Chapter 1

∞-category theory

1.1 Motivations

Exercise 1.1.1. We fix a base field k. Let $X = \mathbb{P}^1_k$ and let U_0 and U_1 be the standard open affine cover of $\mathbb{P}^1_k$. For any k-algebra A, we have:

$$U_0(A) \coloneqq \{[x_0 : x_1] \in \mathbb{P}^1_k(A) \mid x_0 \neq 0\}, \qquad U_1(A) \coloneqq \{[x_0 : x_1] \in \mathbb{P}^1_k(A) \mid x_1 \neq 0\}.$$

Let $U_{01} = U_0 \cap U_1$ be their intersection. Show that the canonical functor

$$\mathrm{h}(\mathcal{D}(\mathbb{P}^1_k)) \to \mathrm{h}(\mathcal{D}(U_0)) \times_{\mathrm{h}(\mathcal{D}(U_{01}))} \mathrm{h}(\mathcal{D}(U_1))$$

is essentially surjective but not fully faithful.

Exercise 1.1.2. Let $\mathcal{C}$ be a triangulated category where countable products and countable direct sums exist. Show that if there exists a functor Tr from the category of arrows $\mathcal{C}^{\Delta^1}$ to the category of exact triangles in $\mathcal{C}$, then every triangle in $\mathcal{C}$ is split. (See [4, Proposition II.1.2.13].)

1.2 Reminders on simplicial sets

Exercise 1.2.1. Show that the nerve functor $\mathrm{N}\colon \mathrm{Cat} \to \mathrm{sSet}$ is fully faithful and its essential image is spanned by those simplicial sets K satisfying the following lifting condition: for every $n \geq 2$ and for every $0 < i < n$ every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a unique solution.

Exercise 1.2.2. Let S, S' be sets, seen as discrete simplicial set. Show that any morphism $f\colon S \to S'$ is a Kan fibration, and that f is a trivial Kan fibration if and only if f is a bijection.

Exercise 1.2.3. Let G and H be simplicial groups and let $f\colon G \to H$ be a surjective group homomorphism. Show that f is a Kan fibration.

Exercise 1.2.4. Let $\partial\Delta^2$ be the smallest full subsimplicial set of Δ^2 spanned by its non-degenerate edges $\Delta^1 \to \Delta^2$. Show that $\partial\Delta^2$ fits into a coequalizer diagram

$$(\Delta^0)^{\amalg 6} \rightrightarrows (\Delta^1)^{\amalg 3} \to \partial\Delta^2.$$

(Hint: Have a look at [2, Theorem III.3.1].)

Exercise 1.2.5. Let S be a set, seen as a discrete simplicial set. Show that $\operatorname{cosk}_n(S)$ satisfies the following property: for every $m \geq n$ and every $0 \leq i \leq m$ the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \operatorname{cosk}_n(S) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

has a solution. In particular, deduce that $\operatorname{cosk}_0(S)$ is a Kan complex.

1.3 ∞-categories

Exercise 1.3.1. Show that every Kan complexes and 1-categories are ∞-categories (quasicategories).

Exercise 1.3.2. A morphism $f\colon X \to Y$ in an ∞-category $\mathcal{C}$ is said to be an equivalence if its image in $\mathrm{h}(\mathcal{C})$ is an isomorphism. Define $S^\infty := \operatorname{cosk}_0(\{0,1\})$ and let $j\colon \Delta^1 \to S^\infty$ be the map classified by

$$\operatorname{sk}_0(\Delta^1) = \{0,1\} \xrightarrow{\mathrm{id}} \{0,1\}.$$

To give a morphism $f\colon X \to Y$ in an ∞-category $\mathcal{C}$ it is equivalent to specify a morphism of simplicial sets $e_f\colon \Delta^1 \to \mathcal{C}$. Show that f is an equivalence if and only if the lifting problem

$$\begin{tikzcd} \Delta^1 \arrow[r, "e_f"] \arrow[d, hook, "j"] & \mathcal{C} \\ S^\infty \arrow[ur, dashed] & \end{tikzcd}$$

has at least one solution. Next, show that any two such solution are homotopic. (Hint: have a look at Exercises 1.2.5 and 1.4.1.)

Exercise 1.3.3. In [3] a functor of ∞-categories $f\colon \mathcal{C} \to \mathcal{D}$ is said to be a *categorical equivalence* if and only if the induced functor $\mathfrak{C}[f]\colon \mathfrak{C}[\mathcal{C}] \to \mathfrak{C}[\mathcal{D}]$ is an equivalence of simplicial categories. Show that f is a categorical equivalence if and only if it is fully faithful and essentially surjective.

Exercise 1.3.4. Let E denote the walking isomorphism (i.e. the 1-category with two objects and an isomorphism between them). Recall the definition of S^∞ from the previous exercise. Show that there is a canonical map $E \to S^\infty$ and that this is a categorical equivalence. In particular, for every ∞-category $\mathcal{C}$, the functor

$$\operatorname{Fun}(S^\infty, \mathcal{C}) \to \operatorname{Fun}(E, \mathcal{C})$$

is a categorical equivalence. (This is a very simple example of what an “internal rectification theorem” looks like.)

