Chapter 1

∞-category theory

Talk by Mauro Porta.

1.1 Why ∞-categories?

Our main reason for studying ∞-categories in this seminar is that derived schemes form an ∞-category. Some other applications of ∞-categories are the following.

1. Formal moduli problems over a field k of characteristic 0 are equivalent to dgLie_k, but this is an equivalence of ∞-categories. We can see explicitly why this equivalence is plausible. For F a formal moduli problem, $T_xF[1]$ is a dgLie algebra. Conversely, Maurer-Cartan elements on the RHS determine $F(k[\epsilon])$, i.e. infinitesimal formal moduli problems. Brackets then allow the complete recovery of F.

2. The ∞-category of rational homotopy types is equivalent to that of dgLie algebras over $\mathbb{Q}$, concentrated in positive degrees:
$$S_*^{\mathrm{rat}} \cong \mathrm{dgLie}_{\mathbb{Q}}^{\geq 1}$$
This statement is related to item 1: Lurie gives nice proof using formal moduli problems, see [4].

3. To $X \in \mathrm{Sch}_k$, we associate its derived category of quasi-coherent sheaves, $D(X) = D(QCoh(X))$. It's a powerful invariant of X, especially when X is not smooth. For example, it contains the cotangent complex and dualizing complex, $\mathbb{L}_X, \omega_X \in D(X)$, which are not necessarily bounded if X is not smooth.

   The problem is that we cannot reconstruct $D(X)$, the derived category in the classical sense, by patching: $D(X) \not\simeq \lim_{\{U\} \text{ Zariski cover}} D(U)$. For example, take $X = \mathbb{P}^1_k$, and its standard cover by 2 open affines U_0, U_1. We show that the functor:
$$D(\mathbb{P}^1) \to D(U_0) \times_{D(U_{01})} D(U_1)$$
   is not faithful, by exhibiting a morphism in $D(\mathbb{P}^1)$ which gets mapped to 0. Start from the observation that morphisms from the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ are the same as sections of the target sheaf, which implies:
$$\mathbb{R}(\mathrm{Hom})\big(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)[1]\big) \cong \mathbb{R}\Gamma\big(\mathcal{O}_{\mathbb{P}^1}(-2)[1]\big).$$
   This complex has nontrivial cohomology in degree 0:
$$H^0\mathbb{R}\Gamma\big(\mathcal{O}_{\mathbb{P}^1}(-2)[1]\big) \cong H^1\big(\mathcal{O}_{\mathbb{P}^1}(-2)\big) \cong k. \tag{1.1.1}$$

However, when passing to the affine patches, $D(U_i) \simeq D(k[T]-Mod)$, and the complexes corresponding to the restrictions of $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-2)[1]$ are the following.

$$0 \longrightarrow 0 \longrightarrow k[T] \longrightarrow 0$$

$$0 \longrightarrow k[T] \longrightarrow 0 \longrightarrow 0$$

As such, there are no non-zero morphisms between the restrictions. Equivalently, when restricting to affine opens, the first cohomology in equation 1.1.1 is 0.

On the other hand, we will see that the ∞-derived category of X (which we temporarily denote by $L_{\mathrm{qcoh}}(X)$) can be patched using the homotopy fiber product:

$$L_{qcoh}(\mathbb{P}^1_k) \simeq L_{qcoh}(U_0) \times_{L_{qcoh}(U_{01})} L_{qcoh}(U_1).$$

4. Let $\mathcal{M}_{ell}$ be the moduli stack of elliptic curves, i.e. the functor F sending Spec(A) to the classes of elliptic curves over Spec(A). It is not a sheaf, because two elliptic curves can become isomorphic after a base extension. The problem here is that we were trying to take $F : \mathcal{A}ff^{\mathrm{op}} \to \mathcal{S}et$, and we can't talk about isomorphisms in $\mathcal{S}et$. Classically one solves this problem by replacing sets by groupoids, which are equivalent to 1-homotopy types.

$$\begin{tikzcd} & \mathcal{G}pd \cong \mathcal{S}^{\leq 1} \arrow[d] \\ \mathcal{A}ff^{\mathrm{op}} \arrow[ur, "\text{stacks}"] \arrow[r, "\text{naive moduli problems}"'] & \mathcal{S}et \cong \mathcal{S}^{\leq 0} \end{tikzcd}$$

We can define higher stacks by extending the tower to higher homotopy types, and ultimately to the category of spaces.

$$\begin{tikzcd} & \mathcal{S} \arrow[d] \\ & \vdots \arrow[d] \\ & \mathcal{S}^{\leq 1} \arrow[d] \\ \mathcal{A}ff^{\mathrm{op}} \arrow[uuur, "\text{higher stacks}"] \arrow[ur, "\text{stacks}"] \arrow[r, "\text{naive moduli problems}"'] & \mathcal{S}et \cong \mathcal{S}^{\leq 0} \end{tikzcd}$$

In later talks, we'll see that the perfect complexes $\mathcal{P}erf$ form an ∞-stack which doesn't factor through finite homotopy types.

1.2 Three ways of working with ∞-categories

To be attempted in order of desperation:

1. Reason model-independently to get a clean proof. The trick is that there are key statements (not proven model independently; some are proven by Lurie and can be found in [3]) which behave like a "non-minimal set of axioms". One should learn a roadmap to [3], in order to know where to find these statements.

