

Derived Algebraic Geometry Seminar: Upenn 2016

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Introduction

This contains notes livetexed for the Derived Algebraic Geometry Seminar currently being held at the University of Pennsylvania math department in the 2016-17 academic year. We are following Mauro Porta's plan to introduce derived algebraic geometry, and then to look at derived geometric objects with extra structure (initially we will be looking at the case of symplectic and Poisson structures).

This is a draft and errors should be expected.

Chapter 1

∞ -category theory

Talk by Mauro Porta.

1.1 Why ∞ -categories?

Our main reason for studying ∞ -categories in this seminar is that derived schemes form an ∞ -category. Some other applications of ∞ -categories are the following.

1. Formal moduli problems over a field k of characteristic 0 are equivalent to dgLie_k , but this is an equivalence of ∞ -categories. We can see explicitly why this equivalence is plausible. For F a formal moduli problem, $T_x F[1]$ is a dgLie algebra. Conversely, Maurer-Cartan elements on the RHS determine $F(k[\epsilon])$, i.e. infinitesimal formal moduli problems. Brackets then allow the complete recovery of F .
2. The ∞ -category of rational homotopy types is equivalent to that of dgLie algebras over \mathbb{Q} , concentrated in positive degrees:

$$S_*^{\mathrm{rat}} \cong \mathrm{dgLie}_{\mathbb{Q}}^{\geq 1}$$

This statement is related to item 1: Lurie gives nice proof using formal moduli problems, see [6].

3. To $X \in \mathrm{Sch}_k$, we associate its derived category of quasi-coherent sheaves, $D(X) = D(\mathrm{QCoh}(X))$. It's a powerful invariant of X , especially when X is not smooth. For example, it contains the cotangent complex and dualizing complex, $\mathbb{L}_X, \omega_X \in D(X)$, which are not necessarily bounded if X is not smooth.

The problem is that we cannot reconstruct $D(X)$, the derived category in the classical sense, by patching: $D(X) \not\cong \lim_{\{U\} \text{ Zariski cover}} D(U)$. For example, take $X = \mathbb{P}_k^1$, and its standard cover by 2 open affines U_0, U_1 . We show that the functor:

$$D(\mathbb{P}^1) \rightarrow D(U_0) \times_{D(U_{01})} D(U_1)$$

is not faithful, by exhibiting a morphism in $D(\mathbb{P}^1)$ which gets mapped to 0. Start from the observation that morphisms from the structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ are the same as sections of the target sheaf, which implies:

$$\mathbb{R}(\mathrm{Hom})(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)[1]) \cong \mathbb{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(-2)[1]).$$

This complex has nontrivial cohomology in degree 0:

$$H^0 \mathbb{R}\Gamma(\mathcal{O}_{\mathbb{P}^1}(-2)[1]) \cong H^1(\mathcal{O}_{\mathbb{P}^1}(-2)) \cong k. \quad (1.1.1)$$

However, when passing to the affine patches, $D(U_i) \simeq D(k[T] - \text{Mod})$, and the complexes corresponding to the restrictions of $\mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{O}_{\mathbb{P}^1}(-2)[1]$ are the following.

$$0 \longrightarrow 0 \longrightarrow k[T] \longrightarrow 0$$

$$0 \longrightarrow k[T] \longrightarrow 0 \longrightarrow 0$$

As such, there are no non-zero morphisms between the restrictions. Equivalently, when restricting to affine opens, the first cohomology in equation 1.1.1 is 0.

On the other hand, we will see that the ∞ -derived category of X (which we temporarily denote by $L_{\text{qcoh}}(X)$) can be patched using the homotopy fiber product:

$$L_{\text{qcoh}}(\mathbb{P}_k^1) \simeq L_{\text{qcoh}}(U_0) \times_{L_{\text{qcoh}}(U_{01})} L_{\text{qcoh}}(U_1).$$

4. Let \mathcal{M}_{ell} be the moduli stack of elliptic curves, i.e. the functor F sending $\text{Spec}(A)$ to the classes of elliptic curves over $\text{Spec}(A)$. It is not a sheaf, because two elliptic curves can become isomorphic after a base extension. The problem here is that we were trying to take $F : \mathcal{A}ff^{\text{op}} \rightarrow \text{Set}$, and we can't talk about isomorphisms in Set . Classically one solves this problem by replacing sets by groupoids, which are equivalent to 1-homotopy types.

$$\begin{array}{ccc} & & \mathcal{G}pd \cong \mathcal{S}^{\leq 1} \\ & \nearrow \text{stacks} & \downarrow \\ \mathcal{A}ff^{\text{op}} & \xrightarrow{\text{naive moduli}} & \text{Set} \cong \mathcal{S}^{\leq 0} \\ & \text{problems} & \end{array}$$

We can define higher stacks by extending the tower to higher homotopy types, and ultimately to the category of spaces.

$$\begin{array}{ccc} & & \mathcal{S} \\ & \nearrow \text{higher stacks} & \downarrow \\ & & \vdots \\ & \nearrow \text{stacks} & \downarrow \\ \mathcal{A}ff^{\text{op}} & \xrightarrow{\text{naive moduli}} & \mathcal{S}^{\leq 1} \\ & \text{problems} & \downarrow \\ & & \text{Set} \cong \mathcal{S}^{\leq 0} \end{array}$$

In later talks, we'll see that the perfect complexes $\mathcal{P}erf$ form an ∞ -stack which doesn't factor through finite homotopy types.

1.2 Three ways of working with ∞ -categories

To be attempted in order of desperation:

1. Reason model-independently to get a clean proof. The trick is that there are key statements (not proven model independently; some are proven by Lurie and can be found in [5]) which behave like a “non-minimal set of axioms”. One should learn a roadmap to [5], in order to know where to find these statements.

2. Internal rectification. Cut the number of homotopies necessary to define the object. Example: an ∞ -category with products, see it as a symmetric monoidal category with products. $Mon_{E_1}(\mathcal{C}) \simeq Fun^\times(\Delta^{op}, \mathcal{C}) \rightarrow Fun(\Delta^{op}, \mathcal{C})$. The reference is [7], 4.1.2.6.
3. Try a “real rectification” result, i.e. work with a model-categorical presentation. For example, take \mathcal{S} , the ∞ -category of spaces, fix $x, y \in X$, want diagram. $\mathcal{S} = \infty(sSet_{Kan})$. Theorem 4.2.4.1 in [5] says that, in this situation, ∞ -categorical limits correspond to homotopy limits. We use this theorem to show that $Path_X(x, y)$ is the ∞ -limit of the diagram.

Rectification: something is defined up to homotopy, and we try to reduce the necessary homotopies. Suppose we have \mathcal{M} , compare $Fun(\Delta^2, \mathcal{M})$ to $Fun(\Delta^2, \infty\mathcal{M})$. The first one gives 3 objects and 3 morphisms, while the other gives a homotopy between composition and the other map. The theorem is that you can forget the homotopy, and just remember the data on the LHS. (HTT 4.2.4.4)

In what follows we give examples where we can get by with procedure 1.

Definition 1.2.1. An ∞ -category is a simplicial set \mathcal{C} such that all inner horns have fillers. In other words, for all $0 < i < n$, the dotted arrow in the following diagram exists.

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^n & & \end{array}$$

Note that this achieves what we want: inner horn fillings act as composition of morphisms, but this composition is not unique. “Higher Topos Theory is the book where all of category theory is carried out without ever talking about composition.” A few problems arise from here:

1. How do we define Yoneda? A morphism $X \rightarrow Y$ is supposed to determine a morphism $h_X \rightarrow h_Y$ by composition, which is not well-defined.
2. Let \mathcal{C} be an ∞ -category. We want $f : x \rightarrow y$ in \mathcal{C} to determine a functor $f_* : \mathcal{C}_{/X} \rightarrow \mathcal{C}_{/Y}$ between over-categories, where, morally speaking, $g : Z \rightarrow X$ is sent to the composition $f \circ g$. Again, this composition is not well-defined.

To the rescue comes Corollary 2.4.7.12 in [5].

Theorem 1.2.2. Let $f : \mathcal{C} \rightarrow \mathcal{D}$ be an ∞ -functor between ∞ -categories. Then the projection

$$\mathcal{P} : Fun(\Delta^1, \mathcal{D}) \times_{Fun(\{1\}, \mathcal{D})} \mathcal{C} \rightarrow Fun(\{0\}, \mathcal{D})$$

is a **cartesian fibration**. Moreover, a morphism in the source is **\mathcal{P} -cartesian** iff its image in \mathcal{C} is an **equivalence**.

Note that the ∞ -functors $Fun(\mathcal{C}, \mathcal{D})$ are nothing but the internal Hom in $sSet$.

$$Fun(\mathcal{C}, \mathcal{D})_n = sSet(\mathcal{C} \times \Delta^n, \mathcal{D})$$

It’s standard to prove that, if \mathcal{C}, \mathcal{D} are ∞ -categories, then so is $Hom(\mathcal{C}, \mathcal{D})$.

We will spend much of section 1.3 defining the terms in bold in Theorem 1.2.2. In Example 1.3.5, we will use Theorem 1.2.2 to obtain the desired pushforward map between overcategories.

1.3 Equivalences and Cartesian fibrations

Definition 1.3.1. $g : x \rightarrow y$ in \mathcal{C} is an **equivalence** if any of the following equivalent conditions hold.

- (a) The outer horn which maps $01 \mapsto g$ and $02 \mapsto 1_x$ has a filler.

$$\begin{array}{ccc} \Lambda_0^2 & \xrightarrow{g, 1_x} & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^2 & & \end{array}$$

Morally speaking, the restriction of the dotted arrow to the 12 face is the right inverse of g .

Moreover, the outer horn which maps $12 \mapsto g$ and $02 \mapsto 1_y$ has a filler.

$$\begin{array}{ccc} \Lambda_2^2 & \xrightarrow{1_y, g} & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ \Delta^2 & & \end{array}$$

Morally speaking, the restriction of the dotted arrow to the 01 face is the left inverse of g .

- (b) The same as variant a, but with higher homotopies included. Formally, we introduce the Kan complex S^∞ , defined as 0-coskeleton of the discrete simplicial set with 2 vertices. (For more details see the exercises [8].) We say g is equivalence if there is a lift in the following diagram.

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{g} & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \\ S^\infty & & \end{array}$$

- (c) We say that g is an equivalence if its image in the homotopy category $h(\mathcal{C})$ is an isomorphism.¹

In the definition, going from version b to version a of is a rectification result, in the sense of procedure 3 described above.

Next, we recall the notions of cartesian morphism and cartesian fibration in the context of 1-categories.

