A \to A \oplus M$.

There are two functors $F_1, F_2 : \tilde{\mathcal{D}er}_A : \mathrm{Fun}(\Delta^1, cdga_{/A}^{\leq 0})$ obtained by restricting to $\Delta_1 \times \{1\}$ and $\{0\} \times \Delta^1$, respectively. Note that their essential images are A-derivations and square-zero extensions of A, respectively, so that we have:

$$\begin{tikzcd} & \tilde{\mathcal{D}er}_A \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ \mathcal{D}er_A & & \mathrm{Fun}(\Delta_1, cdga_{/A}). \end{tikzcd}$$

We prove that F_1 is a trivial Kan fibration, which implies that it has a section s. This will allow us to define $\Phi = F_2 \circ s$.

Lemma 5.1.5. *F_1 is a trivial Kan fibration.*

Proof. Consider the decomposition:

$$\begin{tikzcd} \mathrm{Fun}(\Delta^1 \times \Delta^1, cdga_{/A}) \arrow[rr, "R"] \arrow[dr, "R_1"'] & & \mathrm{Fun}(\Delta^1 \times \{1\}, cdga_{/A}) \\ & \mathrm{Fun}(\Lambda^2_2, cdga_{/A}) \arrow[ur, "R_2"'] & \end{tikzcd}$$

F_1 is the restriction of R to $\tilde{\mathcal{D}}er_A$. Then we have:

1. Using Proposition 4.3.2.15 in [9], a restriction functor $\mathrm{Fun}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$ is a trivial Kan fibration as long as all functors in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ are Kan extensions of those in $\mathrm{Fun}(\mathcal{C}_0, \mathcal{D})$. We apply this twice.

2. The pullback squares in $\mathrm{Fun}(\Delta^1 \times \Delta^1, cdga_{/A}^{\leq 0})$ are Kan extensions, because all limits are Kan extensions. It follows that $R_1|_{\tilde{\mathcal{D}}er_A}$ is a trivial Kan fibration.

3. $R_1(\tilde{\mathcal{D}}er_A)$, the images of extended derivations in $\mathrm{Fun}(\Lambda_2^2, cdga_{/A}^{\leq 0})$, are left Kan extensions. It follows that R_2 restricted to the images of extended derivations is a trivial Kan fibration.

□

Then we invoke the theorem saying that every trivial Kan fibration has a section (Todo: reference this), and define $\Phi = F_2 \circ s$.

5.2 n-small extensions

Let $f : A \to B$ be a map in $cdga_k^{\leq 0}$, and let $I = \mathbf{hofib}(f)$. In other words, I is the homotopy pullback of the following diagram of non-unital commutative monoid objects in $A - \mathcal{M}od$:

$$\begin{tikzcd} I \arrow[r] \arrow[d] & A \arrow[d, "f"] \\ 0 \arrow[r] & B. \end{tikzcd}$$

This induces a non-unital commutative monoid structure on I; in particular, I is an A-module, and there is a multiplication map $I \otimes_A I \to I$. The following is proposition 7.4.1.14. in [12].

Proposition 5.2.1. *The multiplication map $I \otimes_{A^\eta} I \to I$ is nullhomotopic.*

This motivates our definition of n-small extensions. The following definition and remarks are 7.4.1.18-7.4.1.21 in [12].

Definition 5.2.2. Let $f : A \to B$ be a map in $cdga_k^{\leq 0}$, and let $n \geq 0$. We say that f is an **n-connective extension** if $hofib(f) \in cdga_k^{\leq -n}$. We say that f is an **n-small extension** if it is an n-connective extension and, moreover:

1. $hofib(f) \in cdga_k^{\geq -2n}$;

2. the multiplication map $hofib(f) \otimes hofib(f) \to hofib(f)$ is nullhomotopic.

Remark 5.2.3. If $f : A \to B$ is an n-connective extension, from the long exact sequence on homotopy groups we see that $\pi_0(A) \to \pi_0(B)$ is surjective.

Remark 5.2.4. Suppose that $f : A \to B$ is an n-connective extension with $hofib(f) \in cdga_k^{\geq -2n}$. Since $hofib(f) \in cdga_k^{\leq -n}$, we also have that $hofib(f) \otimes hofib(f) \in cdga_k^{\leq -2n}$. It follows that, at the level of homotopy groups, the only potentially nonzero map is:

$$\pi_{2n}\big(hofib(f) \otimes hofib(f)\big) \to \pi_{2n}\big(hofib(f)\big). \tag{5.2.1}$$

Therefore condition 1 in the definition of an n-small extension simply requires that the map 5.2.1 is 0.

Example 5.2.5. Let A be a commutative ring, which we regard as a discrete commutative dga. A map $\tilde{A} \to A$ in $cdga_k^{\leq 0}$ is a 0-small extension if and only if:

1. $\tilde{A}$ is also discrete;

2. $f : \tilde{A} \to A$ is a surjective commutative ring homomorphism;

3. if I is the kernel of f, then $I^2 = 0$, as a consequence of 5.2.1.

So we recover the square-zero extensions in classical AG.

We want to prove that all n-small extensions are square-zero extensions. (But not vice-versa!) First, we identify what n-smallness should correspond to in terms of derivations. It's what you'd expect.

Definition 5.2.6. Let $\mathcal{D}er_{\text{n-con}}(A)$ denote the full subcategory of derivations $\eta : A \to M[1]$ such that $M \in A - \mathcal{M}od^{\leq -n}$. Let $\mathcal{D}er_{\text{n-sm}}(A)$ denote the full subcategory of derivations $\eta : A \to M[1]$ such that $M \in A - \mathcal{M}od^{\leq -n} \cap A - \mathcal{M}od^{\geq -2n}$.

The following is Theorem 7.4.1.23 in [12], and is the main result of this talk.

Theorem 5.2.7. *Let $\Phi : \mathcal{D}er(A) \to \mathrm{Fun}(\Delta^1, cdga_k^{\leq 0})$ be the functor constructed in section* **??**. *For each $n \geq 0$, it induces an equivalence of categories:*

$$\Phi_{n\text{-}sm} : \mathcal{D}er_{n\text{-}sm} \to \mathrm{Fun}_{n\text{-}sm}(\Delta^1, cdga_k^{\leq 0}).$$

Proof. We just give a sketch. First, note that for a derivation $d_\eta : A \to A \oplus M[1]$, there is an equivalence between the homotopy fiber of the square-zero extension $A^\eta \to A$ and M. Moreover, multiplication on the fiber of a square-zero extension is nullhomotopic, by Proposition 5.2.1. It follows that the functor Φ restricts indeed to a functor $\Phi_{\text{n-sm}} : \mathcal{D}er_{\text{n-sm}} \to \mathrm{Fun}_{\text{n-sm}}(\Delta^1, cdga_k^{\leq 0})$.

Φ admits a left adjoint Ψ, which sends a square-zero extension $A^\eta \to A$ to the derivation classified by $\mathbb{L}_A \to \mathbb{L}_{A/A^\eta}$. This restricts to a left adjoint of $\Phi_{\text{n-conn}}$, but we need to truncate in order to get an adjoint $\tau \circ \Psi_{\text{n-conn}}$ for $\Phi_{\text{n-sm}}$. Then we prove that this adjoint pair is an equivalence. □

Corollary 5.2.8. *Every n-small extension is a square-zero extension.*

Corollary 5.2.9. *Let $A \in cdga_k^{\leq 0}$, then every map in the Postnikov tower:*

$$\cdots \to \tau^{\geq -2}(A) \to \tau^{\geq -1}(A) \to \tau^{\geq 0}(A)$$

is a square-zero extension.

This is because the n-th stage is obviously an n-small extension, the homotopy fiber being equal to $\pi_n(A)[n]$ concentrated in degree n. This corollary is highly important, as it allows statements about derived affines A to be proved by induction on the Postnikov tower. The base step, for $\pi_0(A)$, is a classical AG statement, which is proved by classical methods. The inductive step reduces to a linear problem involving the derivation associated to the square-zero extension $\tau^{\geq i}(A) \to \tau^{\geq i-1}(A)$. The next section exemplifies this philosophy.

5.3 Induction on Postnikov tower

Proposition 5.3.1. *Let $A \in sCA_k$, Assume we are given $j : \mathrm{Spec}(\pi_0(A)) \to \mathbb{A}^n$, then there exists a lift of the map j to a map $\mathrm{Spec}(A) \to \mathbb{A}^n$.*

Proof. We use induction on the Postnikov tower. Suppose that there is a map $j_n : Spec(\tau^{\leq n}A) \to \mathbb{A}^n$, we show that there is a lifting j_{n+1} from $Spec(\tau^{\leq n+1}A)$ to $\mathbb{A}^n$. If we can prove this then as $A = lim(\tau^{\leq n}A)$ and there thus exists a lifting of the map j to a map from $Spec(A)$.

We show how this works for the first stage, the construction of $j_1 : \mathrm{Spec}(\tau^{\leq 1}(A)) \to \mathbb{A}^n$.

$$\begin{tikzcd}
k[x_1, \ldots, k_n] \arrow[dr, dashed] \arrow[drr, "j"] \arrow[ddr, "j"'] & & & \\
& \tau^1(A) \arrow[r] \arrow[d] & \pi_0(A) \arrow[d, "d_\eta"] \arrow[ddr, bend left, "="] & \\
& \pi_0(A) \arrow[r, "d_0"] \arrow[drr, bend right, "="'] & \pi_0(A) \oplus \pi_1(A)[2] \arrow[dr, "p"] & \\
& & & \pi_0(A).
\end{tikzcd}$$

In order to invoke the universal property of the homotopy pullback, we need a homotopy between two maps $k[x_1, \ldots, k_n] \to \pi_0(A) \oplus \pi_1(A)[2]$:

$$d_\eta \circ j \cong d_0 \circ j.$$

It suffices to show that this homotopy exists, i.e. that the space X of such homotopies is nonempty. However, we will accomplish more: we find the homotopy type of X, which allows us to comment on the (non-)uniqueness of the lift j_1. Note that, in an ∞-category, the derivations d_0 and d_η come with the data which expresses them as sections of the projection $p : \pi_0(A) \oplus \pi_1(A)[2] \to \pi_0(A)$, i.e. homotopies $p \circ d_0 \Rightarrow \mathrm{id}_A$ and $p \circ d_\eta \Rightarrow id_A$; this is the bottom-right part of diagram 5.3. Moreover, giving a map $k[x_1, \ldots, k_n] \to \pi_0(A) \oplus \pi_1(A)[2]$ (such as $d_\eta \circ j$ or $\cong d_0 \circ j$) is, using the universal property of pullbacks, equivalent to giving a section of the pullback map $B \to k[x_1, \ldots, k_n]$:

$$\begin{tikzcd}
k[x_1, \ldots, k_n] \arrow[dr] & & k[x_1, \ldots, k_n] \arrow[dl] & \pi_0(A) \arrow[dr] & & \pi_0(A) \arrow[dl] \\
& B \arrow[d] & & & \pi_0(A) \oplus \pi_1(A)[2] \arrow[d] & \\
& k[x_1, \ldots, k_n] \arrow[rrr, "j"] & & & \pi_0(A) &
\end{tikzcd}$$

Note that $\pi_0(A) \oplus \pi_1(A)[2] \to \pi_0(A)$ is a degreewise surjection, hence a fibration in the model structure of $cdga_k^{\leq 0}$. (See 2.1.4 for how this model structure is obtained, via the free-forgetful adjunction.) It follows that the homotopy pullback by j is the same as the naive pullback, so $B = k[x_1, \ldots, k_n] \oplus j_* \pi_1(A)[2]$.

Putting everything together, a lift $j_1 : \mathrm{Spec}\, \tau^{\leq 1}(A) \to \mathbb{A}^1$ is the same as a homotopy between d_0' and d_η', the images of d_0 and d_η under pullback. But $d_0', d_\eta' \in Map_{cdga_k^{\leq 0}/k[x_1,\ldots,x_n]}(k[x_1, ..., x_n]), k[x_1, ..., x_n] \oplus j_* \pi_1(A)[2]) \cong Map_{k[x_1,\ldots,x_n]-Mod}(\mathbb{L}_{k[x_1,\ldots,x_n]}, j_* \pi_1(A)[2])$; denote this space by X. We compute its homotopy type.

Since $k[x_1, \ldots, x_n]$ is discrete and smooth over k, $\mathbb{L}_{k[x_1,\ldots,x_n]} \cong \Omega^1_{k[x_1,\ldots,x_n]}[0]$. On the other hand, the functor j_* is t-exact (see 3.4.11), so $j_* \pi_1(A)[2]$ is concentrated in degree 2. It follows that we have:

$$\pi_i(X) = \begin{cases} 0 \text{ if } i \neq 2, \\ \mathrm{Hom}\left(\Omega^1_{k[x_1,\ldots,x_n]}, \pi_1(A)\right) \cong \pi_1(A)^{\oplus n} \text{ if } i = 2 \end{cases}.$$

So $X \simeq K(\pi_1(A)^{\oplus n}, 2)$. In particular, X is path connected, so the required path between d_0' and d_η' exists. We can say more:

$$\{\text{homotopies between } d_0' \text{ and } d_\eta'\} \simeq \Omega X \simeq K(\pi_1(A)^{\oplus n}, 1).$$

We conclude that any two lifts $j_1, j_1' : k[x_1, \ldots, x_n] \to \tau^{\leq 1}(A)$ are homotopic, but not coherently homotopic. $\square$

Chapter 6

Perfect Complexes

Talk by Benedict Morrissey.

Half of this is about perfect complexes in classical AG, and the second half about what we do in the derived setting. In 2-3 weeks we will see that perfect complexes actually form a stack.

6.1 Classical

Let X be a scheme, then we look at $Ch^\bullet(QCoh(X))$.

Definition 6.1.1. $E^\bullet \in Ch^\bullet(QCoh(X))$ is **perfect** if it's Zariski locally quasi-isomorphic to an object of $Ch^b(Vect_X)$.

Remark 6.1.2. This is not the same as requiring cohomology to be finitely supported.

Definition 6.1.3. $E^\bullet \in Ch^\bullet(QCoh(X))$ has **Tor amplitude** in $[a, b]$ if for all $\mathcal{F} \in \mathcal{O}_X - Mod$,

$$H^k(E^\bullet \otimes^{\mathbb{L}}_{\mathcal{O}_X} \mathcal{F}) = 0$$

for $k \notin [a, b]$. In particular, if E is in the heart, this is just saying that the Tor with any given sheaf is bounded.

Remark 6.1.4. For $\mathcal{F} = \mathcal{O}_X$, we just get the cohomology of $E^\bullet$.

This is sometimes difficult to work with, so we have:

Definition 6.1.5. $E^\bullet \in Ch^\bullet(QCoh(X))$ is **almost perfect** if Zariski locally there is a n-quasi-isomorphic to something in $Ch^b(Vect_X)$. n-quasi-isomorphism means isomorphism on cohomologies for $k \geq n+1$, and surjection for degree n.(Todo: Figure out cohomological convention)

Perfect obviously implies almost perfect; this descends to the derived category of quasi-coherent sheaves.

Theorem 6.1.6. *$E^\bullet$ is perfect iff $E^\bullet$ is almost perfect (for some n) and has finite Tor amplitude.*

Proof. Locally free means flat, so tensoring it with anything preserves the tor amplitude. The other direction is in TT, Higher algebraic K-theory of schemes, 2.2.12. □

Alternative definition: $E^\bullet \in D(Coh(X))$. Locally on some U we have an $\mathcal{O}_X(U)$-module $E|^\bullet_U$. We require that this is bounded above and has coherent cohomologies. Equivalently, $\tau_{\leq n}(E^\bullet|_n)$ is compact in $\tau_{\leq n} Mod_{\mathcal{O}_X(U)}$.

Theorem 6.1.7. *For X affine, $E^\bullet$ is perfect if and only if it's globally quasi-isomorphic to an object of $Ch^b(Vect_X)$.*

Theorem 6.1.8. *If X is smooth and Noetherian, then $D(Coh^b(X)) \simeq D(Perf(X))$.*

We'll prove this by Serre regularity.

Definition 6.1.9. A is **regular** if $\dim_k(m/m^2) = \dim_{Krull} A$. The **global dimension** of A is:

$$gldim(A) = \sup_{M \in A-Mod} \operatorname{projdim}(M),$$

where the latter is the minimum length of a projective resolution of M.

Theorem 6.1.10 (Serre regularity)**.** *If A is a Noetherian local ring, TFAE:*

1. *A is regular;*
2. *$gldim(A) < \infty$;*
3. *$gldim(A) = \dim_{Krull} A$.*

Going back to X smooth and Noetherian, we know that all local rings are regular. $E^\bullet \in Coh^b(X)$, then $E^\bullet_p$ is a $\mathcal{O}_{X,p}$ module.

Proof. We actually just do the case of E in the heart, because it's easier. (For the other one, we probably resoluve to a double complex, and take the total complex.)

Take a projective resolution of $E^\bullet_p$ as a $\mathcal{O}_{X,p}$ module; we know it must be finite:

$$0 \to P_n \to \cdots \to P_2 \to P_1 \to E^\bullet_p \to 0.$$

We can do this in an open set around p:(Todo: draw this from paper)

This takes care of one direction. We then use Tor dimension to show that, if E is perfect, it's in $D(Coh^b(X))$. □

For the non-smooth case, we look at the ind-completion $Ind(Perf(X)) \simeq QCoh(X)$. (Always true for X quasi-compact, quasi-separated.) On the other side, $Ind(Coh^b(X)) = IndCoh(X)$. The quotient $IndCoh(X)/QCoh(X) = D_{sing}(X)$, which really sees the singularities of X.

6.2 derived

Let $A \in SCR_k$; recall that $\mathcal{M}od_A$ is a stable ∞-category.

Definition 6.2.1. $\mathcal{M}od_A^{\text{perf}} \subset \mathcal{M}od_A$ is the smallest stable subcategory containing A and closed under retracts. Recall that A is a retract of B if there exist maps $i : A \to B, r : B \to A$ such that $r \circ i = \mathrm{id}_A$.

Definition 6.2.2. $N \in \mathcal{C}$ is **compact** if $\mathcal{H}om_{\mathcal{C}}(N, -)$ commutes with filtered colimits. The latter means that the index category is nonempty, and for all $i, j \in I$, there exists k such that $i \to k \leftarrow j$, and coequalizers exist.

Example 6.2.3. Consider $(\mathcal{M}od_A)^\heartsuit$. The compact objects are the finitely presented ones. We have a map $A^n \to M$, so:

$$\operatorname{Hom}(M, \varinjlim_I B_i) \simeq \varinjlim_I \operatorname{Hom}(M, B_i),$$

because if we have a map $M \to B$, we can fully describe it by the composition $A^n \to B$. Each of the n generators goes to some B_{i_k}, so by the definition of filtered index category, there exists some B_j such that $A^n \to B_j$.

Conversely, starting with compact M, we look at finitely generated submodules M_i, and we have:

$$\operatorname{Hom}(M, \varinjlim_I M_i) \simeq \varinjlim_I (M, M_i).$$

In particular, the identity map $M \to M$ factors through some M_j, so $M = M_j$.

Theorem 6.2.4. *$M \in \mathcal{M}od_A$ is perfect iff it's compact.*

Proof. $\mathcal{M}od_A^{\mathrm{perf}} \subset \mathcal{M}od_A^{\mathrm{cpct}}$. (Todo: add diagram from paper)

Since DK is an equivalence, we only need to argue that truncation and the forgetful functor preserve filtered colimits. For the first one: filtered colimits are t-exact. For the second one: it does.

For the other direction, we have the inclusion $\mathcal{M}od_A^{\mathrm{perf}} \to \mathcal{M}od_A$, we factor this though Ind, which is just the completion with respect to filtered colimits.

$$\begin{tikzcd} \mathcal{M}od_A^{\mathrm{perf}} \arrow[r] & \mathcal{M}od_A \\ & Ind(\mathcal{M}od_A^{\mathrm{perf}}) \arrow[u, "\phi"] \end{tikzcd}$$

f is obviously fully faithful, because $Ind(\mathcal{M}od_A^{\mathrm{perf}})^{\omega} = \text{-Mod}_A^{perf}$.[1] Mapping spaces in Ind are computed by:

$$\mathcal{M}ap_{Ind(\mathcal{C})}(\operatorname*{colim}_{i\in I} \mathcal{F}_i, \operatorname*{colim}_{j\in J} \mathcal{G}_j) = \lim_{i\in I} \operatorname*{colim}_{j\in J} \mathcal{M}ap_{\mathcal{C}}(\mathcal{F}_i, \mathcal{G}_j).$$

This means we're computing mapping spaces by the same formula, so ϕ is fully faithful. □

The following is 7.2.4.5 in [12]:

Theorem 6.2.5. *$M \in \text{-Mod}_A^{perf}$, we have:*

1. *$\pi_n M = 0$ for $n >> 0$;*
2. *If $\pi_m M \cong 0$ for all $m > k$, then $\pi_k M$ is finitely presented as a $\pi_0(M)$-module.*

Proof. For M perfect, we use compactness to get $M \simeq \lim_{n\to\infty}(\tau_{\leq n} M)$. In fact, the map must factor through one of the terms in the limit, so $M \simeq \tau_{\leq n} M$ for some n.

Next, we have the adjunction:

$$\mathcal{M}od_A^{\heartsuit} \longrightarrow \mathcal{M}od_A^{connective} \xrightarrow{[k]} \mathcal{M}od_A^{support \leq k}$$

The adjoint truncates $\geq k$ and then shifts. Both of these preserve (Todo: finish) □

Example 6.2.6. Think about $\mathrm{Sym}(k[2])$ as a module over itself. It is perfect by definition, but it's not bounded below.