Exercise 1.3.5. Let $\mathcal{C}$ be an ∞-category. Let S_0 be a collection of *objects* in $\mathcal{C}$. Let $\mathcal{C}_0$ be the smallest full subsimplicial set of $\mathcal{C}$ containing S_0 (explicitly, an n-simplex $\sigma\colon \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}$ if and only if for every morphism $\Delta^0 \to \Delta^n$ the composition $\Delta^0 \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0.) Show that $\mathcal{C}_0$ is an ∞-category. Furthermore, show that the inclusion $\mathcal{C}_0 \hookrightarrow \mathcal{C}$ of simplicial sets is a fully faithful functor of ∞-categories.

Exercise 1.3.6. Let $\mathcal{C}$ be an ∞-category. Let S_0 be a collection of *morphisms* in $\mathcal{C}$, and suppose that S_0 is closed under composition, in the sense that for every 2-simplex

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

is $\mathcal{C}$, if f and g belong to S_0 then so does h. Let $\mathcal{C}_0$ be the smallest full subsimplicial set of $\mathcal{C}$ containing S_0 (explicitly, an n-simplex $\sigma\colon \Delta^n \to \mathcal{C}$ belongs to $\mathcal{C}$ if and only if for every morphism $\Delta^1 \to \Delta^n$ the composition $\Delta^1 \to \Delta^n \xrightarrow{\sigma} \mathcal{C}$ belongs to S_0). Show that $\mathcal{C}_0$ is an ∞-category.

Exercise 1.3.7. Let $\mathcal{C}$ be an ∞-category. Show that the collection of equivalences in $\mathcal{C}$ is closed under composition, in the sense of the previous exercise. Let $\mathcal{C}^{\simeq}$ be the ∞-subcategory of $\mathcal{C}$ spanned by equivalences in $\mathcal{C}$. Show that $\mathcal{C}^{\simeq}$ is a Kan complex.

1.4 Localization of ∞-categories

Exercise 1.4.1. Let $\mathcal{C}$ be an ∞-category (seen as a quasicategory). Let $\mathcal{C} \to \widetilde{\mathcal{C}}$ be a fibrant replacement for the Kan model structure on sSet. Show that $\widetilde{\mathcal{C}}$ enjoys the following universal property: for every ∞-category $\mathcal{D}$ the functor of ∞-categories

$$\mathrm{Fun}(\widetilde{\mathcal{C}}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful and its essential image is spanned by those morphisms $f\colon \mathcal{C} \to \mathcal{D}$ that send every morphism in $\mathcal{C}$ into an equivalence in $\mathcal{D}$. Thus, there is a categorical equivalence $\tilde{\mathcal{C}} \simeq \mathcal{C}[W^{-1}]$, where W denotes the collection of all arrows in $\mathcal{C}$. Deduce that if $\mathcal{C}$ is an ∞-category where every morphism is invertible, then $\mathcal{C}$ is categorically equivalent to a Kan complex.

Exercise 1.4.2. Let $\mathcal{C}$ be an ∞-category and let S be a (small) collection of arrows in $\mathcal{C}$. Show that $\mathrm{h}(\mathcal{C}[S^{-1}]) \in \mathrm{Cat}$ is canonically equivalent to the 1-categorical localization of $\mathrm{h}(\mathcal{C})$ at $\overline{S}$, the collection of morphism which is the image of S via the canonical functor $\mathcal{C} \to \mathrm{h}(\mathcal{C})$.

Exercise 1.4.3. Let $\mathcal{C}$ be an ∞-category with finite limits and let S be a (small) collection of arrows in $\mathcal{C}$. Suppose that $\mathcal{C}$ is stable under pullbacks. Then the ∞-categorical localization $\mathcal{C}[S^{-1}]$ has finite limits and the localization functor $L\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ commutes with them.

1.5 Limits and colimits

Exercise 1.5.1. Let $\mathcal{S}$ be the ∞-category of spaces and let X be an object in $\mathcal{S}$. Using [3, Theorem 4.2.4.1] show that the colimit of the diagram

$$* \longleftarrow X \longrightarrow *$$

can be canonically identified with $\Sigma(X)$.

Now fix two points $p, q\colon * \to X$. Show that the limit of the diagram

$$* \xrightarrow{p} X \xleftarrow{q} *$$

can be canonically identified with the path space $\mathrm{Path}_X(p, q)$.

Exercise 1.5.2. n-cofinality...

