Chapter 1

∞-category theory

1.1 Motivations

Exercise 1.1.1. We fix a base field k. Let $X = \mathbb{P}^1_k$ and let U_0 and U_1 be the standard open affine cover of $\mathbb{P}^1_k$. For any k-algebra A, we have:

$$U_0(A) \coloneqq \{[x_0 : x_1] \in \mathbb{P}^1_k(A) \mid x_0 \neq 0\}, \qquad U_1(A) \coloneqq \{[x_0 : x_1] \in \mathbb{P}^1_k(A) \mid x_1 \neq 0\}.$$

Let $U_{01} = U_0 \cap U_1$ be their intersection. Show that the canonical functor

$$\mathrm{h}(\mathcal{D}(\mathbb{P}^1_k)) \to \mathrm{h}(\mathcal{D}(U_0)) \times_{\mathrm{h}(\mathcal{D}(U_{01}))} \mathrm{h}(\mathcal{D}(U_1))$$

is essentially surjective but not fully faithful.

Exercise 1.1.2. Let $\mathcal{C}$ be a triangulated category where countable products and countable direct sums exist. Show that if there exists a functor Tr from the category of arrows $\mathcal{C}^{\Delta^1}$ to the category of exact triangles in $\mathcal{C}$, then every triangle in $\mathcal{C}$ is split. (See [**?**, Proposition II.1.2.13].)

1.2 Reminders on simplicial sets

Exercise 1.2.1. Show that the nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful and its essential image is spanned by those simplicial sets K satisfying the following lifting condition: for every $n \geq 2$ and for every $0 < i < n$ every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a unique solution.

Solution. The nerve of a category $\mathcal{C}$ is:

$$(N\mathcal{C})_n = \{(f_1, \dots, f_n) | \text{ composable morphisms}\}.$$

The face maps are:

$$d_j(f_1, \dots, f_n) = \begin{cases} (f_1, \dots, f_{n-1}), & j = 0 \\ (f_1, \dots, f_j \circ f_{j-1}, \dots, f_n), & 0 < j < n \\ (f_2, \dots, f_n), & j = n. \end{cases}$$

The degeneracy s_j is obtained by inserting an identity map in the j^{th} slot.

A functor $F : \mathcal{C} \to \mathcal{D}$, induces a simplicial map:

$$\begin{aligned} N(F)_n : (N\mathcal{C})_n &\to (N\mathcal{D})_n \\ (f_1, \dots, f_n) &\mapsto \big(F(f_1), \dots, F(f_n)\big). \end{aligned}$$

If two functors F, F' induce simplicial maps $N(F) = N(F')$ which agree, then $F(f) = F'(f)$ for every morphism f. Hence N is faithful. Given a simplicial map $G : N\mathcal{C} \to N\mathcal{D}$, we define a functor $F : \mathcal{C} \to \mathcal{D}$ to be G_0 on objects and G_1 on morphisms. We show that F respects composition. Let f_1, f_2 be two composable morphisms in $\mathcal{C}$ and denote by x the 2-simplex (f_1, f_2). Then:

$$F(f_2 \circ f_1) = G(d_1 x) = d_1 G(x) = F(f_2) \circ F(f_1).$$

This proves that N is also full.

We move on to the essential image. Let K be the nerve of a category. The data of a map $\Lambda^n_i \to K$ is the same as the data of maps $y_j : \Delta^{n-1} \to K$ for $j \neq i$, which are compatible along their faces. By Yoneda, this is the same as simplices $\{y_j \in K_{n-1}\}_{j \neq i}$ compatible along faces. Given this data, we define the horn filler $x \in K_n$ by:

$$x = \left((d_0)^{n-2} y_{n-1}, (d_0)^{n-3} d_n y_{n-1}, \dots, d_0 (d_n)^{n-3} y_{n-1}, (d_n)^{n-2} y_0\right).$$

The simplicial identities ensure that $d_j x = y_j$ for $j \neq i$. [1] Using the compatibility of the y_j along faces, x is the unique solution to the lifting problem.

Conversely, given a K which has unique solutions to all lifting problems of inner horns, we define a category $\mathcal{C}$ such that $K \cong \mathcal{C}$. Let K_0 be the objects of $\mathcal{C}$, and for $X, Y \in K_0$, define:

$$\mathrm{Hom}(X, Y) := \{f \in K_1 | d_1 f = X, d_0 f = Y\}.$$

Given $f_1 : X \to Y$ and $f_2 : Y \to Z$, define a lifting problem by mapping the 1-simplices $0 \to 1$ and $1 \to 2$ in Λ^2_1 to f_1 and f_2, respectively. We define $f_2 \circ f_1$ to be d_1 of the unique lift. Associativity of this composition follows from the unique filling of the horn Λ^3_1; we don't give the details here. ☐

Exercise 1.2.2. Let S, S' be sets, seen as discrete simplicial set. Show that any morphism $f : S \to S'$ is a Kan fibration, and that f is a trivial Kan fibration if and only if f is a bijection.

Solution. Since S and S' are sets, all k-simplices are of the form $s^k x$, for x a 0-simplex. Given a lifting problem:

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & S \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S' \end{tikzcd}$$

all k-simplices of Λ^n_i, for $k > 0$, must map to degenerate k-simplices in S. Hence Λ^n_i maps to a point $s \in S$. Similarly, Δ^n maps to $f(s)$. The constant map from Δ^n to s is then the unique solution to the lifting problem. It follows that f is a Kan fibration, and moreover that all sets S are ∞-groupoids.

By definition, f is a weak equivalence if it induces a weak equivalence on geometric realizations. $|S|$ and $|S'|$ are discrete topological spaces, therefore $|f|$ is a weak equivalence iff it is a bijection. ☐

Exercise 1.2.3. Let G and H be simplicial groups and let $f : G \to H$ be a surjective group homomorphism. Show that f is a Kan fibration.

Solution. There is an algorithm for constructing fillers on nLab. [2] We don't have any intuition for it, so we should work on building that.

The algorithm produces unique fillers for all horns, so in particular simplicial groups are ∞-groupoids. ☐

[1] Note that it's essential that both y_0 and y_{n-1} are available to use in the definition of x, i.e. that Λ^n_i is an inner horn.
[2] `https://ncatlab.org/nlab/show/simplicial+group`

Exercise 1.2.4. Let $\partial\Delta^2$ be the smallest full subsimplicial set of Δ^2 spanned by its non-degenerate edges $\Delta^1 \to \Delta^2$. Show that $\partial\Delta^2$ fits into a coequalizer diagram

$$(\Delta^0)^{\amalg 6} \rightrightarrows (\Delta^1)^{\amalg 3} \to \partial\Delta^2.$$

(Hint: Have a look at [**?**, Theorem III.3.1].)