2. Internal rectification. Cut the number of homotopies necessary to define the object. Example: an ∞-category with products, see it as a symmetric monoidal category with products. $Mon_{E_1}(\mathcal{C}) \simeq Fun^{\times}(\Delta^{\text{op}}, \mathcal{C}) \to Fun(\Delta^{\text{op}}, \mathcal{C})$. The reference is [5], 4.1.2.6.

3. Try a "real rectification" result, i.e. work with a model-categorical presentation. For example, take $\mathcal{S}$, the ∞-category of spaces, fix $x, y \in X$, want diagram. $\mathcal{S} = \infty(sSet_{Kan})$. Theorem 4.2.4.1 in [3] says that, in this situation, ∞-categorical limits correspond to homotopy limits. We use this theorem to show that $Path_X(x, y)$ is the ∞-limit of the diagram.

   Rectification: something is defined up to homotopy, and we try to reduce the necessary homotopies. Suppose we have $\mathcal{M}$, compare $Fun(\Delta^2, \mathcal{M})$ to $\mathrm{Fun}(\Delta^2, \infty\mathcal{M})$. The first one gives 3 objects and 3 morphisms, while the other gives a homotopy between composition and the other map. The theorem is that you can forget the homotopy, and just remember the data on the LHS. (HTT 4.2.4.4)

In what follows we give examples where we can get by with procedure 1.

Definition 1.2.1. An ∞-cateogry is a simplicial set $\mathcal{C}$ such that all inner horns have fillers. In other words, for all $0 < i < n$, the dotted arrow in the following diagram exists.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dotted] & \end{tikzcd}$$

Note that this achieves what we want: inner horn fillings act as composition of morphisms, but this composition is not unique. "Higher Topos Theory is the book where all of category theory is carried out without ever talking about composition." A few problems arise from here:

1. How do we define Yoneda? A morphism $X \to Y$ is supposed to determine a morphism $h_X \to h_Y$ by composition, which is not well-defined.

2. Let $\mathcal{C}$ be an ∞-category. We want $f : x \to y$ in $\mathcal{C}$ to determine a functor $f_* : \mathcal{C}_{/X} \to \mathcal{C}_{/Y}$ between over-categories, where, morally speaking, $g : Z \to X$ is sent to the composition $f \circ g$. Again, this composition is not well-defined.

To the rescue comes Corollary 2.4.7.12 in [3].

Theorem 1.2.2. *Let $f : \mathcal{C} \to \mathcal{D}$ be an ∞-functor between ∞-categories. Then the projection*

$$\mathcal{P} : Fun(\Delta^1, \mathcal{D}) \times_{Fun(\{1\}, \mathcal{D})} \mathcal{C} \to Fun(\{0\}, \mathcal{D})$$

*is a **cartesian fibration**. Moreover, a morphism in the source is $\mathcal{P}$-**cartesian** iff its image in $\mathcal{C}$ is an **equivalence**.*

Note that the ∞-functors $Fun(\mathcal{C}, \mathcal{D})$ are nothing but the internal Hom in $s\mathcal{S}et$.

$$Fun(\mathcal{C}, \mathcal{D})_n = s\mathcal{S}et(\mathcal{C} \times \Delta^n, \mathcal{D})$$

It's standard to prove that, if $\mathcal{C}, \mathcal{D}$ are ∞-categories, then so is $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$.

We will spend much of section 1.3 defining the terms in bold in Theorem 1.2.2. In Example 1.3.5, we will use Theorem 1.2.2 to obtain the desired pushforward map between overcategories.

1.3 Equivalences and Cartesian fibrations

Definition 1.3.1. $g : x \to y$ in $\mathcal{C}$ is an **equivalence** if any of the following equivalent conditions hold.

(a) The outer horn which maps $01 \mapsto g$ and $02 \mapsto 1_x$ has a filler.

$$\begin{tikzcd} \Lambda^2_0 \arrow[r, "{g,1_x}"] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

Morally speaking, the restriction of the dotted arrow to the 12 face is the right inverse of g.

Moreover, the outer horn which maps $12 \mapsto g$ and $02 \mapsto 1_y$ has a filler.

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "{1_y,g}"] \arrow[d, hook] & \mathcal{C} \\ \Delta^2 \arrow[ur, dotted] & \end{tikzcd}$$

Morally speaking, the restriction of the dotted arrow to the 01 face is the left inverse of g.

(b) The same as variant a, but with higher homotopies included. Formally, we introduce the Kan complex S^∞, defined as 0-coskeleton of the discrete simplicial set with 2 vertices. (For more details see the exercises.) (Todo: Figure out a way to reference the exercises.) We say g is equivalence if there is a lift in the following diagram.

$$\begin{tikzcd} \Delta^1 \arrow[r, "g"] \arrow[d, hook] & \mathcal{C} \\ S^\infty \arrow[ur, dotted] & \end{tikzcd}$$

(c) We say that g is an equivalence if its image in the homotopy category $h(\mathcal{C})$ is an isomorphism. [1]

In the definition, going from version b to version a of is a rectification result, in the sense of procedure 3 described above.