Recall from the last talk that we have $A \in SCR_k$, and a stable∞-category $\mathcal{M}od_A$. We defined the stable ∞-subcategory $\mathcal{M}od_A^{\mathrm{perf}}$. We proved that $\mathcal{M}od_A^{\mathrm{perf}} \simeq \mathcal{M}od_A^{\mathrm{cpt}}$.

Definition 6.2.7. M is almost perfect if it's almost compact, i.e. M is bounded above and $\forall n \leq 0$, $\tau_{\geq n} M$ is compact in $\mathcal{M}od_A^{\geq n}$.

Remark 6.2.8. In the classical setting, due to Tor amplitude, perfect complexes need to be bounded below. This is no longer the case.

Theorem 6.2.9 (7.2.4.11 in [12]).

1. *$\mathcal{M}od_A^{aperf} \subset \mathcal{M}od_A$ closed under translation, finite colimits, so it's a stable subcategory of $\mathcal{M}od_A$;*
2. *$\mathcal{M}od_A^{aperf}$ is closed under retracts;*
3. *$\mathcal{M}od_A^{perf} \subset \mathcal{M}od_A^{aperf}$;*

[1] We'll talk more about this equality later, it follows because Perf is idempotent complete.

4. $(\mathcal{M}od_A^{aperf})^{\leq 0}$ *closed under geometric realizations;*

5. *Every* $M \in \mathcal{M}od_A^{aperf}$ *is* $M = |P_\bullet|$*, a geometric realization of a simplicial* A*-module. Each* P_i *finite rank and free.*[2]

Proof. Look at 2.4.1 □

A note about geometric realizations, which are colimits of simplicial objects. Simplicial resolutions are the classical description of the cotangent complex. One starts with a simplicial resolution in the category of A-modules, and the realization is the cotangent complex.

Now assume that $X \in \mathrm{Aff}^{classical}$ and that $E_\bullet$ is perfect. We want to show that $E_\bullet$ is equivalent to a finite complex of vector bundles, globally. This follows from the proof before, but we show that D_i are actually vector bundles, using the finite Tor amplitude.

Theorem 6.2.10 (7.2.4.17 in [12])**.** *Say* A *is left coherent, i.e.* A *is connective,* $\pi_n A$ *is a finitely presented* $\pi_0(A)$*-module, and that every finitely generated left ideal of* A *is a finitely presented left* A*-module. Then* $M \in \mathcal{M}od_A$ *is almost perfect if and only if* $\exists m >> 0$ *such that* $\pi_k M = 0$ *for all* $k \geq m$*, and* $\pi_k M$ *is a finitely presented* $\pi_0(A)$*-module.* [3]

Remark 6.2.11. In particular, from this it's obvious that not all almost perfect modules are perfect. Think about the discrete case, this allows almost perfect to be unbounded below, whereas perfect have to be bounded below, due to finite Tor amplitude.

Definition 6.2.12. The tor amplitude $Toramp(M) \leq n$ if for all discrete A-modules N, $\pi_i(M \otimes_A N) = 0$ for all $i \leq -n$.

Theorem 6.2.13 (7.2.4.23 in [12])**.** $A \in SCR_k$*:*

1. $M \in \mathcal{M}od_A$*, tor amplitude* $\leq n$ $M[k]$ *has tor amplitude* $\leq n + k$*;*

2. $M' \to M \to M''$ *a fiber sequence,* M' *and* M'' *have tor amplitude* $\leq n$*, then so does* M*;*

3. M *has tor amplitude* $\leq n$*, so does any retract;*

4. M *is almost perfect, then* M *is perfect iff it has finite Tor amplitude;*

5. M *has* $Toramp \leq n$*, then* $\forall N \in (\mathcal{M}od(A)_{supp \leq 0})$*,* $\pi_i(N \otimes_A M) = 0$ *for* $i \leq -n$*.*

Proof. For 4, the inductive hypothesis uses 2, for the base case we want that almost perfect and flat implies perfect. The latter follows from 7.2.4.20. □

[2] In fact, the almost perfect ones are precisely these geometric realizations - we think. Write about this in more detail.

[3] This formulation is using homological convention, I think. Figure out the signs.

Chapter 7

Descent

Talk by Antonijo Mrcela.

7.1 Statement

We recall the construction of overcategories, introduced in 1.3. In particular, the ∞-category of commutative algebras over A is the homotopy pullback in the following diagram.

$$\begin{tikzcd}
\mathcal{C}Alg_{/A} \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^1, \mathcal{C}Alg_k) \arrow[d, "ev_1"] \\
\{A\} \arrow[r] & \mathcal{C}Alg_k
\end{tikzcd}$$

This is a co-Cartesian fibration. Next, given $f : B \to A$, we get a Cartesian morphism:

$$\begin{tikzcd}
\mathcal{C}Alg_{/B} & \mathcal{C}Alg_{/A} \arrow[l] \arrow[d] \\
\{B\} \arrow[u, leftarrow] \arrow[r, "f"] & \mathcal{C}\{A\}
\end{tikzcd}$$

So we can apply the Grothendieck construction to get a functor $\mathcal{F} : \mathcal{C}Alg_k^{\mathrm{op}} \to \mathcal{C}at_\infty$. We then take the spectralization to $\mathcal{C}at_\infty^{st}$.

Moreover, Mod lands in the presentable category: Pr^R_{st}, where the functors are right adjoint functors. There's also a version Pr^L_{st}, and we have, by [9] 5.5.3.4, $[\mathcal{S}, Pr^R_{st}] \simeq [\mathcal{S}^{\mathrm{op}}, Pr^L]$. So we can take the adjoint of $Mod : \mathcal{C}Alg_k^{\mathrm{op}} \to Pr^R_{st}$ to get $QCoh^\times \mathcal{C}Alg_k \to Pr^L_{st}$.

Note also that at the level of Mod we have a contravariant functoriality, where $f : A \to B$ gets sent to the forgetful functor $f_* : B - Mod \to A - Mod$. At the level of $QCoh^\times$ we have covariant functoriality.

We'll define Grothendieck topologies for ∞-categories, and surprisingly get just Grothendieck topologies on the homotopy category.

Definition 7.1.1. A **sieve** on an ∞-cat $\mathcal{C}$ is a full subcategory such that, if $f : C \to D$ and $D \in \mathcal{C}^0$, then $C \in \mathcal{C}^0$. If $C \in \mathcal{C}$, then a **sieve over** C is a sieve on $\mathcal{C}_{/C}$.

By Remark 6.2.2.3 in [9], a Grothendieck topology on $\mathcal{C}$ is just one on $h\mathcal{C}$.

Note, though, that $\eta : h(\mathcal{C}_{/C}) \to h(\mathcal{C})_{/C}$ is not normally an equivalence. This is because in $h(\mathcal{C}_{/C})$ we also need to specify hte homotopy that makes $A \to B \to C$ commute. However, η induces a bijection on sieves.

In $\mathcal{C}Alg_k$, S the set of faitfully flat morphisms. The following is DAG VII 5.4, 5.1: S determines a Grothendieck topology on $\mathcal{C}Alg_k$. This is called **the flat topology**.

Definition 7.1.2. The **Cech nerve functor** $B^\bullet : \Delta \to \mathcal{C}Alg_k$, is informally described by $i \to B_i = B^{\otimes_A^{i+1}}$. To construct it as an ∞-functor, we take the left Kan extension of the functor $\Delta^{\leq 1} \to \mathcal{C}Alg_k$, given by the morphism $B \to B \otimes_A B$. (Todo: Doesn't sound right, figure this out)

Since A maps to each B_i, we obtain an ∞-functor $\phi : A - \mathcal{M}od \to \varinjlim_\Delta B^{\otimes_A^{n+1}} - \mathcal{M}od$. The **descent problem** asks if this is an equivalence, and if it is, whether we can construct some sort of inverse. We give an affirmative answer to the first question, using the following strategy:

- The category $A - \mathcal{M}od$ has a standard t-structure given by the degrees of the modules. We use Lemma 7.2.1 to put a t-structure on $\varinjlim_\Delta B^{\otimes_A^n} - \mathcal{M}od$ as well.
- Due to Lemma 7.2.3, ϕ is an equivalence if and only if it induces an equivalence on the hearts of the given t-structures. Due to the equivalence 3.4.1, ϕ is an equivalence if and only if it induces an equivalence $\pi_0(A)\mathcal{M}od^\heartsuit \to \varinjlim_\Delta \pi_0(B)^{\otimes_{\pi_0(A)}^{n+1}} - \mathcal{M}od^\heartsuit$.
- We use a version of Quillen's Theorem A, reproduced in 7.2.6, to show that we can replace the infinite Cech nerve with a 3-term Cech nerve, without changing the limit. This reduces the problem to Grothendieck's classical formulation of descent, which we know to be true.

7.2 Proof

The following lemma is 3.20 in Shennon - Porta - Vezzosi, Formal Gluing along non-linear flags. (Todo: cite this once it appears)

Lemma 7.2.1. *Let $p : \mathcal{X} \to \mathcal{S}$ be a stable filtration, and let $\mathcal{S}^{\mathrm{op}} \to \mathcal{C}at_\infty^{st}$ be the associated ∞-functor. Suppose:*

1. *For all $s \in \mathcal{S}$ there is a t-structure $(\mathcal{X}_s^{\leq 0}, \mathcal{X}_s^{\geq 0})$ on $\mathcal{X}_s$.*
2. *For all edges $f : s \to s'$, the induced functor f^* is t-exact.*

Then the stable ∞-category $\varprojlim F$ has a (unique) t-structure characterized by:

$$\forall s \in S, e_s : \varprojlim F \to \mathcal{X}_s$$

is t-exact.

Proof. We can represent the limit as: $\varprojlim F = \mathrm{Map}_S^\flat(S^\sharp, \mathcal{X})$. (Todo: figure out how to do symbol over $\mathcal{X}$; see [9] 3.3.3.2.) I.e. we map all edges to p-Cartesian edges. Define $\mathcal{C}^{\leq 0}$ to be the full subcategory spanned by $x \in \mathcal{C}$ such that $x(s) \in \mathcal{X}_s^{\leq 0}$. in $\mathcal{X}$.

Let $\mathcal{X}^{\leq 0}$ be the full subcat spanned by $x \in \mathcal{X}$ such that $x \in \mathcal{X}^{\leq 0}$. Let $j : \mathcal{X}^{\leq 0} \to \mathcal{X}$ be the inclusion. $p \circ j$ is again a Cartesian fibration, because f^* is exact. Then the inclusion preserves Cartesian edges. By Proposition 1.2.1.5 in [12] (says that $\mathcal{C}^{\leq n}$ is a localization of $\mathcal{C}$), which we apply fiberwise, we get a left adjoint for each fiber. Then apply [12] 7.3.2.6 which says the following. Suppose that we have a commutative diagram,

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "q"'] & & \mathcal{D} \arrow[ll, "G"'] \arrow[dl, "p"] \\ & \mathcal{E} & \end{tikzcd}$$

where p, q are locally Cartesian categorical fibrations. Then G admits a left adjoint iff

1. for every $E \in \mathcal{E}$, the map $G_E : \mathcal{D}_E \to \mathcal{C}_E$ admits a left adjoint;
2. G carries locally p-Cartesian morphisms in $\mathcal{D}$ to locally q-Cartesian in $\mathcal{C}$.

This is a "gluing result for left-adjoints"; not entirely obvious result. But using this gives a global adjoint $\tau_{\leq 0}: \mathcal{X} \to \mathcal{X}^{\leq 0}$.

Note that, in general, G doesn't have to preserve Cartesian edges. But we used the fact that f^* is t-exact to deal with this. $\square$

Example 7.2.2. Pick $A \to B$ a non-flat morphism. Then we have: (Todo: add from paper)

Now we need to reduce to the problem of descent in the heart. We use Lemma 3.3.7 in the same paper.

Lemma 7.2.3. *Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable ∞-categories. Assume $\mathcal{C}, \mathcal{D}$ have t-structures which are left complete and right bounded, and that with respect to these structures f is t-exact. Then TFAE:*

1. *f is an equivalence;*
2. *$f^\heartsuit : \mathcal{C}^\heartsuit \to \mathcal{D}^\heartsuit$ is an equivalence of abelian categories.*

Note that $1 \Rightarrow 2$ is obvious, while $2 \Rightarrow 1$ is very powerful. This is because we can do many constructions at the level of the hearts that we can't do at the level of ∞-categories.

Proof. The first step is full faithfulness. For $x, y \in \mathcal{C}$, there is a canonical transformation $\psi_{x,y} : Map^{st}_{\mathcal{C}}(x, y) \to Map^{st}_{\mathcal{D}}(f(x), f(y))$. Start by fixing x and defining the full subcategory $\mathcal{C}_x \subset \mathcal{C}$, spanned by those y such that $\psi_{x,y}$ is an equivalence. This is closed under loop and suspension, extensions and retract. So if $\mathcal{C}^\heartsuit \subset \mathcal{C}_X$, we go by induction on non-vanishing cohomology groups, to get $\mathcal{C}^b \subset \mathcal{C}_X$. Here we use the left complete and right complete assumptions. Now for an arbitrary $y \in \mathcal{C}$, $y = \varinjlim \tau_{\geq n} y$. f commutes with this specific colimit, so $f(y) = \varinjlim f(\tau_{\geq n} y)$. Every map from x to y lands in $\tau_{\geq n} y$ for some n, and analogously for maps $f(x)$ to $f(y)$, which reduces the problem to the case of bounded modules, which is already proved.

Step 2 is essential surjectivity. On the heart it's the hypothesis. Pick $y \in \mathcal{D}^b$, we have the exact sequence $\tau_{\leq k} y \to y \to \tau_{> k} y$, with k chosen so that both truncations have fewer cohomology groups than y. Since (Todo: fill in from paper) $\square$

Remark 7.2.4. According to Marci, there are two non-equivalent versions of $D^b Coh$ which have the same heart. We could think about why this lemma doesn't apply for them.

Remark 7.2.5. Mauro says that you can use this statement to prove a bunch of things, for example reduce ∞-GAGA to classical GAGA.

Recall that we were trying to determine whether $A - \mathcal{M}od \to \varinjlim B^{\otimes^n_A} - \mathcal{M}od$ is an equivalence. We use Lemma 7.2.1 to put a t-structure on the limit, and Lemma 7.2.3 to show that the problem is equivalent to that of equivalence of the hearts of the categories. The RHS becomes:

$$\varinjlim (B^{\otimes^n_A} - \mathcal{M}od)^\heartsuit = \varinjlim \pi_0(B^{\otimes^n_A}) - \mathcal{M}od,$$

and the LHS becomes:

$$\pi_0(A) - \mathcal{M}od.$$

This is almost the statement of the classical descent theorem à la Grothendieck. However, in our case the Cech nerve is infinite, instead of having only 3 terms. These two versions are actually equivalent, due to the following theorem.

Theorem 7.2.6 (Quillen, version of Theorem A)**.** *If $\mathcal{C}$ is an n category (it's proven in [9] that n can be ∞), $A : J \to I$ a functor, if for every object $x \in I$ we have $\pi_i(J_{/x}) = 0$ for $i < n$, then* $\lim F = \lim F \circ A$, *for all $F : I \to \mathcal{C}$.*

Here $J_{/x} = J \times_I I_{/x}$. We apply the theorem with $\mathcal{C} = \mathcal{C}at \subset \mathcal{C}at_\infty$, which is a 2-category. So we need $n = 2$.[1] Furthermore, we use the inclusion $\Delta_s^{\leq 3} \to \Delta_s$, where the subscript denotes the subcategories with the same objects, but only monomorphisms as morphisms. Define F as the infinite Cech nerve, $F : \Delta_s \to \mathcal{C}$, $n \mapsto B^{\otimes_A^n}$; then the restriction $F \circ A$ is the Cech nerve à la Grothendieck:

$$\pi_0(B) - \mathcal{M}od \rightrightarrows \pi_0(B) \otimes_{\pi_0(A)} \pi_0(B) - \mathcal{M}od \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \pi_0(B) \otimes_{\pi_0(A)} \pi_0(B) \otimes_{\pi_0(A)} \pi_0(B) - \mathcal{M}od.$$

Remark 7.2.7. We motivate the choice of 3 in $\Delta_s^{\leq 3}$ above. As proved in Exercise 1.5.4, the homotopy type of $(\Delta_S^{\leq m})_{/m+k}$, with $k \geq 0$, is a wedge of a number $N_{m,k}$ of $m-1$-spheres. [2] In order for $J = \Delta_S^{\leq m}$ to satisfy the assumptions of Theorem 7.2.6 with $n = 2$, we need $m \geq 3$. Therefore the Cech nerve can be reduced to a minimum of 3 terms.

Remark 7.2.8. We have:

$$\begin{array}{c} \varinjlim B^{\otimes_A^n} - Mod \\ \big\downarrow \\ A - Mod \end{array}$$

Warning: in non-affine situations, the functor $\lim QCoh(U^n) \to \mathrm{Fun}(\Delta, QCoh(X))$ is highly non-explicit. Given a descent datum $\{\mathcal{F}^n\}$, we get an ∞-functor $\Delta \to QCoh(X)$ which is very lax. In practice one uses rectification to write $\mathrm{Fun}(\Delta, QCoh(X)) \simeq \infty\,\mathrm{Fun}(\Delta, Ch(QCoh(X)))$, and use Reed something. The problem is that the rectification is also very non-explicit.

[1] For $n = 1$, Theorem A is classical; for $n = \infty$, it is proved in [9]. For $1 < n < \infty$, we don't think it's written up anywhere, but it should be true.

[2] Mauro has computed $N_{m,1} = 1$ and $N_{m,2} = 3$; we should see if we can determine all $N_{m,k}$.

Chapter 8

Geometric Stacks and Gluing

Talk by Mauro Porta.

We finally leave the affine world! Only took us 2 months. But first, we mention a correction to Lemma 7.2.3 from the previous talk. The statement "$F^\heartsuit$ is an equivalence" should be taken as "$F^\heartsuit$ is essentially surjective and $F|_{\mathcal{C}^\heartsuit}$ is fully faithful". This means we have to take into account Ext between discrete objects; an equivalence at the level of hearts is not strong enough to guarantee an equivalence on the entire categories.

Otherwise, there's a counterexample to the lemma. Take $A \in cdga^{\leq 0}$, we have the pullback functor $A - \mathcal{M}od \xrightarrow{f^*} H^0(A) - \mathcal{M}od$, $M \mapsto M \otimes_A H^0(A)$, which is not t-exact, so the lemma makes no statement about it. However, the forgetful functor $f_* : H^0(A) - \mathcal{M}od \to A - \mathcal{M}od$ is t-exact, but it fails to be an equivalence, even if $f_*^\heartsuit$ is an equivalence between the categories of discrete modules.

8.1 Gluing: problems and approaches

The problem for today is: how do we patch together derived affines? This is difficult for two reasons:

1. If a derived scheme is to be thought of as a gluing of derived affines, we should be able to produce many gluing diagrams in $d\mathcal{A}ff_k$. This means functors $I \to d\mathcal{A}ff_k$, which is difficult because the latter is an ∞-category, so we need to specify higher coherencies when defining functors. In particular, to define $\mathbb{P}^1$, we have:

$$\begin{tikzcd} & \mathbb{A}^1 \arrow[dr] & \\ \mathbb{G}_m \arrow[ur] \arrow[dr] & & \mathbb{P}^1 \\ & \mathbb{A}^1 \arrow[ur] & \end{tikzcd}$$

   as well as a homotopy between the branches. (Todo: Am I interpreting this correctly?) This is not an existential threat, because we can choose a model-categorical presentation for $d\mathcal{A}ff_k$.

2. But we really need an environment category where the gluing is performed; constructing this is tricky. There are 2 ways.

   - Structured spaces, i.e. the environment is the category of locally ringed topoi, or similar.
   - Functor of points, i.e. the environment is the category of presheaves of spaces, $\mathrm{Fun}(d\mathcal{A}ff \to \mathcal{S})$.

   Today we want to address both and compare them.

First, we look at pros and cons for both:

- Structured spaces. It's fairly easy to think about objects in this way: they are $(X, \mathcal{O}_X)$, where X is some sort of topological space (actually ∞-topos), and $\mathcal{O}_X$ is a sheaf of cdga's on X. Note how similar this is to how we think about underived schemes. Moreover, the subdivision $(X, \mathcal{O}_X)$ makes it easy to distinguish the derived information from the underived one. For example, $\pi_0(X, \mathcal{O}_X) = (X, \pi_0(\mathcal{O}_X))$, and the Postnikov tower discussion carries over to derived schemes:

$$\cdots \to (X, \tau_{\leq 2}\mathcal{O}_X) \to (X, \tau_{\leq 1}\mathcal{O}_X) \to (X, \pi_0\mathcal{O}_X).$$

  Cons: maps are difficult to understand, and only Deligne-Mumford stacks can be described in this way. [1] Note that the stack of perfect complexes, as well as the Eilenberg-Maclane stacks are Artin, but not Deligne-Mumford.

- Functor of points. It can deal with Artin stacks, and then some more. Sometimes in life we want to deal with objects which are stacks, but not geometric stacks; the easiest example is $QCoh : dAff \to Cat_\infty$. We saw last time it has descent, so it's a stack, but it's not geometric.[2] We only have one con, but it's pretty bad: it's not clear at all, in this language, why schemes should be simpler than Artin stacks. In other words, techniques which hold for schemes but don't hold for Artin stacks are obscured.

