

Chapter 1

∞ -category theory

1.1 Motivations

Exercise 1.1.1. We fix a base field k . Let $X = \mathbb{P}_k^1$ and let U_0 and U_1 be the standard open affine cover of \mathbb{P}_k^1 . For any k -algebra A , we have:

$$U_0(A) := \{[x_0 : x_1] \in \mathbb{P}_k^1(A) \mid x_0 \neq 0\}, \quad U_1(A) := \{[x_0 : x_1] \in \mathbb{P}_k^1(A) \mid x_1 \neq 0\}.$$

Let $U_{01} = U_0 \cap U_1$ be their intersection. Show that the canonical functor

$$h(\mathcal{D}(\mathbb{P}_k^1)) \rightarrow h(\mathcal{D}(U_0)) \times_{h(\mathcal{D}(U_{01}))} h(\mathcal{D}(U_1))$$

is essentially surjective but not fully faithful.

Exercise 1.1.2. Let \mathcal{C} be a triangulated category where countable products and countable direct sums exist. Show that if there exists a functor Tr from the category of arrows \mathcal{C}^{Δ^1} to the category of exact triangles in \mathcal{C} , then every triangle in \mathcal{C} is split. (See [4, Proposition II.1.2.13].)

1.2 Reminders on simplicial sets

Exercise 1.2.1. Show that the nerve functor $N: \text{Cat} \rightarrow \text{sSet}$ is fully faithful and its essential image is spanned by those simplicial sets K satisfying the following lifting condition: for every $n \geq 2$ and for every $0 < i < n$ every lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & K \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a unique solution.

Solution. The nerve of a category \mathcal{C} is:

$$(N\mathcal{C})_n = \{(f_1, \dots, f_n) \mid \text{composable morphisms}\}.$$

The face maps are:

$$d_j(f_1, \dots, f_n) = \begin{cases} (f_1, \dots, f_{n-1}), & j = 0 \\ (f_1, \dots, f_j \circ f_{j-1}, \dots, f_n), & 0 < j < n \\ (f_2, \dots, f_n), & j = n. \end{cases}$$

The degeneracy s_j is obtained by inserting an identity map in the j^{th} slot.

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$, induces a simplicial map:

$$\begin{aligned} N(F)_n : (N\mathcal{C})_n &\rightarrow (N\mathcal{D})_n \\ (f_1, \dots, f_n) &\mapsto (F(f_1), \dots, F(f_n)). \end{aligned}$$

If two functors F, F' induce simplicial maps $N(F) = N(F')$ which agree, then $F(f) = F'(f)$ for every morphism f . Hence N is faithful. Given a simplicial map $G : N\mathcal{C} \rightarrow N\mathcal{D}$, we define a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ to be G_0 on objects and G_1 on morphisms. We show that F respects composition. Let f_1, f_2 be two composable morphisms in \mathcal{C} and denote by x the 2-simplex (f_1, f_2) . Then:

$$F(f_2 \circ f_1) = G(d_1 x) = d_1 G(x) = F(f_2) \circ F(f_1).$$

This proves that N is also full.

We move on to the essential image. Let K be the nerve of a category. The data of a map $\Lambda_i^n \rightarrow K$ is the same as the data of maps $y_j : \Delta^{n-1} \rightarrow K$ for $j \neq i$, which are compatible along their faces. By Yoneda, this is the same as simplices $\{y_j \in K_{n-1}\}_{j \neq i}$ compatible along faces. Given this data, we define the horn filler $x \in K_n$ by:

$$x = ((d_0)^{n-2} y_{n-1}, (d_0)^{n-3} d_n y_{n-1}, \dots, d_0 (d_n)^{n-3} y_{n-1}, (d_n)^{n-2} y_0).$$

The simplicial identities ensure that $d_j x = y_j$ for $j \neq i$.¹ Using the compatibility of the y_j along faces, x is the unique solution to the lifting problem.

Conversely, given a K which has unique solutions to all lifting problems of inner horns, we define a category \mathcal{C} such that $K \cong \mathcal{C}$. Let K_0 be the objects of \mathcal{C} , and for $X, Y \in K_0$, define:

$$\text{Hom}(X, Y) := \{f \in K_1 \mid d_1 f = X, d_0 f = Y\}.$$

Given $f_1 : X \rightarrow Y$ and $f_2 : Y \rightarrow Z$, define a lifting problem by mapping the 1-simplices $0 \rightarrow 1$ and $1 \rightarrow 2$ in Λ_1^2 to f_1 and f_2 , respectively. We define $f_2 \circ f_1$ to be d_1 of the unique lift. Associativity of this composition follows from the unique filling of the horn Λ_1^3 ; we don't give the details here. \square

Exercise 1.2.2. Let S, S' be sets, seen as discrete simplicial set. Show that any morphism $f : S \rightarrow S'$ is a Kan fibration, and that f is a trivial Kan fibration if and only if f is a bijection.