Next, we recall the notions of cartesian morphism and cartesian fibration in the context of 1-categories.

Definition 1.3.2. Let $\mathcal{P} : C \to D$ be a functor between 1-categories. If $x \in Ob(C)$ and $f \in \mathrm{Hom}(x, y)$, we use the notation $\bar{x} := \mathcal{P}(x)$, $\bar{f} = \mathcal{P}(f)$. In the following diagram, the first 2 rows are in C, while the third one is in D. However, we would like to think about the "square" as a pullback square.

$$\begin{tikzcd} z \arrow[dd, dashed, "\mathcal{P}"] & & \\ & x \arrow[r, "f"] \arrow[d, dashed, "\mathcal{P}"] & y \arrow[d, dashed, "\mathcal{P}"] \\ \bar{z} & \bar{x} \arrow[r, "\bar{f}"] & \bar{y} \end{tikzcd}$$

We say that f is a $\mathcal{P}$**-cartesian morphism** if the data of a morphism $z \to y$ in C and a morphism $\bar{z} \to \bar{x}$ in D uniquely determine a morphism $z \to x$ in C, such that the "diagram" commutes.

We say that $\mathcal{P}$ is a **cartesian fibration** if for all $y \in C$ and all $\bar{x} \xrightarrow{\bar{f}} \bar{y}$ morphism in D, $\exists f : x \to y \in \mathcal{C}$ such that $\mathcal{P}(f) = \bar{f}$ and f is $\mathcal{P}$-cartesian.

The analogous definitions for ∞-categories are the following.

Definition 1.3.3. Let $\mathcal{P} : \mathcal{C} \to \mathcal{D}$ be an ∞-functor. A 1-morphism in $\mathcal{C}$, which is the same as an edge $f : \Delta^1 \to \mathcal{C}$, is $\mathcal{P}$**-cartesian** if for all $n \geq 2$, the following outer horn has a filler.

$$\begin{tikzcd} \Delta^1 = \Delta^{\{n-1,n\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^n_n \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "\mathcal{P}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

[1] Recall that this is a 1-category with objects $\mathrm{Ob}(\mathcal{C})$ and morphisms $\mathrm{Hom}(x, y) = \pi_0(\mathcal{C}(x, y))$.

Morally speaking, when $n = 2$, this says that for any edge $g : z \to f(1)$ and edge $\bar{h} : \bar{z} \to \overline{f(0)}$, there exist an edge $h : z \to f(0)$ and a homotopy $g \simeq f \circ h$, such that $\mathcal{P}(h) = \bar{h}$.

We say that $\mathcal{P}$ is a **cartesian fibration** if for every edge $a : \bar{x} \to \bar{y}$ of $\mathcal{D}$, and every object y of $\mathcal{C}$ such that $\mathcal{P}(y) = \bar{y}$, there exists a $\mathcal{P}$-cartesian edge $f : x \to y$ such that $\mathcal{P}(f) = a$.

Recall that, in the study of fibered 1-categories, one proves that cartesian fibrations with base D are the same as lax 2-functors from D to the 2-category of 1-categories. (This is known as the "Grothendieck construction", see for example, Proposition I.3.26 in [1].) Explicitly, given a cartesian fibration $\mathcal{P} : C \to D$, the corresponding lax 2-functor maps an object $d \in D$ to the fiber $\mathcal{P}^{-1}(d)$. Theorem 3.2.0.1, the main theorem of Chapter 3 in [3], is the analog of this result for the setting of ∞-categories.

Theorem 1.3.4. *For any ∞-category $\mathcal{C}$, there is an equivalence of ∞-categories:*

$$CartesianFibr/\mathcal{C} \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Cat}_\infty). \tag{1.3.1}$$

Example 1.3.5. Recall that we started out by trying to construct an ∞-functor $f_* : \mathcal{C}_{/x} \to \mathcal{C}_{/y}$ between overcategories, given an 1-morphism $f : x \to y$ in $\mathcal{C}$. Taking $F : \mathcal{C} \to \mathcal{C}$ as the identity, Theorem 1.2.2 gives a Cartesian fibration over $\mathcal{C}$:

$$\big\{(f : x \to y, a) | \{f : x \to y\} \in \mathcal{C}, F(a) \cong y\big\} \to \mathcal{C},$$

where a pair $(f : x \to y, a)$ maps to x. We recognize the fiber over x as the undercategory $\mathcal{C}_{x/}$:

$$\mathrm{Hom}_{sSet}(\Delta^n, \mathcal{C}_{x/}) = \big\{\alpha : \Delta^{n+1} \to \mathcal{C} | \alpha_{\Delta^{[0,\ldots,n}} = x\big\}.$$

Theorem 1.3.4 then produces an ∞-functor:

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} &\to \infty - \mathcal{C}at \\ x &\mapsto \mathcal{C}_{x/} \\ f : x \to y &\mapsto f^* : \mathcal{C}_{y/} \to \mathcal{C}_{x/}. \end{aligned}$$

We have obtained a pullback map on undercategories. To obtain the pushforward on overcategories, start with $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ as the contravariant identity functor instead.

