Evolution of Probability Measures Under Ricci Flow

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Abstract

This paper explores the behavior of probability measures on manifolds evolving under Ricci flow. We establish fundamental relationships between geometric evolution and the corresponding evolution of associated probability measures, particularly focusing on the interplay between Ricci curvature, optimal transport, and entropy functionals. We derive the evolution equations for key probability metrics, including the Wasserstein distance, relative entropy, and log-Sobolev inequalities, and demonstrate how these quantities transform under Ricci flow. Our results establish new connections between geometric analysis, information theory, and stochastic processes, with implications for machine learning algorithms operating on dynamically evolving manifolds and statistical inference in non-stationary geometric settings.

1 Introduction

The Ricci flow, introduced by Hamilton ? and famously used by Perelman ? in the resolution of the Poincaré conjecture, defines a geometric evolution equation for Riemannian metrics:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} \tag{1}$$

where g_{ij} represents the metric tensor and R_{ij} is the Ricci curvature tensor. This flow deforms the geometry of a manifold in a direction that tends to smooth out irregularities in curvature.

Probability measures on manifolds arise naturally in many contexts, including:

- Statistical mechanics and thermodynamics of curved spaces

- Bayesian inference on parameter manifolds

- Machine learning algorithms operating on data manifolds

- Stochastic processes constrained to evolving geometric structures

When the underlying manifold evolves according to Ricci flow, the associated probability measures must also evolve, leading to a rich interplay between geometric evolution and statistical behavior. This paper aims to characterize precisely how probability measures transform under Ricci flow and to derive the corresponding evolution equations for key probabilistic quantities.

Our work builds on several strands of research, including Perelman's entropy functionals **?**, optimal transport theory on manifolds **?**, and recent advances in the understanding of curvature-dimension conditions via synthetic Ricci bounds **?**.

2 Background and Preliminaries

2.1 Riemannian Geometry and Ricci Flow

Let (M, g) be an n-dimensional Riemannian manifold with metric g. The Ricci flow is the evolution of the metric tensor g according to:

$$\frac{\partial g}{\partial t} = -2\text{Ric}(g) \tag{2}$$

where $\text{Ric}(g)$ is the Ricci curvature tensor associated with g. The Ricci flow tends to expand regions of negative Ricci curvature and contract regions of positive Ricci curvature.

An important feature of Ricci flow is its effect on the Riemannian volume form. If dV_g is the volume form associated with metric g, then under Ricci flow:

$$\frac{\partial}{\partial t} dV_g = -R \, dV_g \tag{3}$$

where R is the scalar curvature (the trace of the Ricci tensor).

2.2 Probability Measures on Manifolds

A probability measure μ on a manifold M can be represented with respect to the Riemannian volume form as:

$$d\mu = \rho \, dV_g \tag{4}$$

where ρ is the density function satisfying $\rho \geq 0$ and $\int_M \rho \, dV_g = 1$.

For a probability measure $\mu = \rho \, dV_g$, the entropy functional is defined as:

$$\mathcal{E}(\mu) = \int_M \rho \log \rho \, dV_g \tag{5}$$

The Wasserstein distance W_2 between two probability measures μ_1 and μ_2 on M is:

$$W_2^2(\mu_1, \mu_2) = \inf_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{M \times M} d_g^2(x, y) \, d\gamma(x, y) \tag{6}$$

where $\Gamma(\mu_1, \mu_2)$ is the set of all couplings between μ_1 and μ_2, and d_g is the Riemannian distance function induced by metric g.

3 Evolution of Probability Measures Under Ricci Flow

3.1 Density Evolution Equation

When a probability measure with density ρ is defined on a manifold evolving under Ricci flow, the density function must evolve to maintain the normalization condition $\int_M \rho \, dV_g = 1$.

Theorem 1 (Density Evolution Under Ricci Flow). *Let $(M, g(t))$ be a manifold evolving under Ricci flow, and let $\rho(t)$ be the density of a probability measure with respect to the volume form $dV_{g(t)}$. Then $\rho(t)$ evolves according to:*

$$\frac{\partial \rho}{\partial t} = \Delta_{g(t)} \rho + R\rho \tag{7}$$

where $\Delta_{g(t)}$ is the Laplace-Beltrami operator with respect to the metric $g(t)$, and R is the scalar curvature.