8.2 Structured spaces

We want a category $\mathcal{C}$ with the following properties:

1. $\mathcal{C}$ contains $d\mathcal{A}ff_k$ in a fully faithful way.
2. $\mathcal{C}$ is big enough to contain all the gluings we'll make. For example, we'd be happy with $\mathcal{C}$ closed under colimits.
3. $\mathcal{C}$ is small enough to have a good notion of Grothendieck topology. (Note that any co-complete category has a Grothendieck topology, called the **canonical topology**: for an object X, and a collection $\{U\}$ of objects mapping to X, call it a covering if the geometric realization of the $\check{C}$ech nerve of $\{U\}$ is equivalent to X. But this is not a very useful topology; when we say "good Grothendieck topology", we want to have a better grasp on the coverings: describe them using words such as étale, smooth, flat, open immersion etc.)

To exemplify this philosophy, recall what we do in classical AG. That is, start with $\mathcal{A}ff_k$ instead of $d\mathcal{A}ff_k$. Then we have two choices for $\mathcal{C}$.

- $(X, \mathcal{O}_X)$ locally ringed spaces. Property 1 we all know. For 2, note that LRS doesn't admit all colimits (you can't talk about $\mathbb{P}^\infty$, for example; that's an IndScheme), but admits enough of them to describe schemes. For 3, the Grothendieck topology is as follows. We say $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is an **open immersion** if it's an open inclusion at the level of topological spaces, and the induced map $f^\sharp : f^*\mathcal{O}_Y \to \mathcal{O}_X$ is an isomorphism. Then a collection $\{X_i\}$ is a **covering** of Y if each $X_i \to Y$ is an open immersion, and moreover the induced map $\coprod X_i \to Y$ is surjective at the level of topological spaces. (Todo: is this definition of covering correct?) Then we can define schemes as objects in LRS which are covered, in the above sense, by objects in the essential image of $\mathcal{A}ff_k \to LRS$.

- Alternatively, we can take locally ringed 1-topoi, in which case we get Deligne-Mumford stacks. Below we make a short summary of locally ringed 1-topoi.

[1] Any structured space has connected cotangent complex, but Artin stacks don't need to. The smooth étale site is not canonical, while the étale site is.

[2] I.e. it doesn't have an appropriate atlas, we'll see shortly what this means.

Definition 8.2.1. A **Grothendieck site** is a category $\mathcal{D}$ together with a Grothendieck topology. A **1-topos** X is a category equivalent to $\mathrm{Sh}(\mathcal{D})$ for some Grothendieck site $\mathcal{D}$.

If the gluing environment $\mathcal{C}$ is the category of locally ringed topoi, then 1 and 2 are again easy. For the Grothendieck topology on $\mathcal{C}$, we make the following definitions. [3]

Definition 8.2.2. For $\mathcal{X}$, $\mathcal{Y}$ 1-topoi, a **geometric morphism** $\mathcal{Y} \to \mathcal{X}$ is an adjoint pair:

$$f^{-1} : \mathcal{X} \to \mathcal{Y} : f_*,$$

where moreover f^{-1} preserves finite limits.

Note that, despite the notation, the geometric morphism goes from $\mathcal{Y}$ to $\mathcal{X}$; the "inverse image" f^{-1} happens to be the left adjoint. This is motivated by the following example. If X, Y topological spaces, take $\mathcal{X} = \mathrm{Sh}(X)$ and $\mathcal{Y} = \mathrm{Sh}(Y)$. For any $f : Y \to X$, the usual inverse and direct image functors on sheaves:

$$f^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(Y) : f_*$$

form a geometric morphism. Recall that f^{-1} is defined as a colimit indexed by open subsets containing the image of f, followed by sheafification. Both these operations are filtered colimits, therefore commute with finite limits. Hence f^{-1} commutes with finite limits, and the pair (f^{-1}, f_*) is a geometric morphism.

Definition 8.2.3. A geometric morphism $f^{-1} : \mathcal{X} \to \mathcal{Y} : f_*$ is **étale** if there exists $U \in \mathcal{X}$ and an equivalence $\mathcal{X}_{/U}$ making the following diagram commute:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\times U"] \arrow[dr, shift left, "f^{-1}"] & \mathcal{X}_{/U} \arrow[d, "\simeq"] \\ & \mathcal{Y}. \arrow[ul, shift left, "f_*"] \end{tikzcd}$$

To get a feel for this definition, note the following examples.

Lemma 8.2.4. *If $f : Y \to X$ is a local homeomorphism of topological spaces, then the standard f^{-1}, f_* is étale. Moreover, if $f : Y \to X$ is an étale map of schemes, then $f^{-1} : \mathrm{Sh}_{Set}(X_{\text{ét}}, T_{\text{ét}}) \to \mathrm{Sh}_{Set}(Y_{\text{ét}}, T_{\text{ét}})$ is étale.*[4]

In fact, we can uprgrade the second statement of Lemma 8.2.4 to a characterization of étale morphisms of schemes.

Proposition 8.2.5. *$f : X \to Y$ in Sch_k is étale iff the induced (f^{-1}, f_*) morphism of topoi is étale and $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is an equivalence.*

Remark 8.2.6. We elaborate on the element $U \in \mathcal{X}$ which appears in Definition 8.2.3. If $Y \subset X$ is an open subset of the topological space X, then Y defines by Yoneda a sheaf $h_Y \in \mathcal{X} = \mathrm{Sh}(X)$, such that:

$$h_Y(V) = \begin{cases} * & , V \subset Y, \\ \emptyset & , V \not\subset Y. \end{cases}$$

Then we take $U = h_Y$, and note that we have $\mathrm{Sh}(Y) \simeq \mathrm{Sh}(X)_{/U}$.

We can relax the assumptions and take $f : Y \to X$ a local homeomorphism. Then we define a sheaf h_f by $h_f(V) = \mathrm{Hom}_{/V}(V, V \otimes_X Y)$.[5] This generalizes the sheaf h_Y from the previous paragraph, in the sense

[3] Don't confuse the Grothendieck topology on the category $\mathcal{C}$ of locally ringed topoi, which we're trying to define now, with the Grothendieck topology on the underlying site $\mathcal{D}$ of an individual topos $\mathcal{X}$. Sometimes both topologies have the same name, e.g. étale, but it should be clear that they're completely different notions.

[4] By $X_{\text{ét}}$ we mean the étale site of the scheme X, which is the subcategory of the comma category Sch_X given by étale morphisms $U \to X$. We endow this site with the Grothendieck topology $T_{\text{ét}}$ induced from the étale topology on $\mathcal{A}ff$.

[5] In other words, this is the sheaf of sections of $f : Y \to X$.

that, if f is an open immersion, $\mathrm{Hom}_{/V}(V, V \otimes_X Y)$ is nonempty iff $V \cap Y = V$, i.e. $V \subset Y$. Moreover, it's a fact that local homeomorphisms are characterized by the property $\mathrm{Sh}(Y) \simeq \mathrm{Sh}(X)_{/h_f}$. So we take $U = h_f$.

We move on from the idea of gluing in the ambient category of locally ringed 1-topoi to the derived analog. This will involve the theory of ∞-topoi, which allows us to replace sheaves of sets with sheaves of spaces, and obtain higher DM stacks. This is necessary, for example, in order to talk about $K(G, n)$, for G finite abelian and $n > 1$.[6]

8.3 A primer on ∞-topoi

Note first that the HAG framework [21] involves hypercomplete ∞-topoi, while the DAG framework [**?**] involves non-hypercomplete ∞-topoi. The latter make for a nicer theory, but in practice often need to be reduced to the hypercomplete case.

The main difference is that the following hold if $\mathcal{X}$ is hypercomplete, but don't need to otherwise.

- $\mathcal{X} = \mathrm{Sh}(\mathcal{C}, \mathcal{T})$, where $(\mathcal{C}, \mathcal{T})$ is an ∞-Grothendieck site.[7]
- For $f : \mathcal{F} \to \mathcal{G}$ a morphism in $\mathcal{X}$, f is an equivalence iff $\pi_i(f)$ are isomorphisms for all i.

Remark 8.3.1. 1. If $\mathcal{X}$ is a 1-topos, viewing it as an ∞-category (i.e. taking the nerve) *does not* get us an ∞-topos. Instead, if $\mathcal{X} = \mathrm{Sh}_{Set}(\mathcal{C}, \mathcal{T})$, we need to replace it with $\mathrm{Sh}(\mathcal{C}, \mathcal{T}) := \mathrm{Sh}_{\mathcal{S}}(\mathcal{S}, \mathcal{T})$, which is an ∞-topos. Moreover, we have a fully faithful embedding $\mathcal{X} \subset \mathrm{Sh}(\mathcal{C}, \mathcal{T})$.

2. If $\mathcal{X}$ is an ∞-topos, we can look at $\mathcal{X}^{\leq n}$, the ∞-category of n-truncated objects in $\mathcal{X}$. $\mathcal{F} \in \mathcal{X}$ is **n-truncated** if for all $\mathcal{G} \in \mathcal{X}$, $\mathrm{Map}_{\mathcal{X}}(\mathcal{G}, \mathcal{F})$ is n-truncated as an ∞-category. Every $\mathcal{F} \in \mathcal{X}^{\leq n}$ is hypercomplete, i.e. equivalent to the limit of its Postnikov tower.

   We say that $\mathcal{X}$ is **n-localic** if we can recover it from $\mathcal{X}^{\leq n}$. More precisely, this means that for all ∞-topoi $\mathcal{Y}$, $\mathrm{Map}_*(\mathcal{Y}, \mathcal{X}) \simeq \mathrm{Map}_*(\mathcal{Y}^{\leq n}, \mathcal{X}^{\leq n})$. The equivalence is implemented as follows. Since f_* is a right adjoint, it preserves truncated objects, which gives the bottom arrow in the diagram:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "f_*"] & \mathcal{X} \\ \mathcal{Y}^{\leq n} \arrow[u, hook] \arrow[r, dashed] & \mathcal{X}^{\leq n}. \arrow[u, hook] \end{tikzcd}$$

3. Sometimes, ∞-topoi are "naturally" hypercomplete. For example:
   - If X is a topological space with finite covering dimension, such as locally compact Hausdorff, then $\mathrm{Sh}(X)$ is hypercomplete.
   - If X is a quasi-compact, quasi-separated, locally Noetherian scheme, then sheaves on $(X_{Zar}, \mathcal{T}_{Zar})$ are a hypercomplete ∞-topos.
   - With X as before $\mathrm{Sh}(X_{Nis}, \mathcal{T}_{Nis})$ is hypercomplete.
   - With X as nice a scheme as you want, even a field, $\mathrm{Sh}(X_{\text{ét}}, \mathcal{T}_{\text{ét}})$ is *not* hypercomplete. But it's always 1-localic.

Definition 8.3.2. An **∞-topos** is a left exact, accessible localization of a presentable ∞-category. (For example, a presentable ∞-category of presheaves of spaces.)

[6] In algebraic topology, we obtain $K(G, n)$ by de-looping $K(G, n-1)$, a process which almost never returns a scheme. So when doing geometry we need to replace de-looping with working with higher groupoids; morally speaking, de-looping n times in topology corresponds to increasing the stack level by n. For example, the de-looping BG of a topological group G corresponds to the stack BG in geometry.

[7] From now on, sheaves are sheaves of spaces, unless we explicitly say otherwise.

Remark 8.3.3. There exists an analog of Giraud's characterization, giving necessary and sufficient intrinsic conditions for an ∞-category to be an ∞-topos. They are somewhere in [21].

Remark 8.3.4. $\mathcal{X}$ is n-localic iff there exists an n-cateogry $\mathcal{C}$ such that $\mathcal{X} = \mathrm{Sh}(\mathcal{C}, \mathcal{T})$.

We want to construct the category of locally ringed ∞-topoi; a reference for this is Chapter 3 of [10]. Recall the definition of derived affines via Lawvere theory, explained in 2.1.3: $d\mathcal{A}ff = \mathrm{Fun}^{\times}(T_{disc}(k), \mathcal{S})$, where $T_{disk}(k) = \{\mathbb{A}^n_k\}$ is the ∞-category of affine spaces with morphisms of schemes. To get locally ringed ∞-topoi, we change the target $\mathcal{S}$ to something else. Let ${}^{L}Top$ be the ∞-category of ∞-topoi with geometric morphisms; there is a forgetful functor ${}^{L}Top \to \mathcal{C}at_{\infty}$. Using the Grothendieck construction, this buys us a Cartesian fibration $\overline{{}^{L}Top} \to^{L} Top$.

Definition 8.3.5. The ∞-category of locally ringed ∞-topoi is a subcategory:

$$ {}^{L}Top(T_{disc}(k)) \subset \mathrm{Fun}(T_{disc}(k), \overline{{}^{L}Top}) \times_{\mathrm{Fun}(T_{disc}(k), {}^{L}Top)} {}^{L}Top. $$

Its objects are pairs $(\mathcal{X}, \mathcal{O} : T_{disc}(k) \to \mathcal{X})$, where $\mathcal{O}$ commutes with products and $\pi_0(\mathcal{O})$ is a sheaf of local rings. Its morphisms are pairs $(f, f^{\sharp}) : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, such that $f^{\sharp} : f^{-1}\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ induces a local morphism on π_0.

Definition 8.3.6. A **derived Deligne-Mumford stack** is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in^{L} Top(T_{disc}(k))$, such that locally $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is of the form $\mathrm{Spec}\, A = (\mathrm{Sh}(A_{\acute{e}t}), \mathcal{O}_A)$, for some derived ring A.

In practice, $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a derived DM stack if there exist $U_i \in \mathcal{X}$ such that:

1. $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}}|_{U_i}) \cong \mathrm{Spec}\, A_i$;
2. $U = \coprod_i U_i \to 1_{\mathcal{X}}$ is an effective epimorphism (i.e. $\pi_*(U) \to \pi_*(1_{\mathcal{X}})$ is surjective).

Definition 8.3.7. We say that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a **derived algebraic space** if we can take U_i as above such that $U_i \to 1_{\mathcal{X}}$ is a monomorphism for all i.

A key property of maps of derived schemes:

Proposition 8.3.8. *The following diagram is a homotopy pullback:*

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Sh}_{cdga}(\mathcal{X})}(f^{-1}\mathcal{O}_{\mathcal{Y}}, \mathcal{O}_{\mathcal{X}}) \arrow[r] \arrow[d] & \mathrm{Map}_{dSch}\big((\mathcal{X}, \mathcal{O}_{\mathcal{X}}), (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})\big) \arrow[d] \\
(f^{-1}, f) \arrow[r] & \mathrm{Map}_{{}^{L}Top}(\mathcal{X}, \mathcal{Y})
\end{tikzcd}$$

So far so good; but recall that requirement 1 for the environment gluing category was that it admits a fully faithful embedding from the affine category. The following, Theorem 2.1.12 in [10], addresses this. Note that the proof is difficult.(Todo: understand and say why)

Theorem 8.3.9. *The embedding* $\mathrm{Spec} : sc\mathcal{A}lg_k \to^{L} Top(T_{disc}(k))$ *is fully faithful.*[8]

8.4 Functor of points

Recall that we defined ${}^{R}Top(T_{disc})$ the ∞-category of locally ringed ∞-topoi. Then we defined the full subcategory: $DM - Stacks \subset {}^{R}Top(T_{disc})$ with objects $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ satisfying: there exist objects $U_i \in \mathcal{X}$ such that:

1. $\coprod U_i \to 1_{\mathcal{X}}$ is an effective epimorphism;

[8]Essentially one needs to prove that $\mathrm{Map}(\mathcal{X}, \mathrm{Spec}\, A) \simeq \mathrm{Map}(A, \Gamma(\mathcal{O}_{\mathcal{X}}))$.

2. $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}/U_i}) \simeq \mathrm{Spec}(A_i)$, for $A_i \in sCRing$.

Recall also that, if $A \in sCRing$, then Spec A was defined as:

1. $\mathcal{X}_A = \mathrm{Sh}(A_{et}, \tau_{et})$;

2. $\mathcal{O}_A : A_{et} \to scRing_k$

Theorem 8.4.1 (DAG-V, Theorem 2.12). *If $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \in {}^R\mathcal{T}op^{loc}(T_{disc})$, there is a canonical equivalence:*

$$\mathrm{Map}_{{}^R\mathcal{T}op^{loc}(T_{disc})}\left((\mathcal{X}, \mathcal{O}_\mathcal{X}), \mathrm{Spec}\, A\right) \simeq \mathrm{Map}_{sCRing_k}\left(A, \Gamma(\mathcal{O}_\mathcal{X})\right)$$

Corollary 8.4.2. Spec : $sCRing_k \to {}^R\mathcal{T}op^{loc}(T_{disc})$ *is fully faithful.*

Remark 8.4.3. If $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a dDM, then it is a derived algebraic space if the following equivalent conditions are satisfied:

1. $U_i \to 1_\mathcal{X}$ are homotopy monomorphisms (think inclusion of open sets);

2. $\mathcal{X}$ is the étale topos of an ordinary algebraic space.

Remark 8.4.4. There is no internal characterization of schemes, as opposed to algebraic spaces. The reason is that with these definitions the underlying ∞-topos of Spec A is 1-localic and not 0-localic. However, note the following. What we called Spec is usually denoted $\mathrm{Spec}^{\text{ét}}$, as opposed to Spec^{Zar}, defined as $\mathrm{Spec}^{Zar}(A) = (\mathcal{X}_A^{Zar}, \mathcal{O}_A)$, where $\mathcal{X}_A^{Zar} = \mathrm{Sh}(A_{Zar}, \tau_{Zar})$. If we use Spec^{Zar}, derived schemes can be characterized as derived DM stacks which are covered by monomorphisms.

Remark 8.4.5. There's a way of taking the topology into account inside T_{disc}. That is, there is a modified version of Lawvere theory which takes into consideration also the Grothendieck topology. This is called (pre)geometry. It gives extra flexibility that allows to freely switch between different topologies. This language is also used in the analytic setting. (Complex or non-archimedian.) Then we get:

$$\begin{tikzcd}
{}^R\mathcal{T}op(T_{Zar}) \arrow[r, "\mathrm{Spec}^{\text{ét}}_{Zar}"] & {}^R\mathcal{T}op(T_{\text{ét}}) \arrow[r, "\mathrm{Spec}^{an}_{\text{ét}}"] & {}^R\mathcal{T}op(T_{an}(\mathbb{C})) \\
dSch \arrow[u, hook] & dDM \arrow[u, hook] & dAn_\mathbb{C} \arrow[u, hook]
\end{tikzcd}$$

Moreover, the "étalification" functor takes Spec^{Zar} to $\mathrm{Spec}^{\text{ét}}$. Note also that the analytification functor can be constructed by hand, but we need the abstract machinery to prove that the construction is correct. More details about this are in [10].

(Todo: the stuff so far should be merged with the previous section)

Now we actually move on to the functor of points. Say $\mathcal{C}$ is a category of (derived) affines. We want to enlarge $\mathcal{C}$ in order to allow general gluings. This can be performed in 3 steps:

1. Add all colimits to $\mathcal{C}$, i.e. take the ∞-category $\mathrm{PSh}(\mathcal{C})$.

2. Realize that this destroys all geometric information in $\mathcal{C}$.

3. Replace presheaves $\mathrm{PSh}(\mathcal{C})$ with sheaves $\mathrm{Sh}(\mathcal{C}, \tau)$, for an appropriate Grothendieck topology τ.

For an example of why presheaves lose geometric information in $\mathcal{C}$, consider $X, Y \in \mathcal{C}$, and suppose the coproduct $Z = X \coprod Y$ exists. Then we have $h_X, h_Y, h_Z \in \mathrm{PSh}(\mathcal{C})$. Unfortunately, $h_X \coprod h_Y \neq h_Z$. More concretely, take $X = Y = \mathrm{Spec}\, k[x]$ to be affine lines. Then $h_X \coprod h_Y = \mathrm{Map}(-, \mathbb{A}^1_k) \coprod \mathrm{Map}(-, \mathbb{A}^1_k)$. On the other hand, we have $h_Z = \mathrm{Map}(-, \mathbb{A}^1_k \coprod \mathbb{A}^1_k)$. Evaluating both on $\mathrm{Spec}\, k \coprod \mathrm{Spec}\, k$, $h_X \coprod h_Y$ gives a point on each affine line, while h_Z allows two points on the same affine line. This problem doesn't arise when working with sheaves, rather than presheaves, because then $h_X \coprod h_Y$ is defined by applying sheafification to the coproduct of presheaves.

Remark 8.4.6. A moment's thought should convince you that these "geometric relations" in $\mathcal{C}$ are the same as a Grothendieck topology. The best way to explain this is the following lemma.

Lemma 8.4.7. *Let $(\mathcal{C}, \tau)$ be a Grothendieck site. Then τ is subcanonical (every representable preseheaf is a sheaf) if and only if for all $U^\bullet \to X$ τ-cover of X in $\mathcal{C}$, we have $X \cong \operatorname{colim} U^\bullet$.*

Lemma 8.4.8. *Let $(\mathcal{C}, \tau)$ be a Grothendieck site. Then $\mathrm{Sh}(\mathcal{C}, \tau)$ has the following universal property:*

1. *The functor $j : \mathcal{C} \to \mathrm{Sh}(\mathcal{C}, \tau)$ sends Cech nerves of τ-covers to colimits.*
2. *j is universal with property (1), i.e. for all ∞-categories $\mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ such that F sends Cech nerves of τ-covers to colimits, there exists a unique up to homotopy extension $\tilde{F}$ making the following diagram commute.*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j"] \arrow[dr, "F"'] & \mathrm{Sh}(\mathcal{C}, \tau) \arrow[d, dashed, "\tilde{F}"] \\ & \mathcal{D} \end{tikzcd}$$

Thus, we need to replace $\mathrm{PSh}(\mathcal{C})$ with $\mathrm{Sh}(\mathcal{C}, \tau)$. The case we're primarily interested in is $\mathcal{C} = dAff_k$, and $\tau = \tau_{\text{ét}}$.