Next, we discuss a simpler example. Let $\mathcal{C}$ be an ∞-category, and let $x \in \mathcal{C}$ be an initial object. We want to construct a functor $\mathcal{C} \to \mathcal{C}_{x/}$. Note that this is silly in 1-category theory, since there's a unique morphism $x \to y$. To aid us in the context of ∞-categories, we start by giving a good definition.

Definition 1.3.6. $x \in \mathcal{C}$ **initial** if $\forall y \in \mathcal{C}$, $Map_{\mathcal{C}}(x, y)$ is contractible.

The key result, proved, for example, in [2], is the following.

Proposition 1.3.7. *If $\mathcal{C}$ is an ∞ category, then $x \in \mathcal{C}$ is initial iff the canonical projection $\mathcal{C}_{x/} \to \mathcal{C}$ is a trivial Kan fibration.*

To solve our problem, note that $\mathcal{C}$ is cofibrant in the Kan model structure, so there exists a lift in the diagram:

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \mathcal{C} \arrow[r, "\mathrm{id}"] \arrow[ur, dashed] & \mathcal{C}. \end{tikzcd}$$

In the exercises, we also encounter the following problem. Suppose $\mathcal{C}$ has pushouts and a zero object. Construct an ∞-functor $\mathcal{C} \to \mathcal{C}$ sending x to the pushout of 0 and 0 over x. (Todo: write this up, either here or in the exercises)

Chapter 2

Derived Affines

Talk by Benedict Morrissey.

2.1 3 perspectives on derived affines

First recall the notion of affines in classical AG: $\mathrm{Aff}_k^{Cl} \simeq (\mathcal{C}Ring)^{\mathrm{op}}$. We get schemes by gluing these together. There's also the functor of points viewpoint: $X \in \mathrm{Aff}_k^{Cl}$ defines a sheaf by sending $\operatorname{Spec} R \mapsto \operatorname{Hom}(\operatorname{Spec} R, X)$. The schemes are then precisely the sheaves in the Zariski topology. Already in classical AG, there exist constructions which move us out of this category: both Serre's intersection theorem and Illusie's notion of the cotangent complex use derived functors. So by introducing DAG, we will understand better these structures in classical AG.

We will talk about 3 approaches to derived affines:

1. Simplicial commutative rings;
2. Lawvere theory;
3. Sheaves of commutative differential graded algebras.

Remark 2.1.1. Classically gluing is easy. For example, fiber products are computed by reducing to the affine case, where it's just the tensor product of rings. In DAG, the derived tensor product is only defined up to quasi-isomorphism, so gluing can only be defined in a category which allows homotopy, such as an ∞-category. For today's talk we mostly use the model category description; an application of Dwyer-Kan localization produces an ∞-category.

For approach 1, recall that the simplicial category Δ is:

$$Ob(\Delta) = \{n \in \mathbb{N} \cup \{0\}\}$$

where morphisms are compositions of face maps and degeneracy maps.

Definition 2.1.2. The **category of simplicial commutative rings** is the category of contravariant functors:

$$SCR_k = \operatorname{Hom}(\Delta^{\mathrm{op}}, CRing_k).$$

Remark 2.1.3. There's a model category structure on this: fibrations are Kan fibrations on the underlying simplicial sets, i.e. morphisms $f : A \to B$ of simplicial commutative rings, such that all horns have fillers:

$$\begin{tikzcd} \Lambda^n_j \arrow[r] \arrow[d, hook] & A \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B. \end{tikzcd}$$

Weak equivalences are weak homotopy equivalences on the underlying simplicial sets. Cofibrations are then determined from the axioms of a model category; note that they are *not* the same as cofibrations of the underlying simplicial sets.

Remark 2.1.4. We're using transfer to put the model structure on SCR_k. To explain what that means, under suitable conditions, there's a general procedure for defining a model structure on a category $\mathcal{B}$, given a model category $\mathcal{A}$ and an adjoint functor pair:

$$\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}.$$

The procedure forces the adjoint functor pair to be a Quillen adjunction. In our case, we use the free-forgetful adjunction:

$$s\mathcal{S}et \underset{U}{\overset{F}{\rightleftarrows}} SCR_k$$

to transfer the Kan model structure to SCR_k. The key point which allows this to work is that all objects are fibrant. Cofibrations are more difficult to characterize, but the cofibrant objects are precisely the quasi-free ones. (That is, the ones isomorphic to a free object.)

Next, we introduce the Dold-Kan equivalence in the context of CDGAs. Recall that we have a Quillen equivalence:

$$s\mathcal{V}ect \rightleftarrows dg - \mathcal{V}ect^{\leq 0}$$

between simplicial vector spaces and differential graded vector spaces, concentrated in nonpositive degrees. We want to talk about commutative monoids in these categories, $sc\mathcal{A}lg_k$ and $cdg - \mathcal{A}lg^{\leq 0}$, respectively. The model structure on $cdg - \mathcal{A}lg^{\leq 0}$ can also be obtained by transfer from the free-forgetful adjunction; we obtain that the weak equivalences are quasi-isomorphisms, and the fibrations are degree-wise surjections.