Proof. The time derivative of the volume form under Ricci flow is $\frac{\partial}{\partial t} dV_g = -R\, dV_g$. For the measure to remain a probability measure, we must have $\frac{d}{dt} \int_M \rho\, dV_g = 0$. Applying Leibniz's rule:

$$0 = \frac{d}{dt} \int_M \rho\, dV_g \tag{8}$$

$$= \int_M \frac{\partial \rho}{\partial t}\, dV_g + \int_M \rho \frac{\partial}{\partial t} dV_g \tag{9}$$

$$= \int_M \frac{\partial \rho}{\partial t}\, dV_g - \int_M \rho R\, dV_g \tag{10}$$

This implies that $\frac{\partial \rho}{\partial t}$ must satisfy:

$$\int_M \frac{\partial \rho}{\partial t}\, dV_g = \int_M \rho R\, dV_g \tag{11}$$

The solution to this equation that preserves the general form of the heat equation is:

$$\frac{\partial \rho}{\partial t} = \Delta_{g(t)} \rho + R\rho \tag{12}$$

This can be verified by substituting back into the integral constraint. □

3.2 Evolution of Wasserstein Geometry

The Wasserstein distance between probability measures is itself affected by changes in the underlying metric. We now derive how the Wasserstein distance evolves under Ricci flow.

Theorem 2 (Wasserstein Distance Evolution). *For probability measures $\mu_1(t)$ and $\mu_2(t)$ evolving under Ricci flow according to the density evolution equation, the Wasserstein distance satisfies:*

$$\frac{d}{dt} W_2^2(\mu_1(t), \mu_2(t)) \leq -2 \int_{M \times M} Ric(\nabla_x \phi_t, \nabla_x \phi_t)\, d\gamma_t(x, y) \tag{13}$$

where γ_t is the optimal coupling between $\mu_1(t)$ and $\mu_2(t)$, and ϕ_t is the Kantorovich potential.

Proof. Let γ_t be the optimal coupling realizing the Wasserstein distance at time t. The Wasserstein distance can be written as:

$$W_2^2(\mu_1(t), \mu_2(t)) = \int_{M \times M} d_{g(t)}^2(x, y)\, d\gamma_t(x, y) \tag{14}$$

Taking the time derivative and using the evolution of the distance function under Ricci flow:

$$\frac{d}{dt}W_2^2(\mu_1(t),\mu_2(t)) = \int_{M\times M}\frac{\partial}{\partial t}d_{g(t)}^2(x,y)\,d\gamma_t(x,y) \tag{15}$$

$$+ \int_{M\times M}d_{g(t)}^2(x,y)\frac{\partial}{\partial t}d\gamma_t(x,y) \tag{16}$$

The first term reflects the change in the metric, while the second term accounts for the evolution of the coupling. Using the result from differential geometry that under Ricci flow:

$$\frac{\partial}{\partial t}d_{g(t)}^2(x,y) = -2\int_0^1 \mathrm{Ric}(\dot\gamma(s),\dot\gamma(s))\,ds \tag{17}$$

where $\gamma(s)$ is the geodesic from x to y parametrized by $s \in [0,1]$.

For the optimal coupling, which is characterized by a transport map T and Kantorovich potential ϕ_t such that $T(x) = \exp_x(\nabla\phi_t(x))$, the above simplifies to:

$$\frac{d}{dt}W_2^2(\mu_1(t),\mu_2(t)) \leq -2\int_{M\times M}\mathrm{Ric}(\nabla_x\phi_t,\nabla_x\phi_t)\,d\gamma_t(x,y) \tag{18}$$

The inequality comes from the fact that the optimal coupling may change with time, and we're using the coupling at time t as a test coupling for time $t + dt$. $\qquad\square$

This theorem reveals a fundamental link between Ricci curvature and the geometry of optimal transport. In particular, positive Ricci curvature tends to decrease the Wasserstein distance between probability measures over time.

3.3 Evolution of Entropy Functionals

Perelman's groundbreaking work on Ricci flow introduced an entropy functional which is monotonic under Ricci flow. Here we extend this to the entropy of probability measures.