Definition 1.3.2. Let $\mathcal{P} : C \rightarrow D$ be a functor between 1-categories. If $x \in \text{Ob}(C)$ and $f \in \text{Hom}(x, y)$, we use the notation $\bar{x} := \mathcal{P}(x)$, $\bar{f} = \mathcal{P}(f)$. In the following diagram, the first 2 rows are in C , while the third one is in D . However, we would like to think about the “square” as a pullback square.

$$\begin{array}{ccc} z & & \\ \vdots & & \\ \downarrow \mathcal{P} & & \\ \bar{z} & & \end{array} \quad \begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow \mathcal{P} & & \downarrow \mathcal{P} \\ \bar{x} & \xrightarrow{\bar{f}} & \bar{y} \end{array}$$

We say that f is a **\mathcal{P} -cartesian morphism** if the data of a morphism $z \rightarrow y$ in C and a morphism $\bar{z} \rightarrow \bar{x}$ in D uniquely determine a morphism $z \rightarrow x$ in C , such that the “diagram” commutes.

We say that \mathcal{P} is a **cartesian fibration** if for all $y \in C$ and all $\bar{x} \xrightarrow{\bar{f}} \bar{y}$ morphism in D , $\exists f : x \rightarrow y \in \mathcal{C}$ such that $\mathcal{P}(f) = \bar{f}$ and f is \mathcal{P} -cartesian.

The analogous definitions for ∞ -categories are the following.

Definition 1.3.3. Let $\mathcal{P} : \mathcal{C} \rightarrow \mathcal{D}$ be an ∞ -functor. A 1-morphism in \mathcal{C} , which is the same as an edge $f : \Delta^1 \rightarrow \mathcal{C}$, is **\mathcal{P} -cartesian** if for all $n \geq 2$, the following outer horn has a filler.

$$\begin{array}{ccc} \Delta^1 = \Delta^{\{n-1, n\}} & & \\ \downarrow & \searrow f & \\ \Lambda_n^n & \xrightarrow{\quad} & \mathcal{C} \\ \downarrow & \nearrow \text{dotted} & \downarrow \mathcal{P} \\ \Delta^n & \xrightarrow{\quad} & \mathcal{D} \end{array}$$

¹Recall that this is a 1-category with objects $\text{Ob}(\mathcal{C})$ and morphisms $\text{Hom}(x, y) = \pi_0(\mathcal{C}(x, y))$.

Morally speaking, when $n = 2$, this says that for any edge $g : z \rightarrow f(1)$ and edge $\bar{h} : \bar{z} \rightarrow \overline{f(0)}$, there exist an edge $h : z \rightarrow f(0)$ and a homotopy $g \simeq f \circ h$, such that $\mathcal{P}(h) = \bar{h}$.

We say that \mathcal{P} is a **cartesian fibration** if for every edge $a : \bar{x} \rightarrow \bar{y}$ of \mathcal{D} , and every object y of \mathcal{C} such that $\mathcal{P}(y) = \bar{y}$, there exists a \mathcal{P} -cartesian edge $f : x \rightarrow y$ such that $\mathcal{P}(f) = a$.

Recall that, in the study of fibered 1-categories, one proves that cartesian fibrations with base D are the same as lax 2-functors from D to the 2-category of 1-categories. (This is known as the “Grothendieck construction”, see for example, Proposition I.3.26 in [1].) Explicitly, given a cartesian fibration $\mathcal{P} : C \rightarrow D$, the corresponding lax 2-functor maps an object $d \in D$ to the fiber $\mathcal{P}^{-1}(d)$. Theorem 3.2.0.1, the main theorem of Chapter 3 in [5], is the analog of this result for the setting of ∞ -categories.

Theorem 1.3.4. *For any ∞ -category \mathcal{C} , there is an equivalence of ∞ -categories:*

$$\text{CartesianFibr}/\mathcal{C} \simeq \text{Fun}(\mathcal{C}^{\text{op}}, \text{Cat}_{\infty}). \quad (1.3.1)$$

Example 1.3.5. Recall that we started out by trying to construct an ∞ -functor $f_* : \mathcal{C}_{/x} \rightarrow \mathcal{C}_{/y}$ between overcategories, given an 1-morphism $f : x \rightarrow y$ in \mathcal{C} . Taking $F : \mathcal{C} \rightarrow \mathcal{C}$ as the identity, Theorem 1.2.2 gives a Cartesian fibration over \mathcal{C} :

$$\{(f : x \rightarrow y, a) \mid \{f : x \rightarrow y\} \in \mathcal{C}, F(a) \cong y\} \rightarrow \mathcal{C},$$

where a pair $(f : x \rightarrow y, a)$ maps to x . We recognize the fiber over x as the undercategory $\mathcal{C}_{x/}$:

$$\text{Hom}_{s\text{Set}}(\Delta^n, \mathcal{C}_{x/}) = \{\alpha : \Delta^{n+1} \rightarrow \mathcal{C} \mid \alpha_{\Delta^{[0, \dots, n]}} = x\}.$$

Theorem 1.3.4 then produces an ∞ -functor:

$$\begin{aligned} \mathcal{C}^{\text{op}} &\rightarrow \infty\text{-Cat} \\ x &\mapsto \mathcal{C}_{x/} \\ f : x \rightarrow y &\mapsto f^* : \mathcal{C}_{y/} \rightarrow \mathcal{C}_{x/}. \end{aligned}$$

We have obtained a pullback map on undercategories. To obtain the pushforward on overcategories, start with $F : \mathcal{C}^{\text{op}} \rightarrow \mathcal{C}$ as the contravariant identity functor instead.

Next, we discuss a simpler example. Let \mathcal{C} be an ∞ -category, and let $x \in \mathcal{C}$ be an initial object. We want to construct a functor $\mathcal{C} \rightarrow \mathcal{C}_{x/}$. Note that this is silly in 1-category theory, since there’s a unique morphism $x \rightarrow y$. To aid us in the context of ∞ -categories, we start by giving a good definition.

Definition 1.3.6. $x \in \mathcal{C}$ **initial** if $\forall y \in \mathcal{C}$, $\text{Map}_{\mathcal{C}}(x, y)$ is contractible.

The key result, proved, for example, in [3], is the following.

Proposition 1.3.7. *If \mathcal{C} is an ∞ category, then $x \in \mathcal{C}$ is initial iff the canonical projection $\mathcal{C}_{x/} \rightarrow \mathcal{C}$ is a trivial Kan fibration.*

To solve our problem, note that \mathcal{C} is cofibrant in the Kan model structure, so there exists a lift in the diagram:

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{C}_{x/} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{C} & \xrightarrow{\text{id}} & \mathcal{C}. \end{array}$$

In the exercises, we also encounter the following problem. Suppose \mathcal{C} has pushouts and a zero object. Construct an ∞ -functor $\mathcal{C} \rightarrow \mathcal{C}$ sending x to the pushout of 0 and 0 over x . (Todo: write this up, either here or in the exercises)

Chapter 2

Derived Affines

Talk by Benedict Morrissey.

2.1 3 perspectives on derived affines

First recall the notion of affines in classical AG: $\mathrm{Aff}_k^{Cl} \simeq (\mathcal{C}Ring)^{op}$. We get schemes by gluing these together. There's also the functor of points viewpoint: $X \in \mathrm{Aff}_k^{Cl}$ defines a sheaf by sending $\mathrm{Spec} R \mapsto \mathrm{Hom}(\mathrm{Spec} R, X)$. The schemes are then precisely the sheaves in the Zariski topology. Already in classical AG, there exist constructions which move us out of this category: both Serre's intersection theorem and Illusie's notion of the cotangent complex use derived functors. So by introducing DAG, we will understand better these structures in classical AG.

We will talk about 3 approaches to derived affines — all of these consist of embedding the classical category $\mathcal{C}Ring_k$ into a larger category in which we have a derived tensor product. In this section we assume that we are working over a ring k of characteristic zero¹.

1. Simplicial commutative rings;
2. Commutative differential graded algebras. (CDGA's)
3. Lawvere theory;

Remark 2.1.1. Classically gluing is easy. For example, fiber products are computed by reducing to the affine case, where it's just the tensor product of rings. In DAG, the derived tensor product is only defined up to quasi-isomorphism, so gluing can only be defined in a category which allows homotopy, such as an ∞ -category. For today's talk we mostly use the model category description; an application of Dwyer-Kan localization produces an ∞ -category.

2.1.1 Simplicial Commutative Rings

For approach 1, recall that the simplicial category Δ is:

$$Ob(\Delta) = \{n \in \mathbb{N} \cup \{0\}\}$$

where morphisms are compositions of face maps $\delta_i^n : [n-1] \rightarrow [n]$ for $0 \leq i < n$ and degeneracy maps $s_i^n : [n+1] \rightarrow [n]$, subject to the simplicial identities as can be found in e.g. [2].

¹We note here that we neglect to mention the important generalization of rings given by E- ∞ ring spectra, as described in chapter 7 of [7]. In the case where we are working over k a \mathbb{Q} -algebra this infinity category is equivalent to those described in this section as shown in [4] proposition 4.1.11. When we remove the characteristic zero assumption the statements about the Model structure on CDGA's no longer hold. One can still use simplicial commutative rings or E- ∞ algebras, though these give different ∞ categories.

Definition 2.1.2. The **category of simplicial commutative rings** is the category of contravariant functors:

$$SCR_k = \text{Hom}(\Delta^{\text{op}}, CRing_k).$$

Remark 2.1.3. There's a model category structure on this: fibrations are Kan fibrations on the underlying simplicial sets, i.e. morphisms $f : A \rightarrow B$ of simplicial commutative rings, such that all horns have fillers:

$$\begin{array}{ccc} \Lambda_j^n & \longrightarrow & A \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & B. \end{array}$$

Weak equivalences are weak homotopy equivalences on the underlying simplicial sets. Cofibrations are then determined from the axioms of a model category; note that they are *not* the same as cofibrations of the underlying simplicial sets.

Remark 2.1.4. We're using transfer to put the model structure on SCR_k . To explain what that means, under suitable conditions, there's a general procedure for defining a model structure on a category \mathcal{B} , given a model category \mathcal{A} and an adjoint functor pair:

$$\mathcal{A} \xrightleftharpoons[G]{F} \mathcal{B}.$$

The procedure forces the adjoint functor pair to be a Quillen adjunction. In our case, we use the free-forgetful adjunction:

$$sSet \xrightleftharpoons[U]{F} SCR_k$$

to transfer the Kan model structure to SCR_k . The key point which allows this to work is that all objects are fibrant. Cofibrations are more difficult to characterize, but the cofibrant objects are precisely the quasi-free ones. (That is, the ones isomorphic to a free object.)