Solution. Since S and S' are sets, all k -simplices are of the form $s^k x$, for x a 0-simplex. Given a lifting problem:

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & S \\ \downarrow & \nearrow & \downarrow f \\ \Delta^n & \longrightarrow & S' \end{array}$$

all k -simplices of Λ_i^n , for $k > 0$, must map to degenerate k -simplices in S . Hence Λ_i^n maps to a point $s \in S$. Similarly, Δ^n maps to $f(s)$. The constant map from Δ^n to s is then the unique solution to the lifting problem. It follows that f is a Kan fibration, and moreover that all sets S are ∞ -groupoids.

By definition, f is a weak equivalence if it induces a weak equivalence on geometric realizations. $|S|$ and $|S'|$ are discrete topological spaces, therefore $|f|$ is a weak equivalence iff it is a bijection. \square

Exercise 1.2.3. Let G and H be simplicial groups and let $f : G \rightarrow H$ be a surjective group homomorphism. Show that f is a Kan fibration.

Solution. There is an algorithm for constructing fillers on nLab.² We don't have any intuition for it, so we should work on building that.

The algorithm produces unique fillers for all horns, so in particular simplicial groups are ∞ -groupoids. \square

¹Note that it's essential that both y_0 and y_{n-1} are available to use in the definition of x , i.e. that Λ_i^n is an inner horn.

²<https://ncatlab.org/nlab/show/simplicial+group>

Exercise 1.2.4. Let $\partial\Delta^2$ be the smallest full subsimplicial set of Δ^2 spanned by its non-degenerate edges $\Delta^1 \rightarrow \Delta^2$. Show that $\partial\Delta^2$ fits into a coequalizer diagram

$$(\Delta^0)^{\amalg 6} \rightrightarrows (\Delta^1)^{\amalg 3} \rightarrow \partial\Delta^2.$$

(Hint: Have a look at [2, Theorem III.3.1].)

Exercise 1.2.5. Let S be a set, seen as a discrete simplicial set. Show that $\mathrm{cosk}_n(S)$ satisfies the following property: for every $m \geq n$ and every $0 \leq i \leq m$ the lifting problem

$$\begin{array}{ccc} \Lambda_i^n & \longrightarrow & \mathrm{cosk}_n(S) \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

has a solution. In particular, deduce that $\mathrm{cosk}_0(S)$ is a Kan complex.

1.3 ∞ -categories

Exercise 1.3.1. Show that every Kan complexes and 1-categories are ∞ -categories (quasicategories).

Solution. Kan complexes have fillers for all horns. 1-categories have unique fillers for all inner horns. In particular, both have fillers for all inner horns, which is the definition of ∞ -categories. \square

Exercise 1.3.2. A morphism $f: X \rightarrow Y$ in an ∞ -category \mathcal{C} is said to be an equivalence if its image in $\mathrm{h}(\mathcal{C})$ is an isomorphism. Define $S^\infty := \mathrm{cosk}_0(\{0, 1\})$ and let $j: \Delta^1 \rightarrow S^\infty$ be the map classified by

$$\mathrm{sk}_0(\Delta^1) = \{0, 1\} \xrightarrow{\mathrm{id}} \{0, 1\}.$$

To give a morphism $f: X \rightarrow Y$ in an ∞ -category \mathcal{C} it is equivalent to specify a morphism of simplicial sets $e_f: \Delta^1 \rightarrow \mathcal{C}$. Show that f is an equivalence if and only if the lifting problem

$$\begin{array}{ccc} \Delta^1 & \xrightarrow{e_f} & \mathcal{C} \\ \downarrow j & \nearrow & \\ S^\infty & & \end{array}$$

has at least one solution. Next, show that any two such solution are homotopic. (Hint: have a look at Exercises 1.2.5 and 1.4.1.)

Exercise 1.3.3. In [3] a functor of ∞ -categories $f: \mathcal{C} \rightarrow \mathcal{D}$ is said to be a *categorical equivalence* if and only if the induced functor $\mathcal{C}[f]: \mathcal{C}[\mathcal{C}] \rightarrow \mathcal{C}[\mathcal{D}]$ is an equivalence of simplicial categories. Show that f is a categorical equivalence if and only if it is fully faithful and essentially surjective.