Theorem 2.1.5 (Symmetric monoidal Dold-Kan)**.** *There is a Quillen equivalence:*[1]

$$sc\mathcal{A}lg_k \underset{\Gamma}{\overset{N}{\rightleftarrows}} cdg - \mathcal{A}lg_k^{\leq 0}.$$

Moreover, if the simplicial commutative algebra A_* corresponds to the commutative dg-algebra $B_\bullet$, then $\pi_i(A_*) \cong H^i(B_\bullet)$.

(Todo: Matei's note: I was confused when this was introduced in the talk, so I looked on nLab to find the formulation above. Let me know if this is not the formulation that we want.)

Remark 2.1.6. The functor $\Gamma : cdg - \mathcal{A}lg_k^{\leq 0} \to sc\mathcal{A}lg_k$ is complicated, but we describe N. $A_* \in sc\mathcal{A}lg_k$ maps in the first stage to $\tilde{A}_\bullet$, where $\tilde{A}_{-n} = A_n$, and the differential is the alternating sum of the face maps. $N(A_*)$ is then the quotient of $\tilde{A}_\bullet$ by the images of the degeneracy maps.

We move on to approach 2 to derived affines, the Lawvere Theory description. This is important because it's the only one of the 3 procedures which carries through in the analytic setting. You get holomorphic rings, smooth rings, and much of the theory of DAG can be carried in this setting.(Todo: clear this part up)

For example,

$$AbGps \cong \mathrm{Fun}^{\times}(FAb^{\mathrm{op}}, Set).$$

There's a map $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$, which sends $1 \mapsto 1 \times 1$. Since F preserves products, $F(\mathbb{Z}) \times F(\mathbb{Z}) \cong F(\mathbb{Z} \times \mathbb{Z}) \to F(\mathbb{Z})$. Free commutative rings. $k[x_1, \dots, x_n]$. $FCRing_k^{\mathrm{op}} \cong \{\mathbb{A}^n\}$. $\mathrm{Fun}^{\times}(Affplanes, Set) \cong CRing$. Use the group scheme structure of $\mathbb{A}^1$ to get this.

[1] Note that, in general, a Quillen equivalence is not an equivalence of categories. It does, however, induce an equivalence of homotopy categories.

Now pass to $SCR_k \cong s\,\mathrm{Fun}^{\times}(Affplanes, Set) \cong \mathrm{Fun}^{\times}(Affplanes, sSet) \cong \mathrm{Fun}^{\times}(Affplanes, S)$, where S is the infinity category of spaces. The last step is a very hard rectification theorem, proved by Lurie-Bergner.[2]

To simplify notation, we denote Affplanes by T_{disc}. $\mathrm{Fun}^{\times}(T_{disc}, sSet)_{projective}$.

Finally, we say a few words about approach **??** to derived affines. For $A \in cdg - \mathcal{A}lg_k$, we look at the truncation $\operatorname{Spec} H^0(A)$, which is an affine scheme in the classical sense. We can regard A as a sheaf of cdg-algebras on the truncation, as long as we can understand how localization works for cdg-algebras. We claim that it suffices to localize the commutative algebra A_0. Indeed, we have the multiplication map:

$$\mu : A_0 \times A_i \to A_i,$$

so given a multiplicative subset $S \subset A_0$, we define the localization $S^{-1}A_i$ as $\mu(S^{-1}A_0 \times A_i)$. If this makes sense, we get a sheaf $\mathcal{O}_A$ of cdg-algebras.

We would like to define derived affines as pairs $\left(\operatorname{Spec} H^0(A), \mathcal{O}_A\right)$. There is a subtlety: a priori this only gives a 2-category, and we need ∞-categories.

2.2 Our favorite classes of morphisms

Definition 2.2.1. Given $f : A \to B$ in SCR_k, we get maps:

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \to \pi_*(B)$$

of graded modules. We say that f is **strong** if this is an isomorphism of graded modules.

Definition 2.2.2. We define $f : A \to B$ to be **étale** (resp. **smooth**, **Zariski open immersion**, **flat**) if f is strong and $\pi_0(A) \to \pi_0(B)$ is étale (resp. smooth, Zariski open immersion, flat) in the classical sense.

Remark 2.2.3. The strength condition on f is quite restrictive: for example, a strong map from a non-derived domain must have a non-derived target.

Definition 2.2.4. Let $X = \operatorname{Spec}(A)$ be a derived affine over k. Then the **small étale site** of X is:

$$X_{\text{ét}} = \{\text{étale maps } \operatorname{Spec}(B) \to \operatorname{Spec}(A)\}.$$

In order to obtain the small étale site in the sense of classical AG, one needs to pass to the truncated version of the étale maps: $\pi_0(f) : \operatorname{Spec}(\pi_0(B)) \to \operatorname{Spec}(\pi_0(A))$. Then one can prove there's an equivalence of ∞-categories between the derived and classical étale sites. In particular, this shows that $X_{\text{ét}}$ is a 1-category. This is one of the ingredients in the proof of the easy version of Lurie representability. Moreover, the same holds for the small smooth site and the small Zariski site.

After introducing the cotangent complex $\mathbb{L}_f$ of a morphism f, we will see that f is étale iff $\pi_0(f)$ is of finite presentation and $\mathbb{L}_f \simeq 0$.

Definition 2.2.5. $f : A \to B$ is **of finite presentation** if the functor $\mathrm{Map}_A(B, -) : sc\mathcal{R}ing_k \to \mathcal{S}$ commutes with filtered colimits.