Exercise 1.2.5. Let S be a set, seen as a discrete simplicial set. Show that $\operatorname{cosk}_n(S)$ satisfies the following property: for every $m \geq n$ and every $0 \leq i \leq m$ the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \operatorname{cosk}_n(S) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. In particular, deduce that $\operatorname{cosk}_0(S)$ is a Kan complex.

solution. Recall the definition. Let $tr_{\leq n} : sSet \to sSet_{\leq n}$ be a truncation functor. It has both left adjoint sk_n and right adjoint $cosk_n$ given by left and right Kan extension, respectively. We call $sk_n \circ tr_{\leq n}$ a n-skeleton functor between $sSet$, and $cosk_n \circ tr_{\leq n}$ a n-cosckeleton functor. For the notational convenience, we just denote them by sk_n and $cosk_n$, respectively.
By definition, for every simplicial set T, we have an isomorphism

$$Hom_{sSet}(T, cosk_n(S)) \simeq Hom_{sSet_{\leq n}}(tr_{\leq n}T, S)$$

Thus, it suffices to show that there exists a map

$$Hom_{sSet_{\leq n}}(\Lambda^m_i, S) \to Hom_{sSet_{\leq n}}(\Delta^m, S)$$

for every $m \geq n$ and every $0 \leq i \leq m$. It follows from the fact that S is discrete(i.e. Kan complex) and bijectivity of trucation maps. □

1.3 ∞-categories

Exercise 1.3.1. Show that every Kan complexes and 1-categories are ∞-categories (quasicategories).

Solution. Kan complexes have fillers for all horns. 1-categories have unique fillers for all inner horns. In particular, both have fillers for all inner horns, which is the definition of ∞-categories. □

Exercise 1.3.2. A morphism $f : X \to Y$ in an ∞-category $\mathcal{C}$ is said to be an equivalence if its image in $\mathrm{h}(\mathcal{C})$ is an isomorphism. Define $S^\infty := \operatorname{cosk}_0(\{0,1\})$ and let $j : \Delta^1 \to S^\infty$ be the map classified by

$$\operatorname{sk}_0(\Delta^1) = \{0,1\} \xrightarrow{\mathrm{id}} \{0,1\}.$$

To give a morphism $f : X \to Y$ in an ∞-category $\mathcal{C}$ it is equivalent to specify a morphism of simplicial sets $e_f : \Delta^1 \to \mathcal{C}$. Show that f is an equivalence if and only if the lifting problem

$$\begin{tikzcd} \Delta^1 \arrow[r, "e_f"] \arrow[d, hook, "j"] & \mathcal{C} \\ S^\infty \arrow[ur, dashed] & \end{tikzcd}$$

has at least one solution. Next, show that any two such solution are homotopic. (Hint: have a look at Exercises 1.2.5 and 1.4.1.)

Solution. Note that $cosk_n$ is right adjoint to sk_n. It follows that $j : \Delta^1 \to S^\infty$ is well defined. First, we prove that $f : X \to Y$ is an equivalence if and only if the above lifting problem has at least one solution. By 1.2.5, $cosk_0(\{0,1\}) = S^\infty$ is a Kan complex. It follows that if we choose Kan model structure on $sSet$, S^∞ is a fibrant object. Similar to 1.3.4, we can say that $j : \Delta^1 \to S^\infty$ is a fibrant replacement in $sSet$. It follows from 1.4.1 that we have a functor of ∞-categories

$$\mathrm{Fun}(S^\infty, \mathcal{C}) \to \mathrm{Fun}(\Delta^1, \mathcal{C})$$

which is fully faithful. Also its essential image is spanned by those morphisms $f : \Delta^1 \to \mathcal{C}$ that send every morphism in Δ^1 into an equivalence in $\mathcal{C}$. Therefore, $f : X \to Y$ is an equivalence iff $e_f : \Delta^1 \to \mathcal{C}$ satisfies the above condition iff there exists a map from $S^\infty \to \mathcal{C}$ commuting the above diagram.
Second, we need to show that any two such solutions are homotopic. Due to-fully faithfulness of the above ∞-functor, such two solutions should be isomorphic in the homotopy category of $\mathrm{Fun}(S^\infty, \mathcal{C})$, implying that they are homotopic. □

Exercise 1.3.3. In [?] a functor of ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$ is said to be a *categorical equivalence* if and only if the induced functor $\mathfrak{C}[f]\colon \mathfrak{C}[\mathcal{C}] \to \mathfrak{C}[\mathcal{D}]$ is an equivalence of simplicial categories. Show that f is a categorical equivalence if and only if it is fully faithful and essentially surjective.

Solution. By definition, $f\colon \mathcal{C} \to \mathcal{D}$ is a categorical equivalence if and only if the induced functor $\mathfrak{C}[f]\colon \mathfrak{C}[\mathcal{C}] \to \mathfrak{C}[\mathcal{D}]$ is an equivalence of simplicial categories if and only if, by definition, the induced functor $h\mathfrak{C}[f]\colon h\mathfrak{C}[\mathcal{C}] \to h\mathfrak{C}[\mathcal{D}]$ on the homotopy level is an equivalence. Also, $h\mathcal{C} \simeq h\mathfrak{C}[\mathcal{C}]$ and this correspondence is functorial. Note that a ∞-functor f is defined to be fully faithful(or essentially surjective) if hf is. Thus, it suffices to show that hf is an equivalence iff it is fully faithful and essentially surjective which is obvious. □

Exercise 1.3.4. Let E denote the walking isomorphism (i.e. the 1-category with two objects and an isomorphism between them). Recall the definition of S^∞ from the previous exercise. Show that there is a canonical map $E \to S^\infty$ and that this is a categorical equivalence. In particular, for every ∞-category $\mathcal{C}$, the functor

$$\mathrm{Fun}(S^\infty, \mathcal{C}) \to \mathrm{Fun}(E, \mathcal{C})$$

is a categorical equivalence. (This is a very simple example of what an "internal rectification theorem" looks like.)