2.1.2 CDGA's

Next, we introduce CDGA's and the Dold-Kan equivalence — which shows that this category is the same as that of simplicial commutative rings under our assumption that we are working over a characteristic zero field. Recall that we have a Quillen equivalence:

$$sVect \xrightleftharpoons{\quad} dg - Vect^{\leq 0}$$

between simplicial vector spaces and differential graded vector spaces, concentrated in nonpositive degrees. We want to talk about commutative monoids in these categories, $scAlg_k$ and $cdg - Alg^{\leq 0}$, respectively. The model structure on $cdg - Alg^{\leq 0}$ can also be obtained by transfer from the free-forgetful adjunction; we obtain that the weak equivalences are quasi-isomorphisms, and the fibrations are degree-wise surjections.

Theorem 2.1.5 (Symmetric monoidal Dold-Kan (A proof can be found in [11])). *There is a Quillen equivalence.²*

$$scAlg_k \xrightleftharpoons[\Gamma]{N} cdg - Alg_k^{\leq 0}.$$

Moreover, if the simplicial commutative algebra A_* corresponds to the commutative dg-algebra B_\bullet , then $\pi_i(A_*) \cong H^i(B_\bullet)$.

²Note that, in general, a Quillen equivalence is not an equivalence of categories. It does, however, induce an equivalence of Dwyer-Kan localizations (and hence also of homotopy categories).

Remark 2.1.6. We describe $N: A_* \in scAlg_k$ maps in the first stage to \tilde{A}_\bullet , where $\tilde{A}_{-n} = A_n$, and the differential is the alternating sum of the face maps. $N(A_*)$ is then the quotient of \tilde{A}_\bullet by the images of the degeneracy maps. For C a CDGA we can describe $(\Gamma C)_n := Hom_{ch^-}(N(\Delta^n), C)$ where we are in fact using the above definition of N to give a functor from simplicial abelian groups to ch^- — the category of non positively graded chain complexes, and Δ^n is the simplicial abelian group freely generated by an n -simplex.

We have a similar result for the category of simplicial modules for a given simplicial ring, and the category of dg-modules for its image in CDGA's. Note that the categories of simplicial modules and of non-positively graded modules for a given CDGA both have model structures.

Theorem 2.1.7 ([11]). *If A is a simplicial ring the categories of simplicial A -modules and of negatively graded $N(A)$ -modules are Quillen equivalent.*

If A is a CDGA, the categories of negatively graded A -modules and of simplicial $\Gamma(A)$ -modules are equivalent.

We now define the truncation functor of a CDGA. We can use the above Quillen equivalence to also define truncation functors on the category of Simplicial Commutative Rings.

Let $CDGA_k^{\leq n} \hookrightarrow CDGA_k$ denote the subcategory of $CDGA_k$ consisting of objects A such that $H^i(A) = 0$ for all $i > n$. Note that $CDGA_k^{\leq 0} \cong CRing_k$. The inclusion has a right adjoint $\tau^{\leq n} : CDGA_k \rightarrow CDGA_k^{\leq n}$. For $A = (A_n)$,

$$(\tau^{\leq n}(A))_m = \begin{cases} A_m & 0 > m > -n \\ A/im(d^{m+1}) & m = n \\ 0 & m < n. \end{cases}$$

2.1.3 Lawvere Theories

We move on to approach 3 to derived affines, the Lawvere Theory description. This is important because it's the only one of the 3 procedures which carries through in the analytic setting (see e.g. [10]). You get holomorphic rings, smooth rings, and much of the theory of DAG can be carried in this setting. (Todo: clear this part up)

The idea of Lawvere theory is to describe all objects with some type of algebraic structure as functors between the free objects and the category Set . For example,

$$AbGps \cong \text{Fun}^\times(FAb^{\text{op}}, Set).$$

There's a map $\mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$, which sends $1 \mapsto 1 \times 1$. Since F preserves products, $F(\mathbb{Z}) \times F(\mathbb{Z}) \cong F(\mathbb{Z} \times \mathbb{Z}) \rightarrow F(\mathbb{Z})$. In fact $F(\mathbb{Z})$ has the structure of an abelian group, and the equivalence of categories above is realized on objects by $F \mapsto F(Z)$. We denote by T_{disc} the opposite category of free commutative rings. Free commutative rings are the rings $k[x_1, \dots, x_n]$. Hence T_{disc} is the subcategory of the category of affine schemes with objects the planes $\{\mathbb{A}^n\}$. $\text{Fun}^\times(T_{disc}, Set) \cong CRing$. On objects we map a functor to it's value on the group ring \mathbb{A}^1 , $F \mapsto F(\mathbb{A}^1)$. The addition and multiplication on \mathbb{A}^1 give $F(\mathbb{A}^1)$ the structure of a (commutative) ring.

Now pass to $SCR_k \cong s\text{Fun}^\times(T_{disc}, Set) \cong \text{Fun}^\times(T_{disc}, sSet) \cong \text{Fun}^\times(T_{disc}, S)$, where S is the infinity category of spaces. The last step is a very hard rectification theorem, proved by Lurie-Bergner.³

2.2 Derived Affines as Ringed Spaces

Finally, we take the viewpoint of seeing a scheme as a locally ringed space. For $A \in cdg - Alg_k$, we look at the truncation $\text{Spec } H^0(A)$, which is an affine scheme in the classical sense. We can regard A as a sheaf

³HTT Propositions 5.5.9.2

of cdg-algebras on the truncation, as long as we can understand how localization works for cdg-algebras. We claim that it suffices to localize the commutative algebra A_0 . Indeed, we have the multiplication map:

$$\mu : A_0 \times A_i \rightarrow A_i,$$

so given a multiplicative subset $S \subset A_0$, we define the localization $S^{-1}A_i$ as $\mu(S^{-1}A_0 \times A_i)$. If this makes sense, we get a sheaf \mathcal{O}_A of cdg-algebras.

We would like to define derived affines as pairs $(\mathrm{Spec} H^0(A), \mathcal{O}_A)$. There is a subtlety: a priori this only gives a 2-category, and we need ∞ -categories. The key to resolving this is to define the notion of a sheaf valued in an $(\infty, 1)$ -category.

2.3 Our favorite classes of morphisms

Definition 2.3.1. Given $f : A \rightarrow B$ in SCR_k , we get maps:

$$\pi_*(A) \otimes_{\pi_0(A)} \pi_0(B) \rightarrow \pi_*(B)$$

of graded modules. We say that f is **strong** if this is an isomorphism of graded modules.

Definition 2.3.2. We define $f : A \rightarrow B$ to be **étale** (resp. **smooth**, **Zariski open immersion**, **flat**) if f is strong and $\pi_0(A) \rightarrow \pi_0(B)$ is étale (resp. smooth, Zariski open immersion, flat) in the classical sense.

Remark 2.3.3. The strength condition on f is quite restrictive: for example, a strong map from a non-derived domain must have a non-derived target.

Definition 2.3.4. Let $X = \mathrm{Spec}(A)$ be a derived affine over k . Then the **small étale site** of X is:

$$X_{\mathrm{\acute{e}t}} = \{\text{étale maps } \mathrm{Spec}(B) \rightarrow \mathrm{Spec}(A)\}.$$

In order to obtain the small étale site in the sense of classical AG, one needs to pass to the truncated version of the étale maps: $\pi_0(f) : \mathrm{Spec}(\pi_0(B)) \rightarrow \mathrm{Spec}(\pi_0(A))$. Then one can prove there's an equivalence of ∞ -categories between the derived and classical étale sites. In particular, this shows that $X_{\mathrm{\acute{e}t}}$ is a 1-category. This is one of the ingredients in the proof of the easy version of Lurie representability. Moreover, the same holds for the small smooth site and the small Zariski site.

After introducing the cotangent complex \mathbb{L}_f of a morphism f , we will see that f is étale iff $\pi_0(f)$ is of finite presentation and $\mathbb{L}_f \simeq 0$.

Definition 2.3.5. $f : A \rightarrow B$ is **of finite presentation** if the functor $\mathrm{Map}_A(B, -) : \mathrm{scRing}_k \rightarrow \mathcal{S}$ commutes with filtered colimits.

Unlike in the underived case, being of finite presentation is very strong, because it has a hidden regularity condition. In particular, we have the proposition due to Lurie:

Proposition 2.3.6. $f : A \rightarrow B$ is of finite presentation in the derived sense iff $\pi_0(f)$ is of finite presentation in the classical sense (also called to order 0) and the cotangent complex \mathbb{L}_f is perfect.

Example 2.3.7. Let $X = \mathbb{A}^3$, and Y a closed subscheme of X which is not a local complete intersection. Then the inclusion $\iota : Y \rightarrow X$ is not of finite presentation in the derived sense. Indeed, by a conjecture of Quillen, which is now a theorem of Abramov, for maps between classical schemes, the cotangent complex is either concentrated in degrees 0 and -1, or it's unbounded. Since Y is not lci, the first case is ruled out, and \mathbb{L}_ι is unbounded.

Chapter 3

Stable ∞ -categories

Talk by Michael Gerapetritis.

3.1 Motivation

In the 1-categorical setting, if \mathcal{C} is a category, we may require that $\mathcal{C}(A, B)$ be a set. To get particularly well-behaved categories, namely the additive categories, we require that $\mathcal{C}(A, B)$ is actually an abelian group.

We try to replicate this in the ∞ -category setting. Let \mathcal{C} be an ∞ -category, then $\mathcal{C}(X, Y)$ is a space. We want to discover what is the good extra structure to have on this space; we will call the corresponding ∞ -categories stable.

3.2 Stable ∞ -categories and triangulated 1-categories

Definition 3.2.1. An ∞ -category \mathcal{C} is **stable** if:

- \mathcal{C} is pointed, i.e. it has a zero object;
- every morphism $f : X \rightarrow Y$ admits fibers and cofibers;
- a triangle is a fiber iff it is a cofiber.

Recall that a **triangle** in \mathcal{C} is a map of simplicial sets $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$, i.e. a homotopy commutative diagram with the zero object in the bottom-left corner:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & Z \end{array}$$

The triangle is a **fiber** if it is a pullback square, and a **cofiber** if it is a pushout square. We say that $f : X \rightarrow Y$ admits a fiber (resp. cofiber) when $\exists W$ (resp Z) such that:

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & Y \end{array}$$

is a pullback square (or, respectively:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Z \end{array}$$

is a pushout square).

Remark 3.2.2. Note that the data of a triangle consists not only of homotopy commutative diagrams as above, but also of choices of homotopies between the branches. This is crucial, since it ensures that cones are functorial at the level of the homotopy category. This functoriality does not hold in a general triangulated category. (See Theorem 3.2.5 for the relation between stable ∞ -categories and triangulated 1-categories.)