Unlike in the underived case, being of finite presentation is very strong, because it has a hidden regularity condition. In particular, we have the proposition due to Lurie:

Proposition 2.2.6. *$f : A \to B$ is of finite presentation in the derived sense iff $\pi_0(f)$ is of finite presentation in the classical sense (also called to order 0) and the cotangent complex $\mathbb{L}_f$ is perfect.*

Example 2.2.7. Let $X = \mathbb{A}^3$, and Y a closed subscheme of X which is not a local complete intersection. Then the inclusion $\iota : Y \to X$ is not of finite presentation in the derived sense. Indeed, by a conjecture of Quillen, which is now a theorem of Abramov, for maps between classical schemes, the cotangent complex is either concentrated in degrees 0 and -1, or it's unbounded. Since Y is not lci, the first case is ruled out, and $\mathbb{L}_\iota$ is unbounded.

[2] HTT Propositions 5.5.9.2

Chapter 3

Stable ∞-categories

Talk by Michael Gerapetritis.

3.1 Motivation

In the 1-categorical setting, if $\mathcal{C}$ is a category, we may require that $\mathcal{C}(A, B)$ be a set. To get particularly well-behaved categories, namely the additive categories, we require that $\mathcal{C}(A, B)$ is actually an abelian group.

We try to replicate this in the ∞-category setting. Let $\mathcal{C}$ be an ∞-category, then $\mathcal{C}(X, Y)$ is a space. We want to discover what is the good extra structure to have on this space; we will call the corresponding ∞-categories stable.

3.2 Stable ∞-categories and triangulated 1-categories

Definition 3.2.1. An ∞-category $\mathcal{C}$ is **stable** if:

- $\mathcal{C}$ is pointed, i.e. it has a zero object;
- every morphism $f : X \to Y$ admits fibers and cofibers;
- a triangle is a fiber iff it is a cofiber.

Recall that a **triangle** in $\mathcal{C}$ is a map of simplicial sets $\Delta^1 \times \Delta^1 \to \mathcal{C}$, i.e. a homotopy commutative diagram with the zero object in the bottom-left corner:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d, "g"] \\ 0 \arrow[r] & Z \end{tikzcd}$$

The triangle is a **fiber** if it is a pullback square, and a **cofiber** if it is a pushout square. We say that $f : X \to Y$ admits a fiber (resp. cofiber) when $\exists W$ (resp Z) such that:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

is a pullback square (or, respectively:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a pushout square).

Remark 3.2.2. Note that the data of a triangle consists not only of homotopy commutative diagrams as above, but also of choices of homotopies between the branches. This is crucial, since it ensures that cones are functorial at the level of the homotopy category. This functoriality does not hold in a general triangulated category. (See Theorem 3.2.5 for the relation between stable ∞-categories and triangulated 1-categories.)

Example 3.2.3. Our two main examples are ∞-categories of spectra (see Section 3.5) and of modules over a CDGA or SCR (see Section 3.3).

Recall the data for a triangulated category.

Definition 3.2.4. A category $\mathcal{D}$ is triangulated if:

1. $\mathcal{D}$ is additive;

2. $\mathcal{D}$ admits a translation functor $T : \mathcal{D} \xrightarrow{\simeq} \mathcal{D}$;

3. $\mathcal{D}$ has a collection of distinguished triangles:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

This data is required to satisfy some axioms, but we won't go into details here.

Theorem 3.2.5. *If $\mathcal{C}$ is a stable ∞-category, then $h\mathcal{C}$ is triangulated.*

For a proof see [5]. We won't go over it, let's just say that translation is given by Σ, and distinguished triangles are precisely the images of fiber sequences (or equivalently, cofiber sequences), as resulting from the following diagram.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

Proposition 3.2.6. *$\mathcal{C}$ is stable iff the following hold:*

1. *$\mathcal{C}$ admits finite limits and colimits;*

2. *any square is a pushout iff it is a pullback.*

Proof. Again, we don't give a full proof. Let's just see why products and coproducts must exist in a stable ∞-category. Note first that Σ is an equivalence of ∞-categories. Indeed, Σ is a left adjoint functor; moreover, the unit and counit of the adjunction become isomorphisms in the homotopy category, due to condition 2 in the definition of a triangulated category. Then we use the following diagram.

$$\begin{tikzcd} \Omega(X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & X \arrow[r] & X \oplus Y \end{tikzcd}$$

We have defined $X \oplus Y$ as the cofiber of $\Omega(X) \xrightarrow{0} Y$, which is postulated to exist in a stable ∞ -category. This turns the outer rectangle into a pushout square, and it follows that the square on the right is also a pushout square. Thus $X \oplus Y$ is the coproduct of X and Y. We reason dually to obtain products. $\square$

Definition 3.2.7. Let $\mathcal{C}, \mathcal{C}'$ be stable ∞-categories, and $F : \mathcal{C} \to \mathcal{C}'$ an ∞-functor which maps 0 objects to 0 objects. Equivalently, F maps triangles to triangles. If F maps fiber sequences to fiber sequences, we say that F is **exact**.

