Derived Algebraic Geometry Seminar: UPenn 2017

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Introduction

This contains notes for the Derived Algebraic Geometry Seminar currently being held at the University of Pennsylvania math department in the 2016-17 academic year. Having introduced the machinery of Derived Algebraic Geometry the previous semester, we investigate its applications to producing Virtual Fundamental Cycles. Initially we will focus on moduli spaces of stable maps, with various boundary conditions, and how VFCs for these can be used to construct Gromov-Witten invariants and Floer-type theories.

This is a draft and errors should be expected.

Chapter 1

Stable Maps and Gromov-Witten Invariants

(Talk by Matei Ionita)

1.1 The Counting Problem

Basic idea of ennumerative geometry, as explained in [4] 3.1: set up a moduli space M for the objects, e.g. curves, one wants to count: $\mathcal{M}_{g,n}(X,\beta)$, equipped with (flat) evaluation maps $\nu_i : \mathcal{M}_{g,n}(X,\beta) \to X$, given by $\big(C, p_1, \ldots p_n, \mu\big) \mapsto \mu(p_i)$. Each constraint $\nu_i \in \Gamma_i$, where $\Gamma_i \in H_*(X,\mathbb{Z})$, gives a subscheme, of $\mathcal{M}_{g,n}(X,\beta)$. We take the intersection of all these:

$$\bigcap_{i=1}^{m} \nu_i^* \Gamma_i.^{1} \tag{1.1.1}$$

If the intersections are transverse and the result has dimension 0, can count the number of points. We would like to set up Γ_i such that:

$$\sum_{i=1}^{m} \operatorname{codim} \Gamma_i = \dim \mathcal{M}_{g,n}(X,\beta).$$

Thus the ennumerative problem is reduced to intersection theory in M. In order to do intersection theory successfully, M needs to be compact (proper), and we need to understand its Chow ring, where the subschemes live.

A first modification: in order to drop the transversality assumption on Γ_i, we replace them with the Poincaré dual cohomology classes γ_i, and take cup products then 1.1.1 is replaced by a first naive definition of the **Gromov-Witten invariants**:

$$I_{g,n,\beta} := \int_{[\mathcal{M}_{g,n}(X,\beta)]} \bigwedge_i \nu_i^* \gamma_i. \tag{1.1.2}$$

If $\mathcal{M}_{g,n}(X,\beta)$ is smooth and proper, then $[\mathcal{M}_{g,n}(X,\beta)]$ is the fundamental class, against which it makes sense to evaluate cohomology classes. $I_{g,n,\beta}$ is defined to be 0 unless $\sum_i \deg \gamma_i = \dim \mathcal{M}_{g,n}(X,\beta)$.

1.2 Axiomatic Definition of GW

The axiomatic approach of Kontsevich and Manin in [5] is as follows. Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne-Mumford compactification by stable curves of the moduli stack of genus g curves with n marked points. We take this as a well-understood object and explain the rest.

[1] This pullback is an umkehr map and we need some assumptions; is properness of μ_i enough?

Definition 1.2.1 (2.2 in [5])**.** A **system of Gromov-Witten classes for X** is a family of linear maps:

$$I^X_{g,n,\beta} : H^*(X,\mathbb{Q})^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

defined for $n+2g-3 \geq 0$, and satisfying the following axioms.

1. **Effectivity**: $I_{g,n,\beta} = 0$ for β non-effective, i.e. not in the dual of the Kähler cone.
2. S_n**-covariance**: equivariant with respect to the obvious S_n action on the domain and target.
3. **Grading**: $\deg I_{g,n,\beta} = -2\int_\beta c_1(X) + (2-2g)\dim X$. More precisely, this means that we set $|\gamma| = i$ for $\gamma \in H^i(X,\mathbb{Q})$ and we require that:

$$\left|I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)\right| = \sum_{j=1}^m |\gamma_j| - 2\int_\beta c_1(X) + (2g-2)\dim X.$$

   Some comments on the grading axiom:

   - Following the convention in [5], we use the real, not complex, dimension.
   - Informally we think of $I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)$ as obtained by pushing forward via the natural map:

$$\mathcal{M}_{g,n}(X,\beta) \to \mathcal{M}_{g,n}.$$

     As a result, its degree is an expectation for $\dim \mathcal{M}_{g,n} - \dim \mathcal{M}_{g,n}(X,\beta)$. We know that $\dim \mathcal{M}_{g,n} = 2(3g-3+n)$. By deformation theory we also compute $\operatorname{vdim} \mathcal{M}_{g,n}(X,\beta)$, called the **virtual dimension**, the expected dimension whenever first-order deformations are unobstructed.
     The tangent space to $\mathcal{M}_{g,n}(X,\beta)$ at a point $(C,p_1,\dots,p_n,\mu)$ is:

$$H^1(C,T_C(-p_1-\dots-p_n)) \oplus H^0(C,\mu^*T_X).$$

     By Serre duality this is:

$$H^0(C,\Omega_C^{\otimes 2}(p_1+\dots+p_n))^\vee \oplus H^0(C,\mu^*T_X).$$

     Approximating the dimensions with the Euler characteristic, we get via Riemann-Roch:

$$\operatorname{vdim} \mathcal{M}_{g,n}(X,\beta) = 2(\dim X - 3)(1-g) + 2\int_\beta c_1(T_X) + 2n. \tag{1.2.1}$$

     Substracting these we get what the grading axiom requires:

$$\dim \mathcal{M}_{g,n} - \dim \mathcal{M}_{g,n}(X,\beta) = 2\int_\beta c_1(X) - (2-2g)\dim X.$$

   - Assume that $I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)$ is of **codimension zero**, i.e. that:

$$\sum_{j=1}^n |\gamma_j| = 2\int_\beta c_1(X) - (2-2g)\dim X. \tag{1.2.2}$$

     Then $\left|I^X_{g,n,\beta}(\gamma_1,\dots,\gamma_m)\right| = \dim \overline{\mathcal{M}}_{g,n}$. We can integrate this against the fundamental class of $\overline{\mathcal{M}}_{g,n}$, which is a proper smooth Deligne-Mumford stack. (Todo: reference?) We obtain a finite number, which we take as the result of the curve count.

4. **Fundamental class.** We introduce some more terminology. Call a class **basic** if it has the smallest n which makes sense, namely:

$$I^X_{0,3,\beta}(\gamma_1,\gamma_2,\gamma_3) \qquad\qquad I^X_{1,1,\beta}(\gamma_1) \qquad\qquad I^X_{g,0,\beta} \text{ for } g\geq 2.$$

Let $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$ be the projection that forgets the last marked point. Let $e^0_X \in H^0(X,\mathbb{Q})$ be the identity of the cohomology ring. Unless the class on the LHS is basic, we require that:

$$I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_{n-1},e^0_X) = \pi^* I^X_{g,n-1,\beta}(\gamma_1,\ldots,\gamma_{n-1}).$$

In addition, we set:

$$I^X_{0,3,\beta}(\gamma_1,\gamma_2,e^0_X) = \begin{cases} \int_X \gamma_1\wedge\gamma_2, & \text{if } \beta=0,\\ 0, & \text{if } \beta\neq 0.\end{cases}$$

5. **Divisor.** In the case $|\gamma_n| = 2$, i.e. γ_n is the Poincaré dual class of a divisor, and if the LHS is a non-basic class, we require:

$$\pi_{n*} I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \int_\beta \gamma_n I^X_{g,n-1,\beta}(\gamma_1,\ldots,\gamma_{n-1}).$$

6. **Splitting.** This axiom and the next are very important: they postulate a manageable structure of the boundary of the compactification $\overline{\mathcal{M}}_{g,n}(X,\beta)$, compatible with that of the boundary of $\overline{\mathcal{M}}_{g,n}$. One way to get boundary maps is to let the curves have 2 irreducible components, with genera g_1, g_2 and marked points n_1+1, n_2+1 such that $g = g_1+g_2$, $n = n_1+n_2$. The extra marked point on each irreducible component is where we glue them; they become one singular point in the resulting reducible curve. For S some partition of the n marked points into 2 sets of cardinality n_1 and n_2, we let $\phi_S : \overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$ be the gluing map. Choose a basis $\{\Delta_a\}$ of $H^*(X,\mathbb{Q})$ and define $g_{ab} = \int_V \Delta_a\wedge\Delta_b$; let $(g^{ab}) = (g_{ab})^{-1}$ denote the entries of the inverse matrix. Then:

$$\phi_S^* I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = (-1)^S \sum_{\beta_1+\beta_2=\beta}\sum_{a,b} I^X_{g_1,n_1+1,\beta_1}(\otimes_{j\in S_1}\gamma_j\otimes\Delta_a) g^{ab}\otimes I^X_{g_2,n_2+1,\beta_2}(\Delta_b\otimes\otimes_{j\in S_2}\gamma_j).$$

Roughly speaking, we need to introduce $\sum_{a,b}(\Delta_a\otimes\Delta_b)$ to account for the position of the extra marked points. Integrating over these produces a factor $g_{a,b}$ that wasn't there on the LHS, so we need to multiply by g^{ab} to compensate for it.

7. **Genus reduction.** Let $\psi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ be the map which glues together the last 2 marked points. Then:

$$\psi^* I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = \sum_{a,b} I^X_{g-1,n+2,\beta}(\gamma_1,\ldots,\gamma_n,\Delta_a,\Delta_b) g^{ab}.$$

The splitting and genus reduction axioms motivate the choice of stable maps compactification, see **??**.

8. **Motivic axiom.** The maps $I^X_{g,n,\beta}$ are induced by correspondences in the Chow rings:

$$C^X_{g,n,\beta} \in C^*(X^n\times\overline{\mathcal{M}}_{g,n}).$$

Namely, consider the two projection maps:

$$\begin{tikzcd} & X^n\times\overline{\mathcal{M}}_{g,n} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X^n & & \overline{\mathcal{M}}_{g,n}. \end{tikzcd}$$

We require that:

$$I^X_{g,n,\beta}(\gamma_1,\ldots,\gamma_n) = q_*\big(C^X_{g,n,\beta}\wedge p^*(\gamma_1\otimes\cdots\otimes\gamma_n)\big).$$

This axiom is motivated as follows in [5], 2.3.8. Suppose we construct a good compactification $\overline{\mathcal{M}}_{0,n}(X,\beta)$, together with a virtual fundamental class $[\overline{\mathcal{M}}_{0,n}(X,\beta)]$. Consider then the map:

$$\alpha : \overline{\mathcal{M}}_{0,n}(X,\beta) \to X^n \times \overline{\mathcal{M}}_{0,n}$$
$$(C, x_1, \ldots, x_n, f) \mapsto \big(f(x_1), \ldots, f(x_n), (\bar{C}, x_1, \ldots, x_n)\big).$$

We would like $\bar{C}$ to be C, but we may need to contract certain components to get a stable curve from a stable map. Compare definitions 1.3.2 and **??**. Ignoring this for now, we set $C^X_(g,n,\beta) = \alpha_*\big([\overline{\mathcal{M}}_{0,n}(X,\beta)]\big)$. This means, roughly speaking, we're integrating over $\overline{\mathcal{M}}_{0,n}(X,\beta)$, like the naive definition 1.1.2 suggests.

We are mostly interested in codimension zero invariants, which informally are those where we imposed enough constraints to get a finite number of curves. For example, if we want to count degree d rational curves in $\mathbb{P}^2$, the relevant codimension zero condition says:

$$\sum_{i=1}^n |\gamma_i| = 2\int_{d[H]} c_1(\mathbb{P}^2) - 2\dim \mathbb{P}^2 = 6d - 4.$$

For example, we could ask that the curves pass through n given points in $\mathbb{P}^2$, then $|\gamma_i| = 4$, so we obtain $4n = 6d - 4$. If the computation were done right, this would be $12d - 4$, so that we get $n = 3d - 1$. So the relevant thing to count are degree d rational curves passing through $3d - 1$ points.(Todo: fix this)

1.3 Stable Map Compactification

To give a naive compactification of $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^r, d)$, we could just look at the space $W(r,d)$ of $r+1$-tuples of degree d polynomials in 2 variables, up to scaling, and take the subset of tuples which don't vanish simultaneously. We get a subset of a projective space:

$$W(r,d) \subset \mathbb{P}\left(\bigoplus_{i=0}^r H^0(\mathbb{P}^1, \mathcal{O}(d))\right).$$

We need to quotient by $\mathrm{Aut}(\mathbb{P}^1)$ to identify maps that differ by a reparametrization; ignoring this for the moment, one hopes to take the closure of $W(r,d)$ in $\mathbb{P}\left(\bigoplus_{i=0}^r H^0(\mathbb{P}^1, \mathcal{O}(d))\right)$ to obtain a compactification. However, for $g \neq 0$ and $X \neq \mathbb{P}^r$, this doesn't work and we need a less ad-hoc approach.

The choice of compactification matters; different choice leads to different numbers. That's because the numbers now count things in the boundary as well.

Example 1.3.1. In the stable maps compactification that we introduce shortly, which produces Gromov-Witten invariants, we keep the domain curves well-behaved: they acquire nodal singularities, but no non-reduced structure. However, the maps themselves can be highly non-injective. A different choice is the Donaldson-Thomas compactification via Hilbert schemes: here we work with ideal sheaves, which always represent embeddings, however the domain curve can now be non-reduced or have singularities worse than nodal. Section $3\frac{1}{2}$ of [14] illustrates the differences with the following example. We work locally and consider the family of conics:

$$C_t = \{x^2 + ty = 0\} \subset \mathbb{C}^2,$$

which becomes singular as $t \to 0$. In the DT compactification, we take the limit in the defining equation, and get $x^2 = 0$, which is a thickened y-axis. In the stable map compactification, we parametrize the conics:

$$C_t \longleftrightarrow \xi \mapsto (-\sqrt{t}\xi, \xi^2).$$

This is a parametrization modulo automorphisms of the curve, namely $\xi \leftrightarrow -\xi$. Now as $t \to 0$, the limiting map is $\xi \mapsto (0, \xi^2)$, which is a double cover of the y-axis. You can't see from this example, but the different choices of compactification actually give different answers for the counting problem.