Example 3.2.3. Our two main examples are ∞ -categories of spectra (see Section 3.5) and of modules over a CDGA or SCR (see Section 3.3).

Recall the data for a triangulated category.

Definition 3.2.4. A category \mathcal{D} is triangulated if:

1. \mathcal{D} is additive;
2. \mathcal{D} admits a translation functor $T : \mathcal{D} \xrightarrow{\sim} \mathcal{D}$;
3. \mathcal{D} has a collection of distinguished triangles:

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

This data is required to satisfy some axioms, but we won't go into details here.

Theorem 3.2.5. *If \mathcal{C} is a stable ∞ -category, then $h\mathcal{C}$ is triangulated.*

For a proof see [7]. We won't go over it, let's just say that translation is given by Σ , and distinguished triangles are precisely the images of fiber sequences (or equivalently, cofiber sequences), as resulting from the following diagram.

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

Proposition 3.2.6. *\mathcal{C} is stable iff the following hold:*

1. \mathcal{C} admits finite limits and colimits;
2. any square is a pushout iff it is a pullback.

Proof. Again, we don't give a full proof. Let's just see why products and coproducts must exist in a stable ∞ -category. Note first that Σ is an equivalence of ∞ -categories. Indeed, Σ is a left adjoint functor; moreover, the unit and counit of the adjunction become isomorphisms in the homotopy category, due to condition 2 in the definition of a triangulated category. Then we use the following diagram.

$$\begin{array}{ccccc} \Omega(X) & \longrightarrow & 0 & \longrightarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & X & \longrightarrow & X \oplus Y \end{array}$$

We have defined $X \oplus Y$ as the cofiber of $\Omega(X) \xrightarrow{0} Y$, which is postulated to exist in a stable ∞ -category. This turns the outer rectangle into a pushout square, and it follows that the square on the right is also a pushout square. Thus $X \oplus Y$ is the coproduct of X and Y . We reason dually to obtain products. \square

Definition 3.2.7. Let $\mathcal{C}, \mathcal{C}'$ be stable ∞ -categories, and $F : \mathcal{C} \rightarrow \mathcal{C}'$ an ∞ -functor which maps 0 objects to 0 objects. Equivalently, F maps triangles to triangles. If F maps fiber sequences to fiber sequences, we say that F is **exact**.

Lemma 3.2.8. *TFAE:*

1. F is exact;
2. F is right-exact, i.e. commutes with finite colimits;
3. F is left-exact, i.e. commutes with finite limits.

This is very useful: sometimes it's really easy to check that a functor is right or left exact, e.g. if it's a left or right adjoint, respectively.

3.3 Modules

For a useful example of the result in Lemma 3.2.8, we look at $\mathcal{C} = A - \text{Mod}$, where A is a CDGA or SCR over k . (By $A - \text{Mod}$ we mean the unbounded derived category.) The easiest way to see $A - \text{Mod}$ as an ∞ -category is to put a model structure on chain complexes, say the projective one, and then take the underlying ∞ -category. We claim that $A - \text{Mod}$ is a stable ∞ -category. Using the theorem Mauro talked about in Lecture 1, limits and colimits exists in the ∞ -category iff they exist in the model category. (Todo: reference theorem) It remains to prove the following.

Lemma 3.3.1. *A triangle in $A - \text{Mod}$ is a fiber iff it is a cofiber.*

Proof. We prove one direction; the other argument is dual to this one. Assume that $f : M^\bullet \rightarrow N^\bullet$ is the fiber of a map g . Take a cofibrant replacement of f , get \tilde{M}, \tilde{N} cofibrant and a homotopy pullback square: (Todo: figure out how to do the cartesian symbol in tikz)

$$\begin{array}{ccc} \tilde{M}^\bullet & \xrightarrow{\tilde{f}} & \tilde{N}^\bullet \\ \downarrow & & \downarrow \tilde{g} \\ 0 & \longrightarrow & P^\bullet. \end{array}$$

\tilde{f} is cofibrant, so it's a degree-wise injection. Then g is a degreewise surjection, and it follows that the square is a strict pushout. (Todo: wait, how did this work again?) \square

Now suppose we have $f : A \rightarrow B$ a morphism of $CDGA_k^{\leq 0}$. It induces the adjunction of model categories:

$$A - \text{Mod} \begin{array}{c} \xrightarrow{f^*} \\ \xleftarrow{f_*} \end{array} B - \text{Mod},$$

where f_* is the forgetful functor, and $f^*(M) = M \otimes_A B$. So this gives an adjunction of ∞ -categories: ¹

$$A - \text{Mod} \begin{array}{c} \xrightarrow{Lf^*} \\ \xleftarrow{Rf_*} \end{array} B - \text{Mod}.$$

Explicitly, Lf^* is constructed by first choosing a cofibrant replacement \tilde{M} for M , and then taking $\tilde{M} \otimes_A B$. The answer doesn't depend on cofibrant replacement, up to coherent isomorphism. Then Lf^* is a left adjoint functor, so it follows from general nonsense that it's right exact. Lemma 3.2.8 then implies that Lf^* is also left exact and exact.

Remark 3.3.2. If f is not flat in the sense of Definition 2.3.2, then the exactness of Lf^* comes at the price of losing t-exactness. To explain what we mean, pick $M \in A - \text{Mod}$, such that $H^i(M) = 0$ unless $i = 0$. But then $Lf^*(M) = M \otimes_A^{\mathbb{L}} B$, and $H^{-i}(M \otimes_A^{\mathbb{L}} B) = \text{Tor}_i^A(M, B)$, which is $\neq 0$ in general, because f is not flat. So even though M was homologically concentrated in degree 0, $Lf^*(M)$ may not be. In other words, the failure of a functor of (Grothendieck) abelian categories to preserve limits translates into a lack of t-exactness of the derived functor. In the following section we define t-structures and t-exactness for ∞ -categories.

¹Here we use L and R to indicate that the functors are derived. In later talks derived functors will be the default, and we will omit the symbols L and R .

3.4 t-structures

Definition 3.4.1. If \mathcal{C} is a stable ∞ -category, a **t-structure**² on \mathcal{C} is the data of two full subcategories of \mathcal{C} , $\mathcal{C}^{\leq 0}$ and $\mathcal{C}^{\geq 0}$,³ such that:

1. $\pi_0 \operatorname{Map}_{\mathcal{C}}(X, Y[-1]) = 0$ if $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 0}$.⁴
2. $X \in \mathcal{C}^{\leq 0}, X[1] \in \mathcal{C}^{\leq 0}$;
3. $\forall X, \exists$ fiber sequence $X' \rightarrow X \rightarrow X''$, where $X' \in \mathcal{C}^{\leq 0}, X'' \in \mathcal{C}^{\geq 1}$.

Remark 3.4.2. Condition 1 has the following intuitive meaning in the case $\mathcal{C} = A\text{-Mod}$. 0-morphisms in \mathcal{C} are chain maps which preserve degree, while higher morphisms are homotopies which shift the degree to the left; morphisms that shift degree to the right are not allowed. Then, if $X \in \mathcal{C}^{\leq 0}$ and $Y \in \mathcal{C}^{\geq 0}$, no nonzero morphisms should be allowed between X and $Y[-1]$:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & X_{-2} & \longrightarrow & X_{-1} & \longrightarrow & X_0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & & & & & & & & & & & & \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_2 & \longrightarrow & \dots \end{array}$$

Remark 3.4.3. X' and X'' are uniquely determined by X .

Theorem 3.4.4. The inclusion $\mathcal{C}^{\leq 0} \rightarrow \mathcal{C}$ has a right adjoint, which we denote $\tau_{\leq 0} : \mathcal{C} \rightarrow \mathcal{C}^{\leq 0}$. Similarly we get $\tau_{\geq 0} : \mathcal{C} \rightarrow \mathcal{C}^{\geq 0}$.

Corollary 3.4.5. For all $X \in \mathcal{C}$, the fiber sequence of 3 is just:

$$\tau_{\leq 0} X \rightarrow X \rightarrow \tau_{\geq 1} X.$$

Proposition 3.4.6. Denote by $\mathcal{C}^{\heartsuit} := \mathcal{C}^{\leq 0} \cap \mathcal{C}^{\geq 0}$, the **heart** or **core** of the t-structure. It is an abelian 1-category.

Proposition 3.4.7. Let \mathcal{C} be stable. Then if:

$$X \rightarrow Y \rightarrow Z$$

is a fiber sequence, then we have a long exact sequence of H^i , where $H^i(X) := \tau_{\geq i} \circ \tau_{\leq i}(X)$.

Putting the last few results together, from \mathcal{C} a presentable stable ∞ -category with t-structure, the heart is Grothendieck abelian. Write $A = \mathcal{C}^{\heartsuit}$. Then we can form $\mathcal{D}(A)$, the ∞ -derived category of A . The next theorem describes the relationship between \mathcal{C} and $\mathcal{D}(A)$.

Theorem 3.4.8 (Lurie). $\mathcal{D}(A)$ has a universal property which produces an ∞ -functor:

$$\mathcal{D}(A) \rightarrow \mathcal{C}.$$

In general this is very far from being an equivalence.

Example 3.4.9. Let $A \in CDGA_k^{\leq 0}$. The theorem gives a map:

$$(A\text{-Mod})^{\heartsuit} \rightarrow (H^0(A)\text{-Mod})^{\heartsuit}. \quad (3.4.1)$$

This is one of the most important facts in DAG, because it reduces problems about the ∞ -category of A -modules to problems in classical categories of modules, where one can work with generators and relations. The map in 3.4.1 is an equivalence iff $A \simeq H^0(A)$ are quasi-isomorphic. (Todo: figure out what's the precise relationship here)

²t stands for truncation

³Note that we use cohomological notation, while Lurie in [7] uses homological notation. Therefore gradings have opposite signs in this seminar and in [7].

⁴In a stable ∞ -category, we sometimes use the shift notation $[n]$ to denote the $|n|$ -fold iterated application of the Σ functor (if n is positive) or the Ω functor (if n is negative). This notation is justified by Proposition 3.5.4.

Definition 3.4.10. Let \mathcal{C}, \mathcal{D} be stable ∞ -categories with t -structures. Then an exact functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is:

1. **left t-exact** if $F(\mathcal{C}^{\leq 0}) \subset \mathcal{D}^{\leq 0}$;
2. **right t-exact** if $F(\mathcal{C}^{\geq 0}) \subset \mathcal{D}^{\geq 0}$;
3. **t-exact** if both.