Lemma 3.2.8. *TFAE:*

1. *F is exact;*
2. *F is right-exact, i.e. commutes with finite colimits;*
3. *F is left-exact, i.e. commutes with finite limits.*

This is very useful: sometimes it's really easy to check that a functor is right or left exact, e.g. if it's a left or right adjoint, respectively.

3.3 Modules

For a useful example of the result in Lemma 3.2.8, we look at $\mathcal{C} = A - \mathcal{M}od$, where A is a CDGA or SCR over k. (By $A - \mathcal{M}od$ we mean the unbounded derived category.) The easiest way to see $A - \mathcal{M}od$ as an ∞-category is to put a model structure on chain complexes, say the projective one, and then take the underlying ∞-category. We claim that $A - \mathcal{M}od$ is a stable ∞-category. Using the theorem Mauro talked about in Lecture 1, limits and colimits exists in the ∞-category iff they exist in the model category. (Todo: reference theorem) It remains to prove the following.

Lemma 3.3.1. *A triangle in $A - \mathcal{M}od$ is a fiber iff it is a cofiber.*

Proof. We prove one direction; the other argument is dual to this one. Assume that $f : M^\bullet \to N^\bullet$ is the fiber of a map g. Take a cofibrant replacement of f, get $\tilde{M}, \tilde{N}$ cofibrant and a homotopy pullback square: (Todo: figure out how to do the cartesian symbol in tikz)

$$\begin{tikzcd} \tilde{M}^\bullet \arrow[r, hook, "\tilde{f}"] \arrow[d] & \tilde{N}^\bullet \arrow[d, "\tilde{g}"] \\ 0 \arrow[r] & P^\bullet. \end{tikzcd}$$

$\tilde{f}$ is cofibrant, so it's a degree-wise injection. Then g is a degreewise surjection, and it follows that the square is a strict pushout. (Todo: wait, how did this work again?) □

Now suppose we have $f : A \to B$ a morphism of $CDGA_k^{\leq 0}$. It induces the adjunction of model categories:

$$A - Mod \underset{f_*}{\overset{f^*}{\rightleftarrows}} B - Mod,$$

where f_* is the forgetful functor, and $f^*(M) = M \otimes_A B$. So this gives an adjunction of ∞-categories: [1]

$$A - \mathcal{M}od \underset{Rf_*}{\overset{Lf^*}{\rightleftarrows}} B - \mathcal{M}od.$$

Explicitly, Lf^* is constructed by first choosing a cofibrant replacement $\tilde{M}$ for M, and then taking $\tilde{M} \otimes_A B$. The answer doesn't depend on cofibrant replacement, up to coherent isomorphism. Then Lf^* is a left adjoint functor, so it follows from general nonsense that it's right exact. Lemma 3.2.8 then implies that Lf^* is also left exact and exact.

Remark 3.3.2. If f is not flat in the sense of Definition 2.2.2, then the exactness of Lf^* comes at the price of losing t-exactness. To explain what we mean, pick $M \in A - Mod$, such that $H^i(M) = 0$ unless $i = 0$. But then $Lf^*(M) = M \otimes_A^{\mathbb{L}} B$, and $H^{-i}(M \otimes_A^{\mathbb{L}} B) = \mathrm{Tor}_i^A(M, B)$, which is $\neq 0$ in general, because f is not flat. So even though M was homologically concentrated in degree 0, $Lf^*(M)$ may not be. In other words, the failure of a functor of (Grothendieck) abelian categories to preserve limits translates into a lack of t-exactness of the derived functor. In the following section we define t-structures and t-exactness for ∞-categories.

[1] Here we use L and R to indicate that the functors are derived. In later talks derived functors will be the default, and we will omit the symbols L and R.

3.4 t-structures

Definition 3.4.1. If $\mathcal{C}$ is a stable ∞-category, a **t-structure**[2] on $\mathcal{C}$ is the data of two full subcategories of $\mathcal{C}$, $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$, [3] such that:

1. $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y[-1]) = 0$ if $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 0}$. [4]
2. $X \in \mathcal{C}^{\leq 0}, X[1] \in \mathcal{C}^{\leq 0}$;
3. $\forall X$, $\exists$ fiber sequence $X' \to X \to X''$, where $X' \in \mathcal{C}^{\leq 0}$, $X'' \in \mathcal{C}^{\geq 1}$.

Remark 3.4.2. Condition 1 has the following intuitive meaning in the case $\mathcal{C} = A - \mathcal{M}od$. 0-morphisms in $\mathcal{C}$ are chain maps which preserve degree, while higher morphisms are homotopies which shift the degree to the left; morphisms that shift degree to the right are not allowed. Then, if $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 0}$, no nonzero morphisms should be allowed between X and $Y[-1]$:

$$\dots \longrightarrow X_{-2} \longrightarrow X_{-1} \longrightarrow X_0 \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow Y_0 \longrightarrow Y_1 \longrightarrow Y_2 \longrightarrow \dots$$

Remark 3.4.3. X' and X'' are uniquely determined by X.