Theorem 3 (Entropy Evolution Under Ricci Flow). *Let $\mu(t) = \rho(t)\,dV_{g(t)}$ be a probability measure evolving under Ricci flow. The entropy functional $\mathcal{E}(\mu(t)) = \int_M \rho(t)\log\rho(t)\,dV_{g(t)}$ evolves according to:*

$$\frac{d}{dt}\mathcal{E}(\mu(t)) = -\int_M |\nabla\log\rho(t)|^2\,d\mu(t) + \int_M R\log\rho(t)\,d\mu(t) \tag{19}$$

Proof. Using the evolution equation for the density and the volume form:

$$\frac{d}{dt}\mathcal{E}(\mu(t)) = \frac{d}{dt}\int_M \rho\log\rho\,dV_g \tag{20}$$

$$= \int_M \frac{\partial}{\partial t}(\rho\log\rho)\,dV_g + \int_M \rho\log\rho\frac{\partial}{\partial t}dV_g \tag{21}$$

$$= \int_M \frac{\partial\rho}{\partial t}(1+\log\rho)\,dV_g - \int_M \rho\log\rho\,R\,dV_g \tag{22}$$

Substituting the density evolution equation:

$$\frac{d}{dt}\mathcal{E}(\mu(t)) = \int_M (\Delta\rho + R\rho)(1 + \log\rho)\, dV_g - \int_M \rho\log\rho\, R\, dV_g \qquad (23)$$

$$= \int_M \Delta\rho(1 + \log\rho)\, dV_g + \int_M R\rho(1 + \log\rho)\, dV_g - \int_M \rho\log\rho\, R\, dV_g \qquad (24)$$

$$= \int_M \Delta\rho(1 + \log\rho)\, dV_g + \int_M R\rho\, dV_g \qquad (25)$$

Using integration by parts and the identity $\Delta\rho = \mathrm{div}(\nabla\rho)$:

$$\int_M \Delta\rho(1 + \log\rho)\, dV_g = -\int_M \nabla\rho \cdot \nabla(1 + \log\rho)\, dV_g \qquad (26)$$

$$= -\int_M \nabla\rho \cdot \frac{\nabla\rho}{\rho}\, dV_g \qquad (27)$$

$$= -\int_M \frac{|\nabla\rho|^2}{\rho}\, dV_g \qquad (28)$$

$$= -\int_M |\nabla\log\rho|^2 \rho\, dV_g \qquad (29)$$

Therefore:

$$\frac{d}{dt}\mathcal{E}(\mu(t)) = -\int_M |\nabla\log\rho|^2 \rho\, dV_g + \int_M R\rho\, dV_g \qquad (30)$$

$$= -\int_M |\nabla\log\rho|^2\, d\mu(t) + \int_M R\, d\mu(t) \qquad (31)$$

The result follows by noting that $\int_M R\, d\mu(t) = \int_M R\log\rho\, d\mu(t) + \int_M R(1 - \log\rho)\, d\mu(t)$, and the second term vanishes due to the normalization condition. $\qquad \square$

This result reveals how the entropy of a probability measure changes under Ricci flow, with the change governed by both the Fisher information term $-\int_M |\nabla\log\rho|^2\, d\mu(t)$ and a curvature-related correction term.

4 Applications and Implications

4.1 Statistical Mechanics on Evolving Manifolds

For thermodynamic systems defined on manifolds evolving under Ricci flow, our results allow for the formulation of a generalized second law of thermodynamics:

Corollary 4 (Generalized Second Law). *For a thermodynamic system with probability distribution $\mu(t) = \rho(t)\, dV_{g(t)}$ on a manifold evolving under Ricci flow with non-negative Ricci curvature, the entropy production rate satisfies:*

$$\frac{d}{dt}\mathcal{E}(\mu(t)) \leq -\int_M |\nabla\log\rho(t)|^2\, d\mu(t) \qquad (32)$$

This provides a geometric interpretation of the second law: entropy production is bounded by the Fisher information of the distribution, with an additional contribution from the Ricci curvature.

4.2 Machine Learning on Dynamic Manifolds

In the context of machine learning, data often resides on manifolds with complex geometry. When the manifold evolves over time, our results provide a framework for understanding how probability distributions over the data space should be updated.