Example 3.4.11. For $A, B \in CDGA_k^{\leq 0}$, $f : A \rightarrow B$, we have the adjunction:

$$A - \text{Mod} \begin{matrix} \xrightarrow{Lf^*} \\ \xleftarrow{Rf_*} \end{matrix} B - \text{Mod}.$$

Every object is fibrant, so we don't need to derive the functors. Rf_* is both left and right t-exact. Lf^* is not right t-exact, because of nontrivial Tor^i terms; see 3.3.2. However, Lf^* is right t-exact: morally speaking, Projective resolution only puts stuff in negative degrees. We give an ∞ -categorical proof.

Pick $M \in A - \text{Mod}^{\geq 0}$. We want $Lf^*(M) \in B - \text{Mod}^{\leq 0}$. To check this is the same as checking that $\forall N \in B - \text{Mod}^{\geq 1}$, $\text{Map}_{B - \text{Mod}}(Lf^*M, N) \cong 0$. But this is $\text{Map}_{A - \text{Mod}}(M, Rf_*N) \cong 0$, which follows since Rf_* was t-exact.

3.5 Spectra

Going back to the question left unanswered in Section 3.1, the extra structure we want on morphism spaces of stable ∞ -categories is $\text{Map}_{\mathcal{C}}(X, Y) \in \text{Sp}^{\leq 0}$.

Definition 3.5.1. **Spectra** are sequences $\{F_i\}$ of objects in \mathcal{C} such that $F_n \simeq \Omega F_{n+1}$. Alternatively, we identify them with objects of the homotopy limit:

$$\dots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \dots$$

Remark 3.5.2. We must be careful with defining morphisms between spectra: we want squares to commute up to coherent homotopy. Moreover, it's hard to get a monoidal model structure on the category of spectra: this was done only in the 2000s, after Hovey introduced symmetric spectra. Lurie has a very categorical and very nice way of putting a monoidal structure at the level of the ∞ -category directly. See the last chapter of [3], and also 4.8.2 of [7].

Theorem 3.5.3. $Sp(\mathcal{C})$ is stable.

This gives a canonical stabilization for every ∞ -category. The proof of the theorem follows from the following characterization of stable ∞ -categories, and the fact that $\Omega : Sp(\mathcal{C}) \rightarrow Sp(\mathcal{C})$ is an equivalence.

Proposition 3.5.4. \mathcal{C} is a pointed ∞ -category. TFAE:

1. \mathcal{C} is stable;
2. \mathcal{C} admits colimits and $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence;
3. \mathcal{C} admits limits and $\Omega : \mathcal{C} \rightarrow \mathcal{C}$ is an equivalence;

Chapter 4

The Cotangent Complex

Talk by Sukjoo Lee.

4.1 Motivation

We recall from classical AG: if $f : A \rightarrow B$ is a homomorphism between commutative rings and M is a B -module, an **A-derivation** of B into M is a map $d : B \rightarrow M$ such that:

- $d(f(a)) = 0$, for all $a \in A$;
- $d(bb') = db \cdot b' + b \cdot db'$ (Leibniz rule).

We denote by $\text{Der}_A(B, M)$ the set of all derivations of B into M . There is also an absolute version, where we take $f : 0 \rightarrow A$, and the first condition is automatic.

Definition 4.1.1. The **module of relative Kähler differentials** of B over A is a derivation $(\Omega_{B/A}^1, d_A)$ over A satisfying the universal property:

$$\begin{array}{ccc} B & \xrightarrow{d_A} & \Omega_{B/A}^1 \\ & \searrow d' & \downarrow \exists! \\ & & M. \end{array}$$

Equivalently, $\text{Hom}_{B\text{-Mod}}(\Omega_{B/A}^1, M) \simeq \text{Der}_A(B, M)$. (+ absolute version).

Proposition 4.1.2. *If $A \rightarrow B \rightarrow C$ is a sequence of maps of commutative rings, then the following sequence of C -modules is exact:*

$$\Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1 \rightarrow 0. \quad (4.1.1)$$

One of the goals for this talk is to extend the sequence to the left. If $\mathcal{C}Ring$ was an Abelian category, we would attempt to derive the functor Ω^1 ; however, this is not the case. Instead, what we do is generalize the notion of Kähler differential to the ∞ -categorical setting, and show that this gives an extension to the left of the sequence 4.1.1. Slogan: “ ∞ -category theory allows us to do derived functors in a non-linear setting”.

4.2 Generalization and definition

Note that generalizing the Leibniz rule to the ∞ -category setting is hard, because we’d have to replace the equality with a homotopy. Instead, consider the following idea. For a ring homomorphism $\phi : A \rightarrow B$,

we want a new homomorphism $\phi' : A \rightarrow B$ “sufficiently close” to ϕ . For example, take $I \subset B$ an ideal with $I^2 = 0$. Then “sufficiently close” means that $\phi' : A \rightarrow B$ is congruent to ϕ modulo I , i.e.:

$$\forall a \in A, \phi(a) - \phi'(a) \in I.$$

For a fixed ϕ , we have a bijective correspondence:

$$\left\{ \begin{array}{l} \phi' : A \rightarrow B \text{ such that} \\ \phi' \equiv \phi \pmod{I} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} d : A \rightarrow I \text{ satisfying} \\ \text{the Leibniz rule} \end{array} \right\}. \quad (4.2.1)$$

If M an A -module, take $B := A \oplus M$ equipped with the ring structure such that $M^2 = 0$:

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + a_2 m_1).$$

We fix $\phi : A \rightarrow B$ the natural inclusion of A into the coproduct (as abelian groups); with the given ring structure, ϕ is also a ring homomorphism. Take the ideal $I = M$. Then the correspondence 4.2.1 becomes:

$$\text{Map}_{/A}(A, A \oplus M) \cong \text{Der}(A, M). \quad (4.2.2)$$

This is something we can generalize. We work with $A \in \text{SCR}_k$, and the category of $A - \text{Mod}$; all categories in the rest of the talk are ∞ -categories. Take $M \in A - \text{Mod}$ and construct $A \oplus M$, whose underlying simplicial set is the coproduct, and whose ring structure is defined levelwise (see [12], 1.2.1.1 for details).

We adapt equation 4.2.2 to this setting, by defining:

$$\text{Der}(A, M) = \text{Map}_{/A}(A, A \oplus M) \in \mathcal{S}.$$

Moving from the absolute version of derivations to the relative one, for $f : A \rightarrow B$ in SCR_k and $M \in B - \text{Mod}$, we define:

$$\text{Der}_A(B, M) = \text{Map}_{A - \text{Alg}/B}(B, B \oplus M) \in \mathcal{S}.$$

We obtain functors $\text{Der}(A, -) : A - \text{Mod} \rightarrow \mathcal{S}$ and $\text{Der}_A(B, -) : B - \text{Mod} \rightarrow \mathcal{S}$. We claim that these functors are corepresentable, and call the corepresenting objects the **absolute cotangent complex** \mathbb{L}_A and **relative cotangent complex** $\mathbb{L}_{B/A}$, respectively. Equivalently, this means:

$$\begin{aligned} \text{Map}_{A - \text{Mod}}(\mathbb{L}_A, M) &\cong \text{Map}_{/A}(A, A \oplus M), \\ \text{Map}_{B - \text{Mod}}(\mathbb{L}_{B/A}, M) &\cong \text{Map}_{A - \text{Alg}/B}(B, B \oplus M). \end{aligned}$$

The proof of corepresentability relies on the following lemma.

Lemma 4.2.1.

$\text{Der}_A(B, -)$ and $\text{Der}(A, -)$ commute with limits and K -filtered colimits.¹ (Todo: explain the footnote better)

Then the result follows by Theorem 5.5.2.7 in [5], which we reproduce here.²

Theorem 4.2.2. *Let \mathcal{C} be a presentable ∞ -category and $F : \mathcal{C} \rightarrow \mathcal{S}$ a functor. Then F is corepresentable by an object of \mathcal{C} if and only if F preserves K -filtered colimits and all small limits.*

¹ ω -filtered would mean that the representing guy can be given by presentation with gen and relation. Otherwise we just mean for everything strictly less than cardinality K , which could be continuum or more.

² Related to this is the Adjoint Functor Theorem 5.5.2.9, which is one of the most important theorems in ∞ -category theory. It's also the reason we love presentable ∞ -categories.

Remark 4.2.3. Note that preserving small limits is obviously necessary in order to be corepresentable, since Hom is a right adjoint, and thus preserves small limits. (Todo: think more about the small hypothesis) Let's explain this in more detail for 1-category theory. We have the diagram:

$$\begin{array}{ccc} A - \text{Mod} & \xrightarrow{\simeq} & \text{Ab}(\text{CRing}/A) \\ & \downarrow \text{forget} & \\ & \text{CRing}/A & \xrightarrow{\text{Hom}(A, -)} \text{Set}. \end{array}$$

The equivalence on the first line works by sending an A -module M to $A \oplus M$, and a ring B over A to the kernel of $B \rightarrow A$. (Todo: finish this)

For ∞ -category theory, the relevant diagram is:

$$\begin{array}{ccc} A - \text{Mod} & \xrightarrow{\simeq} & \mathcal{S}p(\text{SCR}/A) \\ & \downarrow \text{forget} & \\ & \text{SCR}/A & \xrightarrow{\text{Map}/A(A, -)} \mathcal{S}. \end{array}$$

The reference for this is [7], section 7.4. (Todo: wait, where exactly? can't find it)

In [12], Chapter 1, we find an explicit (although not useful in practice, according to Mauro) model for \mathbb{L}_A . Take a simplicial resolution $\tilde{A} \rightarrow A$, which is also a cofibrant replacement. Then we have:

$$\Omega_{\tilde{A}}^1 \otimes_{\tilde{A}}^{\mathbb{L}} A \in A - \text{Mod}$$

is a model for \mathbb{L} , where the complex $\Omega_{\tilde{A}}^1$ is build by taking Kähler differentials degree-wise:

$$(\Omega_{\tilde{A}}^1)_{\bullet} := \Omega_{\tilde{A}_{\bullet}}^1.$$

4.3 Examples and Properties

In this section we compute $\mathbb{L}_{k[x]/k}$ and $\mathbb{L}_{k/k[x]}$, where x is in degree -1, as opposed to 1, by our cohomological convention. (See remark 4.3.2 for what this implies; $k[x]$ is not what it seems.) In the process we go over some of the properties of cotangent complexes.