With that in mind, let's finally define stable maps. For reference and comparison we include the definition of stable curves:

Definition 1.3.2. (Todo: write this up)

Think about graphs of curves, such that each "twig" has no infinitesimal automorphisms. This means that twigs of genus g must have at least $3 - 2g$ special points, which means either marked points or singular ones.

(Todo: figure out an easy way to include the pictures of graphs)

Definition 1.3.3 (2.4.1 in [5])**.** A **stable map** to X is a structure $(C, x_1, \ldots, x_n, f)$ where:

- $(C, x_1, \ldots, x_n)$ is a connected reduced curve with n pairwise distinct marked non-singular points, and at worst additional singular double points.
- $f : C \to X$ is a map with no non-trivial infinitesimal automorphisms. This means that every irreducible component of C of genus g which is contracted to a point (of degree 0) must have at least $3 - 2g$ special points.

Remark 1.3.4. Note that, in the definition of stable maps $(C, x_1, \ldots, x_n, f)$, the underlying curve $(C, x_1, \ldots, x_n)$ need not be stable. Therefore the forgetful map $\overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$ must contract components of $(C, x_1, \ldots, x_n)$ which have infinitesimal automorphisms.

In his talk notes, Mauro provides the following construction of the moduli stacks of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$. Start from $\overline{\mathcal{M}}_{g,n}$, which are fine moduli spaces of curves, and therefore admit a universal family $\mathcal{C}_{g,n}$. Then define:

$$\overline{\mathcal{M}}_{g,n}(X) = \mathrm{Map}_{\mathbf{St}/\overline{\mathcal{M}}_{g,n}}(\mathcal{C}_{g,n}, X \times \overline{\mathcal{M}}_{g,n}). \tag{1.3.1}$$

To obtain $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we must take maps α with the additional constraint that $\alpha_*[\mathcal{C}_{g,n}] = [\beta] \times [\overline{\mathcal{M}}_{g,n}]$. (Todo: figure out the actual condition)

Remark 1.3.5. When we introduce a derived structure on $\overline{\mathcal{M}}_{g,n}(X, \beta)$, we follow the same approach, but take maps in **dSt** instead of **St**.

Theorem 1.3.6 (3.14 in [3])**.** *$\overline{\mathcal{M}}_{g,n}(X, \beta)$ are proper, algebraic Deligne-Mumford stacks.*

(Todo: we should say something about the proof, but the paper is very techinical)

Definition 1.3.7. A smooth projective scheme X is **convex** if for every $f : \mathbb{P}^1 \to X$, $H^1(\mathbb{P}^1, f^*T_X) = 0$. [2]

For example, $\mathbb{P}^r$ is convex for every r. This notion is relevant due to:

Proposition 1.3.8. *If X is convex, then $\overline{\mathcal{M}}_{0,n}(X, \beta)$ is a smooth, proper Deligne-Mumford stack.* [3] *(Todo: what's a reference for this? [5] say it's an expectation in 2.4.2, but Mauro's notes imply that it's proved.)*

Thus, in the situation of convex X, $[\mathcal{M}_{g,n}(X, \beta)]$ can be taken to be the fundamental class. Otherwise we will need to build a virtual fundamental class.

One of the most important properties of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is the recursive structure of the boundary; this leads to a proof of the splitting and genus lowering axioms. We first do the case $g = 0$, which is formula 2.7.3.1 in [4].

Choose a partition $S_1 \cup S_2$ of the marked points, and classes β_1, β_2 such that $\beta_1 + \beta_2 = \beta$. Let $D(S_1, S_2; \beta_1, \beta_2) \subset \overline{\mathcal{M}}_{0,n}(X, \beta)$ be the boundary divisor consisting of curves of genus 0 with 2 irreducible components, with marked points S_i and mapping to β_i respectively.

[2] We may want to restrict f to be stable, but we haven't defined this yet, so we'll ignore it for now.

[3] Here we are using the compactification by stable maps; this is defined in **??**.

Lemma 1.3.9. *The boundary divisors are given by:*

$$D(S_1, S_2; \beta_1, \beta_2) = \mathcal{M}_{0,S_1\cup\{x\}}(X, \beta_1) \otimes_X \mathcal{M}_{0,S_2\cup\{x\}}(X, \beta_2).$$

Inducting on this formula, we obtain the structure of the lower dimensional strata as well; we don't write this down though.

Remark 1.3.10. The straight up generalization for curves of any genus would be:

$$\coprod_{g_1+g_2=g} \mathcal{M}_{g_1,S_1\cup\{x\}}(X, \beta_1) \otimes_X \mathcal{M}_{g_2,S_2\cup\{x\}}(X, \beta_2).$$

where $g_1 + g_2 = g$, and $[\beta_1] + [\beta_2] = [\beta]$. I haven't computed the dimensions, though, to see for what values of g_1, g_2 we get codimension 1 strata. Moreover, we have extra contributions from cycles of lower genus curves. (Todo: finish this)

To illustrate the need for virtual fundamental classes, we look at an example where $\overline{\mathcal{M}}_{g,n}(X, \beta)$ contains strata of higher dimension than vdim; in this case, taking the straight up fundamental class would break the grading dimension of Kontsevich-Manin. The following example is worked out in full detail Section 4 of [12].

Example 1.3.11. We compute the dimension and virtual dimension of $\overline{\mathcal{M}_{0,0}}(X, 3\pi^*H)$, where $X = \mathrm{Bl}_p\,\mathbb{P}^2$, $\pi : X \to \mathbb{P}^2$ is the blowup map, and $[H] \in H_2(\mathbb{P}^2, \mathbb{Z})$ is the hyperplane class. Using equation 1.2.1, we have:

$$\operatorname{vdim} \overline{\mathcal{M}_{0,0}}(X, 3\pi^*H) = \int_{3\pi^*H} c_1(T_X) - 1 = 8.$$

One could look, for example, at rational curves of degree 3 in $\mathbb{P}^2$ which avoid p, i.e. $\overline{\mathcal{M}_{0,0}}(\mathbb{P}^2, 3H)$. This is a stratum in $\overline{\mathcal{M}_{0,0}}(X, 3\pi^*H)$ of the correct dimension 8 (the space of cubics in $\mathbb{P}^2$ is 9-dimensional, and we substract 1 for reparametrizations of the domain $\mathbb{P}^1$.) More strata are given by rational cubics in $\mathbb{P}^2$ which pass through p with multiplicity k, and therefore lift to a curve in X of class $3\pi^*H - rE$, where $E \subset X$ is the exceptional divisor. To obtain a stable map in the appropriate class $3\pi^*H$, we add r components isomorphic to $\mathbb{P}^1$ which map to E. The dimension of this stratum is:

$$\dim \overline{\mathcal{M}_{0,0}}(X, 3\pi^*H - rE) + \dim \overline{\mathcal{M}_{0,0}}(\mathbb{P}^1, r) = (8 - r) + (2r - 2) = 6 + r.$$

The farthest we can go while keeping $[\beta]$ effective (that is, $\beta.K_X \leq 0$) is $r = 3$. This gives a stratum (supposedly a boundary stratum!) of dimension $9 > 8$.

Chapter 2

Obstruction Theories and Virtual Fundamental Classes

Talk by Benedict Morrissey.

Given a stack X, our objective is to construct a virtual fundamental class $[X]$ for it, motivated by the discussion in 1. We will see two ways in which a derived enhancement of X helps achieve this. We would like $[X]$ to come from an algebraic cycle, i.e. an element of the Chow group. In this case, given $f : X \to Y$ proper, there is a well-defined pushforward $f_*[X] \to [Y]$, which induces a pushforward $f_*[X]^H \to [Y]^H$ on the images $[X]^H, [Y]^H$ of the VFCs in any Weil cohomology theory H.

However, derived Chow groups have yet to be defined, so we start with a piecemeal approach, by defining a class in G-theory only.

2.1 Construction from G-Theory

Definition 2.1.1. The **G-theory** $G_0(X)$ of a classical stack X is defined as the K-theory of the category of coherent sheaves on X: [1]

$$G_0(X) := K_0(\mathrm{Coh}(X)).$$

If $\tilde{X}$ is a derived stack, we set $G_0(\tilde{X}) = K_0((\mathrm{Coh}\tilde{X})^{\heartsuit})$.

Definition 2.1.2. A **derived enhancement** of a stack X is a derived stack $\tilde{X}$ such that $t_0(\tilde{X}) = X$.

There is a natural inclusion, left-adjoint to the truncation, which we denote $j : X \to \tilde{X}$. Using the fact that pushforwards of coherent sheaves by proper maps are coherent, (Todo: check if there are other conditions, and whether $\tilde{X}$ derived changes anything) we obtain $j_* : G_0(X) \to G_0(\tilde{X})$.

Proposition 2.1.3. *If X is quasi-compact, then $j_* : G_0(X) \to G_0(\tilde{X})$ is a bijection. In this case we define:*

$$[X]^{\mathrm{vir}} := j_*^{-1}[\mathcal{O}_{\tilde{X}}].$$

Proof. The identification actually works on the full spectrum of G-theory. We're using the theorem of the heart for K-theory. The identification is done as follows.

1. Theorem of the heart for K-theory. (Due to Quillen, and Batwick in the DG category setting.) If you have $\mathcal{C}$ a stable ∞-category, idempotent complete, with t-structure, and every object in the heart is bounded, then $K(\mathcal{C}) = K(\mathcal{C}^{\heartsuit})$.

[1] One can also define higher G-theory G_i, but we won't need this.

2. $\mathrm{Coh}(\tilde{X})^{\heartsuit} \simeq \mathrm{Coh}(X)^{\heartsuit}$, which follows from descent and the analogous result for derived affines, which was proved during the first semester, in the talk on Stable ∞-categories. [2]

□

Theorem 2.1.4. *For $\tilde{X}$ quasi-compact,[3] $\mathcal{O}_{\tilde{X}}$ is bounded. It follows that the following sum is finite:*

$$j_*^{-1}[\mathcal{O}_{\tilde{X}}] = \sum_{i=0}^{\infty}(-1)^i [H^i(\mathcal{O}_{\tilde{X}})],$$

so it defines an element in $G_0(X)$.

Remark 2.1.5. Note that the cohomology in Theorem 2.1.4 is just the cohomology of the complex, NOT sheaf cohomology. Moreover it wouldn't make sense to use K theory instead of G theory, because even if $\mathcal{O}_{\tilde{X}}$ is perfect, the kernels and cokernels of the various differentials don't need to be.

Proof. We start with a vague understanding of why the theorem may be true. The counterexample is $\mathrm{Spec}(\mathrm{Sym}\, k[2])$, where the cotangent complex is unbounded. But if it's in amplitude [-1,0], it's like an exterior algebra and it works.

We work locally, $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$, $\mathrm{supp}\, \mathbb{L}_{B/A} \subset [-1, 0]$. B is a derived lci over A, so the cotangent complex is perfect, so there's a theorem that says that B is homotopically of finite type over A. These can be constructed by attaching finitely many cells:

$$A = B_0 \to B_1 \to B_2 \to \cdots \to B_k = B.$$

Attaching the $i + 1^{th}$ cells of B looks like:

$$\begin{tikzcd} B_i \arrow[r] & B_{i+1} \\ \bigotimes A[\partial\Delta^{i+1}] \arrow[u] \arrow[r] & A[\Delta^{i+1}]. \arrow[u] \end{tikzcd}$$

B_1 is obtained by attaching cells in degree 1. The map $B_1 \to B$ is an isomorphism on π_0. (Todo: review this proof)

There's another proof by Lowrey and Schürg, in [**?**], which is more intuitive. Having a quasi-smooth structure allows one to describe the derived space locally as the derived zero locus of a section of a vector bundle. Then the derived intersection can be computed as a Koszul resolution, so $\mathcal{O}_{\tilde{X}}$ behaves like an exterior algebra, which means it's bounded. Here the quasi-compactness is used in order to reduce to finitely many local charts, which means that the bound on $\mathcal{O}_{\tilde{X}}$ is uniform. □

Remark 2.1.6. The idea behind the proof of Lowrey and Schürg is also that of **Kuranishi structures**. These are essentially a machinery for working with derived stacks which remembers the local description as zero locus, in order to avoid using the machinery of derived geometry. In DAG quasi-smoothness is an intrinsic property that one can check at the level of the cotangent complex, so that one doesn't need to remember the local descriptions, which are cumbersome and don't glue well.

The VFC in ordinary cohomology is defined by Konstevich to be:

$$[X]^{\mathrm{vir}} = \mathrm{Ch}([X]_G^{\mathrm{vir}})\,\mathrm{Td}(j^*\mathbb{T}_{\tilde{X}}). \tag{2.1.1}$$

Conjecture 2.1.7. *Definition 2.1.1 agrees with the construction of Behrend-Fantechi,* **??**. *(Todo: ref this)*

The conjecture has been verified for schemes (not stacks) by Ciocan-Fontanine and Kapranov, in [**?**], using the additional assumption (which is made in Behrend Fantechi anyway) that the cotangent complex admits a global resolution by vector bundles.

[2]Throughout when we write Coh we mean Coh^b.

[3]Note that we don't need to assume that X is quasi-compact.

2.2 Obstruction Theories

We introduce the alternative construction of VFCs, following [2]. In the words of Mauro, we want to use this as a black box which achieves:

$$\text{Obstruction Theory} \Longrightarrow \text{VFC}.$$

Throughout we will use X, Y for underived stacks, and $\tilde{X}, \tilde{Y}$ for their derived enhancements.