Exercise 1.5.3. $\star$ Let K be a simplicial set and let $F\colon K^{\mathrm{op}}\mathcal{P}\mathrm{r}^{\mathrm{L}}$ be an ∞-functor. Let $\mathcal{C}$ be a presentable ∞-category and let $\Delta_{\mathcal{C}}\colon K^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the constant ∞-functor associated to F. Let $\varphi\colon \Delta_{\mathcal{C}} \to F$ be a natural transformation in $\mathrm{Fun}(K^{\mathrm{op}}, \mathcal{P}\mathrm{r}^{\mathrm{L}})$. We let

$$\Phi\colon \mathcal{C} \to \varprojlim F$$

be the induced functor. For every $x \in K$, the functor $\varphi_x\colon \mathcal{C} \to F(x)$ admits a right adjoint, which we denote $\psi_x\colon F(x) \to \mathcal{C}$. Show that there exists an ∞-functor

$$\overline{\Psi}\colon \varprojlim F \to \mathrm{Fun}(K, \mathcal{C})$$

which informally sends $Y = \{Y_x\}_{x \in K} \in \varprojlim F$ to the diagram $\overline{\Psi}(Y)\colon K \to \mathcal{C}$ given by

$$\overline{\Psi}(Y)(x) = \psi_x(Y_x).$$

Prove moreover that the composition

$$\varprojlim F \xrightarrow{\overline{\Psi}} \mathrm{Fun}(K, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$$

can be canonically identified with a right adjoint for Φ.

1.6 Left and right fibrations

Exercise 1.6.1. Let X be a connected Kan complex and let F be any other Kan complex. Let us further fix a point $x \in X$. Let $\mathrm{LF}_x(X; F)$ be the full subcategory of left fibrations $\mathrm{LF}(X)$ over X whose homotopy fiber at x is equivalent to F. Let $\mathrm{B}(\mathrm{hAut}(F))$ be the classifying space of the simplicial group of homotopy automorphisms of F. Show that there is a canonical equivalence of ∞-categories

$$\mathrm{LF}_x(X; F) \simeq \mathrm{Fun}(X, \mathrm{B}(\mathrm{hAut}(F))).$$

1.7 Cartesian and coCartesian fibrations

Exercise 1.7.1. Let $\mathcal{C}$ be an ∞-category and let $X \in \mathcal{C}$ be an object. Let $f\colon U \to X$ and $g\colon V \to X$ be two morphisms in $\mathcal{C}$. For every 2-simplex $\sigma\colon \Delta^2 \to \mathcal{C}$ such that $d_0(\sigma) = f$ and $d_1(\sigma) = g$, show that there is a pullback square in $\mathcal{S}$:

$$\begin{tikzcd}
\mathrm{Path}_{\mathrm{Map}_{\mathcal{C}}(U,X)}(f, d_2(\sigma)) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}_{/X}}(f, g) \arrow[d] \\
* \arrow[r, "d_2(\sigma)"] & \mathrm{Map}_{\mathcal{C}}(U, V).
\end{tikzcd}$$

(Hint: Use [3, Propositions 2.1.2.1 and 2.4.4.2].)

1.8 Adjunctions

Exercise 1.8.1. Let $\mathcal{C}$ be an ∞-category with a zero object 0. Suppose that for every object $X \in \mathcal{C}$ the span

$$0 \longleftarrow X \longrightarrow 0$$

has both a limit $\Omega(X)$ and a colimit $\Sigma(X)$. Construct in an explicit way ∞-functors $\Sigma, \Omega\colon \mathcal{C} \to \mathcal{C}$ informally given by $X \mapsto \Sigma(X)$ and $X \mapsto \Omega(X)$, respectively. Show that Σ and Ω are adjoint by explicitly constructing a fibration $\mathcal{D} \to \Delta^1$ which is both Cartesian and coCartesian.

Exercise 1.8.2. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an ∞-functor. Show that the following statements are equivalent:

1. F has a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$;
2. for every $Y \in \mathcal{D}$ there exists an object $X \in \mathcal{C}$ and a morphism $\varepsilon_X\colon F(X) \to Y$ such that for every other $X' \in \mathcal{C}$ the canonical composition

   $$\mathrm{Map}_{\mathcal{C}}(X', X) \xrightarrow{f} \mathrm{Map}_{\mathcal{D}}(f(X'), f(X)) \xrightarrow{\varepsilon_{X*}} \mathrm{Map}_{\mathcal{D}}(f(X'), Y)$$

   is a weak homotopy equivalence.

1.9 Stable ∞-categories

Exercise 1.9.1. Let $\mathcal{C}$ be a stable ∞-category and let $\mathcal{D} \subseteq \mathcal{C}$ be a full stable subcategory of $\mathcal{C}$. Let $S \coloneqq \{f \colon X \to Y \in \mathcal{C} \mid \operatorname{cofib}(f) \in \mathcal{D}\}$. Show that the ∞-categorical localization $\mathcal{C}[S^{-1}]$ is a stable ∞-category.

Exercise 1.9.2. It is shown in [1] that $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ is a presentable ∞-category. Prove directly that cofibers in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$ exist.

Chapter 2

Derived rings

Chapter 3

Derived stacks

Bibliography

[1] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. A universal characterization of higher algebraic k-theory. *Geometry & Topology*, 17(2):733–838, 2013.

[2] P. Gabriel and M. Zisman. *Calculus of fractions and homotopy theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35. Springer-Verlag New York, Inc., New York, 1967.

[3] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[4] Jean-Louis Verdier. Des catégories dérivées des catégories abéliennes. *Astérisque*, 1996.