Lemma 4.3.1. *Let $A \in \text{SCR}_k$ and $M \in A - \text{Mod}$. The cotangent complex of $f : A \rightarrow \text{Sym}_A M$ is:*

$$\mathbb{L}_{\text{Sym}_A M/A} \cong M \otimes_A^{\mathbb{L}} \text{Sym}_A M.$$

Proof. For all $\text{Sym}_A M$ -module N ,

$$\begin{aligned} \text{Map}_{\text{Sym}_A M - \text{Mod}}(\mathbb{L}_{\text{Sym}_A M/A}, N) &\cong \text{Map}_{A - \text{Mod}}(\text{Sym}, \text{Sym} \oplus N) \cong \text{Map}_{A - \text{Mod}}(M, f_* N) \\ &\cong \text{Map}_{\text{Sym}_A M - \text{Mod}}(M \otimes_A^{\mathbb{L}} \text{Sym}_A M, N). \end{aligned}$$

Here the first equivalence is definitional, the second follows from the universal property of $\text{Sym}_A M$, and the third is the adjunction 3.3. \square

For our first example, note that $k[x] \cong \text{Sym}_k(k[1])$. Then the answer is $k[1] \otimes_k^{\mathbb{L}} k[x]$, which is just $k[x]$ concentrated in degree -1.

Remark 4.3.2. Note that, since $k[1]$ is concentrated in degree -1, so is $\text{Sym}_k(k[1])$; it does not have information in all nonnegative degrees, as the notation may mislead one into thinking. We just get a copy of k in degree 0 and one in degree -1, and this is what we call $k[x]$. This is because multiplication in the symmetric algebra is graded commutative:

$$xy = (-1)^{|x||y|}yx,$$

so in particular for x of degree 1 we get $x^2 = -x^2 = 0$. If we started with $k[2]$ instead, $\text{Sym}_k(k[2])$ would be nontrivial in all negative even degrees and commutative in the classical sense. More generally, $\text{Sym}(k[n])$ gives what we would classically call a symmetric algebra if n is even, or an alternating algebra if n is odd.

Some properties of cotangent complex:

Proposition 4.3.3. 1. For $A \rightarrow B \rightarrow C$ in SCR_k , there is a homotopy cofiber sequence in $C - \text{Mod}$:

$$\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C \rightarrow \mathbb{L}_{C/A} \rightarrow \mathbb{L}_{C/B}.$$

2. Base change: given a homotopy pullback square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B', \end{array}$$

there is an equivalence $\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} B' \simeq \mathbb{L}_{B'/A'}$.

To compute $\mathbb{L}_{k/k[x]}$, use the cofiber sequence associated to the sequence of maps $k \rightarrow k[x] \rightarrow k$. We get the cofiber sequence in $k - \text{Mod}$:

$$\mathbb{L}_{k[x]/k} \otimes_{k[x]}^{\mathbb{L}} k \rightarrow \mathbb{L}_{k/k} \rightarrow \mathbb{L}_{k/k[x]}. \quad (4.3.1)$$

By our previous computation, the first term is:

$$\mathbb{L}_{k[x]/k} \otimes_{k[x]}^{\mathbb{L}} k \simeq k[1] \otimes_k^{\mathbb{L}} k[x] \otimes_{k[x]}^{\mathbb{L}} k \simeq k[1].$$

(Using associativity for derived tensor product.) The second term in 4.3.1 is 0, so the cofiber sequence is actually a suspension diagram.

$$\begin{array}{ccc} k[1] & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{L}_{k/k[x]} \end{array}$$

Then $\mathbb{L}_{k/k[x]} \simeq k[2]$.

Going back to Section 4.1, we complete the exact sequence 4.1.1 on the left. Using stability of $C - \text{Mod}$, the cofiber sequence gives a long exact sequence on homology (recall proposition 3.4.7; in particular, $H^i(X) = \tau^{\geq i} \circ \tau^{\leq i}(X)$).

$$\begin{array}{ccccccc} \dots & \longrightarrow & H^{-1}(\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C) & \longrightarrow & H^{-1}(\mathbb{L}_{C/A}) & \longrightarrow & H^{-1}(\mathbb{L}_{C/B}) \\ & & \searrow & & \searrow & & \searrow \\ & & H^0(\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C) & \longrightarrow & H^0(\mathbb{L}_{C/A}) & \longrightarrow & H^0(\mathbb{L}_{C/B}) \\ & & \searrow & & \searrow & & \searrow \\ & & H^1(\mathbb{L}_{B/A} \otimes_B^{\mathbb{L}} C) & \longrightarrow & H^1(\mathbb{L}_{C/A}) & \longrightarrow & H^1(\mathbb{L}_{C/B}) \longrightarrow \dots \end{array}$$

We claim that:

1. For underived rings, $H^i(\mathbb{L}_{B/A}) = 0$ if $i > 0$;
2. For underived rings, $H^0(\mathbb{L}_{B/A}) \simeq \Omega_{B/A}^1$;
3. In general, $H^0(\mathbb{L}_{B/A}) \simeq \Omega_{\pi_0(B)/\pi_0(A)}^1$.

An application of these facts is the desired extension to the left of the sequence 4.1.1. The facts are proved in [7], section 7.4.3. We will say more about claim 1, but first we need to talk about connectivity.

4.4 Connectivity

Definition 4.4.1. A space X is **n-connective** if $\pi_i(X, x) = 0$ for all $x \in X$ and $i < n$. We say X is **connective** if it's 0-connective, **connected** if it's 1-connective. $f : X \rightarrow Y$ is **n-connective** if $\text{fiber}(f)$ is n-connective.

The following is in [7], 7.4.3.2, and it's VERY important.

Theorem 4.4.2 (Connectivity estimate). *Assume $f : A \rightarrow B$ is a map in SCR_k and $\text{cofib}(f)$ is n -connective. Then there exists a map:*

$$\mathcal{E}_f : B \otimes_A^{\mathbb{L}} \text{Cofib}(f) \rightarrow \mathbb{L}_{B/A}$$

in $B - \text{Mod}$, which is $2n$ -connective.

Remark 4.4.3. The proof is not hard; the only difficulty is constructing the map, which we can do after we learn Postnikov towers. (Todo: reference once we have the postnikov notes)

Corollary 4.4.4. *The hypothesis of Theorem 4.4.2 implies $\mathbb{L}_{B/A}$ is n -connective.*

Proof. We look at the fiber sequence:

$$\text{fib}(\mathcal{E}_f) \rightarrow B \otimes_A^{\mathbb{L}} \text{cofib}(f) \rightarrow \mathbb{L}_{B/A},$$

and get a long exact sequence of homotopy groups. So it suffices to show that:

1. $B \otimes_A^{\mathbb{L}} \text{Cofib}(f)$ is n -connective;
2. $\text{fib}(\mathcal{E}_f)$ is $n - 1$ -connective.

2 is implied by Theorem 4.4.2; note that theorem is actually considerably stronger. Property 1 is proved in [9]. The proof there uses a spectral sequence due to Quillen: for $M, N \in A - \text{Mod}$, $A \in SCR_k$,

$$\text{Tor}_p^{\pi_q(A)}(\pi_q M, \pi_q N) \implies \pi_{p+q}(M \otimes_A^{\mathbb{L}} N).$$

□

Remark 4.4.5. In particular, cotangent complexes are 0-connective for commutative rings. This gives a proof of fact 1 at the end of the previous section.

Corollary 4.4.6. *For $A \in SCR_k$, \mathbb{L}_A is 1-connective. Moreover, $f : A \rightarrow \pi_0(A)$ is 1-connective, so $\mathbb{L}_{\pi_0(A)/A}$ is 1-connective.*

The most important corollary:

Corollary 4.4.7. *$f : A \rightarrow B$ is an equivalence iff $\pi_0(f) : \pi_0(A) \rightarrow \pi_0(B)$ is and $\mathbb{L}_{B/A} \simeq 0$. One direction obvious, the other comes from the fact that $\mathbb{L}_{B/A}$ is n -connected for all n .*

Remark 4.4.8. Slogan: “DAG = classical AG + DDT”. Lurie’s representability theorem is a great example of the philosophy: it says that a derived stack is representable iff its truncation is representable and its cotangent complex is nice enough. We won’t get to see this in the seminar, since we’ll change course towards structured DAG instead.

Remark 4.4.9. Cotangent complexes we glue for free, which was not possible before ∞ -categories. This allows to reduce many questions to the affine setting, where we may have to do actual computations if things go wrong.

We have one talk on Postnikov tower, and one on perfect complexes, then we leave the affine setting forever.

(Todo: look at last 2 exercises from stable ∞ -category)

Chapter 5

Square Zero Extensions

Talk by Matei Ionita.

5.1 Square Zero Extensions

Recall that, given $A \in \text{cdga}_k^{\leq 0}$ and $M \in A - \text{Mod}$, we defined derivations from A into M as:

$$\text{Der}_k(A, M) = \text{Map}_{A\text{-Alg}/k}(A, A \oplus M).$$

Alternatively, these are the same as sections of the projection map $A \oplus M \rightarrow A$. Morally speaking, we'd like to define square-zero extensions as homotopy fibers of derivations, i.e. $f : A^\eta \rightarrow A$ is a square-zero extension of A by M if there is a homotopy pullback square:

$$\begin{array}{ccc} A^\eta & \xrightarrow{f} & A \\ \downarrow & & \downarrow d_\eta \\ 0 & \longrightarrow & M[1]. \end{array}$$

The problem is that the above diagram doesn't make sense, because a derivation is not a morphism in $\text{cdga}_k^{\leq 0}$. In section 7.4.1 of [7], Lurie addresses this by using the category of tangent correspondences, which acts like a “tangent bundle” of the category $\text{cdga}_k^{\leq 0}$, with $A - \text{Mod}$ acting as the tangent space $T_A \text{cdga}_k^{\leq 0}$. In this new category the diagram makes sense. However, we don't introduce all this technology here, and instead translate Lurie's (more general) definition of square zero extensions into a more accessible version.

Definition 5.1.1. A map $\tilde{f} : \tilde{A} \rightarrow A$ is a **square-zero extension** of A by M if it's equivalent in the category $\text{cdga}_{/A}^{\leq 0}$ to a map $f : A^\eta \rightarrow A$ such that there is a homotopy pullback diagram in $\text{cdga}_k^{\leq 0}$:

$$\begin{array}{ccc} A^\eta & \xrightarrow{f} & A \\ \downarrow & & \downarrow d_\eta \\ A & \xrightarrow{d_0} & A \oplus M[1]. \end{array}$$

Here d_0 is the zero derivation.