Definition 2.2.1. An **obstruction theory** for X is a morphism $\phi : E \to \mathbb{L}_X$ in $D(\mathrm{Coh}(X))$, such that:

$$\begin{aligned}
&h^0(\phi) : H^0(E) \to H^0(\mathbb{L}_X) \text{ is an isomorphism,}\\
&h^{-1}(\phi) : H^{-1}(E) \to H^{-1}(\mathbb{L}_X) \text{ is surjective,}\\
&H^i(E) = 0 \text{ for } i \neq -1, 0.
\end{aligned}$$

Definition 2.2.2. A **perfect obstruction theory** is an obstruction theory such that E is in perfect amplitude [-1,0], which means that locally E is isomorphic to a 2-term complex of vector bundles $[E^{-1} \to E^0]$.

The link to derived geometry is as follows.

Proposition 2.2.3. *Given a derived enhancement $j : X \to \tilde{X}$, with $\tilde{X}$ a quasi-smooth DM stack, there is a perfect obstruction theory:*

$$j^*\mathbb{L}_{\tilde{X}} \to \mathbb{L}_X.$$

Proof. By descent we reduce this to the case of affines, and we need only consider $A \to t_0(A)$. We have the fiber sequence:

$$j\mathbb{L}_A \to \mathbb{L}_{\pi_0(A)} \to \mathbb{L}_{\pi_0(A)/A}.$$

Due to the connectivity estimates, which we introduced last semester in the talk about the cotangent complex, $\mathbb{L}_{\pi_0(A)/A}$ is 2-connective. Indeed, the fiber of $A \to \pi_0(A)$ is 1-connective, so the cofiber, which is the shift of the fiber by 1, is 2-connective.[4] ☐

Throughout the rest of the talk, the goal is to describe how to construct a VFC, starting with an obstruction theory. In the smooth case, if you take the G-construction we did earlier, you'd get the same answer.

We also want to describe functoriality properties for the VFC, and to that effect we introduce compatibility data between obstruction theories. During the check that Kontsevich-Manin axioms are satisfied, we will need to use functoriality a lot. The following is Definition 5.8 in [2].

Definition 2.2.4. Let $u : X' \to X$ be a morphism. A **compatibility datum between obstruction theories** E for X and F for X' is a choice of embeddings $f : X \to Y$, $g : X' \to Y'$ into smooth stacks, such that the following diagrams commute:

$$\begin{tikzcd}
X' \arrow[r, "u"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y' \arrow[r, "v"] & Y,
\end{tikzcd}$$

$$\begin{tikzcd}
u^*E \arrow[r, "\phi"] \arrow[d] & F \arrow[r, "\psi"] \arrow[d] & g^*\mathbb{L}_{Y'/Y} \arrow[d] \\
u^*\mathbb{L}_X \arrow[r] & \mathbb{L}_{X'} \arrow[r] & \mathbb{L}_{X'/X}.
\end{tikzcd}$$

Moreover, we require the two rows to be fibration sequences in $D(\mathrm{Coh}(X'))$.

[4] In order to relate the fiber and cofiber of the morphism $A \to \pi_0(A)$, we use the fact that we are working in the stable ∞-category $\pi_0(A)$-Mod, and not in $\pi_0(A) - \mathrm{Alg}$.

Behrend and Fantechi prove:

Proposition 2.2.5. *Given compatibility data between obstruction theories E for X and F for X', it follows that $u^*[X]^{\mathrm{vir},E} = [X']^{\mathrm{vir},F}$.*

For us obstruction theories come from derived enhancements $\tilde{X}, \tilde{X}'$. In this case, we obtain the functoriality of VFCs in a cleaner way, by giving a morphism between derived enhancements $w : \tilde{X}' \to \tilde{X}$, fitting in the commutative diagram:

$$\begin{tikzcd}
 & \tilde{X}' \arrow[rr, "w"] \arrow[dl, "\tilde{g}"'] & & \tilde{X} \arrow[dl, "\tilde{f}"] \\
X' \arrow[ur, "j"] \arrow[rr, "u"] \arrow[d, "g"'] & & X \arrow[ur, "i"] \arrow[d, "f"'] & \\
Y' \arrow[rr, "v"] & & Y &
\end{tikzcd}$$

Moreover, we require that the top and back square are homotopy pullbacks.

Remark 2.2.6. I was hoping that the top square would be enough. Unfortunately, we still need the choice of ambient spaces Y, Y', as well as morphisms $\tilde{g}$, $\tilde{f}$, and the data for the homotopy commutativity of the back square. However Mauro says:

1. In the applications we care about (stable maps), the entire back square will be there naturally.
2. Working with the derived compatibility data is still easier, in practice, than with the fibration sequences in Definition 2.2.4.

Let us see why the derived compatibility data implies the diagram between fibration sequences in Definition 2.2.4. The assumption is that $E = i^*\mathbb{L}_{\tilde{X}}$ and $F = j^*\mathbb{L}_{\tilde{X}'}$. We first need the map:

$$\phi : u^*i^*\mathbb{L}_{\tilde{X}} \to j^*\mathbb{L}_{\tilde{X}'}.$$

This is just given by w. More precisely, the commutativity of the top square gives the map on the left in the following diagram, and we define the top map as the composition:

$$\begin{tikzcd}
u^*i^*\mathbb{L}_{\tilde{X}} \arrow[r] \arrow[d] & j^*\mathbb{L}_{\tilde{X}'} \\
j^*w^*\mathbb{L}_{\tilde{X}} \arrow[ur] &
\end{tikzcd}$$

To get ψ, which must be such that the row is a fiber sequence, we make use of the maps $\tilde{g}, \tilde{f}$. Question: how to identify $j^*\mathbb{L}_{\tilde{X}'/\tilde{X}}$ with $g^*\mathbb{L}_{Y'/Y}$? Since the back square is a pullback, we have a canonical identification $\mathbb{L}_{\tilde{X}'/\tilde{X}} \simeq \tilde{g}^*\mathbb{L}_{Y'/Y}$, and this gives:

$$j^*\mathbb{L}_{\tilde{X}'/\tilde{X}} \simeq j^*\tilde{g}^*\mathbb{L}_{Y'/Y} \simeq g^*\mathbb{L}_{Y'/Y}. \tag{2.2.1}$$

We take this composition to be ψ. Note that this chain of equivalences depends very much on the extra data of the homotopy commutative back square.

Remark 2.2.7. Throughout, we want Y', Y to be smooth, and $\tilde{f}, \tilde{g}$ to be quasi-smooth. Therefore, if X, X' are not smooth, we cannot expect f, g to be just identity maps. In fact, the point that Behrend-Fantechi make is that Y and Y' should only be expected to exist locally.

Definition 2.2.8. A **local embedding** (U, M) of X is the data of $U \to X$ an étale map and $U \to M$ a local immersion, where M smooth affine k-scheme of finite type. Given a local embedding, the associated **normal bundle** is $\mathfrak{N}_{U|M} := \mathrm{Spec}_M(\mathrm{Sym}(I/I^2))$. Inside this we have the **normal cone** $\mathfrak{C}_{U/M} = \mathrm{Spec}_M(\oplus_{n\geq 0} I^n/I^{n+1})$. The ring homomorphism $\mathrm{Sym}(I/I^2) \to \oplus_{n\geq 0} I^n/I^{n+1}$ is surjective, so the map $\mathfrak{C}_{U/M} \to \mathfrak{N}_{U|M}$ is a closed embedding.

The normal bundle and normal cone of Definition 2.2.8 depend on a choice of local embedding. We would like to have intrinsic versions, and to obtain them we have to take a limit over all local embeddings, morally speaking. For this, we need a way of associating a cone stack to a complex. Behrend and Fantechi have a construction that achieves this, but instead we use a slightly different version from some unpublished notes of Marco Robalo. (Todo: can we reference these?)

Let $E \in \mathrm{QCoh}^b(X)$. Define $\mathbb{V}(E)$ as follows. For a map $\mu : \operatorname{Spec} A \to X$,

$$\mathrm{Map}_{/X}(\operatorname{Spec} A, \mathbb{V}(E)) = \mathrm{Map}_{QCoh(A)}(\mu^*(E), \mathcal{O}_A) \simeq \mathrm{Map}_{QCoh(A)}(\mathcal{O}_A, \mu^*(E^\vee)).$$

E is perfect for the last thing to make sense. Marci: this is the vector bundle corresponding to E. The idea is that, for $E = [E_0 \to E_1]$, we want $\mathbb{V}(E^\vee) \simeq E_1/E_0$. For example, if $E_1 = 0$, then $E_1/E_0 = BE_0$, which has for each fiber the classifying space of the corresponding abelian group.

The first claim is that:

$$\mathbb{V}(E) = \operatorname{Spec}_X(\operatorname{Sym}_X(E)).$$

This is because:

$$\mathrm{Map}_{/X}(\operatorname{Spec}(A), \operatorname{Spec}_X \operatorname{Sym}_X(E)) = \mathrm{Map}_{\mathrm{QCoh}(A)}(\operatorname{Sym}_A(\mu^*(E)), \mathcal{O}_A) = \mathrm{Map}_{\mathrm{QCoh}(A)}(\mu^* E, \mathcal{O}_A).$$

Next claim: if E is of perfect amplitude $[-1, 0]$ over X, then $\mathbb{V}(E[-1]) = h^1/h^0(E^\vee)$.

$$\mathrm{Map}_{/X}\left(\operatorname{Spec} A, h^1/h^0(E^\vee)\right) = \mathrm{Map}_{/A}\left(\operatorname{Spec} A, \mu^*(h^1/h^0(E^\vee))\right) = \mathrm{Map}_{\mathrm{QCoh}(A)}\left(\mu^*(E_0^\vee \to E_{-1}^\vee), \mathcal{O}_A\right).$$

The meaning of the functor $\mathbb{V}$ is actually more intuitive when done in more generality, see subsection 2.2.1 for Mauro's point of view on it.

Definition 2.2.9. The **intrinsic normal sheaf** of X is $\mathfrak{N}_X := \mathbb{V}(\tau^{\geq -1}\mathbb{L}_X)$.

Compare with the definition of [2]:

$$\mathfrak{N}_X = h^1/h_0(\mathbb{L}_X^\vee) = h^1/h^0(\tau^{\leq 1}\mathbb{L}_X^\vee) = \mathbb{V}(\tau^{\geq -1}\mathbb{L}_X).$$

Now consider a local immersion:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"] & M \\ X & \end{tikzcd}$$

We want to compare $\pi^*\mathfrak{N}_X$ with $\mathfrak{N}_{U/M}$; the answer is:

$$[\mathfrak{N}_{U/M}/f^*T_M] \xrightarrow{\sim} \pi^*\mathfrak{N}_X.$$

Recall that for each local immersion we have an immersed cone $\mathfrak{C}_{U/M} \to \mathfrak{N}_{U/M}$. It turns out that $[\mathfrak{C}_{U/M}/f^*T_M]$ glue nicely, and by descent we get an immersed cone $\mathfrak{C}_X \to \mathfrak{N}_X$.

Definition 2.2.10. $\mathfrak{C}_X$ is the **intrinsic normal cone** of X.

Given a perfect obstruction theory $E^\bullet \to \mathbb{L}_X$, [2] show that we obtain another immersed cone stack $\mathbb{V}(E) \to \mathfrak{N}_X$. We would like to define a VFC for X to be the intersection $\mathbb{V}(E) \cap \mathfrak{C}_X$ in $\mathfrak{N}_X$. The problem is that $\mathfrak{N}_X$ is an Artin stack, for which the Chow group and intersection theory are not properly defined. To avoid this issue we need a choice of atlas, which translates the problem into one about Deligne-Mumford stacks, for which Chow groups are well-understood. (Todo: reference to Vistoli would be nice)

For this we need extra data of a **global presentation** for $E^\bullet$: this is a 2-term complex of vector bundles $[F^{-1} \to F^0]$ such that $F^\bullet \simeq E^\bullet$ as objects of $\mathrm{Perf}(X)$. Let $F_1 := F^{-1\vee}$, which is an atlas for the stack $\mathbb{V}F$. Then define $C(F)$ as the pullback:

$$\begin{tikzcd} C(F^\bullet) \arrow[r] \arrow[d] & F_1 \arrow[d] \\ \mathfrak{C}_X \arrow[r] & \mathfrak{N}_X. \end{tikzcd}$$

Definition 2.2.11. Let $0 : X \to F_1$ be the zero section. The **virtual fundamental class** of X induced by the obstruction theory E is the intersection of $[C(F^\bullet)] \in \mathrm{Chow}(F_1)$ with the zero section, i.e. $[X]^{\mathrm{vir},E} := 0^![C(F^\bullet)]$. (Todo: Note that 0 is not flat, so 0^* would be undefined.)

Remark 2.2.12. [2] prove that $[X]^{\mathrm{vir},E}$ does not depend on the choice of global presentation.

2.2.1 Picard stacks

We follow and generalize [1, Exposé XVIII, §1.4]. Recalling the equivalence (up to homotopy) between groupoids and 1-homotopy type, we can rephrase Definition 1.4.5 in loc. cit. as follows:

Definition 2.2.13. Let $\mathcal{X}$ be an ∞-topos. A *Picard stack* over X is a sheaf

$$\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathrm{sAb}^{\leq 1},$$

where $\mathrm{sAb}^{\leq 1}$ denotes the ∞-category of simplicial abelian groups whose underlying space is a 1-homotopy type. We let $\mathrm{Pic}(\mathcal{X})$ denote the ∞-category of Picard stacks on $\mathcal{X}$.

The main result in loc. cit. can then be summarized as follows:

Proposition 2.2.14 (Proposition 1.4.15 & Corollary 1.4.17 in loc. cit.)**.** *Let $\mathcal{X}$ be an ∞-topos. There is an equivalence of ∞-categories*

$$\mathrm{Pic}(\mathcal{X}) \simeq \mathrm{Sh}_{\mathcal{D}^{[-1,0]}(\mathrm{Ab})}(\mathcal{X}),$$

where $\mathcal{D}^{[-1,0]}(\mathrm{Ab})$ denotes the full ∞-subcategory of $\mathcal{D}(\mathrm{Ab})$ (the ∞-derived category of abelian groups) spanned by objects in cohomological amplitude $[-1,0]$.