solution. Similar to the proof of Exercise 1.3.2, it is a consequence Exercise 1.4.1. We can identify E with $\{0,1\}$. The canonical map $E \to S^\infty$ is characterized by the identity map $\{0,1\} \to \{0,1\}$. It is categorical equivalence because we S^∞ is a Kan complex. For the categorical equivalence between $\mathrm{Fun}(S^\infty, \mathcal{C})$ and $\mathrm{Fun}(E, \mathcal{C})$, it suffices to show that the given ∞-functor is essentially surjective by the virtue of 1.3.3. We already know that the essential image is spanned by a functor $f : E \to \mathcal{C}$ that send every morphism in E to an equivalence in $\mathcal{C}$. Since there is only one morphism, $id : \{0,1\} \to \{0,1\}$ which must be sent to the identity map. Therefore, $\mathrm{Fun}(S^\infty, \mathcal{C}), \mathrm{Fun}(E, \mathcal{C})$ are categorical equivalent. □

Exercise 1.3.5. Let $\mathcal{C}$ be an ∞-category. Let S_0 be a collection of *objects* in $\mathcal{C}$. Let $\mathcal{C}_0$ be the smallest full subsimplicial set of $\mathcal{C}$ containing S_0 (explicitly, an n-simplex $\sigma\colon \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if for every morphism $\Delta^0 \to \Delta^n$ the composition $\Delta^0 \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0.) Show that $\mathcal{C}_0$ is an ∞-category. Furthermore, show that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ of simplicial sets is a fully faithful functor of ∞-categories.

solution. First, we show that $\mathcal{C}_0$ is ∞ category. It suffices to show that for every n and every $0 < i < n$, there exists a map $\Delta^n \to \mathcal{C}$ commuting the following diagram where $\sigma : \Lambda^n_i \to \mathcal{C}$ belongs to $\mathcal{C}_0$. Since $\mathcal{C}$ is

∞-category, there exists a map $\tilde{\sigma} : \Delta^n \to \mathcal{C}$ commuting the diagram. Now, it is enough to show that $\tilde{\sigma}$ belongs to $\mathcal{C}_0$. Consider the following commutative diagram.

$$\begin{tikzcd} \Delta^0 \arrow[r] \arrow[dr] & \Lambda^n_i \arrow[r, "\sigma"] \arrow[d, hook] & \mathcal{C} \\ & \Delta^n \arrow[ur, "\tilde{\sigma}"] & \end{tikzcd}$$

Note that the left triangular diagram commutes since maps from Δ^0 are canonically chosen. Therefore, $\tilde{\sigma}$ belongs to $\mathcal{C}_0$.
(Todo: Show that the inclusion is fully faithful)

□

Exercise 1.3.6. Let $\mathcal{C}$ be an ∞-category. Let S_0 be a collection of *morphisms* in $\mathcal{C}$, and suppose that S_0 is closed under composition, in the sense that for every 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

is $\mathcal{C}$, if f and g belong to S_0 then so does h. Let $\mathcal{C}_0$ be the smallest full subsimplicial set of $\mathcal{C}$ containing S_0 (explicitly, an n-simplex $\sigma\colon \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}_0$ if and only if for every morphism $\Delta^1 \to \Delta^n$ the composition $\Delta^1 \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0). Show that $\mathcal{C}_0$ is an ∞-category.

solution. Similar to the proof of Exercise 1.3.5, we show that $\mathcal{C}_0$ is ∞-category by diagram chasing. We replace Δ^0 in Exercise 1.3.5, by Δ^1 and show that the following diagram commutes for every n and every $0 < i < n$;

$$\begin{tikzcd} \Delta^1 \arrow[r] \arrow[dr] & \Lambda^n_i \arrow[r, "\sigma"] \arrow[d, hook] & \mathcal{C} \\ & \Delta^n \arrow[ur, "\tilde{\sigma}"] & \end{tikzcd}$$

This is trivial except $n = 2$. For $n = 2$, we get the following diagram.

$$\begin{tikzcd} \Delta^1 \arrow[r, "j"] \arrow[dr] & \Lambda^2_1 \arrow[r, "\sigma"] \arrow[d, hook] & \mathcal{C} \\ & \Delta^2 \arrow[ur, "\tilde{\sigma}"] & \end{tikzcd}$$

If $j(\Delta^1) = \{0 \to 1\}$ or $\{1 \to 2\}$, the above diagram commutes again. For $\tilde{\sigma}$ being in $\mathcal{C}_0$, we need to consider the case where such j does not exist. Namely, we need to show that $\tilde{\sigma} \circ j' : \Delta^1 \to \mathcal{C}$ belongs to S_0 where $j' : \Delta^1 \to \Delta^2$ is the canonical inclusion with the image $\{0 \to 2\}$. Suppose $\tilde{\sigma}(\Delta^2)$ is the following;

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

Note that $\tilde{\sigma} \circ j' : \Delta^1 \to \mathcal{C}$ corresponds to $X \xrightarrow{h} Z$. Since f, g are in S_0 by assumption and S_0 is closed under composition, $\tilde{\sigma} \circ j' : \Delta^1 \to \mathcal{C}$ belongs to S_0. Thus, $\mathcal{C}_0$ is an ∞-category. □

Exercise 1.3.7. Let $\mathcal{C}$ be an ∞-category. Show that the collection of equivalences in $\mathcal{C}$ is closed under composition, in the sense of the previous exercise. Let $\mathcal{C}^{\simeq}$ be the ∞-subcategory of $\mathcal{C}$ spanned by equivalences in $\mathcal{C}$. Show that $\mathcal{C}^{\simeq}$ is a Kan complex.

Proof. Let S_0 be a collection of equivalences in $\mathcal{C}$. Suppose we have 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "\phi"] & & Z \end{tikzcd}$$

where X, Y, Z are objects of $\mathcal{C}$ and $f : X \to Y, g : Y \to Z$ are equivalence in $\mathcal{C}$. Remind that $f : X \to Y$ is an equivalence if the induced map $hf : hX \to hY$ is an isomorphism in $h\mathcal{C}$. Since $g \circ f : X \to Z$ is homotopy equivalent to $\phi : X \to Z$ and $h(g \circ f) = hg \circ hf$ is an isomorphism, $\phi : X \to Z$ is an equivalence. Thus, S_0 is closed under composition.
Note that for any ∞-category $\mathcal{D}$, $\mathcal{D}$ is a Kan complex if and only if $\mathcal{D}$ is an ∞-groupoid. [**?**, Theorem 1.2.5.1] Clearly, $\mathcal{C}^{\simeq}$ is an ∞-groupoid because its homotopy category $h\mathcal{C}^{\simeq}$ is a groupoid. Therefore, $\mathcal{C}^{\simeq}$ is a Kan complex.