Remark 5.1.2. We explain why the shift by 1 is necessary in definition 5.1.1, by studying the split square-zero extension. We claim that, with the shift in place, the following diagram is a homotopy pullback. (Todo: replace with better explanation)

$$\begin{array}{ccc} A \oplus M & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \oplus M[1] \end{array}$$

To see this, extend the diagram by considering the map $0 \rightarrow A$, and the resulting pullback square in the category $A\text{-Mod}$:

$$\begin{array}{ccccc} M & \longrightarrow & A \oplus M & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A & \longrightarrow & A \oplus M[1]. \end{array}$$

Indeed, the vertical map $A \oplus M \rightarrow A$ is surjective, hence a fibration in $A\text{-Mod}$, and then the naive pullback M is a homotopy pullback. Moreover, the outer square is also a homotopy pullback in $A\text{-Mod}$, because it's equivalent to:

$$\begin{array}{ccc} M & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & M[1]. \end{array}$$

It follows that the square on the right is a homotopy pullback in $A\text{-Mod}$. But all maps in this square are maps of A -algebras, so we claim that the square is actually a homotopy pullback in $A\text{-Alg}$.

Remark 5.1.3. Definition 5.1.1 is easy and clean, but it is hard to see whether a given map satisfies it. For example, if $A \rightarrow B$ is a square zero extension of commutative rings by a B -module M , in the classical sense, the shift $M[1]$ makes us leave the classical category of modules. Moreover, it's hard to prove that the given map $A \rightarrow B$ comes from the fiber product structure of A .

We would like to construct a functor $\Phi : \mathcal{D}er(A, M) \rightarrow \text{Fun}(\Delta^1, \text{cdga}_k^{\leq 0})$ whose essential image are the square-zero extensions. Morally speaking, Φ sends $d_\eta : A \rightarrow A \oplus M$ to its homotopy fiber. The rest of this section makes this construction precise.

Definition 5.1.4. The ∞ -category $\mathcal{D}er_A$ of **derivations of A** has objects derivations $d : A \rightarrow M$ and spaces of morphisms $\mathcal{D}er_A(M_1, M_2) = A\text{-Mod}_A(M_1, M_2)$. The ∞ -category $\tilde{\mathcal{D}er}_A$ of **extended derivations of A** has objects consisting of homotopy pullback squares:

$$\begin{array}{ccc} A^\eta & \xrightarrow{f} & A \\ \downarrow & & \downarrow d_\eta \\ A & \xrightarrow{d_0} & A \oplus M, \end{array}$$

and spaces of morphisms consisting of morphisms of squares.

Note that $\tilde{\mathcal{D}er}_A$ can be described as the full ∞ -subcategory of $\text{Fun}(\Delta^1 \times \Delta^1, \text{cdga}_{/A}^{\leq 0})$ whose objects are homotopy pullback squares and have prescribed restrictions: $F(\{0, 0\}) = A$ and $F(\{1\} \times \Delta_1) = d_0 : A \rightarrow A \oplus M$.

There are two functors $F_1, F_2 : \tilde{\mathcal{D}er}_A \rightarrow \text{Fun}(\Delta^1, \text{cdga}_{/A}^{\leq 0})$ obtained by restricting to $\Delta_1 \times \{1\}$ and $\{0\} \times \Delta^1$, respectively. Note that their essential images are A -derivations and square-zero extensions of A , respectively, so that we have:

$$\begin{array}{ccc} & \tilde{\mathcal{D}er}_A & \\ F_1 \swarrow & & \searrow F_2 \\ \mathcal{D}er_A & & \text{Fun}(\Delta_1, \text{cdga}_{/A}). \end{array}$$

We prove that F_1 is a trivial Kan fibration, which implies that it has a section s . This will allow us to define $\Phi = F_2 \circ s$.

Lemma 5.1.5. F_1 is a trivial Kan fibration.

Proof. Consider the decomposition:

$$\begin{array}{ccc} \text{Fun}(\Delta^1 \times \Delta^1, \text{cdga}_{/A}) & \xrightarrow{R} & \text{Fun}(\Delta^1 \times \{1\}, \text{cdga}_{/A}) \\ & \searrow R_1 \quad \nearrow R_2 & \\ & \text{Fun}(\Lambda_2^2, \text{cdga}_{/A}) & \end{array}$$

F_1 is the restriction of R to $\tilde{\mathcal{D}}er_A$. Then we have:

1. Using Proposition 4.3.2.15 in [5], a restriction functor $\text{Fun}(\mathcal{C}, \mathcal{D}) \rightarrow \text{Fun}(\mathcal{C}_0, \mathcal{D})$ is a trivial Kan fibration as long as all functors in $\text{Fun}(\mathcal{C}, \mathcal{D})$ are Kan extensions of those in $\text{Fun}(\mathcal{C}_0, \mathcal{D})$. We apply this twice.
2. The pullback squares in $\text{Fun}(\Delta^1 \times \Delta^1, \text{cdga}_{\overline{A}}^{\leq 0})$ are Kan extensions, because all limits are Kan extensions. It follows that $R_1|_{\tilde{\mathcal{D}}er_A}$ is a trivial Kan fibration.
3. $R_1(\tilde{\mathcal{D}}er_A)$, the images of extended derivations in $\text{Fun}(\Lambda_2^2, \text{cdga}_{\overline{A}}^{\leq 0})$, are left Kan extensions. It follows that R_2 restricted to the images of extended derivations is a trivial Kan fibration.

□

Then we invoke the theorem saying that every trivial Kan fibration has a section (Todo: reference this), and define $\Phi = F_2 \circ s$.

5.2 n-small extensions

Let $f : A \rightarrow B$ be a map in $\text{cdga}_{\overline{k}}^{\leq 0}$, and let $I = \mathbf{hofib}(f)$. In other words, I is the homotopy pullback of the following diagram of non-unital commutative monoid objects in $A - \text{Mod}$:

$$\begin{array}{ccc} I & \longrightarrow & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B. \end{array}$$

This induces a non-unital commutative monoid structure on I ; in particular, I is an A -module, and there is a multiplication map $I \otimes_A I \rightarrow I$. The following is proposition 7.4.1.14. in [7].

Proposition 5.2.1. *The multiplication map $I \otimes_{A^n} I \rightarrow I$ is nullhomotopic.*

This motivates our definition of n-small extensions. The following definition and remarks are 7.4.1.18-7.4.1.21 in [7].

Definition 5.2.2. Let $f : A \rightarrow B$ be a map in $\text{cdga}_{\overline{k}}^{\leq 0}$, and let $n \geq 0$. We say that f is an **n-connective extension** if $\mathbf{hofib}(f) \in \text{cdga}_{\overline{k}}^{\leq -n}$. We say that f is an **n-small extension** if it is an n-connective extension and, moreover:

1. $\mathbf{hofib}(f) \in \text{cdga}_{\overline{k}}^{\geq -2n}$;
2. the multiplication map $\mathbf{hofib}(f) \otimes \mathbf{hofib}(f) \rightarrow \mathbf{hofib}(f)$ is nullhomotopic.

Remark 5.2.3. If $f : A \rightarrow B$ is an n-connective extension, from the long exact sequence on homotopy groups we see that $\pi_0(A) \rightarrow \pi_0(B)$ is surjective.

Remark 5.2.4. Suppose that $f : A \rightarrow B$ is an n-connective extension with $\mathbf{hofib}(f) \in \text{cdga}_{\overline{k}}^{\geq -2n}$. Since $\mathbf{hofib}(f) \in \text{cdga}_{\overline{k}}^{\leq -n}$, we also have that $\mathbf{hofib}(f) \otimes \mathbf{hofib}(f) \in \text{cdga}_{\overline{k}}^{\leq -2n}$. It follows that, at the level of homotopy groups, the only potentially nonzero map is:

$$\pi_{2n}(\mathbf{hofib}(f) \otimes \mathbf{hofib}(f)) \rightarrow \pi_{2n}(\mathbf{hofib}(f)). \quad (5.2.1)$$

Therefore condition 1 in the definition of an n-small extension simply requires that the map 5.2.1 is 0.

Example 5.2.5. Let A be a commutative ring, which we regard as a discrete commutative dga. A map $\tilde{A} \rightarrow A$ in $\text{cdga}_{\overline{k}}^{\leq 0}$ is a 0-small extension if and only if:

1. \tilde{A} is also discrete;

2. $f : \tilde{A} \rightarrow A$ is a surjective commutative ring homomorphism;
3. if I is the kernel of f , then $I^2 = 0$, as a consequence of 5.2.1.

So we recover the square-zero extensions in classical AG.

We want to prove that all n -small extensions are square-zero extensions. (But not vice-versa!) First, we identify what n -smallness should correspond to in terms of derivations. It's what you'd expect.

Definition 5.2.6. Let $\mathcal{D}er_{n\text{-con}}(A)$ denote the full subcategory of derivations $\eta : A \rightarrow M[1]$ such that $M \in A - \text{Mod}^{\leq -n}$. Let $\mathcal{D}er_{n\text{-sm}}(A)$ denote the full subcategory of derivations $\eta : A \rightarrow M[1]$ such that $M \in A - \text{Mod}^{\leq -n} \cap A - \text{Mod}^{\geq -2n}$.

The following is Theorem 7.4.1.23 in [7], and is the main result of this talk.

Theorem 5.2.7. Let $\Phi : \mathcal{D}er(A) \rightarrow \text{Fun}(\Delta^1, \text{cdga}_k^{\leq 0})$ be the functor constructed in section ???. For each $n \geq 0$, it induces an equivalence of categories:

$$\Phi_{n\text{-sm}} : \mathcal{D}er_{n\text{-sm}} \rightarrow \text{Fun}_{n\text{-sm}}(\Delta^1, \text{cdga}_k^{\leq 0}).$$

Proof. We just give a sketch. First, note that for a derivation $d_\eta : A \rightarrow A \oplus M[1]$, there is an equivalence between the homotopy fiber of the square-zero extension $A^\eta \rightarrow A$ and M . Moreover, multiplication on the fiber of a square-zero extension is nullhomotopic, by Proposition 5.2.1. It follows that the functor Φ restricts indeed to a functor $\Phi_{n\text{-sm}} : \mathcal{D}er_{n\text{-sm}} \rightarrow \text{Fun}_{n\text{-sm}}(\Delta^1, \text{cdga}_k^{\leq 0})$.

Φ admits a left adjoint Ψ , which sends a square-zero extension $A^\eta \rightarrow A$ to the derivation classified by $\mathbb{L}_A \rightarrow \mathbb{L}_{A/A^\eta}$. This restricts to a left adjoint of $\Phi_{n\text{-conn}}$, but we need to truncate in order to get an adjoint $\tau \circ \Psi_{n\text{-conn}}$ for $\Phi_{n\text{-sm}}$. Then we prove that this adjoint pair is an equivalence. \square

Corollary 5.2.8. Every n -small extension is a square-zero extension.