From a modern point of view, the proof is a direct consequence of the Dold-Kan equivalence

$$\mathrm{sAb} \simeq \mathcal{D}^{\leq 0}(\mathrm{Ab}),$$

combined with the remark that objects in cohomological degree $[-1,0]$ in $\mathcal{D}^{\leq 0}(\mathrm{Ab})$ correspond to 1-homotopy types. Actually, the language of higher stacks, allows us to generalize the above proposition:

Proposition 2.2.15. *Let $\mathcal{X}$ be an ∞-topos. There is an equivalence of ∞-categories*

$$\mathrm{Sh}_{\mathrm{sAb}}(\mathcal{X}) \simeq \mathrm{Sh}_{\mathcal{D}^{\leq 0}(\mathrm{Ab})}(\mathcal{X}).$$

Remark 2.2.16. From the ∞-categorical point of view, it is actually very unnatural to distinguish the two categories. In other words, the above proposition, should be perceived as *tautological* (at least from the reader used to higher categorical reasoning). Indeed, the more natural way of seeing this question is to identify $\mathcal{D}(\mathrm{Ab})$ with $\mathrm{Sp}(\mathrm{sAb})$. The Dold-Kan equivalence is then induced by the forgetful functor $\Omega^\infty\colon \mathrm{Sp}(\mathrm{sAb}) \to \mathrm{sAb}$.

We now consider a special case of interest: namely, we will suppose that $\mathcal{X}$ is the smooth-étale site of some derived Artin stack X. In this case, we have a forgetful functor

$$U\colon \mathrm{QCoh}(X)^{\leq 0} \to \mathrm{Sh}_{\mathcal{D}^{\leq 0}(\mathrm{Ab})}(\mathcal{X}),$$

that allows to see a quasi-coherent sheaf $\mathcal{F}$ on X as a (higher) Picard stack on X.

On the other hand, suppose that $\mathcal{E} \in \mathrm{QCoh}(X)$. We can associate to $\mathcal{E}$ a (higher) Picard stack in the following way:

Definition 2.2.17. Let $\mathcal{E} \in \mathrm{QCoh}(X)$. We set

$$\mathbb{V}(\mathcal{E}) := \mathrm{Spec}_X(\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E})).$$

Remark 2.2.18. In other words, for any $u\colon \operatorname{Spec}(A) \to X$, one has

$$\begin{aligned}\operatorname{Map}_{/X}(\operatorname{Spec}(A), \mathbb{V}(\mathcal{E})) &\simeq \operatorname{Map}_{\mathrm{CAlg}(\mathcal{O}_X)}(\operatorname{Sym}_{\mathcal{O}_X}(\mathcal{E}), u_*\mathcal{O}_A) \\ &\simeq \operatorname{Map}_{\mathrm{QCoh}(X)}(\mathcal{E}, u_*\mathcal{O}_A) \simeq \operatorname{Map}_{\mathrm{QCoh}(A)}(u^*(\mathcal{E}), \mathcal{O}_A).\end{aligned}$$

As $\mathrm{QCoh}(A)$ is naturally enriched in $\mathcal{D}(\mathrm{Ab})$ and since for any $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(A)$ we have

$$\operatorname{Map}_{\mathrm{QCoh}(A)}(\mathcal{F}, \mathcal{G}) \simeq \tau_{\leq 0} \operatorname{Map}^{\mathcal{D}(\mathrm{Ab})}_{\mathrm{QCoh}(A)}(\mathcal{F}, \mathcal{G}),$$

it is then clear that $\mathbb{V}(E)$ defines a (higher) Picard stack on X.

Remark 2.2.19. Suppose that $\mathcal{E} \in \mathrm{QCoh}(X)^{\leq n}$, with $n \geq 0$. Then $\mathbb{V}(\mathcal{E})$ is n-truncated in the sense that $\mathrm{t}_0(\mathbb{V}(\mathcal{E}))$ takes values in n-homotopy types.

In view of the above considerations, the following question is a natural one:

Question 2.2.20. Is there a reasonable full subcategory $\mathcal{C}$ of $\mathrm{QCoh}(X)$ and a functor $F\colon \mathcal{C} \to \mathrm{QCoh}(X)^{\leq 0}$ such that the diagram

$$\begin{tikzcd} & & \mathrm{QCoh}(X)^{\leq 0} \arrow[d, "U"] \\ \mathcal{C} \arrow[r, hook] \arrow[urr, dashed, bend left, "F"] & \mathrm{QCoh}(X) \arrow[r, "\mathbb{V}(-)"] & \mathrm{Sh}_{\mathcal{D}^{\leq 0}(\mathrm{Ab})}(X) \end{tikzcd}$$

commutes?

The answer is positive:

Proposition 2.2.21. *Let $\mathcal{C} = \mathrm{Perf}(X)^{\geq 0}$ be the category of perfect complexes on X that are in positive cohomological amplitude. Let $F := (-)^\vee\colon \mathrm{Perf}(X)^{\geq 0} \to \mathrm{QCoh}(A)^{\leq 0}$ be the duality functor:*

$$\mathcal{E}^\vee := \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X).$$

Then the above diagram commutes.

Proof. Let $u\colon \operatorname{Spec}(A) \to X$ be a fixed map and let $\mathcal{E} \in \mathrm{Perf}(X)^{\geq 0}$. Then, since $\mathcal{E}$ is perfect, we have:

$$\operatorname{Map}_{/X}(\operatorname{Spec}(A), \mathbb{V}(\mathcal{E})) \simeq \operatorname{Map}_{\mathrm{QCoh}(A)}(u^*(\mathcal{E}), \mathcal{O}_A) \simeq \operatorname{Map}_{\mathrm{QCoh}(A)}(\mathcal{O}_A, u^*(\mathcal{E}^\vee)).$$

Observe now that we can identify $\operatorname{Map}_{\mathrm{QCoh}(A)}(\mathcal{O}_A, u^*(\mathcal{E}^\vee))$ with the underlying complex of abelian groups of $u^*(\mathcal{E}^\vee)$. In other words, it coincides by definition with $U(\mathcal{E}^\vee)(\operatorname{Spec}(A))$. □

Let now $\mathcal{F} \in \mathrm{Perf}^{[-1,0]}(X)$ be a perfect complex in tor-amplitude $[-1, 0]$. Then $\mathcal{F}[-1] \in \mathrm{Perf}^{\geq 0}(X)$ and therefore the above proposition supplies us with an equivalence

$$\mathbb{V}(\mathcal{F}[-1]) \simeq U(\mathcal{F}^\vee[1]).$$

Proposition 2.2.22. *The stack $U(\mathcal{F}^\vee[1])$ coincides with the stack $(h^1/h^0)(\mathcal{F}^\vee)$ of [2].*

Proof. This follows tautologically if one believes to the claim at the beginning of [2, §2] that $(h^1/h^0)(\mathcal{F}^\vee)$ coincides with the construction $\mathrm{ch}(-)$ performed in [1, Exposé XVIII, §1.4]. (Todo: Understand why this claim is true.) □

2.3 Stable maps

Following [3], let $\mathcal{M}(V, \tau, \beta)$ the moduli space of maps from a Riemann surface of type τ to an algebraic variety V, such that $\mu^*[C] \cong \beta$. Here τ is a graph with edges labeled by g, n; think of it as a type of degeneracy for Riemann surfaces, which becomes a boundary stratum in the moduli space of stable maps

$\mathcal{M}_{g,n}(X)$. Each edge in the graph τ corresponds to an irreducible Riemann surface with genus g and number of makred points n, while nodes of the graph correspond to intersections of such.

Consider the following diagram of stacks over $\mathcal{M}_{g,n}(X)$.

$$\begin{tikzcd}
\mathcal{C}(V,\tau,\beta) \arrow[d, "\pi"] \arrow[dr, "f"] & \\
\mathcal{M}(V,\tau,\beta) & V.
\end{tikzcd}$$

Proposition 2.3.1. $\mathbb{R}\pi_*(f^*T_V)^\vee \to \mathbb{L}_{\mathcal{M}(\tau,V,\beta)}$ *is a perfect obstruction theory.*

Remark 2.3.2. Note that if we take V a point, then $\mathcal{M}(\tau,V,\beta) = \mathcal{M}(\tau)$, and we obtain $0 \to \mathbb{L}_{\mathcal{M}(\tau)}$. But an obstruction theory is supposed to be an isomorphism in degree 0: does this hold here? (Todo: someone figure this out)

Proof. We have the following sequence of maps:

$$f^*\mathbb{L}_V \to \mathbb{L}_C \to \pi^*\mathbb{L}_{\mathcal{M}(\tau,V,\beta)}.$$

Upon tensoring with the canonical sheaf of C, this becomes:

$$f^*\mathbb{L}_V \otimes \omega_C \to$$

(Todo: someone finish this argument) ☐

Behrend proves that these satisfy a bunch of axioms, and then Behrend-Manin in [3] prove that a family of VFCs with said axioms give a system of GW invariants a la Konstsevich-Manin.(Todo: someone write this up)

2.4 Orientations and Gromov–Witten Invariants

(Todo: Explain Thm 9.3 in Behrend Manin, so basically the idea that we can get invariants from and Orientation, Define an Orientation)

Chapter 3

Geometricity of Mapping Stacks

(Talk by Matei Ionita) This chapter is somewhat tangential to our concrete goals for the semester. However we thought that $\mathbb{R}\operatorname{Map}_{g,n}(X,\beta)$ provides a good opportunity to understand Artin-Lurie representability and how it can be used to prove that certain mapping stacks are geometric.

3.1 Using Artin-Lurie representability for Mapping Stacks

The representability theorem says:

Theorem 3.1.1 (Artin-Lurie representability, Theorem 3.2.1 in [7])**.** *[**?**] Let $X : \mathrm{cdga}_k^{\leq 0} \to \mathcal{S}$ be a functor, and suppose we are given a natural transformation $f : X \to \operatorname{Spec} R$. Then X is representable by a derived Deligne-Mumford n-stack locally almost of finite presentation over R if and only if the following are satisfied:*

1. *For every discrete commutative ring A, the space $X(A)$ is n-truncated.*
2. *X is a sheaf for the étale topology.*
3. *X is nilcomplete, infinitesimally cohesive and integrable. These mean:*
   - *X commutes with Postnikov towers;*
   - *X commutes with pullback squares $B \times_A C$, under the assumption that $\pi_0(B) \to \pi_0(A)$ and $\pi_0(C) \to \pi_0(A)$ are surjective with nilpotent kernel;*
   - *for A a complete local ring, $X(A) \simeq \varprojlim X(A/\mathfrak{m}^n)$; loosely speaking, every formal A-point of X integrates to give a point of X.*
4. *$f : X \to R$ admits a connective relative cotangent complex $\mathbb{L}_{X/R}$.*
5. *$f : X \to R$ is locally almost of finite presentation.*

Remark 3.1.2. (2) is obvious, (5) ensures that the DM stack is locally almost of finite presentation. (3) and (4) ensure that X has good local behavior, in particular a good deformation theory. The existence of the relative cotangent complex is conceptually the most important condition, and the one we will put the most effort into verifying. Finally, (1) encodes the geometricity of the representing DM stack. n-stacks are defined to be those for which condition (1) holds; it is then true that: (Todo: Mauro said so; maybe also find a reference)

- n-geometric implies $n+1$-stack;
- m-geometric for some m and n-stack implies that, at worst, $m = n+1$.

(Todo: this is not completely satisfactory: does Lurie representability guarantee that we get m-geometric for some m?)

As an application of this, we want to prove the geometricity of mapping stacks.

Theorem 3.1.3. *Let $g : X \to Z$ be a morphism of derived stacks which is geometric and of finite type. Let $f : Y \to Z$ be a morphism of stacks which is representable by proper flat schemes.* [1] *Then the mapping stack $\mathrm{Map}_{/Z}(Y, X)$ is geometric over Z, i.e. the morphism $\mathrm{Map}_{/Z}(Y, X) \to Z$ is geometric.*

Proof. Recall that the mapping stack is defined by the functor of points:

$$\mathrm{Map}_{/Z}(Y, X)(T) = \mathrm{Map}_{\mathbf{dSt}}(T \times_Z Y, X).$$

We first reduce to Z affine, so that Theorem **??** applies. Then, since g is assumed geometric, it satisfies conditions (3) of the Theorem **??**. It follows by elementary manipulation of the diagrams that $\mathrm{Map}_{/Z}(Y, X) \to Z$ also has these properties; see Proposition 3.3.6 in [7]. The most important issue is the existence of a relative cotangent complex for $\mathrm{Map}_{/Z}(Y, X) \to Z$. Recalling the definition, we need to construct cotangent complexes $\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X),x}$ at each point $x : \mathrm{Spec}\, A \to \mathrm{Map}_{/Z}(Y, X)$, and then make sure that they glue; this will be diagram **??** below.