Exercise 1.3.4. Let E denote the walking isomorphism (i.e. the 1-category with two objects and an isomorphism between them). Recall the definition of S^∞ from the previous exercise. Show that there is a canonical map $E \rightarrow S^\infty$ and that this is a categorical equivalence. In particular, for every ∞ -category \mathcal{C} , the functor

$$\mathrm{Fun}(S^\infty, \mathcal{C}) \rightarrow \mathrm{Fun}(E, \mathcal{C})$$

is a categorical equivalence. (This is a very simple example of what an “internal rectification theorem” looks like.)

Exercise 1.3.5. Let \mathcal{C} be an ∞ -category. Let S_0 be a collection of *objects* in \mathcal{C} . Let \mathcal{C}_0 be the smallest full subsimplicial set of \mathcal{C} containing S_0 (explicitly, an n -simplex $\sigma: \Delta^n \rightarrow \mathcal{C}$ belongs to \mathcal{C}_0 if and only if for every morphism $\Delta^0 \rightarrow \Delta^n$ the composition $\Delta^0 \rightarrow \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0 .) Show that \mathcal{C}_0 is an ∞ -category. Furthermore, show that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ of simplicial sets is a fully faithful functor of ∞ -categories.

Exercise 1.3.6. Let \mathcal{C} be an ∞ -category. Let S_0 be a collection of *morphisms* in \mathcal{C} , and suppose that S_0 is closed under composition, in the sense that for every 2-simplex

$$\begin{array}{ccc} & Y & \\ f \nearrow & & \searrow g \\ X & \xrightarrow{h} & Z \end{array}$$

is \mathcal{C} , if f and g belong to S_0 then so does h . Let \mathcal{C}_0 be the smallest full subsimplicial set of \mathcal{C} containing S_0 (explicitly, an n -simplex $\sigma: \Delta^n \rightarrow \mathcal{C}$ belongs to \mathcal{C} if and only if for every morphism $\Delta^1 \rightarrow \Delta^n$ the composition $\Delta^1 \rightarrow \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0). Show that \mathcal{C}_0 is an ∞ -category.

Exercise 1.3.7. Let \mathcal{C} be an ∞ -category. Show that the collection of equivalences in \mathcal{C} is closed under composition, in the sense of the previous exercise. Let \mathcal{C}^\simeq be the ∞ -subcategory of \mathcal{C} spanned by equivalences in \mathcal{C} . Show that \mathcal{C}^\simeq is a Kan complex.

1.4 Localization of ∞ -categories

Exercise 1.4.1. Let \mathcal{C} be an ∞ -category (seen as a quasicategory). Let $\mathcal{C} \rightarrow \tilde{\mathcal{C}}$ be a fibrant replacement for the Kan model structure on \mathbf{sSet} . Show that $\tilde{\mathcal{C}}$ enjoys the following universal property: for every ∞ -category \mathcal{D} the functor of ∞ -categories

$$\mathrm{Fun}(\tilde{\mathcal{C}}, \mathcal{D}) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and its essential image is spanned by those morphisms $f: \mathcal{C} \rightarrow \mathcal{D}$ that send every morphism in \mathcal{C} into an equivalence in \mathcal{D} . Thus, there is a categorical equivalence $\tilde{\mathcal{C}} \simeq \mathcal{C}[W^{-1}]$, where W denotes the collection of all arrows in \mathcal{C} . Deduce that if \mathcal{C} is an ∞ -category where every morphism is invertible, then \mathcal{C} is categorically equivalent to a Kan complex.

Exercise 1.4.2. Let \mathcal{C} be an ∞ -category and let S be a (small) collection of arrows in \mathcal{C} . Show that $\mathrm{h}(\mathcal{C}[S^{-1}]) \in \mathbf{Cat}$ is canonically equivalent to the 1-categorical localization of $\mathrm{h}(\mathcal{C})$ at \bar{S} , the collection of morphism which is the image of S via the canonical functor $\mathcal{C} \rightarrow \mathrm{h}(\mathcal{C})$.

Exercise 1.4.3. Let \mathcal{C} be an ∞ -category with finite limits and let S be a (small) collection of arrows in \mathcal{C} . Suppose that \mathcal{C} is stable under pullbacks. Then the ∞ -categorical localization $\mathcal{C}[S^{-1}]$ has finite limits and the localization functor $L: \mathcal{C} \rightarrow \mathcal{C}[S^{-1}]$ commutes with them.

