Part 1

1. we say μ_n is unbiased estimator if $E(\mu_n) = \mu$

$$\mathbb{E}(\mu_n) = \mathbb{E}\left(\alpha \cdot \frac{1}{n} \sum y_{1i} + (1-\alpha)\frac{1}{n}\sum y_{2i}\right)$$
$$= \frac{\alpha}{n}\sum \mathbb{E}(y_{1i}) + \frac{1-\alpha}{n}\sum \mathbb{E}(y_{2i})$$
$$= \frac{\alpha}{n}\left[\sum \mathbb{E}(x_i) + \mathbb{E}(\varepsilon_{1i})^{0}\right] + \frac{1-\alpha}{n}\left[\sum \mathbb{E}(x_i) + \cancel{\mathbb{E}(\varepsilon_{2i})}^{0}\right]$$
$$= \frac{\alpha}{n}\sum \mu + \frac{1-\alpha}{n}\sum \mu = \frac{\alpha}{n}(n\mu) + \frac{1-\alpha}{n}(n\mu)$$
$$= \alpha\mu + (1-\alpha)\mu = \mu$$

2. consistent if $\lim_{n\to\infty} \mu_n = \mu$

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \frac{\alpha}{n}\sum y_{1i} + \frac{1-\alpha}{n}\sum y_{2i}$$
$$= \lim_{n\to\infty} \alpha\left[\frac{1}{n}\sum x_i + \frac{1}{n}\sum \varepsilon_{1i}\right] + (1-\alpha)\left[\frac{1}{n}\sum x_i + \frac{1}{n}\sum \varepsilon_{2i}\right]$$
$$= \lim_{n\to\infty} \alpha\left[\mathbb{E}(x_i) + \mathbb{E}(\varepsilon_{1i})\right] + (1-\alpha)\left[\mathbb{E}(x_i) + \mathbb{E}(\varepsilon_{2i})\right]$$
$$= \alpha\mu + 0 + (1-\alpha)\mu + 0 = \mu$$

3. $$\mathrm{Var}(\mu_n) = \mathrm{Var}\left(\frac{\alpha}{n}\sum x_i + \frac{\alpha}{n}\sum \varepsilon_{i1} + \frac{1-\alpha}{n}\sum x_i + \frac{1-\alpha}{n}\varepsilon_{i2}\right)$$

because of $\perp$

$$= \left(\frac{1}{n}\right)^2 \mathrm{Var}\left(\sum x_i\right) + \left(\frac{\alpha}{n}\right)^2 \mathrm{Var}\left[\sum \varepsilon_{i1}\right] + \left(\frac{1-\alpha}{n}\right)^2 \mathrm{Var}\left(\sum \varepsilon_{i2}\right)$$
$$= \frac{1}{n}^2 \sum \mathrm{Var}(x_i) + \frac{\alpha^2}{n^2}\sum \mathrm{Var}(\varepsilon_{i1}) + \left(\frac{1-\alpha}{n}\right)^2 \sum \mathrm{Var}(\varepsilon_{i2})$$
$$= \frac{\sigma_x^2}{n} + \frac{\alpha^2\sigma_1^2}{n} + \frac{(1-\alpha)^2\sigma_2^2}{n} = \frac{1}{n}\left[\sigma_x^2 + \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2\right]$$

4. $$\frac{d}{d\alpha}\mathrm{var}(\mu_n) = 2\alpha^*\sigma_1^2 - 2(1-\alpha^*)\sigma_2^2 = 0$$
$$\cancel{2}\alpha^*\sigma_1^2 = \cancel{2}(1-\alpha^*)\sigma_2^2$$
$$\alpha^*(\sigma_1^2 + \sigma_2^2) = \sigma_2^2$$
$$\boxed{\alpha^* = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$

This means if $\sigma_1^2 > \sigma_2^2$ then $\alpha^* < 0.5$, meaning y_2 is weighted more than y_1

If $\sigma_1^2 < \sigma_2^2$ then $\alpha^* > 0.5$, meaning y_1 is weighted more than y_1

if they are equal $\alpha^* = 0.5$, meaning y_1 & y_2 are equally weighted

5. yes, for example we don't need $\varepsilon_1 \perp \varepsilon_2$ to have unbiasedness.