$\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X),x}$ is supposed to be an object that represents the functor of derivations over $\mathrm{Map}_{/Z}(Y, X)$:

$$\mathrm{Map}_{A\text{-Mod}}(\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X),x}, M) = \mathrm{Der}_{\mathrm{Map}_{/Z}(Y,X)}(A, M).$$

The latter is defined as the homotopy pullback:

$$\begin{tikzcd}
\mathrm{Der}_{\mathrm{Map}_{/Z}(Y,X)}(A, M) \arrow[r] \arrow[d] & \mathrm{Map}(\mathrm{Spec}(A \oplus M) \times_Z Y, X) \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Map}(\mathrm{Spec}\, A \times_Z Y, X).
\end{tikzcd}$$

Let $q : \mathrm{Spec}\, A \times_Z Y \to \mathrm{Spec}\, A$ denote the projection. Then $\mathrm{Spec}(A \oplus M) \times_Z Y$ coincides with the extension $(\mathrm{Spec}\, A \times_Z Y)[q^* M]$ by the pullback $q^* M$.[2] Therefore the pullback diagram becomes:

$$\begin{tikzcd}
\mathrm{Der}_X(A, q^* M) \arrow[r] \arrow[d] & \mathrm{Map}((\mathrm{Spec}\, A \times_Z Y)[q^* M], X) \arrow[d] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Map}(\mathrm{Spec}\, A \times_Z Y, X).
\end{tikzcd}$$

Now the top left is equivalent to $\mathrm{Map}_{\mathrm{Spec}\, A \times_Z Y}(f_x^* \mathbb{L}_{X/Z}, q^* M)$. So the existence of a cotangent complex at the point x is reduced to:

$$\mathrm{Map}_{A\text{-Mod}}(\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X),x}, M) \simeq \mathrm{Map}_{\mathrm{Spec}\, A \times_Z Y}(f_x^* \mathbb{L}_{X/Z}, q^* M).$$

Thus, we need a left adjoint for q^*; this is the map q_+ introduced in **??** below. Then we can define:

$$\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X)/Z,x} := q_+ f_x^* \mathbb{L}_{X/Z}. \tag{3.1.1}$$

Finally, we address the gluing of these cotangent complexes. Assume that we have a morphism $g : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$. We define a point $y : \mathrm{Spec}\, B \to \mathrm{Map}_{/Z}(Y, X)$ by requiring the following diagram to commute:

$$\begin{tikzcd}
 & & X \\
Y \times_Z \mathrm{Spec}\, B \arrow[r] \arrow[d, "q_B"] \arrow[urr, "f_y"] & Y \times_Z \mathrm{Spec}\, A \arrow[d, "q_A"] \arrow[ur, "f_x"'] & \\
\mathrm{Spec}\, B \arrow[r, "g"] & \mathrm{Spec}\, A &
\end{tikzcd}$$

[1] We could replace the condition on f with something slightly more general, such as representable by quasi-compact quasi-separated algebraic spaces of finite tor amplitude.

[2] Work locally, take $\mathrm{Spec}(A \otimes_{\mathcal{O}_Z} \mathcal{O}(Y) \oplus M \otimes_{\mathcal{O}_Z} \mathcal{O}(Y)$ over each affine piece and glue.

From the commutativity of the upper triangle we obtain $f_y = f_x \circ 1 \times g$, so that:

$$q_{B+} f_y^* \mathbb{L}_X \simeq q_{B+} (1 \times g)^* f_x^* \mathbb{L}_X.$$

Our goal is to show that gluing works, which means:

$$q_{B+} f_y^* \mathbb{L}_X \simeq g^* q_{A*} f_x^* \mathbb{L}_X.$$

Therefore it suffices to prove that q_+ has the base change property $q_{B+}(1 \times g)^* \simeq g^* q_{A+}$. This is the object of Lemma 3.3.2, while the construction of q_+ is Lemma 3.3.1. ☐

Remark 3.1.4. The tangent complex of the mapping stack, when it exists, can be obtained more easily from the diagram:

$$\begin{tikzcd}
\mathrm{Map}_{/Z}(Y,X) \times_Z Y \arrow[d, "\pi"] \arrow[dr, "ev"] & \\
\mathrm{Map}_{/Z}(Y,X) & X.
\end{tikzcd}$$

Then:

$$\mathbb{T}_{\mathrm{Map}_{/Z}(Y,X)/Z} = \pi_* \mathrm{ev}^* \mathbb{T}_{X/Z}.$$

We would like to dualize and obtain:

$$\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X)/Z} = (\pi_* \mathrm{ev}^* \mathbb{L}_{X/Z}^\vee)^\vee = \pi_+ \mathrm{ev}^* \mathbb{L}_{X/Z}. \tag{3.1.2}$$

In Theorem 3.1.3, we actually prove that $\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X)/Z}$ exists, by constructing it locally and then showing that the construction glues. The result of the gluing must then be 3.1.2, as can be seen from the extended diagram:

$$\begin{tikzcd}
\mathrm{Spec}\, A \times_Z Y \arrow[r, "x \times 1"] \arrow[d, "q"] & \mathrm{Map}_{/Z}(Y,X) \times_Z Y \arrow[d, "\pi"] \arrow[dr, "ev"] & \\
\mathrm{Spec}\, A \arrow[r, "x"] & \mathrm{Map}_{/Z}(Y,X) & X.
\end{tikzcd}$$

Since the square is a homotopy pullback, we apply base change for q_+ (see Lemma 3.3.2):

$$x^* \pi_+ \mathrm{ev}^* \mathbb{L}_{X/Z} \simeq q_+ (x \times 1)^* \mathrm{ev}^* \mathbb{L}_{X/Z} \simeq q_+ f_x^* \mathbb{L}_{X/Z},$$

which agrees with what we called $\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X)/Z,x}$ in 3.1.1. It follows that $\mathbb{L}_{\mathrm{Map}_{/Z}(Y,X)/Z} \simeq \pi_+ \mathrm{ev}^* \mathbb{L}_{X/Z}$.

3.2 Stable Maps

We apply Theorem 3.1.3 and Remark 3.1.4 to the derived moduli space of stable maps on a smooth projective variety X:

$$\mathbb{R}\mathcal{M}_{g,k}(X) = \mathrm{Map}_{dSt/\mathcal{M}_{g,k}}(\mathcal{C}_{g,k}, X \times \mathcal{M}_{g,k}).$$

According to Remark 3.1.4, and using the notation therein:

$$\mathbb{L}_{\mathbb{R}\mathcal{M}_{g,k}(X)/\mathcal{M}_{g,k}} = \pi_+ \mathrm{ev}^* \mathbb{L}_{X \times \mathcal{M}_{g,k}/\mathcal{M}_{g,k}}. \tag{3.2.1}$$

We can simplify this expression using the pullback diagram:

$$\begin{tikzcd}
X \times \mathcal{M}_{g,k} \arrow[r, "p"] \arrow[d] & X \arrow[d] \\
\mathcal{M}_{g,k} \arrow[r] & \mathrm{Spec}\, k.
\end{tikzcd}$$

The diagram implies $\mathbb{L}_{X \times \mathcal{M}_{g,k}/\mathcal{M}_{g,k}} \simeq p^* \mathbb{L}_X$, so 3.2.1 reduces to:

$$\mathbb{L}_{\mathbb{R}\mathcal{M}_{g,k}(X)/\mathcal{M}_{g,k}} = \pi_+ \mathrm{ev}^* p^* \mathbb{L}_X. \tag{3.2.2}$$

Proposition 3.2.1. *The natural map $\mathbb{R}\mathcal{M}_{g,k}(X) \to \mathcal{M}_{g,k}$ is quasi-smooth.*

Proof. We need to show that $\mathbb{L}_{\mathbb{R}\mathcal{M}_{g,k}(X)/\mathcal{M}_{g,k}} = \pi_+ \mathrm{ev}^* p^* \mathbb{L}_X$ has cohomological amplitude $[-1, 0]$. X is a smooth variety, so $\mathbb{L}_X$ is in amplitude $[0, 0]$. Pullbacks preserve cohomological amplitude, because they only involve tensoring with locally free sheaves. (Todo: Need any assumption on the maps?) π_* may increase cohomological amplitude, because of higher direct image sheaves. However, that the fibers of $\pi : \mathbb{R}\mathcal{M}_{g,k}(X) \times_{\mathcal{M}_{g,k}} \mathcal{C}_{g,k} \to \mathcal{M}_{g,k}$ are curves, so the cohomological amplitude of $\pi_*(\mathrm{ev}^* p^* \mathbb{L}_X)^\vee$ is at most $[0, 1]$. Dualizing again brings $\mathbb{L}_{\mathbb{R}\mathcal{M}_{g,k}(X)/\mathcal{M}_{g,k}}$ to amplitude $[-1, 0]$. □

Together with the fact that $\mathcal{M}_{g,k}$ is smooth, this implies that $\mathbb{R}\mathcal{M}_{g,k}(X)$ is quasi-smooth. Proposition 2.2.3 implies the following.

Corollary 3.2.2. *The derived enhancement $j : \mathcal{M}_{g,k}(X) \to \mathbb{R}\mathcal{M}_{g,k}(X)$ determines a perfect obstruction theory on $\mathcal{M}_{g,k}(X)$:*

$$j^* \mathbb{L}_{\mathbb{R}\mathcal{M}_{g,k}(X)} \to \mathbb{L}_{\mathcal{M}_{g,k}(X)}.$$

Expression 3.2.2 for the cotangent complex of $\mathbb{R}\mathcal{M}_{g,k}(X)$ shows that the obstruction theory is the same as that considered by [2] and introduced in (Todo: reference once the chapter is edited).

3.3 The + Pushforward Functor

Lemma 3.3.1 (3.3.22 and 3.3.23 in [6])**.** *Suppose that $q : Y \to S$ is perfect, i.e. q_* preserves perfect complexes. Then $q^* : \mathrm{QCoh}(S) \to \mathrm{QCoh}(Y)$ has a left adjoint q_+.*

Proof. Let $F \in \mathrm{Perf}(Y)$ and $G \in \mathrm{QCoh}(S)$. Then:

$$\begin{aligned} &\mathrm{Map}_{\mathrm{QCoh}(S)}\left((q_* F^\vee)^\vee, G\right) \simeq \mathrm{Map}_{\mathrm{QCoh}(S)}\left(\mathcal{O}_S, q_* F^\vee \otimes G\right) \\ \simeq\, &\mathrm{Map}_{\mathrm{QCoh}(S)}\left(\mathcal{O}_S, q_*(F^\vee \otimes q^* G)\right) \simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(\mathcal{O}_Y, F^\vee \otimes q^* G) \simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(F, q^* G). \end{aligned}$$

We have used the fact that perfect complexes are dualizable and the projection formula $q_*(F^\vee \otimes q^* G) \simeq q_* F^\vee \otimes G$. This means that, for F perfect, we can use $q_+ F = (q_* F^\vee)^\vee$. Now if S and q are quasi-compact and quasi-separated, $\mathrm{QCoh}(Y) = \mathrm{IndPerf}(Y)$. Remarking that $q_+ : \mathrm{Perf}(Y) \to \mathrm{QCoh}(S)$ is a left adjoint, it commutes with colimits, and so there exists a unique extension $q_+ : \mathrm{IndPerf}(Y) \to \mathrm{QCoh}(S)$ which commutes with colimits. It also follows that the extension is a left adjoint. □

Lemma 3.3.2 (3.3.23 in [6])**.** *Suppose given a pullback diagram of DM stacks, (Todo: can we do better?) with f, f' perfect.*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Then the canonical map $\lambda : f'_+ \circ g'^* \to g^* f_+$ is an equivalence. (We say that the + pushforward satisfies base change.)

Proof. On perfect objects F, λ_F is the dual of:

$$g^* f_* F \to f'_* g'^* F.$$

We have used the fact that pullbacks preserve duals. This is an isomorphism due to base change for the pusforward f_*. To conclude, both f_+ and g^* are now left adjoints, which means they preserve all colimits. It follows that $f'_+ \circ g'^* \simeq g^* f_+$ extends to $\mathrm{QCoh} \simeq \mathrm{IndPerf}$. □

Note that we have used the assumption that $q : Y \to S$ is perfect. It remains, then, to prove that the map $q : \operatorname{Spec} A \times_Z Y \to \operatorname{Spec} A$ from the proof of Theorem 3.1.3 is perfect. We do this in Lemma 3.3.5 below, after introducing some terminology.

Definition 3.3.3. A map $q : Y \to S$ is **categorically proper**, also called **of finite cohomological dimension**, if $q_* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(S)$ increases cohomological dimension by a uniform finite amount.

Definition 3.3.4. A map $q : Y \to S$ is **of finite tor amplitude** if locally $q : \operatorname{Spec} B \to \operatorname{Spec} A$ and B is of finite tor amplitude as an object of A-Mod.

Note that in our example $q : \operatorname{Spec} A \times_Z Y \to \operatorname{Spec} A$, q is:

- proper, because we assumed that $Y \to Z$ is proper, and this is stable under base change;
- categorically proper, because we can compute q_* by a (uniformly) finite Čech resolution, due to the assumption that $Y \to Z$ is representable by proper schemes.
- of finite tor amplitude, because this is a consequence of flatness. We have assumed that $Y \to Z$ is flat, and flatness is stable under base change.

Lemma 3.3.5. *Let $q : Y \to S$ be a map which is proper, categorically proper and of finite tor amplitude. Then q is perfect.*

Proof. First, we claim that the first two assumptions imply that $q_*\mathrm{Coh}^-(X) \to \mathrm{Coh}^-(S)$. This argument uses the Leray spectral sequence. (Todo: fill this in) Next, note that $\mathrm{Perf}(X) \subset \mathrm{Coh}^-(X)$ is characterized as the full subcategory of complexes with finite tor amplitude. So it remains to prove that, if $F \in \mathrm{Coh}^-(X)$ has finite tor amplitude, then so does q_*F.

Take a Zariski affine cover for X; due to the quasi-compactness assumption this can be taken finite. Then the Čech nerve $U^\bullet$ is a finite complex. Since q_* is a right adjoint, it commutes with limits, and we have:

$$q_*F = \varprojlim q|_{U^\bullet *}F|_{U^\bullet}.$$

Note that the maps $U_i \to X$ are not, in general, proper, so $q|_{U^\bullet *}F|_{U^\bullet}$ needn't be coherent. However, on affines q_* is just a forgetful functor on modules. Therefore the coherence of $q|_{U^\bullet *}F|_{U^\bullet}$ follows from the fact that $F|_{U^\bullet}$ is coherent and the map on rings is finitely generated. (Todo: explain more)

For finite tor amplitude, it suffices to check that, for every M discrete, $q_*F \otimes_{\mathcal{O}_S} M$ is cohomologically supported in $[-m, \infty)$ for some m. (We already know that q_*F is bounded above.) Since the Čech nerve is finite, we have:

$$q_*F \otimes_{\mathcal{O}_S} M = \varprojlim q|_{U^\bullet *}F|_{U^\bullet} \otimes_{\mathcal{O}_S} M.$$

The inclusion $\mathrm{Coh}^{-,[-m,\infty)}(S) \subset \mathrm{Coh}^-(S)$ commutes with limits, which gives the result. □

3.4 Application: Weil Restriction

Weil restriction is, roughly speaking, an adjoint for base change. We sketch the treatment that Lurie gives in [7].