Remark 8.4.9. $\tau_{\text{ét}}$ is subcanonical thanks to the descent for *QCoh* that we discussed.

Often in $\mathcal{C}$ there are many interesting geometric notions (e.g. properness). It's not clear how to translate them for random objects in $\mathrm{Sh}(\mathcal{C}, \tau)$. Therefore we restrict to a full subcategory $\mathrm{Geom}(\mathcal{C}, \tau, \mathbb{P}) \subset \mathrm{Sh}(\mathcal{C}, \tau)$, spanned by "tame objects", where the geometric notions transport in a painless way. The idea is that every sheaf is a colimit of representable ones, but we want to restrict the diagrams which can idex the colimit.

The original idea of geometric stack is due to Artin; it has been generalized by Simpson. Before starting, we fix a collection of morphisms $\mathbb{P}$ in $\mathcal{C}$, which are **τ-local**. I.e. for $f : X \to Y$ in $\mathcal{C}$, $f \in \mathbb{P}$ iff for all τ-covers $Y_i \to Y$, $X \times_Y Y_i \to Y_i$ is in $\mathbb{P}$. [9] In our case, τ is étale and $\mathbb{P}$ is smooth morphisms.

Definition 8.4.10.

1. $F \in \mathrm{Sh}(\mathcal{C}, \tau)$ is (-1)-geometric if it's representable ($F = \mathrm{Spec}^f(A)$, where f stands for functor).
2. A morphism $f : F \to G$ is (-1)-geometric if for all $X \in \mathcal{C}$ and all $h_X \to G$, $h_X \times_G F$ is representable. In other words, there is some Y making the following diagram cartesian.

$$\begin{tikzcd} h_Y \arrow[r] \arrow[d] & F \arrow[d] \\ h_X \arrow[r] & G \end{tikzcd}$$

3. An n-atlas of $F \in \mathrm{Sh}(\mathcal{C}, \tau)$ is an epimorphism $\pi : U \to F$ such that U is (-1)-geometric, π is $n-1$-geometric and π is $n-1$-$\mathbb{P}$.
4. $F \in \mathrm{Sh}(\mathcal{C}, \tau)$ is n-geometric if it has an n-atlas and $F \to F \times F$ is $n-1$-geometric.
5. A morphism $f : F \to G$ is n-geometric if for all $h_X \to G$, $h_X \times_G F$ is $n-1$-geometric.

Remark 8.4.11. If you start with affine schemes, 0-geometric is algebraic spaces.

Remark 8.4.12. Let F be a geometric stack, and let $\pi : U \to F$ be an n-atlas. Then $U^\bullet = \check{C}(U \to F)$ is a groupoid, and each level U^k is $n-1$-geometric. Moreover, $|U^\bullet| \simeq F$, and the transition morphisms are in $\mathbb{P}$. The converse is also true: given a groupoid where transition morphisms are in $\mathbb{P}$, the geometric realization is a geometric stack.

[9] Actually this is local on target, we may also need local on domain.

Remark 8.4.13. In $\mathcal{C} = d\mathrm{Aff}_k$, it's not necessary to ask $F \to F \times F$ to be $n-1$-geometric.

Remark 8.4.14. There's always a cheap way of extending geometric properties of morphisms in $\mathcal{C}$ to analogous properties of geometric stacks. This goes by induction. But caution: this is not necessarily the best thing to do, e.g. properness has to be dealt with more cleverly.

Remark 8.4.15. In the specific case $\mathcal{C} = d\mathrm{Aff}_k$, given $F : d\mathrm{Aff}_k^{\mathrm{op}} \to \mathcal{S}$, we consider the composition:

$$\begin{tikzcd} \mathrm{Aff}_k^{\mathrm{op}} \arrow[r, "t_0(F)"] \arrow[d, hook] & \mathcal{S} \\ d\mathrm{Aff}_k^{\mathrm{op}} \arrow[ur, "F"'] & \end{tikzcd}$$

We call this **truncation**. The truncation functor preserves geometric stacks, as well as finite limits. Moreover, $t_0(\mathrm{Spec}(A)) = \mathrm{Spec}^f(\pi_0(A))$.

Remark 8.4.16. If F is a derived geometric stack, then in general F is not truncated. For example, take $A = k[x]$, $F = \mathrm{Spec}\, A$, $F(B) = \mathrm{Map}(\mathrm{Spec}\, B, \mathrm{Spec}\, A) = \mathrm{Map}_{sCR_k}(A, B) \cong B$, the underlying topological space of B. This can be arbitrarily complicated. However, $t_0(F)$ always factors through $\mathcal{S}^{\leq n}$ for some n. More precisely, if F is n-geometric, then $t_0(F)$ is $n+1$-truncated. The proof is simple, by induction on the geometric level. [10]

Remark 8.4.17. When $\mathbb{P}$ is smooth, we call $\mathrm{Geom}(d\mathrm{Aff}_k, \tau_{\text{ét}}, \mathbb{P})$ **derived Artin stacks**. When $\mathbb{P}$ is étale, we call $\mathrm{Geom}(d\mathrm{Aff}_k, \tau_{\text{ét}}, \mathbb{P})$ **derived Deligne-Mumford stacks**.

From now on, we only consider the case $\mathcal{C} = d\mathrm{Aff}_k$.

Lemma 8.4.18. *$F \in \mathrm{Geom}(d\mathrm{Aff}_k, \tau_{\text{ét}}, \mathbb{P})$, there exists a functor of the form:*

$$F_{\text{ét}} \to t_0(F)_{\text{ét}}$$
$$\big(G \overset{\text{ét}}{\to} F\big) \mapsto \big(t_0(G) \overset{\text{ét}}{\to} t_0(F)\big),$$

which is a Morita equivalence. [11] *Note that $F_{\text{ét}}$ is the small étale site, which considers derived affines mapping into F via étale maps.*

Corollary 8.4.19. *For $F \in \mathrm{Geom}(d\mathrm{Aff}_k, \tau_{\text{ét}}, \mathbb{P})$, if F is n-geometric, then $F_{\text{ét}}$ is an n-category.*

Remark 8.4.20. Let $F, G \in \mathrm{Geom}(d\mathrm{Aff}_k, \tau_{\text{ét}}, \mathbb{P})$ and $f : F \to G$. The pullback $G_{lis-\text{ét}} \to F_{lis-\text{ét}}$ [12] does not induce a geometric morphism of topoi. There is an adjunction:

$$f^* : \mathrm{Sh}(G_{lis-\text{ét}}, \tau_{\text{ét}}) \to \mathrm{Sh}(F_{lis-\text{ét}}, \tau_{\text{ét}}) : f_*,$$

but f^* does not preserve finite limits. The reason for this, approximately, is that certain pullbacks don't exist in the lisse-étale site. If we use the étale site, given two étale morphisms $X \to G$, $Y \to G$, any morphism between them is forced to be étale as well. But this is no longer true when using the lisse-étale site. This is a thorny problem, but it can be avoided if one is content with working with coherent sheaves.

8.5 Comparison of approaches

Theorem 8.5.1. *Let dDM^{loc} be the ∞-category of structured derived DM-stacks $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, where $\mathcal{X}$ is n-localic for some n. Let $dDM^f := \mathrm{Geom}(d\mathrm{Aff}_k, \tau_{\text{ét}}, \mathbb{P}_{\text{ét}})$ be derived DM-stacks obtained from the functor of points. Then there is an equivalence:*

$$dDM^{loc} \simeq dDM^f.$$

[10] In fact, this is how most proofs go for geometric stacks.

[11] This means that the categories of sheaves on them are equivalent.

[12] These sites are called **lisse-étale**, which means that $\mathbb{P}$ is the class of smooth morphisms but the topology τ is étale.

Proof. The construction $dDM^{loc} \to dDM^{f}$ takes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and sends it to the functor $F : dAff_k^{\mathrm{op}} \to \mathcal{S}$, which maps B to $\mathrm{Map}(\mathrm{Spec}^{\text{ét}}(B), (\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$. $\mathcal{X}$ localic means that F satisfies hyperdescent.

In the other direction, pick F and send it to $\big(\mathrm{Sh}(F_{\text{ét}}, \tau_{\text{ét}}), \mathcal{O}_F\big)$. Thanks to Corrolary 8.4.19 $\mathrm{Sh}(F_{\text{ét}}, \tau_{\text{ét}})$ is n-localic. □

8.6 Descent and infinitesimal theory

How do we define $QCoh(F)$ for a sheaf F? We would like it to be the dashed arrow in the diagram:

$$\begin{tikzcd} \mathrm{dAff}_k^{\mathrm{op}} \arrow[r, "QCoh"] \arrow[d] & \mathrm{Pr}^L \\ \mathrm{Sh}(d\mathrm{Aff}_k, \tau_{\text{ét}}) \arrow[ur, dashed, "\underline{QCoh}"'] & \end{tikzcd}$$

Then one defines $\underline{QCoh}(F)$ as the **mapping stack** $\underline{Map}(F, QCoh)$. In turn, this is defined as the adjoint of $- \times F$. That is, the functor of points of a mapping stack is:

$$\mathrm{Map}(\mathrm{Spec}\, A, \underline{Map}(F, QCoh)) = \mathrm{Map}(\mathrm{Spec}\, A \times F, QCoh).$$

This is the same as setting $\underline{QCoh}(F) := \lim QCoh(A)$, where the limit is taken over $\mathrm{Spec}(A) \to F$.

Remark 8.6.1. This makes sense for every sheaf F. If F is geometric, then we can restrict to the lisse-étale site when taking the limit.

Definition 8.6.2. Let $F \in \mathrm{Sh}(d\mathrm{Aff}_k, \tau_{\text{ét}})$. Then:

1. Let $x : \mathrm{Spec}\, A \to F$, we say that F **has a cotangent complex at** x if there exists an object $\mathbb{L}_{F,x} \in QCoh^{-}(A)$ such that for all $M \in QCoh(A)$, $\mathrm{Map}_{QCoh(A)}(\mathbb{L}_{F,x}, M) \simeq Der_F(A; M)$. Here the F**-linear derivations** $Der_F(A; M)$ are defined as the pullback:

$$\begin{tikzcd} Der_F(A; M) \arrow[r] \arrow[d] & \mathrm{Map}\big(\mathrm{Spec}(A \oplus M), F\big) \arrow[d, "0"] \\ * \arrow[r, "x"'] & \mathrm{Map}(\mathrm{Spec}(A), F). \end{tikzcd}$$

2. F **has a global cotangent complex** if it has a cotangent complex at x for every $x : \mathrm{Spec}(A) \to F$, and moreover there exists $\mathbb{L}_F \in \underline{QCoh}(F)$ such that for all x, $x^*\mathbb{L}_F \cong \mathbb{L}_{F,x}$.

Theorem 8.6.3. *If F is geometric, then it has a global cotangent complex.*

Remark 8.6.4. If F is dDM, then $\mathbb{L}_F$ is connective, i.e. concentrated in non-positive degrees. However, if F is Artin n-geometric, then $\mathbb{L}_F$ is concentrated in degrees $(-\infty, n]$. (Todo: ± 1)

Chapter 9

The Stack of Perfect Complexes

Talk by Sukjoo Lee.

We first define Perf and Perf(X) and prove their geometricity. Then we proceed to describe the tangent complex of these. Finally, we describe the perfect determinant map and construct the Atiyah class.

9.1 Construction

Recall that we defined $\mathrm{QCoh} : d\mathrm{Aff}^{\mathrm{op}} \to \mathcal{S}$, and extended it to $\underline{QCoh} : \mathrm{Sh}(d\mathrm{Aff}, \tau_{\acute{e}t}) \to \mathcal{S}$. In particular, if X is a scheme, we'll write $\mathrm{QCoh}(X)$ for $\underline{Map}(X, \mathrm{QCoh})$. We also defined $\mathrm{Map}(\mathrm{Spec}\, A, \underline{Map}(X, \mathrm{QCoh})) = \mathrm{Map}(X \times \mathrm{Spec}\, A, \mathrm{QCoh})$.

Now for X a smooth and proper k-scheme, we define a functor $\mathrm{Perf} : d\mathrm{Aff}^{\mathrm{op}} \to \mathcal{S}$, sending $A \mapsto (A - mod)^{perf}$. (Recall the definition of perfect complexes over a derived affine from 6.) The action on morphisms is given by base change, which makes sense, since perfect modules are stable under base change. [1] Perf satisfies étale descent for the same reason that QCoh does; it follows that Perf is a derived stack. Furthermore, we define perfect complexes on a given stack X as the mapping stack:

$$\mathrm{Perf}(X) := \mathrm{Map}_{\mathbf{dSt}}(X, \mathrm{Perf}).$$

Remark 9.1.1. For smooth schemes, Perf and Coh^b are the same. But in order to define the stack Perf, it's essential that we don't restrict to A smooth. Moreover, in general Coh^b is not functorial, because base change can kill the boundedness condition. In fact, Perf and Coh^- make sense functorially, while Coh^+ or Coh^b do not.

9.2 Geometricity

In this section we work towards proving:

Theorem 9.2.1. Perf *and* Perf(X) *are locally geometric and locally of finite presentation.*

Definition 9.2.2 (Properties of stacks, Definition 1.3.6.4 in [22])**.**

(a) A stack F is **quasi-compact** if there exists a finite family of representable stacks X_i and an epimorphism $\coprod_i X_i \to F$. [2]

(b) A morphism of stacks $G \to F$ is **quasi-compact** if for every representable X, $X \times_F G$ is quasi-compact.

[1] Note that, as always, to define a functor rigorously we need to use co-cartesian fibrations.

[2] Throughout this definition, "representable" means representable by a derived affine.

(c) An n-geometric stack F is **strongly quasi-compact** if for arbitrary X, Y representable stacks and maps to X, the $n-1$-geometric stack $X \times_F Y$ is strongly quasi-compact. This is an inductive definition, where at level -1 it just means quasi-compact.

(d) A morphism of stacks $G \to F$ is **strongly quasi-compact** if for every representable X, $X \times_F G$ is quasi-compact.

(e) A stack F is **finitely presented** if for every filtered system of objects $B_i \in \mathrm{cdga}_k$ we have:

$$\operatorname*{colim}_i \mathrm{Map}(\mathrm{Spec}\, B_i, F) \simeq \mathrm{Map}\left(\mathrm{Spec}(\operatorname*{colim}_i B_i), F\right).$$

(f) A stack F is **locally finitely presented** if there exists an n-atlas $\{\mathcal{X}_i\}$ such that each $\mathcal{X}_i$ is finitely presented.

(g) A stack F is **strongly finitely presented** if it is locally finitely presented and quasi-compact.

Remark 9.2.3. All these definitions are stable by pull-backs and retracts.

Definition 9.2.4. F is **locally geometric** if it can be written as a filtered colimit $F \simeq \mathrm{colim}_i F_i$, such that each F_i is n-geometric (n can depend on i!), and $F_i \to F$ is a monomorphism (equivalently, $F_i \to F_i \times_F F_i$ is iso.)

We will use without proof the following lemma.(Todo: give a reference for the proof though)

Lemma 9.2.5. *For $f : F \to G$ n-representable, if G is n-geometric, then so is F.*

Let's begin the proof of Theorem 9.2.1.

Proof. We can define $\mathrm{Perf}^{[a,b]} \subset \mathrm{Perf}$ to consist of the complexes which have tor amplitude contained in $[a, b]$. This is an open immersion, which means that amplitude is stable under quasi-isomorphism. Note that $\mathrm{Perf} = \cup_{a\leq b}\mathrm{Perf}^{[a,b]}$, which exhibits Perf as a union of connected components. Moreover, we can define $\mathrm{Perf}(X)^{[a,b]}$ to be the homotopy pullback:

$$\begin{tikzcd}
\mathrm{Perf}(X)^{[a,b]} \arrow[r] \arrow[d] & \mathrm{Perf}^{[a,b]} \arrow[d] \\
\mathrm{Perf}(X) \arrow[r] & \mathrm{Perf}.
\end{tikzcd}$$

But we actually need to be careful with how to define the bottom map. We need to choose a compact generator E of the ∞-category $\mathrm{Perf}(X)$ of perfect complexes on X, see [20]. (Todo: be more specific) We obtain $\mathrm{Perf}(X) = \cup_{a\leq b}\mathrm{Perf}(X)^{[a,b]}$.

So the strategy is to prove:

1. that $\mathrm{Perf}^{[a,b]}$ is n-geometric and locally finitely presented for $n = b - a + 1$. Due to definition 9.2.4, it follows that Perf is locally geometric and locally finitely presented.

2. that $\mathrm{Perf}(X)^{[a,b]} \to \mathrm{Perf}^{[a,b]}$ is n-representable. Due to Lemma 9.2.5, it follows that $\mathrm{Perf}(X)$ is locally geometric and locally finitely presented.

It remains to prove the two assertions.

1. We want to find a cover $\pi : U \to \mathrm{Perf}^{[a,b]}$ such that U is $n-1$-geometric and l.f.p., and π is an $n-1$ representable, smooth epimorphism.

   The functor $U : \mathrm{cdga}_k \to \mathcal{S}$ is defined to map A to the space of morphisms in A-Mod consisting of $u : Q \to R$, where $Q \in \mathrm{Perf}(A)^{[a,b-1]}$ and $R \in \mathrm{Perf}(A)^{[b-1,b-1]}$. Then $\pi : U \to \mathrm{Perf}^{[a,b]}$ is defined to take $u : Q \to R$ to $\mathrm{hofib}(u)$.

In order to show that U is $n-1$ geometric, we build a map p from U to an $n-1$ geometric stack, then show that p is $n-1$ representable, and invoke Lemma 9.2.5. If $a=b$, then $\mathrm{Perf}^{[a,b]} \simeq \underline{Vect}$, which is a 1-geometric l.f.p. stack. This is because $\underline{Vect} = \cup_n BGL(n)$, and each of $BGL(n)$ is Artin, as follows from the groupoid presentation. We make the inductive assumption that $\mathrm{Perf}^{[a,b-1]}$ is n-1-geometric. Then define the map:

$$\begin{aligned} p : U &\to \mathrm{Perf}^{[a,b-1]} \times \underline{Vect} \\ \{u : Q \to R\} &\mapsto \big(Q, R[b-1]\big). \end{aligned}$$

p is $n-1$ representable; in fact, it's representable, which is a consequence of Sub-Lemma 3.11 in [20]. It follows that U is $n-1$ geometric.

Lemma 9.2.6. *The diagonal of* $\mathrm{Perf}^{[a,b]}$ *is* $n-1$*-representable.*

(Todo: this is part of the definition of an Artin stack, but not sure where the result is coming from)

The smoothness of $\pi : U \to \mathrm{Perf}^{[a,b]}$ is proved using the infinitesimal criterion for smoothness, which is Corollary 2.2.5.3 in [22]. One needs to check that, for every $A \in \mathrm{cdga}_k$ and map $x : \mathrm{Spec}\, A \to \mathrm{Perf}$, the cotangent complex $\mathbb{L}_{\mathrm{Perf}^{[a,b]}/U,x}$ is perfect in A-Mod and concentrated in non-negative degrees.

Finally, we show that $\pi : U \to \mathrm{Perf}^{[a,b]}$ is an epimorphism. For all $P \in \mathrm{Perf}^{[a,b]}(A)$, we can find a vector bundle E in $\mathrm{Spec}(A)$ and a morphism $E[-b] \to P$ whose cofiber Q is contained in $\mathrm{Perf}^{[a,b-1]}$. Writing down the resulting fiber sequence, P is a homotopy fiber of $Q \to E[-b+1]$.

2. We need to show that $\mathrm{Perf}(X)^{[a,b]} \to \mathrm{Perf}^{[a,b]}$ is $n-1$ representable and strongly f.p. (Todo: strongly? did I mean to write locally?) First we do a reduction to $X = \mathrm{Spec}\, B$, where B is homotopically f.p. and:

$$\begin{tikzcd} \pi^{-1}p & \mathrm{Spec}\, A \arrow[d, "p"] \\ \mathrm{Perf}(X)^{[a,b]} \arrow[r, "\pi"] & \mathrm{Perf}^{[a,b]} \end{tikzcd}$$

$\pi^{-1}(p) = \underline{Map}(B, \epsilon(p))$, where $\epsilon(p) := End_{A-mod^{perf}}(p,p)$; Lemma from TV 3.13 and 3.14, and 3.9 retraction (Todo: help! someone clear this up)

□

9.3 Tangent complex

Since Perf and $\mathrm{Perf}(X)$ are locally geometric, they admit a cotangent complex. Since they are locally f.p., it is a perfect complex, hence dualizable, and we can talk about a tangent complex. Even though the tangent complex exists globally, we only compute it locally for now.

Theorem 9.3.1. *Let* $E : * = \mathrm{Spec}\, k \to \mathrm{Perf}(X)$ *be a perfect complex on* X*. Then* $T_E \mathrm{Perf}(X) = End_{\mathrm{Perf}(X)}(E,E)[1]$.

Proof. Given E, we construct:

$$\begin{aligned} \Omega_E \mathrm{Perf}(X) : \mathrm{cdga}_k &\to \mathcal{S} \\ A &\mapsto \mathrm{Map}\big(k, \mathrm{End}(E,E) \otimes A\big). \end{aligned}$$

(Todo: finish this) □

Remark 9.3.2. In particular, this recovers the entire deformation theory for vector bundles, by taking E to be in degree 0. The shift means that we recover Ext^1.