1.5 Limits and colimits

Exercise 1.5.1. Let \mathcal{S} be the ∞ -category of spaces and let X be an object in \mathcal{S} . Using [3, Theorem 4.2.4.1] show that the colimit of the diagram

$$* \longleftarrow X \longrightarrow *$$

can be canonically identified with $\Sigma(X)$.

Now fix two points $p, q: * \rightarrow X$. Show that the limit of the diagram

$$* \xrightarrow{p} X \xleftarrow{q} *$$

can be canonically identified with the path space $\mathrm{Path}_X(p, q)$.

Exercise 1.5.2. n -cofinality...

Exercise 1.5.3. ★ Let K be a simplicial set and let $F: K^{\text{op}}\mathcal{P}\mathcal{R}^{\text{L}}$ be an ∞ -functor. Let \mathcal{C} be a presentable ∞ -category and let $\Delta_{\mathcal{C}}: K^{\text{op}} \rightarrow \mathcal{P}\mathcal{R}^{\text{L}}$ denote the constant ∞ -functor associated to F . Let $\varphi: \Delta_{\mathcal{C}} \rightarrow F$ be a natural transformation in $\text{Fun}(K^{\text{op}}, \mathcal{P}\mathcal{R}^{\text{L}})$. We let

$$\Phi: \mathcal{C} \rightarrow \varprojlim F$$

be the induced functor. For every $x \in K$, the functor $\varphi_x: \mathcal{C} \rightarrow F(x)$ admits a right adjoint, which we denote $\psi_x: F(x) \rightarrow \mathcal{C}$. Show that there exists an ∞ -functor

$$\overline{\Psi}: \varprojlim F \rightarrow \text{Fun}(K, \mathcal{C})$$

which informally sends $Y = \{Y_x\}_{x \in K} \in \varprojlim F$ to the diagram $\overline{\Psi}(Y): K \rightarrow \mathcal{C}$ given by

$$\overline{\Psi}(Y)(x) = \psi_x(Y_x).$$

Prove moreover that the composition

$$\varprojlim F \xrightarrow{\overline{\Psi}} \text{Fun}(K, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

can be canonically identified with a right adjoint for Φ .

1.6 Left and right fibrations

Exercise 1.6.1. Let X be a connected Kan complex and let F be any other Kan complex. Let us further fix a point $x \in X$. Let $\text{LF}_x(X; F)$ be the full subcategory of left fibrations $\text{LF}(X)$ over X whose homotopy fiber at x is equivalent to F . Let $\text{B}(\text{hAut}(F))$ be the classifying space of the simplicial group of homotopy automorphisms of F . Show that there is a canonical equivalence of ∞ -categories

$$\text{LF}_x(X; F) \simeq \text{Fun}(X, \text{B}(\text{hAut}(F))).$$

1.7 Cartesian and coCartesian fibrations

Exercise 1.7.1. Let \mathcal{C} be an ∞ -category and let $X \in \mathcal{C}$ be an object. Let $f: U \rightarrow X$ and $g: V \rightarrow X$ be two morphisms in \mathcal{C} . For every 2-simplex $\sigma: \Delta^2 \rightarrow \mathcal{C}$ such that $d_0(\sigma) = f$ and $d_1(\sigma) = g$, show that there is a pullback square in \mathcal{S} :

$$\begin{array}{ccc} \text{Path}_{\text{Map}_{\mathcal{C}}(U, X)}(f, d_2(\sigma)) & \longrightarrow & \text{Map}_{\mathcal{C}/X}(f, g) \\ \downarrow & & \downarrow \\ * & \xrightarrow{d_2(\sigma)} & \text{Map}_{\mathcal{C}}(U, V). \end{array}$$

(Hint: Use [3, Propositions 2.1.2.1 and 2.4.4.2].)

1.8 Adjunctions

Exercise 1.8.1. Let \mathcal{C} be an ∞ -category with a zero object 0 . Suppose that for every object $X \in \mathcal{C}$ the span

$$0 \longleftarrow X \longrightarrow 0$$

has both a limit $\Omega(X)$ and a colimit $\Sigma(X)$. Construct in an explicit way ∞ -functors $\Sigma, \Omega: \mathcal{C} \rightarrow \mathcal{C}$ informally given by $X \mapsto \Sigma(X)$ and $X \mapsto \Omega(X)$, respectively. Show that Σ and Ω are adjoint by explicitly constructing a fibration $\mathcal{D} \rightarrow \Delta^1$ which is both Cartesian and coCartesian.