Definition 3.4.1. Let $\phi : Y \to Z$ and $X \to Y$ be maps of derived stacks. A **Weil restriction** for X along ϕ is a stack $\mathrm{Res}_{Y/Z} X \to Z$, equipped with a morphism $\rho_X : \mathrm{Res}_{Y/Z} X \times_Z Y \to X$ over Y, such that composition with ρ determines a homotopy equivalence:

$$\mathrm{Map}_{\mathbf{dSt}/Z}(-, \mathrm{Res}_{Y/Z} X) \simeq \mathrm{Map}_{\mathbf{dSt}/Y}(- \times_Z Y, X). \tag{3.4.1}$$

Example 3.4.2. In arithmetic geometry $\phi : Y \to Z$ is taken to be a field extension $\phi : \operatorname{Spec} L \to \operatorname{Spec} k$; then the adjunction 3.4.1 gives a bijection between L-points of X and k-points of $\operatorname{Res}_{\operatorname{Spec} L/\operatorname{Spec} k} X$. To illustrate this without getting too much out of our comfort zone, we take $k = \mathbb{R}$, $L = \mathbb{C}$, and start with X an affine variety over $\mathbb{C}$, given as a subset of $\mathbb{C}^n$ by equations $f_i(z_1, \dots, z_n) = 0$. Then $\operatorname{Res} \operatorname{Spec} L/\operatorname{Spec} kX$ is an affine variety over $\mathbb{R}$, given as a subset of $\mathbb{R}^{2n}$ by equations $\Re f_i(x_1 + iy_1, \dots, x_n + iy_n) = 0$, $\Im f_i(x_1 + iy_1, \dots, x_n + iy_n) = 0$.

Lurie proves the following existence result.

Theorem 3.4.3.

Proof. The basic idea is to define $\operatorname{Res}_{Y/Z} X$ as the homotopy pullback:

$$\begin{tikzcd}
\operatorname{Res}_{Y/Z} X \arrow[r] \arrow[d] & \operatorname{Map}_{/Z}(Y, X) \arrow[d] \\
Z \arrow[r] & \operatorname{Map}_{/Z}(Y, Y)
\end{tikzcd}$$

□

Chapter 4

Reduced Gromov Witten Invariants for K3 Surfaces

(Talk by Benedict Morrissey) In chapter 3, we constructed a quasismooth derived enhancement of the moduli stack of stable maps. The machinery of chapter 2 can then be used to construct virtual fundamental classes, and from there Gromov–Witten invariants. In this chapter we consider the case where the target varietry X is a K3 surface. In this case Gromov–Witten invariants defined in this fashion vanish. In this chapter, following [15], we provide an alternative quasismooth derived enhancement. The virtual fundamental classes obtained were earlier described in [10, 11, 13] (henceforth referred to as OMPT).

4.1 Non Reduced Gromov–Witten Invariants for K3 Surfaces

Note that GW invariants are deformation invariant, but in the moduli space of K3 surfaces there's a dense locus of non-algebraic K3s. These don't admit any (1,1) classes in H_2. It follows that the deformation of the class $[\beta]$ is not effective, and the corresponding invariants must vanish.

Somehow, passing to the reduced obstruction theory restricts to algebraic deformations only, and this problem disappears. (Todo: understand why this happens)

(Todo: Work out where I can find a proof that all these invariants disappear!)

4.2 $\mathbb{R}Pic$ for K3 Surfaces

The point of this section is to give a canonical identification $\mathbb{R}Pic(X) \xrightarrow{\cong} Pic(X) \times \mathbb{R}Spec(Sym(H^0(X, K_X)[1]))$. Recall that $\mathbb{R}Pic(X)$ is a (locally of finite presentation) derived group stack.

Theorem 4.2.1 (Prop. 4.5 in [15])**.** *A locally of finite presentation group stack G over a field k, with identity $e : \operatorname{Spec} k \to G$, and Lie algebra $\mathfrak{g} = T_eG$,*[1] *has a canonical map*

$$\gamma(G) : t_0(G) \times \mathbb{R}\operatorname{Spec}(A) \to G,$$

where $A = k \oplus (\mathfrak{g}^\vee)_{<0}$.

Proof. The projection map $A \to k$ gives a k-point $x_0 : \operatorname{Spec}(k) \to \mathbb{R}\operatorname{Spec}(A)$.

We wish to find a commuting diagram

$$\begin{tikzcd} B \arrow[rr, "a"] & & C \\ & A \arrow[ul, "x_0"] \arrow[ur, "d"'] & \end{tikzcd}$$

[1] This is the tangent space at the identity, so it's a dg Lie algebra.

this is equivalent to giving a morphism $a' : \mathbb{L}_{G,e} \cong \mathfrak{g}^\vee \to (\mathfrak{g}^\vee)_{<0}$ (due to our choice of A), hence taking the truncation map $\tau_{<0}$ (using the standard t-structure on the stable category of (dg) vector spaces) gives this map.

We then take the composition

$$t_0(G) \times \mathbb{R}\operatorname{Spec}(A) \xrightarrow{j \times a} G \times G \xrightarrow{\times} G,$$

where the final map uses the group product in G. □

We now apply Theorem 4.2.1 to the group stack $\mathbb{R}\operatorname{Pic}(X)$ for a K3 surface X.

Theorem 4.2.2. *The map $\gamma_{\mathbb{R}\operatorname{Pic}(X)}$ for X a K3 surface gives an isomorphism of derived stacks*

$$\mathbb{R}\operatorname{Pic}(X) \xrightarrow{cong} \operatorname{Pic}(X) \times \mathbb{R}\operatorname{Spec}(\operatorname{Sym}(H^0(X,K_X)[1])).$$

Proof. We note first that this map clearly provides an isomorphism on truncations. Hence as $\mathbb{R}Pic(X)$ is a derived group stack, we need only show that it is étale at e, that is to say

$$\mathbb{T}_{t_0(e),x_0}(\gamma_{\mathbb{R}\operatorname{Pic}(X)}) : \mathbb{T}_{t_0(e),x_0}(\operatorname{Pic}(X) \times \mathbb{R}\operatorname{Spec}(\operatorname{Sym}(H^0(X,K_X)[1]))) \to T_eG$$

is an isomorphism of dg k-vector spaces.

Note that, since $H^1(X,\mathcal{O}_X) = 0$:

$$T_eG = \mathfrak{g} = \mathbb{R}\Gamma(X,\mathcal{O}_X)[1] \cong H^0(X,\mathcal{O}_X)[1] \oplus H^2(X,\mathcal{O}_X)[-1].$$

Hence

$$A = \mathbb{C} \oplus (\mathfrak{g}^\vee)_{<0} = \mathbb{C} \oplus H^2(X,\mathcal{O}_X)[1] \cong \mathbb{C} \oplus H^0(X,K_X)[1] \cong Sym(H^0(X,K_X)[1]$$

where the final step follows because $H^0(X,K_X)$ is free of dimension 1.

Clearly $T_{t_0(e),x_0}(\gamma_{\mathbb{R}Pic(X)})$ is an isomoprhism. □

This identification allows the definition of a projection $pr_{der} : \mathbb{R}Pic(X) \to \mathbb{R}Spec(Sym(H^0(X,K_X)[1])$.

4.3 The reduced Moduli Space $\mathbb{R}M_{g,n}(X,\beta)^{red}$.

Recall the map $x_0 : Spec(\mathbb{C}) \to \mathbb{R}Spec(Sym(H^0(X,K_X)[1])$.

We defined the reduced derived enhancement as follows:

Definition 4.3.1. The **reduced stack of n-pointed stable maps** of genus g, class β to a K3 surface X is given by the pullback

$$\begin{tikzcd} \mathbb{R}M^{red}_{g,n}(X,\beta) \arrow[r] \arrow[d] & \mathbb{R}M_{g,n}(X,\beta) \arrow[d, "\delta_1^{der}(X,\beta)"] \\ Spec(\mathbb{C}) \arrow[r, "x_0"] & \mathbb{R}Spec(Sym(H^0(X,K_X)[1]) \end{tikzcd}$$

where we define the map $\delta_1^{der}(X,\beta)$ by the composition

$$\mathbb{R}M_{g,n}(X,\beta) \to \mathbb{R}M_{g,n}(X) \xrightarrow{a} Perf(X) \xrightarrow{det} \mathbb{R}Pic(X) \xrightarrow{pr_{det}} \mathbb{R}Spec(Sym(H^0(X,K_X)[1]),$$

where the map a is induced by the perfect complex $\mathbb{R}\pi_*(\mathcal{O}_{\mathbb{R}\mathcal{C}_{g,X}})$, (that is this specifies a perfect complex over $X \times M_{g,n}(X,\beta)$, and hence a map $M_{g,n}(X,\beta) \to Perf(X)$).

Remark 4.3.2. This derived enhancement is quasismooth, and we will show that the induced obstruction theory on $M_{g,n}(X,\beta)$ agrees with that of OMPT.

Proposition 4.3.3. *The derived stack $\mathbb{R}M^{red}_{g,n}(X,\beta)$ is quasismooth.*

Proof. The cotangent complex relative to $M_{g,n}$ is:

$$\mathbb{T}_{\mathbb{R}M_{g,n}(X,\beta)/M_{g,n}, f:C\to X} = \mathbb{R}\Gamma(C, f^*\mathbb{T}_X).$$

The absolute one is:

$$\mathbb{T}_{\mathbb{R}M_{g,n}(X,\beta), f:C\to X} = \mathbb{R}\Gamma\big(C, \mathrm{Cone}(\mathbb{T}_C(-\sum x_i) \to f^*\mathbb{T}_X)\big).$$

We use the pullback square:

$$\begin{tikzcd} \mathbb{R}M^{red}_{g,n}(X,\beta) \arrow[r] \arrow[d] & \mathbb{R}M_{g,n}(X,\beta) \arrow[d] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathbb{R}\,\mathrm{Spec}(\mathrm{Sym}\,H^0(X,K_X)[1]). \end{tikzcd}$$

This gives the following distinguished triangle of complexes:

$$\mathbb{T}^{red} := \mathbb{T}(\mathbb{R}M^{red}_{g,n}(X,\beta)) \to \mathbb{R}\Gamma\big(C, \mathrm{Cone}(\mathbb{T}_C(-\sum x_i) \to f^*\mathbb{T}_X)\big) \to H^2(X, \mathcal{O}_X)[-1].$$

Looking at the associated long exact sequence of cohomology, it suffices to show that:

$$H^1\mathbb{R}\Gamma\big(C, \mathrm{Cone}(\mathbb{T}_C(-\sum x_i) \to f^*\mathbb{T}_X)\big) \to H^2(X, \mathcal{O}_X)$$

is surjective. We do this by precomposing with the map:

$$H^1(C, f^*\mathbb{T}_X) \to H^1\mathbb{R}\Gamma\big(C, \mathrm{Cone}(\mathbb{T}_C(-\sum x_i) \to f^*\mathbb{T}_X)\big),$$

and proving that the resulting map is surjective. For the latter, we use the commutative diagram:

$$\begin{tikzcd} H^1(X,\mathbb{T}_X) \arrow[r] \arrow[d] & \mathrm{Ext}^2_X(\mathbb{R}f_*\mathcal{O}_C, \mathbb{R}f_*\mathcal{O}_C) \arrow[r] & H^2(X,\mathcal{O}_X)[-1] \\ H^1(C, f^*\mathbb{T}_X) \arrow[r] & H^1(X, \mathrm{Cone}(\mathbb{T}_C(-\sum x_i) \to f^*\mathbb{T}_X)) \arrow[u] & \end{tikzcd}$$

(Todo: Finish This) □

4.4 The Resultant Obstruction Theories

In this section we compare the obstruction theories obtained from the reduced derived enhancement with those obtained in OMPT.

(Todo: Want THM 4.8 from STV)

(Todo: Write an explicit description)

The explicit description is as follows.

$$H^0(\mathbb{T}^{red}_f) \cong H^0\big(C, \mathrm{Cone}(\mathbb{T}_C \to f^*\mathbb{T}_X)\big)$$
$$H^1(\mathbb{T}^{red}_f) \cong \mathrm{Ker}\,\big(H^1(\Theta_f) : H^1\big(C, \mathrm{Cone}(\mathbb{T}_C \to f^*\mathbb{T}_X)\big) \to H^2(X, \mathcal{O}_X)\big)$$

4.5 The Link to Donaldson–Thomas Theory

Recall that in section **??** we made use of a map $\mathbb{R}Map(X,\beta) \to Perf(X)$. Recall that the stack $Perf(X)$ is used in Donaldson–Thomas theory, a curve counting theory that uses a different compactification of the space of curves in X that the stable curves used in Gromov–Witten invariants. There are conjectured to be various relationships between Gromov–Witten and Donaldson–Thomas invariants as developed in [8, 9].

The compactification for DT is as follows. For $C \subset X$ an embedded curve, $f_*\mathcal{O}_C \in \mathrm{Perf}(X)$. More precisely, $f_*\mathcal{O}_C$ lands in a component (or union thereof) $\mathrm{Perf}^{si,\geq 0,\beta}$. The latter is the DT stack, and we want to have a theory of integration over it.