9.4 The Determinant Functor

Definition 9.4.1. The **Picard stack** is $\mathrm{Pic}(X) = \mathrm{Map}(X, BG_m)$.

Note that $\mathrm{Pic} \simeq \mathrm{Vect}_1 \subset \mathrm{Perf}$. We want to construct $\det : \mathrm{Perf} \to \mathrm{Pic}$, by using Waldhausen K-theory. The steps are as follows.

1. We have $\det : \mathrm{Vect} \to \mathrm{Pic}$.

2. Construct the simplicial stacks:

$$\begin{aligned} B_\bullet \,\mathrm{Pic} &: \Delta^{\mathrm{op}} \ni [n] \mapsto \mathrm{Pic}^n, \\ B_\bullet \,\mathrm{Vect} &: \Delta^{\mathrm{op}} \ni [n] \mapsto wS_n(\mathrm{Vect}), \\ B_\bullet \mathrm{Perf} &: \Delta^{\mathrm{op}} \ni [n] \mapsto wS_n(\mathrm{Perf}). \end{aligned}$$

   The simplicial set $wS_n(\mathrm{Vect})$ is the nerve of the category which has objects sequences of split monomorphisms:
$$0 \to M_1 \to \cdots \to M_k \to 0,$$
   and morphisms are weak equivalences. $wS_n(\mathrm{Perf})$ is defined analogously.

3. Extend det to a map $B_\bullet \,\mathrm{Vect} \to B_\bullet \,\mathrm{Pic}$.

4. Pass to Waldhaussen K-theory, applying the functor $K = \Omega \circ |-|$, where the latter denotes geometric realization.

5. We obtain:
$$K(\det) : K(B_\bullet \,\mathrm{Vect}) \to K(B_\bullet \,\mathrm{Pic}) \simeq \mathrm{Pic}\,.$$
   The latter isomorphism is an analog of the familiar $\Omega|B_\bullet G| \simeq G$.

6. The inclusion $\mathrm{Vect} \to \mathrm{Perf}$ determines a map $\mu : K(B_\bullet \,\mathrm{Vect}) \to K(B_\bullet \mathrm{Perf})$; we claim that this is an isomorphism. The general principle at play is: given categories C and D, if each object of C has a resolution by objects of D, then the K-theory of the two categories is the same.

7. The perfect determinant map is the composition of:

$$\mathrm{Perf} \longrightarrow K(B_\bullet \mathrm{Perf}) \xrightarrow{\mu} K(B_\bullet \mathrm{Vect}) \xrightarrow{K(\det)} K(B_\bullet \,\mathrm{Pic}) \simeq \mathrm{Pic}\,.$$

Infinitesimally the perfect determinant induces a trace map, as follows. For $x_E : \mathrm{Spec}\, k \to \mathrm{Perf}(X)$, there is an induced morphism on tangent spaces:

$$\begin{tikzcd} T_{\mathrm{Perf}(X),E} \arrow[r] \arrow[d, "\cong"'] & T_{\mathrm{Pic}(X),\det(E)} \arrow[d, "\cong"] \\ \mathrm{End}(E,E)[1] \arrow[r, "\mathrm{tr}_E[1]"] & \mathrm{End}(\det(E),\det(E))[1]. \end{tikzcd}$$

Definition 9.4.2. Let $\mathcal{Y}$ be a geometric stack having a perfect cotangent complex, E a perfect complex on $\mathcal{Y}$, which is the same as $\phi_E : \mathcal{Y} \to \mathrm{Perf}$. There is an induced map:

$$T_{\phi_E} : \mathbb{T}_{\mathcal{Y}} \longrightarrow \phi_E^* \mathbb{T}_{\mathrm{Perf}} \simeq E^* \otimes E[1].$$

The **Atiyah class** is the morphism dual to the above:

$$\mathrm{at}_E : E \longrightarrow \mathbb{L}_{\mathcal{Y}} \otimes E[1].$$

For example, $\mathrm{at}_{\det(E)} = \mathrm{tr}_{E[1]}$.

Theorem 9.4.3. *There is an equivalence $FMP_k \to dgLie_k$, which sends a formal moduli problem F to $T_F[-1]$. In particular, if we take the formal completion $\mathcal{Y}_x$, it gets sent to $x^*T_y[-1]$.*

Fact: the Lie algebra structure on $x^*T_Y[-1]$ is given by the Atiyah class. Nmely, the bracket is (Todo: finish this section)

Chapter 10

D-modules

Talk by Benedict Morrissey.

10.1 Introduction and motivation

We want to talk about differential equations in the setting of AG. It's all classical AG today, but things will go south starting next week. We also want to talk about quantization: replace functions on the classical phase space ($\mathcal{O}(T^*X)$) with differential operators on X ($\mathcal{D}(X)$).

There's also motivation from representation theory: Beilinson-Bernstein says roughly that $Rep(\mathcal{U}(\mathfrak{g})) \simeq \mathcal{D}_{G/B} - mod$.

The setting is X smooth complex projective variety. We introduce sheaf of noncommutative algebras: $\mathcal{D}_X = Der_{\mathcal{C}_X}(\mathcal{O}_X)$. Say $U \subset X$ is a coordinate chart, with coordinates $y_1, \ldots, y_n$, $\mathcal{D}_X$ is generated by $[y_i, \partial_j] = \delta_{ij}$.

First thing to notice is that $\mathcal{D}_X$ acts on $\mathcal{O}_X$ on the left. Therefore $\mathcal{D}_X(U)$ also acts on the distributions $\mathcal{O}_X(U)^*$ on the right.[1]

Say f is the solution of a differential equation on X, e.g.:

$$(\partial_x - \lambda)f = 0.$$

then $\mathcal{D}_X \dot{f} = \mathcal{D}_X/(\partial_x - \lambda)f$, and we call this $\mathcal{D}$-module $M_{e^{\lambda x}}$. Then $\mathrm{Hom}_{\mathcal{D}_X - mod}(M_{e^{\lambda x}}, C^\infty(\mathbb{A}^1))$ is in bijective correspondence with solutions to the differential equation.

Note also that this sheaf of solutions is a local system.

10.2 Operations on D-modules

Example 10.2.1. We start with pullback. If $f : X \to Y$, and $N \in \mathcal{D}_Y$ is a D-module, we define:

$$p^*N = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}N.$$

So far, this is just a sheaf of $\mathcal{O}_X$ modules. We also need to define an ation by derivations $\theta \in \mathcal{D}_X$, as follows. (Todo: paper) Taking coordinates $\{y_i\}$ on Y,

$$\theta(\psi \otimes s) = \theta(\psi) \otimes s + \psi \sum_{i=1}^{n} \theta(y_i \circ f) \otimes \partial_i(s).$$

Note that, if the D-module is one of functions, this is the same as pulling back the function to X, and then acting by differentition.

[1] $\mathcal{O}_X(U)$ should be the analytic sheaf.

Direct image is more annoying; we need to use complexes of D-modules for it to be meaningful. We define:

$$f_*M = \mathcal{D}_{Y,X} \otimes^{\mathbb{L}}_{\mathcal{D}_X} M.$$

Here $\mathcal{D}_{Y,X}$ is a $\mathcal{D}_Y - \mathcal{D}_X$ bimodule defined as follows. (Todo: paper)

Fourier transform of $\mathcal{D}$-modules. Think about functions first: we pull back functions from $\mathbb{A}^1$ to $\mathbb{A}^2$, multiply with a function (kernel) there, and then push forward to the second factor of $\mathbb{A}^1$. For $\mathcal{D}$-modules, this is done as follows:

$$F(M) = p_{2*}\big((p_1^*M) \otimes M_{e^{2\pi i x y}}\big).$$

10.3 Characteristic Variety

We next investigate how the D-module relates to the geometry of the solution set of the associated differential equation. We take the associated graded (Todo: enter more details here), and $\mathrm{Spec}(\mathcal{D}_X^{gr}) = T^*X$, as long as X is smooth. For any D-module M, the associated graded M^{gr} is a coherent $\mathcal{O}(T^*X)$-module. Then we associate the **character variety** $ch(M) = sup(M^{gr}) \subset T^*X$.

We have a functor $D_X - mod \to \mathcal{O}_X - mod$, $sol_X(M) = \mathbb{R}\,\mathrm{Hom}_{\mathcal{D}_X}(M, \mathcal{O}_X)$. There's an easier functor to work with, the deRHam functor, which does: $DR_X(M) = M \otimes \Omega^\bullet_X$. There's a duality $DR_X(M) = sol(D_X M)$, where (Todo: paper)

Definition 10.3.1. $\eta \in T^*_x X$ is **non-characteristic** if for any division of X into level sets of h such that $dh = \eta$, $dR(M)$ is locally constant transverse to level sets.

Theorem 10.3.2 (Cauchy-Kovalevski). *The set* $\{\eta \in T^*X \;\; characteristic\} = ch(M)$.

We have $\dim(ch(M)) \geq \dim(X)$; if equality holds, we call M **holonomic**.

If M is holonomic, $dR(M)$ is really nice, in the sense that $H^i(dR(M))$ is locally ocnstant on the strata of some stratification of X, $X = \bigsqcup_{\alpha \in \Lambda} X_\alpha$. Think about solutions to the logarithm, whose associated D-module is locally constant on the strata $\mathbb{C} = \mathbb{C}^\times \sqcup \{0\}$.

Definition 10.3.3. A D-module M is **regular** if for all $x \in X$, $\mathbb{R}\,\mathrm{Hom}_D(M_x, \mathcal{O}_x) \cong \mathbb{R}\,\mathrm{Hom}_D(M_x, \hat{\mathcal{O}_x})$.

Remark 10.3.4. This can be promoted to a categorical statement, whereby regular holonomic $\mathcal{D}$-modules correspond under the deRham functor to constructible complexes. THis is the Riemann-Hilbert correspondence.

10.4 More general spaces

We can embed a singular variety X in a nonsingular Y. A theorem of Kashiwara tells us that the following is well-defined:

$$\mathcal{D}_X - mod = \{U \in \mathcal{D}_Y - mod | supp_{\mathcal{O}_Y}(U) \subset X\}.$$

Quotient stacks: $X = Y/G$, where Y is smooth and G is a reductive algebraic group. We want to relate D-modules on X to D-modules on Y with a strong equivariance property.

Definition 10.4.1. A D-module M on Y is **strongly equivariant** if there's an equivalence $\alpha : act^*M \to p^*M$ of $\mathcal{D}_G \otimes \mathcal{D}_Y$-modules, where the latter is $p_1^*\mathcal{D}_G \otimes p_2^*\mathcal{D}_Y$. [2]. This has to satisfy some compatibility condition (Todo: paper)

[2] Change to square tensor

10.5 Representation theory

Beilinson-Bernstein theorem. G acts on $G/B = Fl$. Infinitesimally this gives a map $\mathfrak{g} \to T_p Fl$, ie. $\mathfrak{g} \to$ Vector fields on Fl. This extends to a map:

$$\mathcal{U}(\mathfrak{g}) \to \mathcal{D}_{Fl}.$$

Then given a line bundle $\mathcal{L}^\chi$ on G/B, we get $\mathcal{D}^\chi$ acting on the sections of $\mathcal{L}$. Note that Fl is the moduli space of Borels. Given a character $\chi : T \to \mathbb{C}$, I get a character on B by precomposing with the projection $B \to T$, i.e. the character is trivial on strictly upper-triangular matrices.

There's a theorem by Harish-Chandra saying that, to such a character, we can associate (Todo: paper). Then the theorem says that, for χ regular, we have $\mathcal{U}(\mathfrak{g})^\chi - mod = \mathcal{D}^\chi_{FL} - mod$.

10.6 Nearby and vanishing cycles

Say I have a family of algebraic varieties which degenerate at some point. Say I have the family of tori over $\mathbb{C}$ given by $y^2 = x(x-a)(x-2a)$. But over 0 I have the cuspidal cubic $y^2 = x^3$. We want to use data about the nearby nice objects to talk about the singular object.

The general setup is:

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] & X^\times \arrow[l] \arrow[d] \\ \{0\} \arrow[r] & \mathbb{C} & \mathbb{C}^\times . \arrow[l] \end{tikzcd}$$

Nearby cycle functor: (Todo: paper)

Theorem: if f is a proper map, then there exists U of X_0 in X such that $q : U \to X_0$ is continuous and homotopes to identity on X_0.

Silly example: if $X = \mathbb{C}$, M is a $\mathcal{D}$-module on $\mathbb{C}^\times$, $K = DR(M)$. If we take f to be the identity, then $\psi_f(K) \cong K_{X_\epsilon}$. We have a monodromy operator $T : \psi_f(K) \to \psi_f(K)$, sending $a \mapsto a + c$. If K comes from logarithm, this is $a \mapsto a + 2\pi$.

If M is holonomic, from adjunction we have a unit map $M \mapsto p_* p^* M$ (need holonomic to have $p^* = p^!$). Then we have an exact triangle in the derived category, $i^* M \to \psi_f(M) \to \Psi_f[K](1) \to \dots$, the latter is the **vanishing cycle**.

An application to geometric representation theory: (Todo: paper).

More generally, can take parabolic $P \subset G$. We have the projection $q : G[[t]] \to G$, and take $\bar{P} = q^{-1}(P)$. Then $\bar{P}$-equivariant constructible sheaves on $G((t))/\bar{P}$ should be equivalent to representations of some other group. Now take a family over X a Riemann surface; this is related to the vanishing cycle, but none of us understand this yet.

Chapter 11

D-modules: de Rham space perspective

Talk by Benedict Morrissey.

We will relate these to crystals, and use this to describe D-modules in DAG. In particular, we work towards the following.

Theorem 11.0.1. *If X is a smooth scheme, then $Crys(\mathrm{QCoh}(X)) \simeq D_X - mod$ is an equivalence of categories.*

11.1 Classical setting

Definition 11.1.1. A **crystal** of quasi-coherent sheaves on X is $\mathcal{F} \in \mathrm{QCoh}(X)$ with the following data. For every map from an affine scheme $x : \operatorname{Spec} R \to X$, let $\mathcal{F}_x = x^*\mathcal{F}$. Let I be the nilpotent ideal of R, and consider the sequence of maps:

$$\operatorname{Spec}(R/I) \to \operatorname{Spec} R \overset{x,y}{\rightrightarrows} X.$$

We say x, y are **arbitrarily close** if the associated R/I points are equal. We also need the choice of:

$$\eta_{x,y} : \mathcal{F}_x \to \mathcal{F}_y$$

an isomorphism of R-modules, satisfying a cocycle condition: $\eta_{x,z} = \eta_{y,z} \circ \eta_{x,y}$.

Definition 11.1.2. The **de Rham prestack** is defined via the functor of points:

$$X_{dR}(R) = X(R/I).$$

Note that crystals are very similar to quasi-coherent sheaves on X_{dR}. That's because the latter have stalks over each $\bar{x}$, and we can pull these back to get stalks at each x, y which are canonically isomorphic.

Remark 11.1.3. Recall that we haven't defined QCoh for a prestack, so what exactly we mean is problematic.

Proof.

$$\operatorname{Spec}(R/I) \to \operatorname{Spec}(R) \overset{x,y}{\rightrightarrows} X \times X$$

The two maps factor through the diagonal. However, the maps from $\operatorname{Spec} R$ don't necessarily factor through the diagonal: we can get fat points pointing in directions perpendicular to the diagonal. However, it factors through the completion of the diagonal.

Let's define this more precisely. $\Delta_X \in X \times X$ is defined as the vanishing of some sheaf of ideals (such as $x - y$, but globally). In other words, $\Delta_X = \mathrm{Spec}_{X\times X}(\mathcal{O}_{X\times X}/I)$. Then we define the partial formal completions:

$$\Delta_X^n = \mathrm{Spec}_{X\times X}(\mathcal{O}_{X\times X}/I^n).$$

Finally, **the formal completion of the diagonal** is defined as the ind-scheme:

$$\hat{\Delta}_X = \operatorname*{colim}_n \Delta_X^n.$$

Then we have that $Crys(\mathrm{QCoh}(X)) \cong \mathrm{QCoh}(\hat{\Delta}_X)$. (Todo: not sure how to finish this, but the derived case we do in a bit should take care of it). For each n, consider the 2 projections:

$$\begin{tikzcd} & \Delta_X^n & \\ X & & X. \end{tikzcd}$$

Pulling back and pushing forward gives:

$$\mathcal{F} \to \pi_{1*}^{(n)}\pi_2^{(n)*}\mathcal{F} = \mathcal{O}_X^{(n)} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Where $\mathcal{O}_X^{(n)}$ is by definition $\pi_{1*}^{(n)}\mathcal{O}_{\Delta_X^{(n)}}$.

Now let $\mathcal{D}_X^{\leq n}$ be differential operators of order $\leq n$.

$$\mathcal{D}_X^{\leq n} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^n} \to \mathcal{O}_X$$

given by $(Dg)(x) = Dg(x, x)$. g is a function of x and y, consider X fixed, and y is varying locally around it. Act the differnetial operators as if it's just a function of y. Get a new function, and evaluate it at (x, x). □

Example 11.1.4. $X = \mathbb{A}^1$, $\mathcal{O}_X = k[x]$. $\mathcal{O}_X^{(n)} = k[x, y]/(x - y)^{n+1}$. The latter has a basis as a $k[x]$-module, given by $\{(x - y)^k\}_{0\leq k\leq n}$. Moreover:

$$\mathcal{D}_X^{\leq n} = \left\{ \frac{1}{k!}\frac{\partial^k}{\partial y^k} \right\}$$

as $k[x]$-module.

Lemma 11.1.5. *If you act by differential operators twice, you get a commutative diagram:*

$$\begin{tikzcd} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_X \mathcal{F} & \mathcal{D}_X \otimes_X \mathcal{F} \\ \mathcal{D}_X \otimes_X \mathcal{F} & \mathcal{F}. \end{tikzcd}$$

Summing up, we have the chain of equivalences $Crys(\mathrm{QCoh}(X)) \cong \mathrm{QCoh}(\hat{\Delta}_X) \cong \mathcal{D}_X - mod$.

11.2 Derived setting

Recall X_{dR}, which we redefine as:

$$X_{dR}(R) = X(R^{red}),$$

where $R^{red} = H^0(R)/nil(H^0(R))$.

Definition 11.2.1. Left D-modules are:

$$D - mod^l(X) := \mathrm{QCoh}(X_{dR}),$$

where the latter are restricted to be eventually co-connective. Similarly **right D-modules** are:

$$D - mod^r(X) := IndCoh(X_{dR}),$$

wher the latter are restricted to be locally almost of finite type.

How about the definition of QCoh(X)? We'll just take $\mathrm{QCoh}(X) = \lim \mathrm{QCoh}(S)$, where the limit is taken over all $S \to X$ derived affines. Recall also the definition of IndCoh: it's the colimit completion of Coh(Y) in PSh(Y). Then we can define a map ψ_Y as a left Kan extension:

$$\begin{tikzcd} \mathrm{Coh}(Y) \arrow[r] & \mathrm{QCoh}(X) \\ \mathrm{IndCoh}(Y) \arrow[ur, "\psi_Y"] & \end{tikzcd}$$

These are all stable ∞-categories with t-structure.

We have a natural transformation $p_{DR} : I \to DR$, which gives natural transofrmations:

$$D - mod^l \to \mathrm{QCoh}$$
$$D - mod^r \to IndCoh.$$

(Todo: here I and DR are reversed, what's up with that?)

11.3 Properties

Let's talk about descent. $D - mod^r$ satisfies fpppf, h- and etale descent on $DGSch^{aff}_{aft}$, where the subscript says almost of finite type. [1]

We also have an adjunction:

$$obliv^r_x : D - mod^r(X) \to IndCoh(X) : ind^r_X,$$

which is monadic. Therefore $IndCoh(X_{dR}) \cong T - mod(IndCoh(X))$.

T-structure. QCoh(Y) for Y a prestack has a t-structure, such that for $S \to Y$, $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(S)$ preserves the t-structure. Same holds for $IndCoh$.

Proposition 11.3.1. *For X a stack,*

$$D(D - mod^r(X)^{\heartsuit}) \simeq D - mod(X).$$

Remark 11.3.2. This is not that surprising, since when passing from X to X_{dR} we're throwing out all the derived information.

11.4 Differential operators in the derived setting

This relation follows closely the proof from the classical case, in section 11.1

Proposition 11.4.1 (GR 3.1.3)**.** *There's an equivalence:*

$$IndCoh(X_{dR}) \simeq Tot(IndCoh(X^{\bullet}/X_{dR})).$$

The latter is defined by the Čech nerve of the natural map $X \to X_{dR}$.

Let $X, Y \in DGSch_{aft}$.

Proposition 11.4.2. *$IndCoh(X)$ is cannonically self-dual, in the sense that:*

$$Funct_{cont}(IndCoh(X), IndCoh(Y)) \simeq IndCoh(X) \otimes IndCoh(Y) \simeq IndCoh(X \times Y).$$

$Funct_{cont}$ means limit-preserving functors.

[1] We have no idea what h- is, but it's apparently important. Also, fpppf is not a typo; Gaitsgory and Rozenblyum add another condition which gives a third f.