Exercise 1.8.2. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be an ∞ -functor. Show that the following statements are equivalent:

1. F has a right adjoint $G: \mathcal{D} \rightarrow \mathcal{C}$;
2. for every $Y \in \mathcal{D}$ there exists an object $X \in \mathcal{C}$ and a morphism $\varepsilon_X: F(X) \rightarrow Y$ such that for every other $X' \in \mathcal{C}$ the canonical composition

$$\mathrm{Map}_{\mathcal{C}}(X', X) \xrightarrow{f} \mathrm{Map}_{\mathcal{D}}(f(X'), f(X)) \xrightarrow{\varepsilon_X^*} \mathrm{Map}_{\mathcal{D}}(f(X'), Y)$$

is a weak homotopy equivalence.

1.9 Stable ∞ -categories

Exercise 1.9.1. Let \mathcal{C} be a stable ∞ -category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full stable subcategory of \mathcal{C} . Let $S := \{f: X \rightarrow Y \in \mathcal{C} \mid \mathrm{cofib}(f) \in \mathcal{D}\}$. Show that the ∞ -categorical localization $\mathcal{C}[S^{-1}]$ is a stable ∞ -category.

Exercise 1.9.2. It is shown in [1] that $\mathrm{Cat}_{\infty}^{\mathrm{Ex}}$ is a presentable ∞ -category. Prove directly that cofibers in $\mathrm{Cat}_{\infty}^{\mathrm{Ex}}$ exist.

Chapter 2

Derived rings

2.1 Derived rings

Exercise 2.1.1. Show that a discrete commutative ring A over k is finitely presented if and only if its associated corepresentable functor

$$\mathrm{Hom}_{\mathrm{CAlg}_k}(A, -): \mathrm{CAlg}_k \rightarrow \mathrm{Set}$$

commutes with filtered colimits.

Exercise 2.1.2. Let $A \in \mathrm{sCAlg}_k$ and let $M \in A\text{-Mod}^{\leq 0}$. Show that the diagram

$$\begin{array}{ccc} \mathrm{Sym}_A(M) & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & \mathrm{Sym}_A(M[1]) \end{array}$$

is a (homotopy) pushout square (where the two maps $\mathrm{Sym}_A(M) \rightarrow A$ are both classified by the zero map $M \rightarrow A$, and where both the maps $A \rightarrow \mathrm{Sym}_A(M[1])$ are the structure morphisms).

Exercise 2.1.3. Let $A \in \mathrm{sCAlg}_k$ and let $M \in A\text{-Mod}^{\leq 0}$. Let $A \oplus M$ denote the split square-zero extension of A by M . Show that the diagram

$$\begin{array}{ccc} A \oplus M & \longrightarrow & A \\ \downarrow & & \downarrow d_0 \\ A & \xrightarrow{d_0} & A \oplus M[1] \end{array}$$

is a homotopy pullback, where $d_0: A \rightarrow A \oplus M[1]$ is the morphism classifying the zero derivation.

2.2 Modules

Exercise 2.2.1. Let A be a discrete commutative ring over k . Show that $M \in A\text{-Mod}^{\heartsuit}$ is finitely generated if and only if its associated corepresentable functor

$$\mathrm{Hom}_{A\text{-Mod}^{\heartsuit}}(M, -): A\text{-Mod}^{\heartsuit} \rightarrow \mathrm{Set}$$

commutes with filtered colimits of monomorphisms.

Exercise 2.2.2. Let A be a discrete commutative ring over k . Show that $M \in A\text{-Mod}^{\heartsuit}$ is finitely presented if and only if its associated corepresentable functor

$$\mathrm{Hom}_{A\text{-Mod}^{\heartsuit}}(M, -): A\text{-Mod}^{\heartsuit} \rightarrow \mathrm{Set}$$

commutes with filtered colimits.

2.3 Cotangent complex

Exercise 2.3.1. Compute the cotangent complex of the following morphisms:

1. $k \rightarrow k[\varepsilon]/(\varepsilon^2)$, $\deg(\varepsilon) = 0$;
2. $k[X, Y] \rightarrow k[X, Y]/(Y^3 - X^2)$;
3. $k \rightarrow k[X, Y]/(Y^3 - X^2)$.

Exercise 2.3.2. Find all the square-zero extensions (up to homotopy) of $R := k[\varepsilon]/(\varepsilon^2)$ by $k \simeq R/(\varepsilon)$. What happens if we replace k by $k[n]$, $n \geq 0$?

Chapter 3

Derived stacks

Bibliography

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