For a start, fix $\mathcal{L}$ a line bundle on X, corresponding to a map $\mathrm{Spec}\,\mathbb{C} \to \mathbb{R}\,\mathrm{Pic}(X)$. Then we define the stack of perfect complexes with fixed determinant:

$$\begin{tikzcd} \mathrm{Perf}(X)_{\mathcal{L}} \arrow[r] \arrow[d] & \mathrm{Perf}(X) \arrow[d, "\det"] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathbb{R}\,\mathrm{Pic}(X). \end{tikzcd}$$

We have another constraint:

$$\mathrm{Ext}^i(F,F) = 0, \forall i < 0,$$

and moreover $\mathrm{Ext}^0(F,F) \simeq H^0(X, \mathcal{O}_X)$, which gives $\mathrm{Perf}(X)^{si\geq 0}$.

For a K3 surface, we can consider again the reduced theory:

$$\begin{tikzcd} \mathrm{Perf}(X)^{red} \arrow[r] \arrow[d] & \mathrm{Perf}(X) \arrow[d] \\ \mathrm{Spec}\,\mathbb{C} \arrow[r] & \mathbb{R}\,\mathrm{Spec}(\mathrm{Sym}\, H^0(X, K_X)[1]). \end{tikzcd}$$

In the end we put all of these constraints together to form $\mathrm{Perf}(X)^{si\geq 0,\mathcal{L},red}$.

(Todo: sb draw big diagram, I got lazy)

These triangles are precisely the diagram necessary for functoriality of obstruction theories. Of course, this is rather silly: we showed that if you restrict to the open subset, the choice of compactification doesn't influence the computation.

(Todo: talk about all of S5)

Chapter 5

Abelian Threefold

Let's denote by $\eta : \mathcal{O}_M \to \mathbb{L}_M \otimes \mathbb{T}_M$ the counit and $\epsilon : \mathbb{L}_M \otimes \mathbb{T}_M \to \mathcal{O}_M$ the unit. The most natural expressions for ω and ι_ξ are as follows.

$$\begin{tikzcd} p^*\mathfrak{g} \otimes \wedge^2\mathbb{L}_M \arrow[r, "act\otimes 1"] \arrow[rr, bend left, "\iota"] & \mathbb{T}_M \otimes \wedge^2\mathbb{L}_M \arrow[r, "\epsilon"] & \mathbb{L}_M \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{T}_M \otimes \mathbb{T}_M \arrow[r] \arrow[rrrr, bend left, "\omega"] & p_*\mathcal{A} \otimes p_*\mathcal{A}[2] \arrow[r] & p_*(\mathcal{A} \otimes \mathcal{A})[2] \arrow[r] & p_*\mathcal{O}_{M\times A}[2] \arrow[r] & \mathcal{O}_M[2-3] \end{tikzcd}$$

The diagram we want to find a lift in can therefore be written as:

$$\begin{tikzcd} & & & \mathcal{O}_M[-1] \arrow[d, "d_{dR}"] \\ p^*\mathfrak{g} \otimes \mathcal{O}_M \arrow[r, "\wedge^2\eta"'] \arrow[urrr, dashed, "\mu"] & p^*\mathfrak{g} \otimes \wedge^2\mathbb{T}_M \otimes \wedge^2\mathbb{L}_M \arrow[r, "\omega"'] & p^*\mathfrak{g} \otimes \wedge^2\mathbb{L}_M \otimes \mathcal{O}_M[-1] \arrow[r, "\iota"'] & \mathbb{L}_M \otimes \mathcal{O}[-1]. \end{tikzcd}$$

5.1 An Introduction to Derived Symplectic Reduction

Chapter 6

DT-Theory and stable pairs

In this chapter, we introduce two other ways of curve counting, especially in projective 3-fold, called Donaldson-Thomas theory and Pandharipande-Thomas theory(Stable pairs). Comparison of these invariants with GW-invariants is mainstream of enumerative geometry and we explain several conjectures and results. Next, we formalize it in the derived setting. More generally, we introduce compactly supported integration along the fiber with which we could define shifted symplectic structure on the derived mapping stack with non-proper source.

6.1 Donaldson-Thomas Theory

The idea of Donaldson-Thomas theory is to count curves "embedded" in X, instead of counting stable maps from curve to X. One might be able to take Hilbert scheme to compactify such embedded curves as follows. Let $n \in \mathbb{Z}, \beta \in H_2(X;\mathbb{Z})$. There exists a projective scheme $Hilb_n(X,\beta)$ compactifying the moduli space of embedded curves in X.

$$Hilb_n(X,\beta) = \{C \hookrightarrow X | dimC \leq 1, \chi(\mathcal{O}_C) = n, [C] = \beta\}$$

However, it does not admit perfect obstruction theory because deformation and obstruction space don't behave nicely enough. In order to remedy this, we should interpret it as a moduli of (stable) sheaves on X.

Lemma 6.1.1. *Let $I_n(X,\beta)$ be a moduli space of rank 1 torsion free sheaf I in X with chern character $ch(I) = (1, 0, -\beta, -n) \in H^*(X;\mathbb{Z})$ and trivial determinant. Then, $Hilb_n(X;\beta) \simeq I_n(X,\beta)$*

Proof. The map $Hilb_n(X;\beta) \to I_n(X,\beta)$ is given by $Z \mapsto I_Z$ where I_Z is an ideal sheaf of the subscheme Z. I_Z is torsion-free because it is a subsheaf of coherent locally free sheaf $\mathcal{O}_X$. The Chern character part is trivial. Conversely, any element $I \in I_n(X,\beta)$ gives an ideal sheaf of $\mathcal{O}_X$ from a natural map to reflexive hull $I^{\vee\vee} \simeq \mathcal{O}_X$. Since I is torsion-free, $det(I) \simeq I^{\vee\vee} \simeq \mathcal{O}_X$. ☐

Remark 6.1.2. This is a special case of a moduli of stable sheaves. If we choose polarization ω on X, then we can put (ω)-stability condition on the category of coherent sheaves on X. To be specific, $E \in \mathrm{Coh}(X)$ is ω-(semi)stable if

- E is pure
- For any subsheaf $0 \neq F \subsetneq E$, $P_F(m) < (\leq) P_E(m)$ where $P_E(m) = \chi(E \otimes \mathcal{O}_X(m))$ modulo a leading coefficent.

Hence, for fixed Chern character $\mu \in H^*(X;\mathbb{Q})$, $\mathfrak{M}_\omega^{s(s)}(\mu)$ is defined to be a moduli stack of ω-(semi)stable sheaf with fixed Chern character. Clearly, $\mathfrak{M}_\omega^s(\mu) \subset \mathfrak{M}_\omega^{ss}(\mu)$ and there exists a quasi-projective scheme $M_\omega(\mu)$ with a $\mathbb{C}^*$-gerbe $\mathfrak{M}_\omega^s(\mu)$ because automorphism in $\mathfrak{M}_\omega^s(\mu)$ is just $\mathbb{C}^*$. In particular, if $\mathfrak{M}^s_\omega(\mu) = \mathfrak{M}^{ss}_\omega(\mu)$, then $M_\omega(\mu)$ is a projective scheme. In our case, for $\mu = (1, 0, -\beta, -n)$, this condition holds and $M_\omega(\mu)$ is isomorphic to $Hilb_n(X, \beta)$.

The moduli space of sheaves $I_n(X, \beta)$ admits a virtual class. The main point is that deformations and obstructions are governed by $\mathrm{Ext}^1(I_C, I_C)_0, \mathrm{Ext}^2(I_C, I_C)_0$ respectively, where the subscript 0 denotes the trace-free part governing deformations with fixed determinant $\mathcal{O}_X$. Since $\mathrm{Hom}(I_C, I_C) = \mathbb{C}$ consists of only the scalars, the trace-free part vanishes. By Serre duality,

$$\mathrm{Ext}^3(I_C, I_C) \cong \mathrm{Hom}(I_C, I_C \otimes K_X)^* \cong H^0(K_X) \cong H^3(\mathcal{O}_X)$$

It vanishes when we take trace-free parts. Hence, there is no higher obstruction spaces except $\mathrm{Ext}^1(I_C, I_C)_0, \mathrm{Ext}^2(I_C, I_C)_0$ and they govern a perfect obstruction theory of virtual dimension equals to

$$ext^1(I_C, I_C)_0 - ext^2(I_C, I_C)_0 = \int_\beta c_1(X)$$

In CY-case, we get the following definition.

Definition 6.1.3. Let X be a projective Calabi-Yau 3-fold. Donaldson-Thomas invariant $I_{n,\beta}$ is defined to be

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1$$

Remark 6.1.4. By Serre duality, we can see that it indeed admits a symmetric obstruction theory. By Berhend, when the moduli space is equipped with symmetric obstruction theory, we can construct a function $\nu : I_n(X, \beta) \to \mathbb{Z}$ which is called Berhend function. Then

$$I_{n,\beta} = \int_{[I_n(X,\beta)]} \nu de = \sum_{m \in \mathbb{Z}} me(\nu^{-1}(m))$$

In particular, if $I_n(X, \beta)$ is non-singular and connected, then it is just the topological Euler characteristic of $I_{n,\beta}$

Example 6.1.5. Consider $C_t = \{x = z = 0\} \cup \{y = 0, z = t\} \subset \mathbb{C}^3$. As a subscheme, the limit goes to C_0 whose ideal sheaf is generated by $(x, z)(y, z) = (xy, xz, yz, z^2) \subsetneq (xy, z)$. This limit ideal does not contain (z), so as a scheme, the limit curve C_0 is given by $\{xy = 0 = z\}$ with a scheme-theoretic point added at the origin. Moreover, this embedded point can break off as follows. Consider the flat family $C_\epsilon = \{xy = \epsilon, z = 0\}$. C_0 can be smoothed to C_ϵ with higher genus. In this case, the origin is not an embedded point anymore, but free point not on the curve.

This example reveals some disadvantages of DT-invariant in counting curves. Free and embedded points distract us from counting curves.

Definition 6.1.6. For fixed curve class $\beta \in H_2(X, \mathbb{Z})$, the DT partition function is

$$\mathsf{Z}^{DT}_\beta(q) = \sum_n I_{n,\beta} q^n.$$

Since $I_n(X, \beta)$ is easily seen to be empty for sufficiently negative n, the partition function is a Laurent series in q. In order to count just curves, not points and curves, we define the reduced generating function by dividing by contribution of just points:

$$\mathsf{Z}^{red}_\beta(q) = \frac{\mathsf{Z}^{DT}_\beta(q)}{\mathsf{Z}^{DT}_0(q)}$$

Note that $I_n(X, 0)$ is a moduli space of points with length n. Here, we gather some properties of partition functions which is not obvious at all.

Proposition 6.1.7. *1.* $\mathsf{Z}_0^{DT}(q) = M(-q)^{e(x)}$ *where M is the MacMahon function, $M(q) = \prod_{n\geq 1}(1-q^n)^{-n}$ the generating function for 3d partitions.*

2. $\mathsf{Z}_\beta^{red}(q)$ *is the Laurent expansion of a rational function in q, invariant under $q \leftrightarrow q^{-1}$.*

Therefore, we can substitute $q = -e^{iu}$ to get a real-valued function of u. The main conjecture of MNOP in the CY-case is the following,

Conjecture 6.1.8. $\mathsf{Z}_\beta^{GW}(u) = \mathsf{Z}_\beta^{red}(-e^{iu})$

6.2 Stable Pairs

Let's start with an example.

Example 6.2.1. Unlike 6.1.5, we take different a limit of C_t as follows. Denote each component by C_t^1, C_t^2. Consider the sheaf map $s_t : \mathcal{O}_{\mathbb{C}^3} \to \mathcal{O}_{C_t^1} \oplus \mathcal{O}_{C_t^2}$ and take the limit $t \to 0$ of this map. Then we get the following exact sequence of sheaves.

$$0 \to \mathrm{Ker}(s) \to \mathcal{O}_{\mathbb{C}^3} \xrightarrow{s} \mathcal{O}_{C^1} \oplus \mathcal{O}_{C^2} \to \mathrm{Coker}(s) \to 0$$

In this case, $\mathrm{Ker}(s)$ is ideal of subscheme without an embedded point and $Im(s)$ is indeed a structure sheaf of $C_0 = \{xy = 0 = z\}$. The lost data of intersection is encoded in $\mathrm{Coker}(s)$ which is a skyscraper sheaf at the origin. As you can see, the difference from 6.1.5 is the surjectivity of s.

In 6.2.1, $Im(s)$ is a pure sheaf of rank 1 which means that any subsheaf of $Im(s)$ has 1-dimensional support(i.e. no embedded points). Also, the origin can not break off the curve because it is controlled by the sheaf map. (different limit determines different choice of point.) It justifies the following definition.

Definition 6.2.2. A stable pair on X is (F, s) such that

1. F is a pure sheaf of rank 1

2. $s : \mathcal{O}_X \to F$ has 0-dimensional cokernel.

Example 6.2.3. Consider a divisor $D \subset C$ where C is a curve embedded in X. Then, the natural map

$$\mathcal{O}_X \hookrightarrow i_*\mathcal{O}_C \hookrightarrow i_*\mathcal{O}_C(D)$$

is a stable pair.

Lemma 6.2.4. *Giving a stable pair (F, s) is equivalent to choosing 1-dimensional subscheme C of X with a maximal ideal $m \subset \mathcal{O}_C$*

Proof. Given a stable pair (F, s), $\ker(s)$ determines ideal sheaf associated a certain 1-dimensional subscheme C whose struture sheaf is given by $Im(s)$. The support $\mathrm{Coker}(s)$ corresponds to a set of points on C, hence determining a maximal ideal $m \subset \mathcal{O}_C$ associated to these points. The converse also holds. (Todo: reference) □

Remark 6.2.5. Similar to moduli space of sheaves, we can put stability condition to justify the word "stable" in the definition.