Part 2

1. $y_1 = \mu + \epsilon_{1i}$
$y_2 = \mu + \epsilon_{2i}$

$$Cov(y_1, y_2) = \underbrace{\mathbb{E}(y_1 y_2)}_{\mu^2 \text{ because } E(y_{iu}) = \mu \text{ since } E(\varepsilon_{iu}) = 0} - \mathbb{E}(y_1)\mathbb{E}(y_2)$$
$$= \mathbb{E}(\mu^2) + \mu\mathbb{E}(\varepsilon_{1i}) + \mu\mathbb{E}(\varepsilon_{2i}) + \mathbb{E}(\varepsilon_{1i}\varepsilon_{2i})$$
$$- \mathbb{E}(\mu)^2 - \mathbb{E}(\varepsilon_{1i})\mathbb{E}(\varepsilon_{i2}) - E(\mu)E(\varepsilon_{1i}) - E(\mu)E(\varepsilon_{2i})$$
$$= 0 + \mathbb{E}(\varepsilon_{1i}\varepsilon_{2i}) - \mathbb{E}(\varepsilon_{1i})\mathbb{E}(\varepsilon_{2i}) = 0 + Cov(\varepsilon_1, \varepsilon_2)$$
$$= 0$$

$\therefore$ under these conditions, $y_1 \perp y_2$

if $\sigma_x > 0$, then:

$$Cov(y_1, y_2) = \underbrace{\mathbb{E}(y_1 y_2)}_{= \mathbb{E}((x_i + \varepsilon_{1i})(x_i + \varepsilon_{2i}))} - \underbrace{\mathbb{E}(y_1)\mathbb{E}(y_2)}_{= \mathbb{E}(x_i + \varepsilon_{1i})\mathbb{E}(x_i + \varepsilon_{2i})}$$
$$= \mathbb{E}(x_i^2) + \mathbb{E}(x_i\varepsilon_{1i}) + \mathbb{E}(x_i\varepsilon_{2i}) + \mathbb{E}(\varepsilon_{1i}\varepsilon_{2i})$$
$$- \mathbb{E}(x_i)^2 - \mathbb{E}(x_i)E(\varepsilon_{1i}) - \mathbb{E}(x_i)\mathbb{E}(\varepsilon_{2i}) - \mathbb{E}(\varepsilon_{1i})\mathbb{E}(\varepsilon_{2i})$$
$$= [\mathbb{E}(x_i^2) - \mathbb{E}(x_i)^2] + [\mathbb{E}(x_i\varepsilon_{1i}) - \mathbb{E}(x_i)E(\varepsilon_{1i})$$
$$+ [\mathbb{E}(x_i\varepsilon_{2i}) - \mathbb{E}(x_i)\mathbb{E}(\varepsilon_{2i})] - \underbrace{[\mathbb{E}(\varepsilon_{1i}\varepsilon_{2i}) - \mathbb{E}(\varepsilon_{1i})E(\varepsilon_{2i})]}_{= 0 \text{ bc } = Cov(\varepsilon_1, \varepsilon_2) = 0}$$
$$\boxed{= Var(x) + cov(x, \varepsilon_1) + Cov(x, \varepsilon_2)}$$