Proof. Take $Q \in IndCoh(X \times Y)$. Then consider the projections:

$$\begin{tikzcd}
 & X \times Y \arrow[d, "\Delta \times Id"] & \\
 & X \times X \times Y & \\
X & & X \times Y \\
 & & Y.
\end{tikzcd}$$

Run Q through this to get a map from X to Y. You need to say the words $(\Delta \times id)^!(\mathcal{F} \otimes Q)$, but the tensor product is square. $\square$

Recall the monad T, define $D_X^r = \alpha^{-1}(T) \in IndCoh(X \times X)$. Moreover $|X^\bullet / X_{dR}| \to X \times X$ is a derived analog of the completion of the diagonal. Now we have $D_X^r \simeq \Delta_*^{Indcoh}(\omega_{X \times_{X_{dR}} X})$. [2] ω is some notion of dualizing sheaf, which gives a generalization of Serre duality $- \otimes \omega_Y : IndCoh(Y) \to QCoh(Y)$.

Let $X \in DGSch_{aft}$; by Kashiwara's theorem, it suffices to take it smooth. Let $Diff_X$ be the sheaf of diff operators in the sense ofthe previous talk; we would like to identify its action with that of D_X^r.

(Todo: paper)

Taking $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{O}_X$, we get $\mathcal{D}_X^l \to Diff_X$.

[2] GR need to modify the pushforward slightly to make it play nicely with Indcoherent sheaves. That's what the IndCoh superscript means. Don't worry about it too much.

Chapter 12

HKR isomorphism and shifted symplectic structures

Talk by Matei Ionita.

We have reached the "structured" part of our "structured DAG" plan for the semester. We want to talk about shifted symplectic structures and give examples, following [15]. But first we discuss some prerequisites.

12.1 Affine Stacks

For this section we follow [25], Exposé 10. Recall that we defined derived affine schemes as the category opposite to $\mathrm{cdga}_k^{\leq 0}$. We have the following adjoint pair:

$$dSt \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathrm{cdga}_k^{\leq 0}\,.$$

The functor of global sections $\mathcal{O}$ is defined as the unique functor such that $\mathcal{O}(\mathrm{Spec}(A)) = A$ and commutes with colimits. We want to extend this adjunction to cdga's that are not necessarily connective:

$$dSt \underset{\mathrm{Spec}}{\overset{\mathcal{O}}{\rightleftarrows}} \mathrm{cdga}_k\,.$$

This Spec functor is defined by $\mathrm{Spec}(A)(R) = \mathrm{Map}_{\mathrm{cdga}_k}(R, A)$, and then $\mathcal{O}$ is defined as a left adjoint, like before.

Definition 12.1.1. A derived stack is **affine** if it is in the essential image of Spec. Some authors use **co-affine** to refer to the essential image of $\mathrm{cdga}_k^{\geq 0}$ under Spec.

In particular, we can regard topological spaces as constant sheaves, and thus as elements of dSt. Certain topological spaces are co-affine. Recall, for example, the Eilenber-Maclane spaces $K(\mathbb{Z}, n) \simeq B^n\mathbb{G}_a$.

Proposition 12.1.2. $B^n\mathbb{G}_a \simeq \mathrm{Spec}(\mathrm{Sym}\, k[-n])$.

Proof. The proof in [25] is by induction on pushout squares. □

This leads to a description of $\mathcal{O}(X)$ for X the constant sheaf associated to a CW complex.

Theorem 12.1.3. *If X is a CW complex, then there is a quasi-isomorphism $\mathcal{O}(X) \simeq C^*(X)$, where the latter is the singular cochain complex.*

Proof. Write X as the colimit of some diagram of points: $X = \operatorname{colim} \operatorname{Spec} k$. Then consider the map:

$$C^*(X) \simeq C^*(\operatorname{colim} \operatorname{Spec}(k)) \to \lim C^*(\operatorname{Spec}(k)) \simeq \lim k \simeq \mathcal{O}(X).$$

This induces an isomorphism in cohomology:

$$H^n(X) \cong [X, B^n\mathbb{G}_a] \cong \pi_0 \operatorname{Map}_{\mathrm{cdga}_k}(\operatorname{Sym} k[-n], \mathcal{O}(X)) \simeq H^n(\mathcal{O}(X)).$$

□

12.2 HKR isomorphism

In this section we follow [23]; a less technical exposition of these ideas is in [3]. We want to prove an equivalence between cdga's with an S^1 action and, roughly speaking, cdga's with an extra derivation.

Definition 12.2.1. Let $S^1 - \mathrm{cdga}_k^{\leq 0}$ denote connective cdga's with an action of S^1. This definition can be made rigorous by realizing the connetive cdga's as simplicial commutative algebras via Dold-Kan, and realizing S^1 as a simplicial group, which can act on the simplicial algebras.

Example 12.2.2. For $A \in \mathrm{cdga}_k^{\leq 0}$, the free S^1-algebra is $S^1 \otimes A \simeq A \otimes_{A \otimes A} A$, also known as the Hochschild chain complex. The functor $A \mapsto S^1 \otimes A$ is left adjoint to the forgetful functor $S^1 - \mathrm{cdga}_k^{\leq 0} \to \mathrm{cdga}_k^{\leq 0}$.

The second category of interest is the following.

Definition 12.2.3. Let $\epsilon - \mathrm{cdga}_k^{\leq 0}$ be the category of cdga's equipped with an extra differential $\epsilon : A \to A[-1]$. This definition can be made more rigorous by considering the over-category $k[\epsilon] - dg - mod_k$, where $|\epsilon| = -1$. (These are called **mixed complexes** in [15].) This category has a symmetric monoidal structure, arising from the co-multiplication $k[\epsilon] \to k[\epsilon] \otimes k[\epsilon]$. Then we define $\epsilon - \mathrm{cdga}_k^{\leq 0}$ to be the monoids in $k[\epsilon] - dg - mod_k$.

Example 12.2.4. For $A \in \mathrm{cdga}_k^{\leq 0}$, the free ϵ-algebra is **the de Rham complex** $DR(A) = \operatorname{Sym}_A(\mathbb{L}_A[1])$, where the extra differential is the de Rham differential.

$$\begin{tikzcd}
A^{-3} \arrow[d] & \mathbb{L}^{-2} \arrow[d] & (\wedge^2\mathbb{L})^{-1} \arrow[d] & (\wedge^3\mathbb{L})^0 \\
A^{-2} \arrow[d] \arrow[ur, "d_{dR}"] & \mathbb{L}^{-1} \arrow[d] \arrow[ur, "d_{dR}"] & (\wedge^2\mathbb{L})^0 \arrow[ur, "d_{dR}"] & \\
A^{-1} \arrow[d] \arrow[ur, "d_{dR}"] & \mathbb{L}^0 \arrow[ur, "d_{dR}"] & & \\
A^0 \arrow[ur, "d_{dR}"] & & &
\end{tikzcd}$$

The functor $A \mapsto DR(A)$ is left adjoint to the forgetful functor $\epsilon - \mathrm{cdga}_k^{\leq 0} \to \mathrm{cdga}_k^{\leq 0}$.

We can now state and sketch the proof of the main theorem in [23].

Theorem 12.2.5. *There is an equivalence ϕ which commutes with the forgetful functors:*

$$\begin{tikzcd}
S^1 - \mathrm{cdga}_k^{\leq 0} \arrow[rr, "\phi"] \arrow[dr] & & \epsilon - \mathrm{cdga}_k^{\leq 0} \arrow[dl] \\
& \mathrm{cdga}_k^{\leq 0} &
\end{tikzcd}$$

Proof. 1. Note first that $S^1 - \mathrm{cdga}_k^{\leq 0} \simeq \operatorname{Fun}(BS^1, \mathrm{cdga}_k^{\leq 0})$, where BS^1 is the category with a single object and 1-morphisms for each element of S^1.

2. Passing to global sections gives a functor ϕ_3:

$$\begin{tikzcd}
\mathrm{Fun}(BS^1, \mathrm{cdga}_k^{\leq 0}) \arrow[r, "\phi_3"] \arrow[d, "\mathrm{eval}"'] & k[u] - \mathrm{cdga}_k \arrow[d, "k\otimes_{k[u]}-"] \\
\mathrm{cdga}_k^{\leq 0} \arrow[r, hook] & \mathrm{cdga}_k\,.
\end{tikzcd}$$

   Some comments are necessary here. First, the global sections of BS^1 are $k[u]$ with $|u| = 2$, due to Theorem 12.1.3. Second, working with non-connective cdga's is necessary in the right hand column, because $\mathrm{cdga}_k^{\leq 0}$ is not closed under limits. Third, the diagram is *not* commutative, but becomes so when restricted to cdga_k^{+}, the category of cdga's which are bounded on the left. The proof of commutativity uses an increasing induction on Postnikov towers, and the existence of a base case for the induction requires this extra constraint. Moreover, the restriction ϕ_3^+ to left-bounded cdga's is fully faithful, with essential image given by $k[u] - \mathrm{cdga}_k^{+,\leq 0}$.

3. $k[\epsilon]$ is Koszul dual to $k[u]$, which gives a map:

$$\mathrm{Ext}_{k[\epsilon]}(k, -) : k[\epsilon] - dg - mod_k \to k[u] - dg - mod_k.$$

   Moreover, this map preserves the monoidal structure on the two categories, so it induces a map $\phi_4 : \epsilon - \mathrm{cdga}_k^{\leq 0} \to k[u] - \mathrm{cdga}_k$, which fits into a diagram:(Todo: do this argument more carefully)

$$\begin{tikzcd}
\epsilon - \mathrm{cdga}_k^{\leq 0} \arrow[r, "\phi_4"] \arrow[d] & k[u] - \mathrm{cdga}_k \arrow[d, "k\otimes_{k[u]}-"] \\
\mathrm{cdga}_k^{\leq 0} \arrow[r, hook] & \mathrm{cdga}_k\,.
\end{tikzcd}$$

   The comments about commutativity, full faithfulenss and essential image from the previous step also apply here.

4. Putting these together, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}(BS^1, \mathrm{cdga}_k^{+,\leq 0}) \arrow[r, "\phi_3^+"] \arrow[d, "\mathrm{eval}"'] & k[u] - \mathrm{cdga}_k^{+} \arrow[d, "k\otimes_{k[u]}-"] & \epsilon - \mathrm{cdga}_k^{+,\leq 0} \arrow[l, "\phi_4^+"'] \arrow[d, "\mathrm{forget}"] \\
\mathrm{cdga}_k^{+,\leq 0} \arrow[r, hook] & \mathrm{cdga}_k^{+} & \mathrm{cdga}_k^{+,\leq 0}\,. \arrow[l, hook']
\end{tikzcd}$$

   ϕ_3^+ and ϕ_4^+ are fully faithful with the same essential image, which gives an equivalence ϕ which fits into the commutative diagram:

$$\begin{tikzcd}
\mathrm{Fun}(BS^1, \mathrm{cdga}_k^{+,\leq 0}) \arrow[r, "\phi^+"] \arrow[d, "\mathrm{eval}"'] & \epsilon - \mathrm{cdga}_k^{+,\leq 0} \arrow[d, "\mathrm{forget}"] \\
\mathrm{cdga}_k^{+,\leq 0} \arrow[r, hook] & \mathrm{cdga}_k^{+,\leq 0}\,.
\end{tikzcd}$$

   ϕ^+ is an equivalence, so in particular it commutes with colimits. Therefore it extends to an equivalence between $\mathrm{Fun}(BS^1, \mathrm{cdga}_k^{\leq 0})$ and $\epsilon - \mathrm{cdga}_k^{\leq 0}$.

□

Theorem 12.2.6. *ϕ also commutes with the free functors, i.e. $\phi(S^1 \otimes A) \simeq DR(A)$.*

$$\begin{tikzcd}
S^1 - \mathrm{cdga}_k^{\leq 0} \arrow[rr, "\phi"] & & \epsilon - \mathrm{cdga}_k^{\leq 0} \\
& \mathrm{cdga}_k^{\leq 0} \arrow[ul, "S^1\otimes"] \arrow[ur, "DR"'] &
\end{tikzcd}$$

Proof. This just follows from category theory and the fact that ϕ commutes with the forgetful functors. □

Corollary 12.2.7 (HKR isomorphism)**.** *Let $X \in dSt$. (Todo: in the paper they state this just for classical schemes, but it clearly works in more generality. what assumptions do we need on the stack? finite presentation?)*

1. *$\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} \mathcal{O}_X \simeq \mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[1])$.*
2. *If X is a smooth truncated scheme, this specializes to $\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} \mathcal{O}_X \simeq \mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_X[1])$. This is the classical version of the HKR isomorphism, enhanced to an isomorphism of algebras, as opposed to just complexes.*
3. *Let hS^1 denote homotopy fixed points of the S^1 action, and ev denote the even part of de Rham cohomology. Then:*
$$(\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_X} \mathcal{O}_X)^{hS^1} \simeq H^{ev}_{dR}(X).$$

Proof. The first statement is just saying that we can apply the equivalence ϕ at the level of sheaves. (Todo: say something about the last one, maybe even negative cyclic complexes) □

Remark 12.2.8. For $X \in dSt$, we can define the **derived loop stack** $\mathscr{L}X$ as the mapping stack $\mathrm{Map}_{dSt}(S^1, X)$. Using tensor-hom adjunction, $\mathcal{O}(\mathscr{L}X) \simeq S^1 \otimes X$. On the other side of the HKR isomorphism, $\mathrm{Sym}_{\mathcal{O}_X}(\mathbb{L}_X[1])$ has an interpretation as $\mathcal{O}(T[1]X)$, global sections of the shifted tangent bundle of X. This gives an identification $\mathscr{L}X \simeq T[1]X$. The slogan is that "forms on X are functions on $\mathcal{L}X$". More about this in [3].

12.3 Shifted symplectic structures

In all remaining sections we follow [15]. For X a smooth scheme over k, a symplectic structure is a closed 2-form $\omega \in H^0(X, \Omega^{2,cl}_X)$, which induces an isomorphism $\Theta_\omega : T_X \simeq \Omega^1_X$. We want to generalize this for X a derived Artin stack over k. Morally speaking, we want an n-shifted symplectic structure to be an element $\omega \in H^0(X, \wedge^2 \mathbb{L}_X[2])$, which is closed and induces an isomorphism $\Theta_\omega : \mathbb{T}_X \to \mathbb{L}_X[n]$. The main technical difficulty is defining what it means for a form to be closed: instead of $d_{dR}\omega = 0$, we want $d_{dR}\omega$ to be exact, meaning that it's in the image of the internal differential of the complex $\wedge^3 \mathbb{L}_X[3]$. (See the diagram in 12.2.4; the internal differentials are the vertical maps there.) In particular, "closed" is now a structure, instead of a property.(Todo: explain this better)

Using the last part of Corollary 12.2.7, as well as remark 12.2.8 we can translate the closed condition into the condition of being a homotopy fixed point for the action of S^1 on $\mathscr{L}X$. However, the difficulty in working with $\mathscr{L}X$ is that loops do not satisfy descent in the étale (or smooth) topology. As such, one needs to work with infinitesimal loops, which is achieved in [3] by introducing the completion of $\mathscr{L}X$ around the constant loops, i.e. the zero section $X \to \mathscr{L}X$.

The authors of [15] prefer a different approach, based on the negative cyclic complex, where descent is immediate from the definition of forms and closed forms, see 12.3.8.

Definition 12.3.1. Let E be an ϵ-dg-module (or mixed complex) over k. The **negative cyclic complex** of E is a dg-module $NC(E)$ over k, given by:

$$NC^n(E) = \prod_{i \geq 0} E^{n-2i}.$$

The differential D is the sum of ϵ and the internal differential d:

$$D(\{m_i\})_j = \epsilon m_{j-1} + dm_j.$$

Consider now the **weight grading** on E, which is $E = \oplus_p E(p)$ such that $\epsilon : E(p) \to E(p+1)$. We consider a variant of the negative cyclic complex which uses the weight grading: $NC^n(E)(p) = \prod_{i\geq 0} E^{n-2i}(p+i)$.[1]

Definition 12.3.2. The **weighted negative cyclic complex** is:

$$NC^w(E) = \oplus_p NC(E)(p).$$

In other words, we have a double complex whose degree n, weight p part is $\prod_{i\geq 0} E^{n-2i}(p+i)$.

(Todo: give some intuition about how this is related to the corollary in HKR)

Remark 12.3.3. There is a natural map $NC^w(E) \to E$, given by projection to the $i = 0$ component:

$$NC^w(E)^n(p) = \prod_{i\geq 0} E^{n-2i}(p+i) \to E^n(p).$$

Shortly we will interpret this as assigning to a closed p-form its underlying p-form.

Recall that, for $A \in \mathrm{cdga}_k^{\leq 0}$, we have the de Rham complex $DR(A) = \oplus_p (\wedge^p \mathbb{L}_A)[p]$.

Definition 12.3.4. The space of **p-forms of degree n** on A is[2]:

$$\mathcal{A}^p(A,n) = \big| \wedge^p \mathbb{L}_A[n-p] \big| \in \mathcal{S}.$$

The space of **closed p-forms of degree n** on A is:

$$\mathcal{A}^{p,cl}(A,n) = \big| NC^w(DR(A))[n-p](p) \big| \in \mathcal{S}.$$

Remark 12.3.5. One may expect n-shifted p-forms to require a shift by n, instead of the $n-p$ that appears in the definition. But note that a shift in p is already present in the de Rham complex, as well as its associated weighted negative cyclic complex. This brings the total shift to $n-p$.(Todo: wait, this doesn't actually work yet, fix it)

Remark 12.3.6. The map in Remark 12.3.3 induces a map on the geometric realizations $\mathcal{A}^{p,cl}(A,n) \to \mathcal{A}^p(A,n)$, which associates to a closed p-form its underlying p-form.

Definition 12.3.7. For $w \in \mathcal{A}^p(A,n)$, the homotopy fiber $K(w)$ of $\mathcal{A}^{p,cl}(A,n) \to \mathcal{A}^p(A,n)$ is **the space of keys** of w.

As advertised, this formalism makes it easy to prove:

Proposition 12.3.8. *$\mathcal{A}^p(-,n)$ and $\mathcal{A}^{p,cl}(-,n)$ are derived stacks for the étale topology.*

Proof. Backtracking through two pairs of adjoint functors we have:

$$\mathcal{A}^p(A,n) \simeq \mathrm{Map}_{dg-mod_k}(k, \wedge^p \mathbb{L}_A[n]).$$

This reduces the problem to showing that $A \mapsto \wedge^p \mathbb{L}_A[n]$ satisfies descent, which it does, because it's quasi-coherent. Similarly, letting $k(p)$ denote the mixed complex with just a copy of k in degree 0 and weight p,

$$\mathcal{A}^{p,cl}(A,n) = |NC^w(DR(A))[n-p](p)| \simeq \mathrm{Hom}_{\epsilon-dg-mod_k}\big(k(p), DR(A)[n-p]\big) = \mathrm{Hom}_{\epsilon-dg-mod_k}\big(k(p), \oplus_q \wedge^q \mathbb{L}_A[n-p+q]\big).$$

The middle equivalence follows from Corollary 1.4 in [15]. Hence this problem is also reduced to descent for the cotangent complex. ☐

[1] When talking about complexes, round parantheses always refer to the weight grading, while square brackets refer to the cohomological grading. The confusing bit is that (p) means that we're isolating the weight p subspace, while $[n]$ means that we're shifting the cohomological degree by n.

[2] We're using Dold-Kan to identify a connective complex with a simplicial set, and then taking geometric realization to get a space.

This proposition allows us to globalize the definitions. For $F \in dSt_k$, let:

$$\mathcal{A}^p(F,n) = \mathrm{Map}_{dSt_k}\left(F, \mathcal{A}^p(-,n)\right),$$
$$\mathcal{A}^{p,cl}(F,n) = \mathrm{Map}_{dSt_k}\left(F, \mathcal{A}^{p.cl}(-,n)\right).$$

We also have the following nice description for $\mathcal{A}^p(F,n)$, but unfortunately no such thing exists for $\mathcal{A}^{p,cl}(F,n)$.

Proposition 12.3.9. *Let $F \in dSt_k$, then $\mathcal{A}^p(F,n) \simeq \mathrm{Map}_{QCoh(F)}\left(\mathcal{O}_F, \wedge^p \mathbb{L}_F[n]\right)$.*

Proof. The idea is to induct on the m-geometricity level of F. That is, choose an atlas for F in terms of $m-1$-geometric stacks $\{X_\alpha\}$, and look at the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Map}_{QCoh(F)}\left(\mathcal{O}_F, \wedge^p \mathbb{L}_F[n]\right) \arrow[r] \arrow[d] & \lim \mathrm{Map}_{QCoh(X_\alpha)}\left(\mathcal{O}_{X_\alpha}, \wedge^p \mathbb{L}_{X_\alpha}[n]\right) \arrow[d] \\
\mathcal{A}^p(F,n) \arrow[r] & \lim \mathcal{A}^p(X_\alpha, n).
\end{tikzcd}$$

The bottom map is an equivalence by descent, the right map is an equivalence by the inductive hypothesis. The top map is also an equivalence, but this is more subtle, see [15] for the details. Therefore the left map is an equivalence. □

Let F be a derived Artin stack, locally finitely presented over k. Then $\mathbb{L}_F$ is perfect, so in particular it's dualizable in $QCoh(F)$. The dual is the tangent complex $\mathbb{T}_F$. Using Proposition 12.3.9, $\omega \in \mathcal{A}^2(F,n)$ determines a map $\mathcal{O}_F \to \wedge^2 \mathbb{L}_F[n]$, which is dual to a map $\Theta_\omega : \mathbb{T}_F \to \mathbb{L}_F[n]$.