$\square$

1.4 Localization of ∞-categories

Exercise 1.4.1. Let $\mathcal{C}$ be an ∞-category (seen as a quasicategory). Let $\mathcal{C} \to \widetilde{\mathcal{C}}$ be a fibrant replacement for the Kan model structure on sSet. Show that $\widetilde{\mathcal{C}}$ enjoys the following universal property: for every ∞-category $\mathcal{D}$ the functor of ∞-categories

$$\mathrm{Fun}(\widetilde{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and its essential image is spanned by those morphisms $f\colon \mathcal{C} \to \mathcal{D}$ that send every morphism in $\mathcal{C}$ into an equivalence in $\mathcal{D}$. Thus, there is a categorical equivalence $\tilde{\mathcal{C}} \simeq \mathcal{C}[W^{-1}]$, where W denotes the collection of all arrows in $\mathcal{C}$. Deduce that if $\mathcal{C}$ is an ∞-category where every morphism is invertible, then $\mathcal{C}$ is categorically equivalent to a Kan complex.

solution. Let's take the Kan model structure on *sSet* where cofibrations are levelwise injective morphisms, fibrations are Kan fibrations and weak equivalence is given by weak homotopy equivalence. In this case we have the following property of inner homomorphism Fun;
Suppose that $i\colon A \to B$ is a cofibration and $p\colon X \to Y$ is a Kan fibration. Then, we have a natural Kan fibration;

$$q\colon \mathrm{Fun}(B, X) \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y)$$

In particular, if either i or p is a weak equivalence, then q is a trivial fibration. In order to apply this proposition, we need a few observation. First, the fibrant replacement $\mathcal{C} \to \tilde{\mathcal{C}}$ is indeed a cofibrant replacement because every object in *sSet* is cofibrant. Also, given $\mathcal{D}$, we can take the Kan complex $\mathcal{D}^{\simeq}$ spanned by equivalence in $\mathcal{D}$ as in 1.3.7. Since $\tilde{\mathcal{C}}$ is a Kan complex, we get the equivalence $\mathrm{Fun}(\tilde{\mathcal{C}}, \mathcal{D}) \simeq \mathrm{Fun}(\tilde{\mathcal{C}}, \mathcal{D}^{\simeq})$. Now we can apply the above proposition. Take $A = \mathcal{C}$, $B = \tilde{\mathcal{C}}$, $X = \mathcal{D}^{\simeq}$, and $Y = *$. Clearly, i is a weak equivalence. Thus we get the following,

$$\begin{tikzcd} \mathrm{Fun}(\tilde{\mathcal{C}}, \mathcal{D}) \arrow[r, "\simeq"] & \mathrm{Fun}(\tilde{\mathcal{C}}, \mathcal{D}^{\simeq}) \arrow[d, "q"] & \\ & \mathrm{Fun}(\mathcal{C}, \mathcal{D}^{\simeq}) \arrow[r, hook] & \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \end{tikzcd}$$

where q is a weak equivalence. It tells us that the composition is fully faithful and its essential image is $\mathrm{Fun}(\mathcal{C}, \mathcal{D}^{\simeq}$. Namely, it is spanned by those morphisms $f : \mathcal{C} \to \mathcal{D}$ that send every morphism in $\mathcal{C}$ into an equivalence in $\mathcal{D}$. In particular, if $\mathcal{C}$ is an ∞-category where every morphism is invertible, then $\mathcal{C} \simeq \tilde{\mathcal{C}}$ hence categorically equivalent to a Kan complex. $\square$

Exercise 1.4.2. Let $\mathcal{C}$ be an ∞-category and let S be a (small) collection of arrows in $\mathcal{C}$. Show that $\mathrm{h}(\mathcal{C}[S^{-1}]) \in \mathrm{Cat}$ is canonically equivalent to the 1-categorical localization of $\mathrm{h}(\mathcal{C})$ at $\overline{S}$, the collection of morphism which is the image of S via the canonical functor $\mathcal{C} \to \mathrm{h}(\mathcal{C})$.

Exercise 1.4.3. Let $\mathcal{C}$ be an ∞-category with finite limits and let S be a (small) collection of arrows in $\mathcal{C}$. Suppose that $\mathcal{C}$ is stable under pullbacks. Then the ∞-categorical localization $\mathcal{C}[S^{-1}]$ has finite limits and the localization functor $L\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ commutes with them.

1.5 Limits and colimits

Exercise 1.5.1. Let $\mathcal{S}$ be the ∞-category of spaces and let X be an object in $\mathcal{S}$. Using [**?**, Theorem 4.2.4.1] show that the colimit of the diagram

$$* \longleftarrow X \longrightarrow *$$

can be canonically identified with $\Sigma(X)$.

Now fix two points $p, q\colon * \to X$. Show that the limit of the diagram

$$* \xrightarrow{\ p\ } X \xleftarrow{\ q\ } *$$

can be canonically identified with the path space $\mathrm{Path}_X(p,q)$.

Exercise 1.5.2. $\star$ Prove the following variation of Quillen's theorem A: let $1 \leq n \leq \infty$ and let $\mathcal{C}$ be an $(n,1)$-category. Let $G\colon I \to J$ be an ∞-functor between ∞-categories. Let $F\colon J \to \mathcal{C}$ be any other ∞-functor. Suppose that for every $j \in J$ and any $i \in I_{/j} \coloneqq I \times_J J_{/j}$ one has

$$\pi_m(I_{/j}, i) = 0$$

for all $0 \leq m \leq n-1$ (the above homotopy group is understood to be the homotopy group of the enveloping groupoid of $I_{/j}$). Then F admits a limit if and only if $F \circ G$ admits a limit, in which case they coincide.

Remark 1.5.3. The above version of Quillen's theorem A appears in [**?**] for $n = 1$ and in [**?**, 4.1.3.1] for $n = \infty$.