Corollary 5.2.9. Let $A \in \text{cdga}_k^{\leq 0}$, then every map in the Postnikov tower:

$$\dots \rightarrow \tau^{\geq -2}(A) \rightarrow \tau^{\geq -1}(A) \rightarrow \tau^{\geq 0}(A)$$

is a square-zero extension.

This is because the n -th stage is obviously an n -small extension, the homotopy fiber being equal to $\pi_n(A)[n]$ concentrated in degree n . This corollary is highly important, as it allows statements about derived affines A to be proved by induction on the Postnikov tower. The base step, for $\pi_0(A)$, is a classical AG statement, which is proved by classical methods. The inductive step reduces to a linear problem involving the derivation associated to the square-zero extension $\tau^{\geq i}(A) \rightarrow \tau^{\geq i-1}(A)$. The next section exemplifies this philosophy.

5.3 Induction on Postnikov tower

Proposition 5.3.1. Let $A \in sCA_k$, Assume we are given $j : \text{Spec}(\pi_0(A)) \rightarrow \mathbb{A}^n$, then there exists a lift of the map j to a map $\text{Spec}(A) \rightarrow \mathbb{A}^n$.

Proof. We use induction on the Postnikov tower. Suppose that there is a map $j_n : \text{Spec}(\tau^{\leq n} A) \rightarrow \mathbb{A}^n$, we show that there is a lifting j_{n+1} from $\text{Spec}(\tau^{\leq n+1} A)$ to \mathbb{A}^n . If we can prove this then as $A = \lim(\tau^{\leq n} A)$ and there thus exists a lifting of the map j to a map from $\text{Spec}(A)$.

Chapter 6

Perfect Complexes

Talk by Benedict Morrissey.

Half of this is about perfect complexes in classical AG, and the second half about what we do in the derived setting. In 2-3 weeks we will see that perfect complexes actually form a stack.

6.1 Classical

Let X be a scheme, then we look at $Ch^\bullet(QCoh(X))$.

Definition 6.1.1. $E^\bullet \in Ch^\bullet(QCoh(X))$ is **perfect** if it's Zariski locally quasi-isomorphic to an object of $Ch^b(Vect_X)$.

Remark 6.1.2. This is not the same as requiring cohomology to be finitely supported.

Definition 6.1.3. $E^\bullet \in Ch^\bullet(QCoh(X))$ has **Tor amplitude** in $[a, b]$ if for all $\mathcal{F} \in \mathcal{O}_X - Mod$,

$$H^k(E^\bullet \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{F}) = 0$$

for $k \notin [a, b]$. In particular, if E is in the heart, this is just saying that the Tor with any given sheaf is bounded.

Remark 6.1.4. For $\mathcal{F} = \mathcal{O}_X$, we just get the cohomology of E^\bullet .

This is sometimes difficult to work with, so we have:

Definition 6.1.5. $E^\bullet \in Ch^\bullet(QCoh(X))$ is **almost perfect** if Zariski locally there is a n -quasi-isomorphic to something in $Ch^b(Vect_X)$. n -quasi-isomorphism means isomorphism on cohomologies for $k \geq n + 1$, and surjection for degree n . (Todo: Figure out cohomological convention)

Perfect obviously implies almost perfect; this descends to the derived category of quasi-coherent sheaves.

Theorem 6.1.6. E^\bullet is perfect iff E^\bullet is almost perfect (for some n) and has finite Tor amplitude.

Proof. Locally free means flat, so tensoring it with anything preserves the tor amplitude. The other direction is in TT, Higher algebraic K-theory of schemes, 2.2.12. \square

Alternative definition: $E^\bullet \in D(Coh(X))$. Locally on some U we have an $\mathcal{O}_X(U)$ -module $E|_U^\bullet$. We require that this is bounded above and has coherent cohomologies. Equivalently, $\tau_{\leq n}(E^\bullet|_n)$ is compact in $\tau_{\leq n}Mod_{\mathcal{O}_X(U)}$.

Theorem 6.1.7. For X affine, E^\bullet is perfect if and only if it's globally quasi-isomorphic to an object of $Ch^b(Vect_X)$.

Theorem 6.1.8. *If X is smooth and Noetherian, then $D(\text{Coh}^b(X)) \simeq D(\text{Perf}(X))$.*

We'll prove this by Serre regularity.

Definition 6.1.9. A is **regular** if $\dim_k(m/m^2) = \dim_{K_{\text{rull}}} A$. The **global dimension** of A is:

$$\text{gldim}(A) = \sup_{M \in A\text{-Mod}} \text{projdim}(M),$$

where the latter is the minimum length of a projective resolution of M .

Theorem 6.1.10 (Serre regularity). *If A is a Noetherian local ring, TFAE:*

1. A is regular;
2. $\text{gldim}(A) < \infty$;
3. $\text{gldim}(A) = \dim_{K_{\text{rull}}} A$.

Going back to X smooth and Noetherian, we know that all local rings are regular. $E^\bullet \in \text{Coh}^b(X)$, then E_p^\bullet is a $\mathcal{O}_{X,p}$ module.

Proof. We actually just do the case of E in the heart, because it's easier. (For the other one, we probably resolve to a double complex, and take the total complex.)

Take a projective resolution of E_p^\bullet as a $\mathcal{O}_{X,p}$ module; we know it must be finite:

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow E_p^\bullet \rightarrow 0.$$

We can do this in an open set around p : (Todo: draw this from paper)

This takes care of one direction. We then use Tor dimension to show that, if E is perfect, it's in $D(\text{Coh}^b(X))$. \square

For the non-smooth case, we look at the ind-completion $\text{Ind}(\text{Perf}(X)) \simeq \text{QCoh}(X)$. (Always true for X quasi-compact, quasi-separated.) On the other side, $\text{Ind}(\text{Coh}^b(X)) = \text{IndCoh}(X)$. The quotient $\text{IndCoh}(X)/\text{QCoh}(X) = D_{\text{sing}}(X)$, which really sees the singularities of X .

6.2 derived

Let $A \in \text{SCR}_k$; recall that Mod_A is a stable ∞ -category.

Definition 6.2.1. $\text{Mod}_A^{\text{perf}} \subset \text{Mod}_A$ is the smallest stable subcategory containing A and closed under retracts. Recall that A is a retract of B if there exist maps $i : A \rightarrow B, r : B \rightarrow A$ such that $r \circ i = \text{id}_A$.

Definition 6.2.2. $N \in \mathcal{C}$ is **compact** if $\mathcal{H}om_{\mathcal{C}}(N, -)$ commutes with filtered colimits. The latter means that the index category is nonempty, and for all $i, j \in I$, there exists k such that $i \rightarrow k \leftarrow j$, and coequalizers exist.

Example 6.2.3. Consider $(\text{Mod}_A)^\heartsuit$. The compact objects are the finitely presented ones. We have a map $A^n \rightarrow M$, so:

$$\text{Hom}(M, \varinjlim_I B_i) \simeq \varinjlim_I \text{Hom}(M, B_i),$$

because if we have a map $M \rightarrow B$, we can fully describe it by the composition $A^n \rightarrow B$. Each of the n generators goes to some B_{i_k} , so by the definition of filtered index category, there exists some B_j such that $A^n \rightarrow B_j$.

Conversely, starting with compact M , we look at finitely generated submodules M_i , and we have:

$$\text{Hom}(M, \varinjlim_I M_i) \simeq \varinjlim_I \text{Hom}(M, M_i).$$

In particular, the identity map $M \rightarrow M$ factors through some M_j , so $M = M_j$.

Theorem 6.2.4. $M \in \mathcal{M}od_A$ is perfect iff it's compact.

Proof. $\mathcal{M}od_A^{\text{perf}} \subset \mathcal{M}od_A^{\text{cpct}}$. (Todo: add diagram from paper)

Since DK is an equivalence, we only need to argue that truncation and the forgetful functor preserve filtered colimits. For the first one: filtered colimits are t-exact. For the second one: it does.

For the other direction, we have the inclusion $\mathcal{M}od_A^{\text{perf}} \rightarrow \mathcal{M}od_A$, we factor this through Ind , which is just the completion with respect to filtered colimits.

$$\begin{array}{ccc} \mathcal{M}od_A^{\text{perf}} & \longrightarrow & \mathcal{M}od_A \\ & \uparrow \phi & \\ & \text{Ind}(\mathcal{M}od_A^{\text{perf}}) & \end{array}$$

f is obviously fully faithful, because $\text{Ind}(\mathcal{M}od_A^{\text{perf}})^{\omega} = \mathcal{M}od_A^{\text{perf}}$.¹ Mapping spaces in Ind are computed by:

$$\mathcal{M}ap_{\text{Ind}(\mathcal{C})}(\text{colim}_{i \in I} \mathcal{F}_i, \text{colim}_{j \in J} \mathcal{G}_j) = \lim_{i \in I} \text{colim}_{j \in J} \mathcal{M}ap_{\mathcal{C}}(\mathcal{F}_i, \mathcal{G}_j).$$

This means we're computing mapping spaces by the same formula, so ϕ is fully faithful. \square

The following is 7.2.4.5 in [7]:

Theorem 6.2.5. $M \in \mathcal{M}od_A^{\text{perf}}$, we have:

1. $\pi_n M = 0$ for $n \gg 0$;
2. If $\pi_m M \cong 0$ for all $m > k$, then $\pi_k M$ is finitely presented as a $\pi_0(M)$ -module.

Example 6.2.6. Think about $\text{Sym}(k[2])$ as a module over itself. It is perfect by definition, but it's not bounded below.

¹We'll talk more about this equality later, it follows because Perf is idempotent complete.

Bibliography

- [1] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck’s FGA explained.
- [2] W D Gillam. *Simplicial Methods In Algebra And Algebraic Geometry*.
- [3] Moritz Groth. A short course on ∞ -categories. Preprint, 2010.
- [4] Jacob Lurie. *Derived Algebraic Geometry V: Structurd Spaces*.
- [5] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.
- [6] Jacob Lurie. DAG XIII: Rational and p-adic homotopy theory. Preprint, 2011.
- [7] Jacob Lurie. Higher algebra. Preprint, August 2012.
- [8] Mauro Porta. Exercises. 2016.
- [9] Mauro Porta and Gabriele Vezzosi. Infinitesimal and square-zero extensions of simplicial algebras. Preprint, 2013.
- [10] Mauro Porta and Tony Yue Yu. Higher analytic stacks and GAGA theorems. *arXiv preprint arXiv:1412.5166*, 2014.
- [11] Stefan Schwede and Brooke Shipley. Equivalences of monoidal model categories. *Algebraic & Geometric Topology*, 3(1):287–334, 2003.
- [12] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008.