Theorem 3.4.4. *The inclusion $\mathcal{C}^{\leq 0} \to \mathcal{C}$ has a right adjoint, which we denote $\tau_{\leq 0} : \mathcal{C} \to \mathcal{C}^{\leq 0}$. Similarly we get $\tau_{\geq 0} : \mathcal{C} \to \mathcal{C}^{\geq 0}$.*

Corollary 3.4.5. *For all $X \in \mathcal{C}$, the fiber sequence of 3 is just:*

$$\tau_{\leq 0} X \to X \to \tau_{\geq 1} X.$$

Proposition 3.4.6. *Denote by $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$, the* ***heart*** *or* ***core*** *of the t-structure. It is an abelian 1-category.*

Proposition 3.4.7. *Let $\mathcal{C}$ be stable. Then if:*

$$X \to Y \to Z$$

is a fiber sequence, then we have a long exact sequence of H^i, where $H^i(X) := \tau_{\geq i} \circ \tau_{\leq i}(X)$.

Putting the last few results together, from $\mathcal{C}$ a presentable stable ∞-category with t-structure, the heart is Grothendieck abelian. Write $A = \mathcal{C}^{\heartsuit}$. Then we can form $\mathcal{D}(A)$, the ∞-derived category of A. The next theorem describes the relationship between $\mathcal{C}$ and $\mathcal{D}(A)$.

Theorem 3.4.8 (Lurie)**.** *$\mathcal{D}(A)$ has a universal property which produces an ∞-functor:*

$$\mathcal{D}(A) \to \mathcal{C}.$$

In general this is very far from being an equivalence.

Example 3.4.9. Let $A \in CDGA_k^{\leq 0}$. The theorem gives a map:

$$(A - \mathcal{M}od)^{\heartsuit} \to (H^0(A) - Mod)^{\heartsuit}. \tag{3.4.1}$$

This is one of the most important facts in DAG, because it reduces problems about the ∞-category of A-modules to problems in classical categories of modules, where one can work with generators and relations. The map in 3.4.1 is an equivalence iff $A \simeq H^0(A)$ are quasi-isomorphic.(Todo: figure out what's the precise relationship here)

[2] t stands for truncation

[3] Note that we use cohomological notation, while Lurie in [5] uses homological notation. Therefore gradings have opposite signs in this seminar and in [5].

[4] In a stable ∞-category, we sometimes use the shift notation $[n]$ to denote the $|n|$-fold iterated application of the Σ functor (if n is positive) or the Ω functor (if n is negative). This notation is justified by Proposition 3.5.4.

Definition 3.4.10. Let $\mathcal{C}, \mathcal{D}$ be stable ∞-categories with t-structures. Then an exact functor $F : \mathcal{C} \to \mathcal{D}$ is:

1. **left t-exact** if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$;
2. **right t-exact** if $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$;
3. **t-exact** if both.

Example: for $A, B \in CDGA_k^{\leq 0}$, $f : A \to B$, we have the adjunction:

$$A - \mathcal{M}od \underset{Rf_*}{\overset{Lf^*}{\rightleftarrows}} B - \mathcal{M}od.$$

Every object is fibrant, so we don't need to derive the functors. Rf_* is both left and right t-exact. Lf^* is not right t-exact, because of nontrivial Tor^i terms; see 3.3.2. However, Lf^* is right t-exact: morally speaking, Projective resolution only puts stuff in negative degrees. We give an ∞-categorical proof.

Proof. Pick $M \in A - Mod^{\geq 0}$. We want $Lf^*(M) \in B - Mod^{\leq 0}$. To check this is the same as checking that $\forall N \in B - Mod^{\geq 1}$, $\mathrm{Map}_{B-Mod}(Lf^*M, N) \cong 0$. But this is $Map_{A-Mod}(M, Rf_*N) \cong 0$, which follows since Rf_* was t-exact. □

3.5 Spectra

Going back to the question left unanswered in Section 3.1, the extra structure we want on morphism spaces of stable ∞-categories is $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathrm{Sp}^{\leq 0}$.

Definition 3.5.1. Spectra are sequences $\{F_i\}$ of objects in $\mathcal{C}$ such that $F_n \simeq \Omega F_{n+1}$. Alternatively, we identify them with objects of the homotopy limit:

$$\ldots \overset{\Omega}{\to} \mathcal{C} \overset{\Omega}{\to} \mathcal{C} \overset{\Omega}{\to} \ldots$$

Remark 3.5.2. We must be careful with defining morphisms between spectra: we want squares to commute up to coherent homotopy. Moreover, it's hard to get a monoidal model structure on the category of spectra: this was done only in the 2000s, after Hovey introduced symmetric spectra. Lurie has a very categorical and very nice way of putting a monoidal structure at the level of the ∞-category directly. See the last chapter of [2], and also 4.8.2 of [5].

Theorem 3.5.3. *$Sp(\mathcal{C})$ is stable.*

This gives a canonical stabilization for every ∞-category. The proof of the theorem follows from the following characterization of stable ∞-categories, and the fact that $\Omega : Sp(\mathcal{C}) \to Sp(\mathcal{C})$ is an equivalence.

Proposition 3.5.4. *$\mathcal{C}$ is a pointed ∞-category. TFAE:*

1. *$\mathcal{C}$ is stable;*
2. *$\mathcal{C}$ admits colimits and $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence;*
3. *$\mathcal{C}$ admits limits and $\Omega : \mathcal{C} \to \mathcal{C}$ is an equivalence;*

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