Definition 12.3.10. $\omega \in \mathcal{A}^2(F,n)$ is **non-degenerate** if Θ_ω is a quasi-isomorphism. The space of **n-shifted symplectic structures** on F is the pullback of the diagram:

$$\begin{tikzcd}
Symp(F,n) \arrow[r] \arrow[d] & \mathcal{A}^{2,cl}(F,n) \arrow[d] \\
\mathcal{A}^2(F,n)^{nd} \arrow[r] & \mathcal{A}^2(F,n).
\end{tikzcd}$$

Remark 12.3.11. $\mathcal{A}^2(F,n)^{nd}$ is a union of path components of $\mathcal{A}^2(F,n)$, so symplectic structures are the same as closed 2-forms whose underlying 2-forms live in these particular components.

Remark 12.3.12. If $\mathbb{L}_X$ has amplitude $(-m,n)$, then shifted symplectic structures on X can only exist in degree $m-n$.

Example 12.3.13. If X is a smooth underived scheme, then $\mathbb{L}_X = \Omega^1_X[0]$, so the only shifted symplectic structures on X are symplectic structures in the usual sense.

Remark 12.3.14. For low values of n, an n-shifted symplectic structure can be seen as pairing the negative degree terms in $\mathbb{L}_X$ (the derived structure) with the positive degree terms (the stacky structure).

12.4 Examples: BG and Perf

For our first first nontrivial example, we take $X = BG$, for G reductive. Let $\mathfrak{g}$ be the Lie algebra of G.

Proposition 12.4.1. *There exist 2-shifted symplectic structures on BG. Moreover, homotopy classes of such correspond to non-degenerate invariant bilinear forms on $\mathfrak{g}$.*

Proof. Recall that $\mathbb{L}_{BG} \simeq g^*[-1]$, so that $\mathbb{T}_{BG} \simeq g[1]$. It's immediate that non-degenerate p-forms of degree n cannot exist unless $n = 2$. Let's see what the spaces of forms look like.

We proved that $\mathcal{A}^p(-,n)$ satisfies descent; this is equivalent to the fact that $DR(-)$ satisfies descent. Therefore we can talk about $DR(BG)$, obtained by taking global sections on BG, which corresponds to taking invariants under G. We obtain $DR(BG) \simeq (\mathrm{Sym}_k\, \mathfrak{g}^*)^G$, concentrated in degree 0, and where $\mathfrak{g}^*$ has weight 1.

$$\begin{tikzcd}
0 & 0 & 0 & 0 \\
k \arrow[ur, "d_{dR}"] & \mathfrak{g}^* \arrow[ur, "d_{dR}"] & \mathrm{Sym}^2\, \mathfrak{g}^* \arrow[ur, "d_{dR}"] & \\
0 & 0 & 0 &
\end{tikzcd}$$

Now recall that $\mathcal{A}^p(BG,n) = |DR(BG)[n-p](p)|$; in particular:

$$\pi_i\big(\mathcal{A}^p(BG,n)\big) = H^{-i}\tau^{\leq 0} DR(BG)[n-p](p) = H^{-i}\tau^{\leq 0}(\mathrm{Sym}^p\, \mathfrak{g}^*)^G[n-p].$$

This means that we have:

$$\mathcal{A}^p(BG,n) = \begin{cases} * & \text{if } n < p, \\ K\big((\mathrm{Sym}^p\, \mathfrak{g}^*)^G, n-p\big) & \text{if } n \geq p. \end{cases} \tag{12.4.1}$$

The interesting case is $n = p$, where $\pi_0\big(\mathcal{A}^p(BG,n)\big) \cong (\mathrm{Sym}^p\, \mathfrak{g}^*)^G$; in all other cases, $\pi_0\big(\mathcal{A}^p(BG,n)\big) = 0$.

The de Rham differential is identically 0 for degree reasons. This means that all forms admit a closed structure. However, this need not be canonical. For example, $\pi_0\big(\mathcal{A}^1(BG,3)\big) = 0$, but $\pi_0\big(\mathcal{A}^{1,cl}(BG,3)\big) \cong (\mathrm{Sym}^2\, \mathfrak{g})^G$. This can be seen from the global sections of the de Rham complex:

$$\begin{tikzcd}
k & \mathfrak{g}^* & \mathrm{Sym}^2\, \mathfrak{g}^* \arrow[d] \\
0 & 0 & 0 \\
0 & 0 \arrow[ur, "d_{dR}"] & 0
\end{tikzcd}$$

Let's determine the spaces of closed forms in more detail:

$$\mathcal{A}^{p,cl}(BG,n) = \big|NC^w(DR(BG))[n-p](p)\big|.$$

Using the definition of NC^w, and then the particular form that $DR(BG)$ takes:

$$\begin{aligned} NC^w(DR(BG))^m = \oplus_p \prod_{i\geq 0} DR(BG)^{m-2i}(p+i) &= \begin{cases} 0 & \text{for m odd,} \\ \oplus_{p\geq -\frac{m}{2}} DR(BG)^0(p+\frac{m}{2}) & \text{for m even} \end{cases} \\ &= \begin{cases} 0 & \text{for m odd,} \\ \oplus_{p\geq -\frac{m}{2}} \big(\mathrm{Sym}^{p+j}\, \mathfrak{g}^*\big)^G & \text{for m even} \end{cases} \end{aligned}$$

We may as well write $m = 2j$, and remember that for odd m the result is 0:

$$NC^w(DR(BG))^{2j} = \oplus_{p\geq -j}\big(\mathrm{Sym}^{p+j}\, \mathfrak{g}^*\big)^G[-2j].$$

The final form of the weighted negative cyclic complex is, then:

$$NC^w(DR(BG)) = \oplus_j \oplus_{p\geq -j} \big(\mathrm{Sym}^{p+j}\, \mathfrak{g}^*\big)^G[-2j].$$

Finally, we need to shift by $n-p$, take the weight p part, and take geometric realization:

$$\mathcal{A}^{p,cl}(BG,n) = \big| \oplus_{j\geq -p} \big(\mathrm{Sym}^{p+j}\, \mathfrak{g}^*\big)^G[n-p-2j]\big|.$$

It's a bit hard to give a complete description of these spaces, analogous to 12.4.1. But we make a few comments:

1. If $n < p$, the entire negative cyclic complex gets shifted into positive degrees, which are killed by geometric realization. Hence for $n < p$, $\mathcal{A}^{p,cl}(BG, n) = * = \mathcal{A}^{p}(BG, n)$.
2. To have nontrivial π_0 we need $n - p \geq 0$ and even. In this case, $\pi_0\big(\mathcal{A}^{p,cl}(BG, n)\big) \cong \big(\mathrm{Sym}^{\frac{p+n}{2}}\, \mathfrak{g}^*\big)^G$. In particular, $\pi_0\big(\mathcal{A}^{2,cl}(BG, 2)\big) \cong \big(\mathrm{Sym}^2\, \mathfrak{g}^*\big)^G$.

All this was quite messy, but at least the non-degeneracy condition is what you'd expect. □

Remark 12.4.2. If $G \subset GL(n)$, there's a canonical 2-shifted symplectic structure coming from the trace $\mathrm{Tr}(\mathrm{mult})$.

We move on to an example which is universal in some sense. Recall the stack Perf from Chapter 9, and in particular the description of its tangent spaces in Theorem 9.3.1: $T_E\mathrm{Perf} \simeq \mathrm{End}(E)[1]$. We use here without proof the global description of the tangent bundle, which is as follows. Let $\mathcal{E}$ be the universal perfect complex over Perf, which is classified by the identity map $\mathrm{Perf} \to \mathrm{Perf}$. Then $\mathbb{T}_{\mathrm{Perf}} \simeq \mathrm{End}(\mathcal{E}, \mathcal{E})[1]$.

Theorem 12.4.3. *There is a 2-shifted symplectic structure on* Perf*.*

Proof. The closed 2-form on Perf is given by the Chern character, which is constructed in [24]. We first discuss the underlying 2-form and the nondegeneracy condition.

Recall from **??** the Atiyah class:

$$a_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{Perf}}} \mathbb{L}_{\mathrm{Perf}}[1].$$

Iterating on this construction, we obtain:

$$a^i_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{Perf}}} \wedge^i \mathbb{L}_{\mathrm{Perf}}[i].$$

Since $\mathcal{E}$ is dualizable (Todo: say sth about this), we can consider the dual map, which we denote by the same symbol:

$$a^i_{\mathcal{E}} : \mathcal{O}_{\mathrm{Perf}} \to \mathcal{E}^* \otimes \mathcal{E} \otimes_{\mathcal{O}_{\mathrm{Perf}}} \wedge^i \mathbb{L}_{\mathrm{Perf}}[i].$$

Composing with the trace map $\mathcal{E}^* \otimes \mathcal{E} \to \mathcal{O}_{\mathrm{Perf}}$, we obtain:

$$\mathrm{Ch}_i(\mathcal{E}) = \frac{\mathrm{Tr}(a^i_{\mathcal{E}})}{i!} \in \mathrm{Hom}\left(\mathcal{O}_{\mathrm{Perf}}, \wedge^i \mathbb{L}_{\mathrm{Perf}}[i]\right) \cong H^i\big(\wedge^i\, \mathbb{L}_{\mathrm{Perf}}\big).$$

Taking $i = 2$ we otain an element of $H^2(\wedge^2 \mathbb{L}_{\mathrm{Perf}})$, which is the underlying 2-form that we are looking after.

Now we use the fact that $\mathbb{T}_{\mathrm{Perf}} \simeq \mathrm{End}(\mathcal{E})[1]$, so that $\mathbb{L}_{\mathrm{Perf}}[1] \simeq \mathrm{End}(\mathcal{E})$. The Atiyah class is, then, the adjoint of the multiplication morphism:

$$\mathrm{mult} : \mathrm{End}(\mathcal{E}) \otimes \mathrm{End}(\mathcal{E}) \to \mathrm{End}(\mathcal{E}),$$

and we can identify $\mathrm{Ch}_2(\mathcal{E})$ with $\mathrm{Tr}(\mathrm{mult})/2$, which is nondegenerate because the trace is. (Recall that the trace is defined as adjoint to the identity map $\mathcal{E} \to \mathcal{E}$.)

It remains to lift $\mathrm{Ch}_2(\mathcal{E})$ to the negative cyclic complex:

$$\begin{tikzcd} & NC \arrow[d] \\ \mathrm{Perf} \arrow[ur, dotted] \arrow[r, "\mathrm{Ch}_2"] & DR. \end{tikzcd}$$

(Todo: finish this) □

Remark 12.4.4. Note that there is an embedding $i : BGL(n) \to \mathrm{Perf}$, obtaining by regarding vector bundles as a sub-category of the perfect complexes. Theorem 12.4.3 gives a 2-shifted symplectic structure ω_{Perf} on Perf, while 12.4.2 gives an explicit form for a 2-shifted symplectic structure $\omega_{BGL(n)}$ on $BGL(n)$. Up to a numerical factor, both expressions are the trace of a multiplication map between endomorphisms. It follows that $\omega_{BGL(n)} = i^*\omega_{\mathrm{Perf}}$.

12.5 Examples: Mapping Stacks

For the next example, we build towards the following AKSZ-type statement, made precise in Theorem 12.5.7. If $F \in dSt_k$ is equipped with an n-shifted symplectic structure, and $X \in dSt_k$ is equipped with, roughly, a fundamental class in degree d, then the mapping stack $\mathrm{Map}_{dSt}(X, F)$ admits an $n-d$ shifted symplectic structure. Together with Theorem 12.4.3, this proves that various moduli stacks of bundles and complexes admit shifted symplectic structures. (See Corollary 12.5.9.)

Remark 12.5.1. Let us first give a heuristic idea for the AKSZ construction, by working with C^∞ manifolds. Let M be a compact manifold of dimension d, and (N, ω) a symplectic manifold. Consider the evaluation map:

$$M \times C^\infty(M, N) \xrightarrow{eval} N.$$

Pullback and integration along M give a map:

$$\begin{aligned} \Omega^p_N &\to \Omega^{p-d}_{C^\infty(M,N)} \\ \alpha &\mapsto \int_M eval^* \alpha. \end{aligned}$$

In particular applying it to ω gives $\int_M eval^*\omega \in \Omega^{2-d}$. (Todo: how can this be symplectic if it's no longer a 2-form?)

To imitate this strategy in the derived context, we need to define appropriate notions of orientability and integration along the fibers.

Definition 12.5.2. For any $X \in dSt_k$, let X_A denote $X \times \operatorname{Spec} A$. [3] We say that X is **$\mathcal{O}$-compact** if:

1. $\mathcal{O}_{X_A}$ is a compact object in $QCoh(X_A)$;
2. for any perfect complex E on X_A, the A-dg-module $C(X, E) = \mathrm{Hom}(\mathcal{O}_{X_A}, E)$ is perfect. [4]

The property of being $\mathcal{O}$-compact buys us the following.

Lemma 12.5.3. *If X is $\mathcal{O}$-compact, there is a natural transformation:*

$$\kappa_X : DR(- \times X) \to DR(-) \otimes_k C(X, \mathcal{O}_X).$$

Proof. A brief explanation for this is as follows. $C(X, \mathcal{O}_X)$ is perfect over k by the $\mathcal{O}$-compactness hypothesis, so the functor $E \mapsto \otimes_k C(X, \mathcal{O}_X)$ commutes with limits. Similarly, DR sends colimits to limits (this is descent). Hence both functors send colimits to limits. Now, since every object of dSt_k is a colimit of objects in $d\mathrm{Aff}_k$, it suffices to construct a natural transformation between the functors restricted to $d\mathrm{Aff}_k$, and then take a left Kan extension in the following diagram.

$$\begin{tikzcd} dSt_k^{\mathrm{op}} \arrow[r, shift left, "DR(-\times X)"] \arrow[r, shift right, "DR(-)\otimes_k C(X,\mathcal{O}_X)"'] & \epsilon - dg - mod_k^{gr} \\ d\mathrm{Aff}_k^{\mathrm{op}} \arrow[u, hook] \arrow[ur, shift left] \arrow[ur, shift right] & \end{tikzcd}$$

At the level of derived affines, the natural transformation is obtained essentially from a Kunneth formula: $DR(B) \otimes_k DR(C) \simeq DR(B \otimes_k C)$. □

[3] If, like me, you're not very good with stacks, here's some intuition for considering X_A. When making statements about the "points" of a stack, e.g. in the proof of Theorem 12.5.7, it's not sufficient to consider k-valued points, but rather those valued in an arbitrary derived affine. That's because stacks are defined as functors on $d\mathrm{Aff}$, so information about them is complete only when we've probed with all derived affines.

[4] This is the same as $p_* E$, where $p : X \times \operatorname{Spec} A \to \operatorname{Spec} A$ is the projection.

An application of the functor NC^w to κ_X, together with the fact that $C(X, \mathcal{O}_X)$ is perfect, gives another natural transformation:

$$NC^w(- \times X) \to NC^w\big(DR(-) \otimes_k C(X, \mathcal{O}_X)\big) \simeq NC^w(F) \otimes_k C(X, \mathcal{O}_X).$$

Moreover these commute with the canonical maps from NC^w to DR:

$$\begin{array}{ccc} NC^w(F \times X) & \xrightarrow{\kappa_{F,X}} & NC^w(F) \otimes_k C(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ DR(F \times X) & \xrightarrow[\kappa_{F,X}]{} & DR(F) \otimes_k C(X, \mathcal{O}_X). \end{array}$$

Definition 12.5.4. Let $X, F \in dSt_k$ with X $\mathcal{O}$-compact. Assume given a morphism $\eta : C(X, \mathcal{O}_X) \to k[-d]$ of perfect complexes over k. The **integration map along** η is the morphism:

$$\textstyle\int_\eta : NC^w(F \times X) \xrightarrow{K_{F,X}} NC^w(F) \otimes_k C(X, \mathcal{O}_X) \xrightarrow{id \otimes \eta} NC^w(F)[-d].$$

Remark 12.5.5. We can define the same for DR, and the two integration maps are compatible.

Finally, an additional constraint on the morphism η makes it an $\mathcal{O}$-orientation on X, which we define now. For any $E \in \mathrm{Perf}(X)$, we have a natural pairing, which we can compose with η:

$$C(X, E) \otimes_k C(X, E^*) \longrightarrow C(X, \mathcal{O}_X) \longrightarrow k[-d].$$

Dually this gives a "cap product" morphism:

$$\cap\eta : C(X, E) \to C(X, E^*)^*[-d].$$

Definition 12.5.6. Let $X \in dSt_k$ $\mathcal{O}$-compact. An $\mathcal{O}$**-orientation of degree d** on X is a morphism of complexes:

$$[X] : C(X, \mathcal{O}_X) \to k[-d]$$

such that, for every $A \in \mathrm{cdga}_k^{\leq 0}$ and any $E \in \mathrm{Perf}(X_A)$, the morphism

$$\cap[X]_A : C(X_A, E) \to C(X_A, E^*) * [-d]$$

is a quasi-isomorphism of A-dg-modules.

The main theorem of this section is:

Theorem 12.5.7 (Theorem 2.5 in [15])**.** *Let F be a derived Artin stack equipped with $\omega \in \mathrm{Symp}(F, n)$. Let X be an $\mathcal{O}$-compact derived stack equipped with an $\mathcal{O}$-orientation of degree d:*

$$[X] : C(X, \mathcal{O}_X) \to k[-d].$$

Assume, moreover, that $\mathrm{Map}_{dSt}(X, F)$ is a derived Artin stack, locally of finite presentation. Then there is a canonical $n - d$ shifted symplectic structure on $\mathrm{Map}_{dSt}(X, F)$.

Proof. Note that $\omega \in \mathcal{A}^{2,cl}(F, n)$ is the same as:

$$\omega : k[2 - n](2) \to NC^w(F).$$

Pulling back along the evaluation morphism $\pi : X \to \mathrm{Map}(X, F)$ and integrating on X gives:

$$k[2 - n](2) \xrightarrow{\omega} NC^w(F) \xrightarrow{\pi^*} NC^w\big(X \times \mathrm{Map}(X, F)\big) \xrightarrow{\int_{[X]}} NC^w\big(\mathrm{Map}(X, F)\big)[-d].$$

It remains to see that the underlying 2-form of $\int_{[X]} \pi^*\omega$ is non-degenerate. We can check this condition locally. Let $f : \operatorname{Spec} A \to \operatorname{Map}(X, F)$ be an A-point of $\operatorname{Map}(X, F)$. The tangent complex at f is:

$$\mathbb{T}_f \operatorname{Map}(X, F) \simeq C\big(X \times \operatorname{Spec} A, f^*(\mathbb{T}_F)\big).$$

The underlying 2-form of ω determines a non-degenerate pairing:

$$\mathbb{T}_F \wedge \mathbb{T}_F \to \mathcal{O}_F[n].$$

By pull-back we obtain a non-degenerate pairing of A-dg-modules:

$$C\big(X \times \operatorname{Spec} A, f^*(\mathbb{T}_F)\big) \wedge C\big(X \times \operatorname{Spec} A, f^*(\mathbb{T}_F)\big) \to C\big(X \times \operatorname{Spec} A, \mathcal{O}_{X \times \operatorname{Spec} A}[n]\big).$$

Composing with the orientation $[X_A]$ gives a non-degenerate pairing:

$$C\big(X \times \operatorname{Spec} A, f^*(\mathbb{T}_F)\big) \wedge C\big(X \times \operatorname{Spec} A, f^*(\mathbb{T}_F)\big) \to A[n-d].$$

But this is just the pairing induced by the underlying 2-form of $\int_{[X]} \pi^*\omega$. □

The following are examples of stacks X which satisfy the $\mathcal{O}$-compactness and $\mathcal{O}$-orientability hypotheses of Theorem 12.5.7.

Example 12.5.8.

1. (Calabi-Yau) Let X be a smooth and proper DM stack over $\operatorname{Spec} k$ with relative dimension d, with connected geometric fibers. Assume given an isomorphism of line bundles $u : \omega_X \simeq \mathcal{O}_X$. X is $\mathcal{O}$-compact automatically (Todo: why?). Moreover, the isomorphism u together with Serre duality give an isomorphism:

$$H^d(X, \mathcal{O}_X) \xrightarrow{u} H^d(X, \omega_X) \cong k,$$

which lifts to a quasi-isomorphism of complexes:

$$C(X, \mathcal{O}_X) \to k[-d].$$

Theorem 12.5.7 also requires that $\operatorname{Map}(X, F)$ be a derived Artin stack when F is one. This follows from Artin-Lurie representability.

2. (Betti)

3. (de Rham)

4. (Dolbeault) We omit this one.

We have the following existence statements for various moduli spaces of bundles and complexes.

Corollary 12.5.9 (Corollaries 2.6 and 2.13 in [15]). *Let G be a reductive group scheme over k, and fix $\omega \in (\operatorname{Sym}^2 \mathfrak{g})^G$ non-degenerate.*

1. *(Betti) Let M be a compact, orientable topological manifold of degree d. A choice of fundamental class $[M] \in H_d(M, k)$ determines a $2-d$ shifted symplectic structure on:*

$$\begin{aligned} LocSys(M) &= \operatorname{Map}(M, BG), \\ \operatorname{Perf}(M) &= \operatorname{Map}(M, \operatorname{Perf}). \end{aligned}$$

2. *(de Rham) Let Y be a smooth and proper DM stack with connected geometric fibers of relative dimension d. A choice of fundamental class $[Y] \in H^{2d}_{dR}(Y)$ determines a $2-2d$ shifted symplectic structure on the stacks of bundles/perfect complexes with flat connections:*

$$\begin{aligned} Loc_{dR}(Y) &= \operatorname{Map}(Y_{dR}, BG), \\ \operatorname{Perf}_{dR}(Y) &= \operatorname{Map}(Y_{dR}, \operatorname{Perf}). \end{aligned}$$

3. *(Dolbeault) Let Y be as before. A choice of fundamental class $[Y] \in H^{2d}_{\partial}(Y)$ determines a $2-2d$ shifted symplectic structure on the stacks of bundles/perfect complexes with Higgs fields:*

$$Higgs(Y) = \mathrm{Map}(Y_{\partial}, BG),$$
$$\mathrm{Perf}_{\partial}(Y) = \mathrm{Map}(Y_{\partial}, \mathrm{Perf}).$$

4. *(Calabi-Yau) Let Y be as before. A choice of trivialization $\omega_{Y/k} \simeq \mathcal{O}_Y$ determines a canonical $2-d$ shifted symplectic structure on:*

$$Bun_G(Y) = \mathrm{Map}(Y, BG),$$
$$\mathrm{Perf}(Y) = \mathrm{Map}(Y, \mathrm{Perf}).$$

Remark 12.5.10. For Y a K3 or an elliptic surface, we obtain a 0-shifted symplectic structure on $Bun_G(Y)$. This recovers a result that was known before the paper [15], for the locus of simple bundles.[5] But the fact that this classical symplectic structure extends to the entire moduli space is new.