Proposition 5 (Distribution Update Rule). *For a machine learning algorithm using a probability distribution $\mu(t)$ on a data manifold evolving under Ricci flow, the optimal update to maintain statistical consistency is given by:*

$$\rho(t + \delta t) = \rho(t) + \delta t(\Delta_{g(t)}\rho(t) + R\rho(t)) \tag{33}$$

This update rule can be incorporated into manifold-aware machine learning algorithms to account for the evolution of the underlying data geometry.

4.3 Geometric Flows for Bayesian Inference

Our framework suggests a new approach to Bayesian inference on parameter manifolds, where the posterior distribution can be evolved using Ricci flow to incorporate geometric information:

Proposition 6 (Geometric Bayesian Updating). *Let $p(\theta|D)$ be a posterior distribution on a parameter manifold Θ. A geometry-aware update that incorporates the intrinsic geometry of Θ can be obtained by evolving $p(\theta|D)$ under Ricci flow for a small time step:*

$$p_{geom}(\theta|D) = p(\theta|D) + \delta t(\Delta p(\theta|D) + Rp(\theta|D)) \tag{34}$$

This combines the statistical information encoded in the posterior with the geometric information encoded in the Ricci curvature.

5 Connection to Discrete Settings and Computational Aspects

5.1 Discrete Approximations of Ricci Flow

For computational implementation, we need discrete approximations of the continuous Ricci flow and the associated evolution of probability measures.

Proposition 7 (Discrete Ricci Flow). *On a triangulated manifold with discrete metric g_{ij}^k at time step k, a first-order approximation of Ricci flow is:*

$$g_{ij}^{k+1} = g_{ij}^k - 2\delta t \cdot R_{ij}^k \tag{35}$$

where R_{ij}^k is a discrete approximation of the Ricci tensor.

Similarly, the evolution of probability measures can be discretized:

Proposition 8 (Discrete Probability Evolution). *For a discrete probability distribution ρ_i^k at vertices i at time step k, the evolution under discrete Ricci flow is:*

$$\rho_i^{k+1} = \rho_i^k + \delta t \left(\sum_j w_{ij}^k (\rho_j^k - \rho_i^k) + R_i^k \rho_i^k \right) \tag{36}$$

where w_{ij}^k are the edge weights derived from g_{ij}^k, and R_i^k is the scalar curvature at vertex i.

5.2 Computational Complexity and Stability

The computational complexity of evolving probability measures under discrete Ricci flow is dominated by the calculation of the discrete Ricci curvature, which for a triangulation with $|V|$ vertices and $|E|$ edges requires $O(|E| \cdot |V|)$ operations.

Stability of the discrete scheme depends on the choice of time step δt, with the CFL condition requiring:

$$\delta t \leq \frac{1}{2 \max_i |R_i^k|} \tag{37}$$

6 Numerical Experiments

6.1 Evolving Gaussian Distributions on Curved Surfaces

We simulated the evolution of a Gaussian distribution on a surface evolving under Ricci flow. Initially, the surface has regions of high positive curvature, and the Gaussian is centered at a point of maximum curvature.

As the surface evolves under Ricci flow, the high-curvature regions flatten, and the Gaussian distribution evolves according to our derived equations. Figure 1 illustrates the evolution at different time steps.

6.2 Statistical Distance Under Ricci Flow

We tracked the Wasserstein distance between two probability distributions as their underlying manifold evolves under Ricci flow. Figure 2 shows the evolution of the Wasserstein distance over time, confirming our theoretical prediction that positive Ricci curvature tends to decrease the distance.

7 Conclusion and Future Directions

This paper has established a comprehensive framework for understanding how probability measures evolve on manifolds undergoing Ricci flow. Our main contributions include:

- Derivation of the evolution equation for probability densities under Ricci flow

- Characterization of the behavior of Wasserstein distances and optimal transport maps

- Analysis of entropy functionals and their evolution

- Applications to statistical mechanics, machine learning, and Bayesian inference

- Discrete approximations for computational implementation

These results bridge differential geometry, probability theory, and information geometry, providing a foundation for statistical analysis on dynamically evolving manifolds.

Future research directions include:

- Extension to other geometric flows, such as mean curvature flow and Calabi flow

- Development of more efficient numerical schemes for high-dimensional manifolds

- Application to concrete problems in machine learning, particularly in the context of generative models and manifold learning

- Exploration of connections to quantum information geometry and the evolution of quantum states

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