Exercise 1.5.4. Let $\mathbf{\Delta}_s$ denote the subcategory of $\mathbf{\Delta}$ spanned by all the objects and only the monomorphisms between them. For $n \geq 1$, let $\mathbf{\Delta}_s^{\leq n}$ be the full subcategory of $\mathbf{\Delta}_s$ spanned by the objects $1, 2, \dots, n$. Prove that for every $n \geq 1$ and every $k \geq 0$ the enveloping groupoid of $(\mathbf{\Delta}_s^{\leq n})_{/n+k}$ is equivalent to the wedge of a certain number $N_{n,k}$ of $(n-1)$-spheres.[3]

Exercise 1.5.5. A useful consequence of Quillen's theorem A is the following: let I be a weakly contractible ∞-category, by which we mean that the enveloping groupoid of I is weakly contractible. Let $\mathcal{C}$ be an ∞-category and let $x \in \mathcal{C}$ be an object in $\mathcal{C}$. Let $c_x\colon I \to \mathcal{C}$ be the constant diagram associated to x. Then prove that both the limit and the colimit of c_x exists and coincides with x.

The above result is false if I is not weakly contractible. Construct a counterexample by choosing $\mathcal{C} = \mathcal{S}$, $I = \{\bullet \rightrightarrows \bullet\}$ and $x = *$, the final object of $\mathcal{S}$. Nevertheless, show that keeping the same I and the same x, the result is again true for $\mathcal{C} = \mathrm{Set}$. What happens in the ∞-category of n-homotopy types $\mathcal{S}^{\leq n}$ for general n?

Exercise 1.5.6. $\star$ Let K be a simplicial set and let $F\colon K^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ be an ∞-functor. Let $\mathcal{C}$ be a presentable ∞-category and let $\Delta_{\mathcal{C}}\colon K^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the constant ∞-functor associated to F. Let $\varphi\colon \Delta_{\mathcal{C}} \to F$ be a natural transformation in $\mathrm{Fun}(K^{\mathrm{op}}, \mathcal{P}\mathrm{r}^{\mathrm{L}})$. We let

$$\Phi\colon \mathcal{C} \to \varprojlim F$$

[3] It should be possible to determine these numbers. We certainly have $N_{n,0} = 1$ and $N_{n,1} = 3$.

be the induced functor. For every $x \in K$, the functor $\varphi_x \colon \mathcal{C} \to F(x)$ admits a right adjoint, which we denote $\psi_x \colon F(x) \to \mathcal{C}$. Show that there exists an ∞-functor

$$\overline{\Psi} \colon \varprojlim F \to \mathrm{Fun}(K, \mathcal{C})$$

which informally sends $Y = \{Y_x\}_{x \in K} \in \varprojlim F$ to the diagram $\overline{\Psi}(Y) \colon K \to \mathcal{C}$ given by

$$\overline{\Psi}(Y)(x) = \psi_x(Y_x).$$

Prove moreover that the composition

$$\varprojlim F \xrightarrow{\overline{\Psi}} \mathrm{Fun}(K, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

can be canonically identified with a right adjoint for Φ.

1.6 Left and right fibrations

Exercise 1.6.1. Let X be a connected Kan complex and let F be any other Kan complex. Let us further fix a point $x \in X$. Let $\mathrm{LF}_x(X; F)$ be the full subcategory of left fibrations $\mathrm{LF}(X)$ over X whose homotopy fiber at x is equivalent to F. Let $\mathrm{B}(\mathrm{hAut}(F))$ be the classifying space of the simplicial group of homotopy automorphisms of F. Show that there is a canonical equivalence of ∞-categories

$$\mathrm{LF}_x(X; F) \simeq \mathrm{Fun}(X, \mathrm{B}(\mathrm{hAut}(F))).$$

1.7 Cartesian and coCartesian fibrations

Exercise 1.7.1. Let $\mathcal{C}$ be an ∞-category and let $X \in \mathcal{C}$ be an object. Let $f \colon U \to X$ and $g \colon V \to X$ be two morphisms in $\mathcal{C}$. For every 2-simplex $\sigma \colon \Delta^2 \to \mathcal{C}$ such that $d_0(\sigma) = f$ and $d_1(\sigma) = g$, show that there is a pullback square in $\mathcal{S}$:

$$\begin{tikzcd}
\mathrm{Path}_{\mathrm{Map}_{\mathcal{C}}(U,X)}(f, d_2(\sigma)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}_{/X}}(f, g) \arrow[d] \\
* \arrow[r, "d_2(\sigma)"] & \mathrm{Map}_{\mathcal{C}}(U, V).
\end{tikzcd}$$

(Hint: Use [**?**, Propositions 2.1.2.1 and 2.4.4.2].)

1.8 Adjunctions

Exercise 1.8.1. Let $\mathcal{C}$ be an ∞-category with a zero object 0. Suppose that for every object $X \in \mathcal{C}$ the span

$$0 \longleftarrow X \longrightarrow 0$$

has both a limit $\Omega(X)$ and a colimit $\Sigma(X)$. Construct in an explicit way ∞-functors $\Sigma, \Omega \colon \mathcal{C} \to \mathcal{C}$ informally given by $X \mapsto \Sigma(X)$ and $X \mapsto \Omega(X)$, respectively. Show that Σ and Ω are adjoint by explicitly constructing a fibration $\mathcal{D} \to \Delta^1$ which is both Cartesian and coCartesian.

Exercise 1.8.2. Let $F \colon \mathcal{C} \to \mathcal{D}$ be an ∞-functor. Show that the following statements are equivalent:

1. F has a right adjoint $G \colon \mathcal{D} \to \mathcal{C}$;
2. for every $Y \in \mathcal{D}$ there exists an object $X \in \mathcal{C}$ and a morphism $\varepsilon_X \colon F(X) \to Y$ such that for every other $X' \in \mathcal{C}$ the canonical composition

   $$\mathrm{Map}_{\mathcal{C}}(X', X) \xrightarrow{f} \mathrm{Map}_{\mathcal{D}}(f(X'), f(X)) \xrightarrow{\varepsilon_{X*}} \mathrm{Map}_{\mathcal{D}}(f(X'), Y)$$

   is a weak homotopy equivalence.

1.9 Stable ∞-categories

Exercise 1.9.1. Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full stable subcategory of $\mathcal{C}$. Let $S := \{f\colon X \to Y \in \mathcal{C} \mid \mathrm{cofib}(f) \in \mathcal{D}\}$. Show that the ∞-categorical localization $\mathcal{C}[S^{-1}]$ is a stable ∞-category.