As before, we can define the moduli space of stable pairs on X, denoted by $P_n(X, \beta)$. However, again, it does not admit virtual class. The way to remedy this is to consider a pair as an object $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F]$ in derived category $D^b(X)$. Deformation and obstruction spaces with fixed determinant is given by $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0, \mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ governing a perfect obstruction theory of virtual dimension equals to

$$ext^1(I^\bullet, I^\bullet)_0 - ext^2(I^\bullet, I^\bullet)_0 = \int_\beta c_1(X)$$

Definition 6.2.6. Let X be a projective Calabi-Yau 3-fold. Pandharipande-Thomas invariant $P_{n,\beta}$ is given by

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1$$

Also, for fixed curve class $\beta \in H_2(X, \mathbb{Z})$, the stable pairs partition function is defined to be

$$\mathsf{Z}_\beta^{PD}(q) = \sum_n P_{n,\beta} q^n$$

Theorem 6.2.7.

$$\mathsf{Z}_\beta^{PD}(q) = \mathsf{Z}_\beta^{red}(q)$$

6.3 Derived Version

In this section, we introduce derived variants of $I_n(X,\beta), P_n(X,\beta)$ and prove that they admit (-1)-shifted symplectic structure. More generally, we construct shifted symplectic structure on the mapping stack with non-proper source and apply this to the case X is non-proper CY variety.

Definition 6.3.1. Let X be a variety and $\mathcal{L} = \mathcal{O}_X[+d] \in \mathrm{Pic}^{gr} X$ be the trivial line in grading $d \neq 0$. The stack of perfect complexes on X with fixed determinant $\mathcal{L}$, is $\mathrm{Perf}^{\mathcal{L}}(X) = \mathrm{Perf}(X) \times_{\mathrm{Pic}^{gr}(X)} \{\mathcal{L}\}$.

In particular, if X is a CY 3-fold and $d = 1$, then we can define derived analogue of $I_n(X,\beta), P_n(X,\beta)$. For simplicity, we ignore n, β and denote them by $I(X), P(X)$, respectively.

1. $I(X)$ is a derived stack of torsion free sheaves of rank 1 with fixed determinant. It is open substack of $\mathrm{Perf}^{\mathcal{O}_X[+1]}$.
2. $P(X)$ is a derived stack of stable pairs. It is open substack of $\mathrm{Perf}^{\mathcal{O}_X[+1]}$ as well.

(Todo: There must be concrete ways to define both stacks, but I don't know at this point.) For grading, we can think it as follows. Consider a vector bundle E of rank d on X. The associated determinant bundle of E is $\det(E) \cong \bigwedge^d E$. In the derived setting, taking wedge product is the same as taking symmetric product followed by shifting the degree by 1. So, $\det(E) \cong \mathrm{Sym}^d(E[1])$ which is place at degree $-d$. In order to compare with $\mathcal{O}_X$ we should take shifting by d.

If X is compact, then (-1) shifted symplectic structure ω on $\mathrm{Perf}(X)$ can be restricted to a closed 2-form on $\mathrm{Perf}^{\mathcal{O}_X[+1]}$ which is still non-degenerate. The proof is similar to our main theorem6.3.4. Now, we are interested in the case X is not proper. The first ingredient we need is the notion of compactly supported integration along the fiber.

Definition 6.3.2. Suppose that $\mathcal{X}$ is a derived pre-stack and $K \subset \mathcal{X}$ a closed subset.

1. Define the (filtered) chain complex of relative de Rham cochains to be

$$F^k\mathbf{DR}(\mathcal{X}, \mathcal{X} \setminus K) := fib\{i^* : F^k\mathbf{DR}(\mathcal{X}) \to F^k\mathbf{DR}(\mathcal{X} \setminus K)\}$$

2. The compactly supported de Rham cochains are defined as the directed colimit

$$F^k\mathbf{DR}_c(\mathcal{X}) = \varinjlim_{K \subset \mathcal{X}} F^k\mathbf{DR}(\mathcal{X}, \mathcal{X} \setminus K)$$

over all closed subsets $K \subset \mathcal{X}$ which are proper over the base.

3. If $\mathcal{X}$ is an S-prestack, then a relative variant is defined to be

$$F^k\mathbf{DR}_{c/S}(\mathcal{X}) = \varinjlim_{K \subset \mathcal{X}} F^k\mathbf{DR}(\mathcal{X}, \mathcal{X} \setminus K)$$

Theorem 6.3.3. *Suppose that X is a smooth d-dimensional scheme, and that $\mathcal{F}$ is a derived pre-stack almost of finite presentation over k. A choice of volume form $vol_X : \mathcal{O}_X \to \Omega^d_X$ gives rise to an integration map of filtered complexes*

$$\int_X vol_X \wedge - : F^\bullet \boldsymbol{DR}_{c/\mathcal{F}}(X \times \mathcal{F}) \to F^\bullet \boldsymbol{DR}(\mathcal{F})[-d]$$

such that the induced map on associated graded pieces is the Grothendieck-Serre trace map.

(Todo: proof, mention Grothendieck trace map) This map comes from the composition of the following maps:

$$F^\bullet \mathbf{DR}_{c/\mathcal{F}}(X \times \mathcal{F}) \to F^\bullet \mathbf{DR}(\mathcal{F}) \otimes \Gamma(X, \mathcal{O}_X) \to F^\bullet \mathbf{DR}(\mathcal{F}) \otimes \Gamma(X, \Omega^d_X) \to F^\bullet \mathbf{DR}(\mathcal{F}) \otimes k[-d]$$

By assumption this map factors through the space of global sections with compact support, denoted by $F^\bullet \mathbf{DR}(\mathcal{F}) \otimes \Gamma_c(X, \mathcal{O}_X)$. Now the issue is the existence of relative version of Grothendieck-Serre trace map. (Todo: reference)

Theorem 6.3.4. *Suppose that X is a variety and that $\mathcal{L} = \mathcal{O}_X[+d] \in \operatorname{Pic}^{gr} X$ is trivial line in grading $d \neq 0$. Let $\mathfrak{U} \subset \operatorname{Perf}^{\mathcal{L}}(X)$ be an oper sub-stack statisfying the following properness condition:*

For any ring R and R-point $\eta : \operatorname{Spec} R \to \mathfrak{U}$, let $\mathcal{F} \in \operatorname{Perf}(X_R)$ be the perfect complex classified by η. Then, we require that the cone of the trace map of sheaves on $X_R := X \times \operatorname{Spec} R$

$$\mathcal{RH}om_{X_R}(\mathcal{F}, \mathcal{F}) \xrightarrow{tr} \mathcal{O}_{X_R}$$

have support proper over $\operatorname{Spec} R$.

Then, $\mathfrak{U} \subset \operatorname{Perf}^{\mathcal{L}}(X)$ carries a $(2-d)$ shifted symplectic structure. Furthermore, this is natural for open inclusion of substacks satisfying the above condition.

Now we get our motivating examples.

Corollary 6.3.5. *Suppose that X is a (not necessarily compact) 3-CY variety and $\mathcal{L} = \mathcal{O}_X[1+]$. Let $\mathfrak{U} \subset \operatorname{Perf}^{\mathcal{L}}(X)$ be the locus classifying ideal sheaves of proper subvarieties. In our case $\mathfrak{U} = I(X)$. Then, $\mathfrak{U}$ admits (-1) shifted symplectic structure.*

Proof. It is enough to show that the condition holds. Take $\mathcal{E} \in \operatorname{Perf}(X_R)$ with an identification $\det(\mathcal{E}) \cong \mathcal{O}_X[+1]$ corresponding to a point $\eta : \operatorname{Spec} R \to \mathfrak{U}$. The natural map

$$\mathcal{E} \to \mathcal{E}^{\vee\vee} \cong (\det(\mathcal{E})[-1] \cong \mathcal{O}_X$$

exhibits it as an ideal sheaf with the cone having proper support by assumption. Since this map is the same with the trace map

$$tr : \operatorname{RHom}(\mathcal{E}, \mathcal{E}) \to \mathcal{O}_X$$

is isomorphism away from this proper support as well. ☐

Corollary 6.3.6. *Under the same assumption, consider a derived moduli of stable pairs $P(X) \subset \operatorname{Perf}^{\mathcal{L}}(X)$. It admits (-1) shifted symplectic structure as well.*

Proof. In this case, the trace map is an isomorphism.(Todo: finish) Let $I^\bullet = [\mathcal{O}_X \xrightarrow{s} F]$ be a stable pair in $\operatorname{Perf}^{\mathcal{L}}$. We have the following exact triangles associated to $I^\bullet$:

$$F[-1] \to I^\bullet \to \mathcal{O}_X \xrightarrow{s} F \to \cdots \tag{6.3.1}$$

Applying $\mathcal{H}om(-, \mathcal{O}_X)$ to 6.3.1 yields

$$\mathcal{H}om(F, \mathcal{O}_X) \to \mathcal{H}om(\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{H}om(I^\bullet, \mathcal{O}_X) \to \mathcal{E}xt^1(F, \mathcal{O}_X)$$

The first and last term vanish because F has support of codimension 2. The canonical map $I^\bullet \to \mathcal{O}_X$ generates the third term, so we get $\mathcal{H}om(I^\bullet, \mathcal{O}_X) \cong \mathcal{O}_X$. In fact, it is the image of the identity in the exact sequence

$$\mathcal{E}xt^{-1}(I^\bullet, F) \to \mathcal{H}om(I^\bullet, I^\bullet) \to \mathcal{H}om(I^\bullet, \mathcal{O}_X)$$

obtained from 6.3.1 by applying $\mathcal{H}om(I^\bullet, -)$. Therefore, in order to show $\mathcal{H}om(I^\bullet, I^\bullet \cong \mathcal{O}_X$, we need only prove the vanishing of $\mathcal{E}xt^{-1}(I^\bullet, F)$. But, it turns out to be $\mathcal{H}om(\mathrm{Coker}(s), F)$ which vanishes due to purity of F. $\square$

proof of 6.3.4. We divide this into four steps.

1. Pull back the universal form from Perf to $X \times \mathfrak{U}$
   Consider the followng sequence of derived stacks

$$X_{\mathfrak{U}} = X \times \mathfrak{U} \xrightarrow{j} X \times \mathrm{Perf}^{\mathcal{L}}(X) \xrightarrow{i} X \times \mathrm{Perf}(X) \xrightarrow{ev} \mathrm{Perf}$$

   Let $\mathcal{E}$ be the universal perfect complex and

$$\omega_{\mathrm{Perf}} = ch(\mathcal{E})_2 \in H^0(F^2\mathbf{DR}(\mathrm{Perf})[2])$$

   be the 2-shifted symplectic form constructed in PTVV. Since pullback on derived de Rham complexes commutes with filtration, we obtain a class

$$\omega_{X\times\mathfrak{U}} = j^*i^*ev^*(\omega_{\mathrm{Perf}}) = ch(j^*i^*ev^*\mathcal{E})_2 \in H^0(F^2\mathbf{DR}(X \times \mathfrak{U})[2])$$

   The last assertion follows from the functoriality of Chern character.

2. Lift the form to compactely supported cochain level over $\mathfrak{U}$.
   Let $\mathcal{F} = j^*i^*ev^*\mathcal{E} \in \mathrm{Perf}(X \times \mathfrak{U})$ and $K \subset X \times \mathfrak{U}$ be the support of the cone of the trace map given by

$$\mathcal{RH}om_{X\times\mathfrak{U}}(\mathcal{F}, \mathcal{F}) \xrightarrow{tr} \mathcal{O}_{X\times\mathfrak{U}}$$

   The assumption require K to be proper over $\mathfrak{U}$. By setting $V = X \times \mathfrak{U} \setminus K$, we claim that the restriction of the symplectic form $\omega_{X\times\mathfrak{U}}|_V$ vanishes. On V, trace map

$$tr : \mathrm{RHom}_V(\mathcal{F}|_V, \mathcal{F}|_V) \xrightarrow{\cong} \mathcal{O}_V$$

   is an isomorphism so that we get the natural isomorphism

$$\mathcal{F}|_V \cong (\det \mathcal{F})|_V \cong \det(\mathcal{F}|_\mathcal{V}) \cong \mathcal{O}_V[+d]$$

   Hence,

$$\omega_{X\times\mathfrak{U}}|_V = ch(\mathcal{F}|_V)_2 = ch(\mathcal{O}_V[+d])_2 = 0$$

3. Integration along the fiber
   Applying 6.3.3 we can integrate $\omega_{X\times\mathfrak{U}} \in H^0(F^2\mathbf{DR}_{c/\mathfrak{U}}(X \times \mathfrak{U})[+2]$ to obtain

$$\omega_{\mathfrak{U}} = \int_{[X]} \omega_{X\times\mathfrak{U}} \in H^0(F^2\mathbf{DR}(\mathfrak{U})[2-d])$$

4. $\omega_{\mathfrak{U}}$ is non-degenerate.
   Fix an R-point classifying a perfect complex $\mathcal{E} \in \mathrm{Perf}(X_R)$ with trivial determinant; the tangent space at this point is

$$\mathrm{RHom}_{X_R}(\mathcal{E}, \mathcal{E})_0 = \mathrm{fib}\{tr : \mathrm{RHom}_{X_R}(\mathcal{E}, \mathcal{E}) \to \Gamma(X_R, O_{X_R})\}$$

At a sheaf level, we can define $\mathcal{RH}om_{X_R}(\mathcal{E},\mathcal{E})_0$ and by assumption, it also admits support proper over R. The 2-form $\omega_{\mathfrak{U}}$ at this point is nothing but a composition map

$$\mathrm{RHom}_{X_R}(\mathcal{E},\mathcal{E})_0^{\otimes 2} \to \mathrm{RHom}_{X_R}(\mathcal{E},\mathcal{E})^{\otimes 2} \to \mathrm{RHom}_{X_R}(\mathcal{E},\mathcal{E}) \to \Gamma(X_R, \mathcal{O}_{X_R})$$

By assumption, it factors through $\Gamma_c(X_R, \mathcal{O}_{X_R})$. Now non-degeneracy of $\omega_{\mathfrak{U}}$ follows from the property of the relative Grothendieck-Serre trace map.

□

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