(Todo: some extensions of the mapping stack result in a paper by Calaque and in the thesis of Ted Spaide)

12.6 Examples: Lagrangian intersections

Probably no time for this

[5]Simple means that the only endomorphisms are constants, and they don't have negative self extensions.

Chapter 13

Stable Maps and Gromov-Witten Invariants

13.1 The Counting Problem

Basic idea of ennumerative geometry, as explained in [7] 3.1: set up a moduli space M for the objects, e.g. curves, one wants to count: $\mathcal{M}_{g,n}(X,\beta)$, equipped with (flat) evaluation maps $\nu_i : \mathcal{M}_{g,n}(X,\beta) \to X$, given by $\big(C, p_1, \dots p_n, \mu\big) \mapsto \mu(p_i)$. Each constraint $\nu_i \in \Gamma_i$, where $\Gamma_i \in H_*(X,\mathbb{Z})$, gives a subscheme, of $\mathcal{M}_{g,n}(X,\beta)$. We take the intersection of all these:

$$\bigcap_{i=1}^{m} \nu_i^* \Gamma_i.^1 \tag{13.1.1}$$

If the intersections are transverse and the result has dimension 0, can count the number of points. We would like to set up Γ_i such that:

$$\sum_{i=1}^{m} \operatorname{codim} \Gamma_i = \dim \mathcal{M}_{g,n}(X,\beta).$$

Thus the ennumerative problem is reduced to intersection theory in M. In order to do intersection theory successfully, M needs to be compact (proper), and we need to understand its Chow ring, where the subschemes live.

A first modification: in order to drop the transversality assumption on Γ_i, we replace them with the Poincaré dual cohomology classes γ_i, and take cup products then 13.1.1 is replaced by a first naive definition of the **Gromov-Witten invariants**:

$$I_{g,n,\beta} := \int_{[\mathcal{M}_{g,n}(X,\beta)]} \bigwedge_i \nu_i^* \gamma_i. \tag{13.1.2}$$

If $\mathcal{M}_{g,n}(X,\beta)$ is smooth and proper, then $[\mathcal{M}_{g,n}(X,\beta)]$ is the fundamental class, against which it makes sense to evaluate cohomology classes. $I_{g,n,\beta}$ is defined to be 0 unless $\sum_i \deg \gamma_i = \dim \mathcal{M}_{g,n}(X,\beta)$.

13.2 Axiomatic Definition of GW

The axiomatic approach of Kontsevich and Manin in [8] is as follows. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification by stable curves of the moduli stack of genus g curves with n marked points. We take this as a well-understood object and explain the rest.

[1] This pullback is an umkehr map and we need some assumptions; is properness of μ_i enough?

Definition 13.2.1 (2.2 in [8]). A **system of Gromov-Witten classes for X** is a family of linear maps:

$$I^X_{g,n,\beta} : H^*(X,\mathbb{Q})^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

defined for $n+2g-3 \geq 0$, and satisfying the following axioms.

1. **Effectivity**: $I_{g,n,\beta} = 0$ for β non-effective, i.e. not in the dual of the Kähler cone.
2. S_n**-covariance**: equivariant with respect to the obvious S_n action on the domain and target.
3. **Grading**: $\deg I_{g,n,\beta} = -2\int_\beta c_1(X) + (2-2g)\dim X$. More precisely, this means that we set $|\gamma| = i$ for $\gamma \in H^i(X,\mathbb{Q})$ and we require that:

$$\left|I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)\right| = \sum_{j=1}^m |\gamma_j| - 2\int_\beta c_1(X) + (2g-2)\dim X.$$

Some comments on the grading axiom:

- Following the convention in [8], we use the real, not complex, dimension.
- Informally we think of $I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)$ as obtained by pushing forward via the natural map:

$$\mathcal{M}_{g,n}(X,\beta) \to \mathcal{M}_{g,n}.$$

As a result, its degree is an expectation for $\dim\mathcal{M}_{g,n} - \dim\mathcal{M}_{g,n}(X,\beta)$. We know that $\dim\mathcal{M}_{g,n} = 2(3g-3+n)$. By deformation theory we also compute $\operatorname{vdim}\mathcal{M}_{g,n}(X,\beta)$, called the **virtual dimension**, the expected dimension whenever first-order deformations are unobstructed.

The tangent space to $\mathcal{M}_{g,n}(X,\beta)$ at a point $(C,p_1,\dots,p_n,\mu)$ is:

$$H^1(C,T_C(-p_1-\cdots-p_n)) \oplus H^0(C,\mu^*T_X).$$

By Serre duality this is:

$$H^0(C,\Omega_C^{\otimes 2}(p_1+\cdots+p_n))^\vee \oplus H^0(C,\mu^*T_X).$$

Approximating the dimensions with the Euler characteristic, we get via Riemann-Roch:

$$\operatorname{vdim}\mathcal{M}_{g,n}(X,\beta) = 2(\dim X - 3)(1-g) + 2\int_\beta c_1(T_X) + 2n. \tag{13.2.1}$$

Substracting these we get what the grading axiom requires:

$$\dim\mathcal{M}_{g,n} - \dim\mathcal{M}_{g,n}(X,\beta) = 2\int_\beta c_1(X) - (2-2g)\dim X.$$

- Assume that $I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)$ is of **codimension zero**, i.e. that:

$$\sum_{j=1}^n |\gamma_j| = 2\int_\beta c_1(X) - (2-2g)\dim X. \tag{13.2.2}$$

Then $\left|I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)\right| = \dim\overline{\mathcal{M}}_{g,n}$. We can integrate this against the fundamental class of $\overline{\mathcal{M}}_{g,n}$, which is a proper smooth Deligne-Mumford stack. (Todo: reference?) We obtain a finite number, which we take as the result of the curve count.

4. **Fundamental class.** We introduce some more terminology. Call a class **basic** if it has the smallest n which makes sense, namely:

$$I^X_{0,3,\beta}(\gamma_1,\gamma_2,\gamma_3) \qquad\qquad I^X_{1,1,\beta}(\gamma_1) \qquad\qquad I^X_{g,0,\beta} \text{ for } g \geq 2.$$

Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ be the projection that forgets the last marked point. Let $e^0_X \in H^0(X,\mathbb{Q})$ be the identity of the cohomology ring. Unless the class on the LHS is basic, we require that:

$$I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_{n-1},e^0_X) = \pi^* I^X_{g,n-1,\beta}(\gamma_1,\ldots,\gamma_{n-1}).$$

In addition, we set:

$$I^X_{0,3,\beta}(\gamma_1,\gamma_2,e^0_X) = \begin{cases} \int_X \gamma_1 \wedge \gamma_2, & \text{if } \beta = 0, \\ 0, & \text{if } \beta \neq 0. \end{cases}$$

5. **Divisor.** In the case $|\gamma_n| = 2$, i.e. γ_n is the Poincaré dual class of a divisor, and if the LHS is a non-basic class, we require:

$$\pi_{n*} I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \int_\beta \gamma_n I^X_{g,n-1,\beta}(\gamma_1,\ldots,\gamma_{n-1}).$$

6. **Splitting.** This axiom and the next are very important: they postulate a manageable structure of the boundary of the compactification $\overline{\mathcal{M}}_{g,n}(X,\beta)$, compatible with that of the boundary of $\overline{\mathcal{M}}_{g,n}$. One way to get boundary maps is to let the curves have 2 irreducible components, with genera g_1, g_2 and marked points n_1+1, n_2+1 such that $g = g_1+g_2$, $n = n_1+n_2$. The extra marked point on each irreducible component is where we glue them; they become one singular point in the resulting reducible curve. For S some partition of the n marked points into 2 sets of cardinality n_1 and n_2, we let $\phi_S : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ be the gluing map. Choose a basis $\{\Delta_a\}$ of $H^*(X,\mathbb{Q})$ and define $g_{ab} = \int_V \Delta_a \wedge \Delta_b$; let $(g^{ab}) = (g_{ab})^{-1}$ denote the entries of the inverse matrix. Then:

$$\phi_S^* I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = (-1)^S \sum_{\beta_1+\beta_2=\beta} \sum_{a,b} I^X_{g_1,n_1+1,\beta_1}(\otimes_{j\in S_1}\gamma_j \otimes \Delta_a) g^{ab} \otimes I^X_{g_2,n_2+1,\beta_2}(\Delta_b \otimes \otimes_{j\in S_2}\gamma_j).$$

Roughly speaking, we need to introduce $\sum_{a,b}(\Delta_a \otimes \Delta_b)$ to account for the position of the extra marked points. Integrating over these produces a factor $g_{a,b}$ that wasn't there on the LHS, so we need to multiply by g^{ab} to compensate for it.

7. **Genus reduction.** Let $\psi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the map which glues together the last 2 marked points. Then:

$$\psi^* I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \sum_{a,b} I^X_{g-1,n+2,\beta}(\gamma_1,\ldots,\gamma_n,\Delta_a,\Delta_b) g^{ab}.$$

The splitting and genus reduction axioms motivate the choice of stable maps compactification, see **??**.

8. **Motivic axiom.** The maps $I^X_{g,n,\beta}$ are induced by correspondences in the Chow rings:

$$C^X_{g,n,\beta} \in C^*(X^n \times \overline{\mathcal{M}}_{g,n}).$$

Namely, consider the two projection maps:

$$\begin{tikzcd} & X^n \times \overline{\mathcal{M}}_{g,n} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X^n & & \overline{\mathcal{M}}_{g,n}. \end{tikzcd}$$

We require that:

$$I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = q_*\big(C^X_{g,n,\beta} \wedge p^*(\gamma_1 \otimes \cdots \otimes \gamma_n)\big).$$

This axiom is motivated as follows in [8], 2.3.8. Suppose we construct a good compactification $\overline{\mathcal{M}}_{0,n}(X,\beta)$, together with a virtual fundamental class $[\overline{\mathcal{M}}_{0,n}(X,\beta)]$. Consider then the map:

$$\begin{aligned} \alpha : \overline{\mathcal{M}}_{0,n}(X,\beta) &\to X^n \times \overline{\mathcal{M}}_{0,n} \\ (C, x_1, \ldots, x_n, f) &\mapsto \big(f(x_1), \ldots, f(x_n), (\bar{C}, x_1, \ldots, x_n)\big). \end{aligned}$$

We would like $\bar{C}$ to be C, but we may need to contract certain components to get a stable curve from a stable map. Compare definitions 13.3.2 and **??**. Ignoring this for now, we set $C_{(}^X g, n, \beta) = \alpha_*\big([\overline{\mathcal{M}}_{0,n}(X,\beta)]\big)$. This means, roughly speaking, we're integrating over $\overline{\mathcal{M}}_{0,n}(X,\beta)$, like the naive definition 13.1.2 suggests.

We are mostly interested in codimension zero invariants, which informally are those where we imposed enough constraints to get a finite number of curves. For example, if we want to count degree d rational curves in $\mathbb{P}^2$, the relevant codimension zero condition says:

$$\sum_{i=1}^n |\gamma_i| = 2\int_{d[H]} c_1(\mathbb{P}^2) - 2\dim \mathbb{P}^2 = 6d - 4.$$

For example, we could ask that the curves pass through n given points in $\mathbb{P}^2$, then $|\gamma_i| = 4$, so we obtain $4n = 6d - 4$. If the computation were done right, this would be $12d - 4$, so that we get $n = 3d - 1$. So the relevant thing to count are degree d rational curves passing through $3d - 1$ points.(Todo: fix this)

13.3 Stable Map Compactification

To give a naive compactification of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, we could just look at the space $W(r,d)$ of $r+1$-tuples of degree d polynomials in 2 variables, up to scaling, and take the subset of tuples which don't vanish simultaneously. We get a subset of a projective space:

$$W(r,d) \subset \mathbb{P}\left(\bigoplus_{i=0}^r H^0(\mathbb{P}^1, \mathcal{O}(d))\right).$$

We need to quotient by $\mathrm{Aut}(\mathbb{P}^1)$ to identify maps that differ by a reparametrization; ignoring this for the moment, one hopes to take the closure of $W(r,d)$ in $\mathbb{P}\left(\bigoplus_{i=0}^r H^0(\mathbb{P}^1, \mathcal{O}(d))\right)$ to obtain a compactification. However, for $g \neq 0$ and $X \neq \mathbb{P}^r$, this doesn't work and we need a less ad-hoc approach.

The choice of compactification matters; different choice leads to different numbers. That's because the numbers now count things in the boundary as well.

Example 13.3.1. In the stable maps compactification that we introduce shortly, which produces Gromov-Witten invariants, we keep the domain curves well-behaved: they acquire nodal singularities, but no non-reduced structure. However, the maps themselves can be highly non-injective. A different choice is the Donaldson-Thomas compactification via Hilbert schemes: here we work with ideal sheaves, which always represent embeddings, however the domain curve can now be non-reduced or have singularities worse than nodal. Section $3\frac{1}{2}$ of [14] illustrates the differences with the following example. We work locally and consider the family of conics:

$$C_t = \{x^2 + ty = 0\} \subset \mathbb{C}^2,$$

which becomes singular as $t \to 0$. In the DT compactification, we take the limit in the defining equation, and get $x^2 = 0$, which is a thickened y-axis. In the stable map compactification, we parametrize the conics:

$$C_t \longleftrightarrow \xi \mapsto (-\sqrt{t}\xi, \xi^2).$$

This is a parametrization modulo automorphisms of the curve, namely $\xi \leftrightarrow -\xi$. Now as $t \to 0$, the limiting map is $\xi \mapsto (0, \xi^2)$, which is a double cover of the y-axis. You can't see from this example, but the different choices of compactification actually give different answers for the counting problem.

With that in mind, let's finally define stable maps. For reference and comparison we include the definition of stable curves:

Definition 13.3.2. (Todo: write this up)

Think about graphs of curves, such that each "twig" has no infinitesimal automorphisms. This means that twigs of genus g must have at least $3 - 2g$ special points, which means either marked points or singular ones.

(Todo: figure out an easy way to include the pictures of graphs)

Definition 13.3.3 (2.4.1 in [8])**.** A **stable map** to X is a structure $(C, x_1, \dots, x_n, f)$ where:

- $(C, x_1, \dots, x_n)$ is a connected reduced curve with n pairwise distinct marked non-singular points, and at worst additional singular double points.
- $f : C \to X$ is a map with no non-trivial infinitesimal automorphisms. This means that every irreducible component of C of genus g which is contracted to a point (of degree 0) must have at least $3 - 2g$ special points.

Remark 13.3.4. Note that, in the definition of stable maps $(C, x_1, \dots, x_n, f)$, the underlying curve $(C, x_1, \dots, x_n)$ need not be stable. Therefore the forgetful map $\overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$ must contract components of $(C, x_1, \dots, x_n)$ which have infinitesimal automorphisms.

In his talk notes, Mauro provides the following construction of the moduli stacks of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Start from $\overline{\mathcal{M}}_{g,n}$, which are fine moduli spaces of curves, and therefore admit a universal family $\mathcal{C}_{g,n}$. Then define:

$$\overline{\mathcal{M}}_{g,n}(X) = \mathrm{Map}_{\mathbf{St}/\overline{\mathcal{M}}_{g,n}}(\mathcal{C}_{g,n}, X \times \overline{\mathcal{M}}_{g,n}). \tag{13.3.1}$$

To obtain $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we must take maps α with the additional constraint that $\alpha_*[\mathcal{C}_{g,n}] = [\beta] \times [\overline{\mathcal{M}}_{g,n}]$. (Todo: figure out the actual condition)

Remark 13.3.5. When we introduce a derived structure on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we follow the same approach, but take maps in **dSt** instead of **St**.

Theorem 13.3.6 (3.14 in [2])**.** *$\overline{\mathcal{M}}_{g,n}(X, \beta)$ are proper, algebraic Deligne-Mumford stacks.*

(Todo: we should say something about the proof, but the paper is very techinical)

Definition 13.3.7. A smooth projective scheme X is **convex** if for every $f : \mathbb{P}^1 \to X$, $H^1(\mathbb{P}^1, f^*T_X) = 0$. [2]

For example, $\mathbb{P}^r$ is convex for every r. This notion is relevant due to:

Proposition 13.3.8. *If X is convex, then $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is a smooth, proper Deligne-Mumford stack.* [3] *(Todo: what's a reference for this? [8] say it's an expectation in 2.4.2, but Mauro's notes imply that it's proved.)*

Thus, in the situation of convex X, $[\mathcal{M}_{g,n}(X, \beta)]$ can be taken to be the fundamental class. Otherwise we will need to build a virtual fundamental class.

One of the most important properties of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the recursive structure of the boundary; this leads to a proof of the splitting and genus lowering axioms. We first do the case $g = 0$, which is formula 2.7.3.1 in [7].

Choose a partition $S_1 \cup S_2$ of the marked points, and classes β_1, β_2 such that $\beta_1 + \beta_2 = \beta$. Let $D(S_1, S_2; \beta_1, \beta_2) \subset \overline{\mathcal{M}}_{0,n}(X, \beta)$ be the boundary divisor consisting of curves of genus 0 with 2 irreducible components, with marked points S_i and mapping to β_i respectively.

[2] We may want to restrict f to be stable, but we haven't defined this yet, so we'll ignore it for now.
[3] Here we are using the compactification by stable maps; this is defined in **??**.

Lemma 13.3.9. *The boundary divisors are given by:*

$$D(S_1, S_2; \beta_1, \beta_2) = \mathcal{M}_{0,S_1\cup\{x\}}(X, \beta_1) \otimes_X \mathcal{M}_{0,S_2\cup\{x\}}(X, \beta_2).$$

Inducting on this formula, we obtain the structure of the lower dimensional strata as well; we don't write this down though.

Remark 13.3.10. The straight up generalization for curves of any genus would be:

$$\coprod_{g_1+g_2=g} \mathcal{M}_{g_1,S_1\cup\{x\}}(X, \beta_1) \otimes_X \mathcal{M}_{g_2,S_2\cup\{x\}}(X, \beta_2).$$

where $g_1 + g_2 = g$, and $[\beta_1] + [\beta_2] = [\beta]$. I haven't computed the dimensions, though, to see for what values of g_1, g_2 we get codimension 1 strata. Moreover, we have extra contributions from cycles of lower genus curves. (Todo: finish this)

To illustrate the need for virtual fundamental classes, we look at an example where $\overline{\mathcal{M}}_{g,n}(X, \beta)$ contains strata of higher dimension than vdim; in this case, taking the straight up fundamental class would break the grading dimension of Kontsevich-Manin. The following example is worked out in full detail Section 4 of [13].

Example 13.3.11. We compute the dimension and virtual dimension of $\overline{\mathcal{M}_{0,0}}(X, 3\pi^*H)$, where $X = \mathrm{Bl}_p\,\mathbb{P}^2$, $\pi : X \to \mathbb{P}^2$ is the blowup map, and $[H] \in H_2(\mathbb{P}^2, \mathbb{Z})$ is the hyperplane class. Using equation 13.2.1, we have:

$$\operatorname{vdim} \overline{\mathcal{M}_{0,0}}(X, 3\pi^*H) = \int_{3\pi^*H} c_1(T_X) - 1 = 8.$$

One could look, for example, at rational curves of degree 3 in $\mathbb{P}^2$ which avoid p, i.e. $\overline{\mathcal{M}_{0,0}}(\mathbb{P}^2, 3H)$. This is a stratum in $\overline{\mathcal{M}_{0,0}}(X, 3\pi^*H)$ of the correct dimension 8 (the space of cubics in $\mathbb{P}^2$ is 9-dimensional, and we substract 1 for reparametrizations of the domain $\mathbb{P}^1$.) More strata are given by rational cubics in $\mathbb{P}^2$ which pass through p with multiplicity k, and therefore lift to a curve in X of class $3\pi^*H - rE$, where $E \subset X$ is the exceptional divisor. To obtain a stable map in the appropriate class $3\pi^*H$, we add r components isomorphic to $\mathbb{P}^1$ which map to E. The dimension of this stratum is:

$$\dim \overline{\mathcal{M}_{0,0}}(X, 3\pi^*H - rE) + \dim \overline{\mathcal{M}_{0,0}}(\mathbb{P}^1, r) = (8 - r) + (2r - 2) = 6 + r.$$

The farthest we can go while keeping $[\beta]$ effective (that is, $\beta.K_X \leq 0$) is $r = 3$. This gives a stratum (supposedly a boundary stratum!) of dimension $9 > 8$.

13.4 Derived Structure

Generalize construction **??** by taking maps in **dSt**. Explain how the cotangent complex, pulled back to the truncation, gives an obstruction theory. Explain, using [1], how this produces a virtual fundamental class.

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