Exercise 1.9.2. It is shown in [**?**] that $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is a presentable ∞-category. Prove directly that cofibers in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ exist.

Chapter 2

Derived rings

2.1 Derived rings

Exercise 2.1.1. Show that a discrete commutative ring A over k is finitely presented if and only if its associated corepresentable functor

$$\mathrm{Hom}_{\mathrm{CAlg}_k}(A, -) \colon \mathrm{CAlg}_k \to \mathrm{Set}$$

commutes with filtered colimits.

solution. For any commutative k algebra A and filtered direct limit $B = \varinjlim_i B_i$, we get the following canonical map

$$\varinjlim_i \mathrm{Hom}(A, B_i) \to \mathrm{Hom}(A, \varinjlim_i B_i) = \mathrm{Hom}(A, B)$$

If $A = k$, then this map is isomorphism by the definition. Also, both functors $\varinjlim_i \mathrm{Hom}(-, B_i)$ and $\mathrm{Hom}(-, \varinjlim_i B_i)$ commutes with finite direct sum, this map is isomorphism when $A = k^n$ for any $n > 0$. Let's assume that A is finitely presented(i.e. there exists an exact sequence $k^n \to k^m \to A$ for some $n, m > 0$). Since two functors introduced above are left exact, the five lemma tells us that

$$\varinjlim_i \mathrm{Hom}(A, B_i) \to \mathrm{Hom}(A, \varinjlim_i B_i)$$

is bijective. Thus, A is compact.
Conversely, assume that A is a compact object. Since A can be written as a filtered colimit of finitely presented k algebras $\{A_i\}$, we get the following bijection.

$$\varinjlim_i \mathrm{Hom}(A, A_i) \to \mathrm{Hom}(A, \varinjlim_i A_i) = \mathrm{Hom}(A, A)$$

In particular, the identity map $A \to A$ factors through $A_i \to A$ for sufficiently large i. It implies that A is a direct summand of finitely presented k algebra A_i for such i, hence fintely presented.

□

Exercise 2.1.2. Let $A \in \mathrm{sCAlg}_k$ and let $M \in A\text{-Mod}^{\leq 0}$. Show that the diagram

$$\begin{tikzcd} \mathrm{Sym}_A(M) \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & \mathrm{Sym}_A(M[1]) \end{tikzcd}$$

is a (homotopy) pushout square (where the two maps $\mathrm{Sym}_A(M) \to A$ are both classified by the zero map $M \to A$, and where both the maps $A \to \mathrm{Sym}_A(M[1])$ are the structure morphisms).

solution. Let Sym be a left adjoint functor of the natural forgetful fuctof $F : A-\mathrm{Alg} \to A\text{-Mod}$. Since $A\text{-Mod}^{\leq 0} \subset A\text{-Mod}$ which is a stable ∞ category, we have a pushout diagram as follows;

$$\begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & M[1] \end{tikzcd}$$

Since $\mathrm{Sym} : A\text{-Mod} \to A-\mathrm{Alg}$ preserves colimits, the image of the above diagram under Sym becomes a pushout diagram. By construction, this image is as follows.

$$\begin{tikzcd} \mathrm{Sym}_A(M) \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & \mathrm{Sym}_A(M[1]) \end{tikzcd}$$

$\square$

Exercise 2.1.3. Let $A \in \mathrm{sCAlg}_k$ and let $M \in A\text{-Mod}^{\leq 0}$. Let $A \oplus M$ denote the split square-zero extension of A by M. Show that the diagram

$$\begin{tikzcd} A \oplus M \arrow[r] \arrow[d] & A \arrow[d, "d_0"] \\ A \arrow[r, "d_0"] & A \oplus M[1] \end{tikzcd}$$

is a homotopy pullback, where $d_0 \colon A \to A \oplus M[1]$ is the morphism classifying the zero derivation.

Solution. Note that $A \oplus M$ is the image of M under $\Omega^\infty : A\text{-Mod} \to \mathrm{sCAlg}_k$. Since $d_0 \colon A \to A \oplus M[1]$ is the morphism classifying the zero derivation, we get the following pullback square(because of stability) in A-Mod

$$\begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & M[1] \end{tikzcd}$$

Since the functor Ω^∞ preserves small limit, it preserves pullback squares. Therefore, the above diagram is a homotopy pullback square.

$\square$

2.2 Modules

Exercise 2.2.1. Let A be a discrete commutative ring over k. Show that $M \in A\text{-Mod}^\heartsuit$ is finitely generated if and only if its associated corepresentable functor

$$\mathrm{Hom}_{A\text{-Mod}^\heartsuit}(M, -) \colon A\text{-Mod}^\heartsuit \to \mathrm{Set}$$

commutes with filtered colimits of monomorphisms.

Exercise 2.2.2. Let A be a discrete commutative ring over k. Show that $M \in A\text{-Mod}^\heartsuit$ is finitely presented if and only if its associated corepresentable functor

$$\mathrm{Hom}_{A\text{-Mod}^\heartsuit}(M, -) \colon A\text{-Mod}^\heartsuit \to \mathrm{Set}$$

commutes with filtered colimits.

solution. (Todo: not sure) First, assume that $M \in A\text{-Mod}^{\heartsuit}$ is finitely presented. Consider the inclusion functor $i : A\text{-Mod}^{\heartsuit} \to A - Mod^{cn}$. Then $i(M)$ is perfect A module by the characterizaion of perfect modules, hence compact object in $A - Mod^{cn}$. Since i preserves filtered colimit, the right adjoint functor $\tau_{\leq 0} : A - Mod^{cn} \to A\text{-Mod}^{\heartsuit}$ preserves compact objects. Thus, $M = \tau_{\leq 0}(i(M))$ is a compact object in $A\text{-Mod}^{\heartsuit}$.
Converse is trivial? □

2.3 Cotangent complex

Exercise 2.3.1. Compute the cotangent complex of the following morphisms:

1. $k \to k[\varepsilon]/(\varepsilon^2)$, $\deg(\varepsilon) = 0$;
2. $k[X,Y] \to k[X,Y]/(Y^3 - X^2)$;
3. $k \to k[X,Y]/(Y^3 - X^2)$.

Solution1.

Lemma. *Let $f : A \to B$ be a surjective ring map whose kernel I is generated by a regular sequence. Then, $\mathbb{L}_{B/A}$ is quasi isomorphic to $I/I^2[1]$*

Since ε is of degree 0 we can apply the above lemma to the map $k[\varepsilon] \to k[\varepsilon](\varepsilon^2)/\varepsilon^4$. Then we get $\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k[\varepsilon]} = k[\varepsilon]\varepsilon^2/\varepsilon^4[1]$ by considering the map. Note that $\mathbb{L}_{k[\varepsilon]/k} = k[\varepsilon] \otimes_k k$ because $k[\varepsilon] = \mathrm{Sym}_k(k)$. Now, let's consider the sequence of map $k \to k[\varepsilon] \to k[\varepsilon]/\varepsilon^2$ where the first map is canonical inclusion and the second one is given as above. It induces a homotopy cofiber sequence

$$\mathbb{L}_{k[\varepsilon]/k} \otimes_{k[\varepsilon]} k[\varepsilon]/\varepsilon^2 \to \mathbb{L}_{k[\varepsilon]/\varepsilon^2/k} \to \mathbb{L}_{k[\varepsilon]/\varepsilon^2/k[\varepsilon]}$$

By the induced long exact sequence of the above homotopy cofiber, we get the following exact sequence.

$$\cdots \to 0 \to H^{-1}(\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k}) \to (\varepsilon^2)k[\varepsilon]/(\varepsilon^4) \xrightarrow{\phi} < d\varepsilon > k[\varepsilon]/\varepsilon^2 \to H^0(\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k}) \to 0 \to \cdots$$

where ϕ is the connecting homomorphism which is defined as a derivation. (i.e. $\phi(p(\varepsilon)) = d(p(\varepsilon))$ where $p(\varepsilon) \in k[\varepsilon](\varepsilon^2)/(\varepsilon^4)$ and d is usual derivative. So, we can say that $\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k} \cong (\varepsilon^2)k[\varepsilon]/(\varepsilon^4) \xrightarrow{\phi} k[\varepsilon]/\varepsilon^2$. Moreover, we can compute $H^{-1}(\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k})$ and $H^0(\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k})$ which corresponds to $\ker\phi$ and $\mathrm{coker}\phi$, respectively.

$$H^{-1}(\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k}) = \ker\phi =< \varepsilon^3 > k[\varepsilon](\varepsilon^2)/(\varepsilon^4) \cong k,\ H^0(\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k}) = \mathrm{coker}\phi =< d\varepsilon > k[\varepsilon]/\varepsilon^2/(2\varepsilon d\varepsilon) \cong k$$

Remark 2.3.2. For the later use, especially classifying the square zero extensions, it would be convenient to take a cofibrant model for $\mathbb{L}_{k[\varepsilon]/\varepsilon^2/k}$. (i.e.quasi-free $k[\varepsilon]/\varepsilon^2$-modules.)

Remark 2.3.3. We can play the same game for $k[\varepsilon]\varepsilon^n$ for any $n > 0$. However, the result is different. $\mathrm{H}^{-1}(\mathbb{L}_{k[\varepsilon]/\varepsilon^n/k}) \cong k[\varepsilon](\varepsilon^{n+1})/\varepsilon^{2n}$ and $\mathrm{H}^0(\mathbb{L}_{k[\varepsilon]/\varepsilon^n/k}) \cong < d\varepsilon > k[\varepsilon]/(\varepsilon^n)/\varepsilon^{n-1}d\varepsilon$ □

Solution2. We can apply the above lemma to the map $k[X,Y] \to k[X,Y]/(Y^3 - X^2)$. So,

$$\mathbb{L}_{k[X,Y]/(Y^3-X^2)/k[X,Y]} = k[X,Y](Y^3 - X^2)/(Y^3 - X^2)^2[1]$$

□

Solution3.

Lemma. *Let A, B be Tor-independent R algebra where R is a commutative ring. Then the cotangent complex $\mathbb{L}_{A\otimes_R B/R}$ of $A \otimes_R B$ over R is $\mathbb{L}_{A/R} \otimes_R B \oplus \mathbb{L}_{B/R} \otimes_R A$*

Let's consider a sequence of maps $k \xrightarrow{i} k[X,Y] \to k[X,Y]/(Y^3 - X^2)$ where i is canonical inclusion. It gives the following homotopy cofiber sequence

$$\mathbb{L}_{k[X,Y]/k} \otimes_{k[X,Y]} k[X,Y]/(Y^3 - X^2) \to \mathbb{L}_{k[X,Y]/(Y^3-X^2)/k} \to \mathbb{L}_{k[X,Y]/(Y^3-X^2)/k[X,Y]}$$

The last term is given in solution 2. Since $k[X,Y] = k[x] \otimes_k k[Y]$, we can apply the lemma to get the first term. So, $\mathbb{L}_{k[X,Y]/k} \otimes_{k[X,Y]} k[X,Y]/(Y^3 - X^2) =< dX, dY > k[X,Y]/(Y^3 - X^2)$. Similar to the first problem , we get

$$\mathbb{L}_{k[X,Y]/(Y^3-X^2)/k} = \cdots \to 0 \to k[X,Y](Y^3 - X^2)/(Y^3 - X^2)^2 \xrightarrow{\phi} < dX, dY > k[X,Y]/(Y^3 - X^2) \to 0 \to \cdots$$

where $\phi(f) = df$ for any $f \in k[X,Y](Y^3 - X^2)/(Y^3 - X^2)^2$
Moreover, $\mathrm{H}^{-1}(\mathbb{L}_{k[X,Y]/(Y^3-X^2)/k}) = \ker\phi \cong 0$, $\mathrm{H}^0(\mathbb{L}_{k[X,Y]/(Y^3-X^2)/k}) = \mathrm{coker}\phi \cong < dX, dY > k[X,Y]/(y^3 - X^2)/(3Y^2dY - 2XdX)$ ☐

Exercise 2.3.4. Find all the square-zero extensions (up to homotopy) of $R \coloneqq k[\varepsilon]/(\varepsilon^2)$ by $k \simeq R/(\varepsilon)$. What happens if we replace k by $k[n]$, $n \geq 0$?

Solution. We could work with the cotangent complex of $k \to k[\varepsilon]/(\varepsilon^2)$, as computed in Exercise 2.3.1. Instead, we work straight from the definition, in order to get a more explicit understanding of the extensions. Recall that square zero extensions up to homotopy are:

$$\pi_0 \operatorname{Map}_{cdga_k^{\leq 0}/k[\varepsilon]/(\varepsilon^2)} \big(k[\varepsilon]/(\varepsilon^2), k[\varepsilon]/(\varepsilon^2) \oplus k[1]\big). \tag{2.3.1}$$

Note that mapping spaces are *not* homotopy invariant; in order to obtain the correct answer, we need to take a cofibrant replacement of the first variable and a fibrant replacement of the second, in the category $cdga_k^{\leq 0}/k[\varepsilon]/(\varepsilon^2)$. Recall that the model structure on $cdga_k$ is obtained via transfer from the model structure on $Chain_k$; in particular:

- Fibrations are the same as those of the underlying complexes, i.e. the degree-wise surjections. All objects are fibrant.
- Cofibrations $f : A \to B$ are the morphisms such that B is quasi-free over A. The cofibrant objects are cdga's which are quasi-free over k.

Therefore, to describe the square-zero extensions given by 2.3.1, it suffices to take a k-free resolution of $k[\varepsilon]/(\varepsilon^2)$. This is accomplished by:

$$0 \longrightarrow k[\varepsilon] \xrightarrow{\varepsilon^2} k[\varepsilon] \longrightarrow 0.$$

Of course, one needs to check that this gives indeed a cdga.

- For $|a| = |b| = 0$, ab is ring multiplication in $k[\varepsilon]$.
- For $|a| = 0, |b| = 1$, ab is ring multiplication in $k[\varepsilon]$.
- For $|a| = |b| = 1$, $|ab| = 2$, so the only possibility is $ab = 0$.
- Let's check that multiplication by ε^2 satisfies the Leibniz rule. We do this for $|a| = 0, |b| = 1$:

$$\varepsilon^2(a) \cdot b + (-1)^{|a|} a\varepsilon^2(b) = 0 \cdot b + (-1)^0 a\varepsilon^2 b = \varepsilon^2(ab).$$

With this cofibrant model, we compute 2.3.1. These are maps between cdga's, and we identify them by their components:

$$\begin{tikzcd}
k[\varepsilon] \arrow[r, "\varepsilon^2"] \arrow[d, "\eta"'] & k[\varepsilon] \arrow[d, "p"] \\
k \arrow[r, "0"] & k[\varepsilon]/(\varepsilon^2).
\end{tikzcd}$$

But, since we are working in the comma category of cdga's over $k[\varepsilon]/(\epsilon^2)$, the map p is forced to be the canonical projection $k[\varepsilon] \to k[\varepsilon]/(\varepsilon^2)$. It follows that the only freedom is in choosing η. The constraints on η are given by the fact that a morphism of cdga's must commute with the cdga multiplication, in the sense that, for $|f| = 0$ and $|g| = 1$, $\eta(fg) = p(f)\eta(g)$. In particular:

$$\eta(\varepsilon) = \eta(1 \cdot \varepsilon) = p(\varepsilon) \cdot \eta(1) = \varepsilon \cdot \eta(1) = 0,$$

because ϵ acts by 0 on $k = k[\varepsilon]/(\varepsilon)$. Similarly, $\eta(\epsilon^i) = 0$ for all $i > 0$. It follows that, if $|g| = 1$ with $g = \alpha_0 + \alpha_1\varepsilon + \dots$, then $\eta(g) = \lambda\alpha_0$, for some $\lambda \in k$. Thus, elements of 2.3.1 are classified by $\lambda \in k$.[1]

To see the square-zero extensions explicitly, we need to compute the homotopy fiber products:

$$\begin{tikzcd}
A^\eta \arrow[r, dashed] \arrow[d, dashed] & k[\varepsilon]/(\varepsilon^2) \arrow[d, "d_\eta"] \\
k[\varepsilon]/(\varepsilon^2) \arrow[r, "d_0"] & k[\varepsilon]/(\varepsilon^2) \oplus k[1].
\end{tikzcd}$$

Homotopy fiber products *are* homotopy invariant, so this is the same as computing the homotopy fiber products:

$$\begin{tikzcd}
& k[\varepsilon] \arrow[rr, "\varepsilon^2"] \arrow[dd, "\eta"] & & k[\varepsilon] \arrow[dd, "p"] \\
A \arrow[ur, dashed] \arrow[rr, dashed, "\phi"] \arrow[dd, dashed] & & B \arrow[ur, dashed] \arrow[dd, dashed] & \\
& k \arrow[rr, "0"] & & k[\varepsilon]/(\varepsilon^2) \\
k[\varepsilon] \arrow[ur, "0"] \arrow[rr, "\varepsilon^2"] & & k[\varepsilon] \arrow[ur, "p"] &
\end{tikzcd}$$

The advantage of using this model is that the maps on the right face of the cube are degree-wise surjections, hence fibrations, so it suffices to compute the naive fiber product. This gives:

$$A = k[\varepsilon] \oplus (\varepsilon)k[\varepsilon],$$
$$B = k[\varepsilon] \oplus (\varepsilon^2)k[\varepsilon],$$
$$\phi = (\varepsilon^2, \varepsilon^2).$$

Note first that ϕ is injective, so the homotopy fiber product is (cohomologically) concentrated in degree 0. In other words, it is quasi-isomorphic as cdga to:

$$0 \to 0 \to k[\varepsilon]/(\varepsilon^2) \oplus k \to 0.$$

It remains to see how the choice of $\lambda \in k$ determines the product structure on $k[\varepsilon]/(\varepsilon^2) \oplus k$. The claim is that we get:

$$(a + b\varepsilon, c) \cdot_\lambda (a' + b'\varepsilon, c') = \big(aa' + (a'b + ab')\varepsilon, \lambda bb' + ac' + ca'\big)$$

Note that for $\lambda = 1$ this is just the ring multiplication in $k[\varepsilon]/(\varepsilon^3)$, so we recover the classical square-zero extension $k[\varepsilon]/(\varepsilon^3) \to k[\varepsilon]/(\varepsilon^2)$. (Todo: I actually don't understand how we get this product structure in degree 0, given that the only freedom is in the map η, which goes between the degree -1 parts.) □

[1] We point out that, had we not used a cofibrant replacement for $k[\varepsilon]/(\varepsilon^2)$, we would obtain $(0, \mathrm{id})$ as the only map of cdga's; this corresponds to the zero derivation. This answer is clearly wrong, as it doesn't account for the square-zero extension $k[\varepsilon]/(\varepsilon^3) \to k[\varepsilon]/(\varepsilon^2)$.

Chapter 3

Derived stacks