2. $$L(\mu, \sigma_1, \sigma_2; (y_{i1}, y_{i2})_{i=1}^n) \overset{\text{because of independence}}{=} L(\mu, \sigma_1, \sigma_2; y_{i1}) \cdot L(\mu, \sigma_1, \sigma_2; y_{2i})$$

from normality
$$= \underbrace{[2\pi\sigma_1^2] e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^n (y_{i1}-\mu)^2} [2\pi\sigma_1^2] e^{-\frac{1}{2\sigma_2^2}\sum_{i=1}^n (y_{i2}-\mu)^2}}_{\text{refer to as } \star}$$

$$\ell(\mu, \sigma_1, \sigma_2; (y_{11}, y_{2i})_{i=1}^n) = \log(\star)$$

$$= -\frac{n}{2}\log[2\pi\sigma_1^2] - \frac{n}{2}\log[2\pi\sigma_2^2] - \frac{1}{2\sigma_1^2}\sum_{i=1}^n (y_{i1}-\mu)^2$$
$$- \frac{1}{2\sigma_2^2}\sum_{i=1}^n (y_{2i}-\mu)^2$$

3. $\frac{\partial \ell}{\partial \mu} = -\frac{1}{2\sigma_1^2} \sum_{i=1}^{n} -2(y_{i1} - \mu^*) - \frac{1}{2\sigma_2^2} \sum_{i=1}^{n} -2(y_{i2} - \mu) = 0$

$= \frac{1}{\sigma_1^2} \sum_{i=1} (y_{i1} - \mu^*) + \frac{1}{\sigma_2^2} \sum_{i=1}^{n} (y_{i2} - \mu) = 0$

$= \frac{\sum y_{i1}}{\sigma_1^2} + \frac{\sum y_{i2}}{\sigma_2^2} - \frac{n\mu^*}{\sigma_1^2} - \frac{n\mu^*}{\sigma_2^2} = 0$

$\Rightarrow \underbrace{\frac{\sum y_{i1}}{n}}_{\bar{y}_1} \left(\frac{1}{\sigma_1^2}\right) + \underbrace{\frac{\sum y_{i2}}{n}}_{\bar{y}_2} \frac{1}{\sigma_2^2} = \frac{\sigma_2^2 \mu^* + \sigma_1^2 \mu^*}{\sigma_1^2 \sigma_2^2}$

$\Rightarrow \mu^* = \left(\frac{\bar{y}_1}{\sigma_1^2} + \frac{\bar{y}_2}{\sigma_2^2}\right)\left(\frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2}\right)$

$= \frac{\sigma_2^2 \bar{y}_1 + \sigma_1^2 \bar{y}_2}{\sigma_1^2 + \sigma_2^2}$

4. Similarities: calculation for both is dependent on σ_1 & σ_2

Differences:

Estimator includes an additional variable, which is the sample means of y_1 & y_2.

This means it is not solely relying on the magnitude of σ_1^2 & σ_2^2 (the variance) but also $\bar{y}_1$ & $\bar{y}_2$ (our means)

Part 3

1. False

let $y = 2$. Then $\mathbb{E}(U^2) = \frac{1}{b-a}\int_a^b u^2\,du = \frac{1}{b-a}\left[\frac{u^3}{3}\right]\Big|_a^b = \frac{b^3 - a^3}{3(b-a)}$

we have $a = 0$ & $b = 1$ $\Rightarrow$ $\mathbb{E}(U^2) = \frac{1}{3} \neq \frac{1}{2}$ as the problem would suggest

2 False

let's take a look at the MLE of variance, where we know $\sigma^2_{MLE} = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_{MLE})^2$

we know $\sum x_i = n\mu_{MLE}$ & $\sum \mu^2_{MLE} = n\mu^2_{MLE}$

$$\mathbb{E}(\sigma^2_{MLE}) = \frac{1}{n}\mathbb{E}\left(x_i^2 - 2x_i\mu_{MLE} + \mu^2_{MLE}\right)$$
$$= \frac{1}{n}\mathbb{E}(x_i^2) - 2n\mu^2_{MLE} + N\mu^2_{MLE}$$
$$= \mathbb{E}(x^2) - \mathbb{E}(\mu^2_{MLE})$$
$$= (\sigma_x^2 + \mu^2) - (\sigma^2_{\mu_{MLE}} + \mu^2) = \sigma_x^2 + \frac{1}{N}\sigma_x^2$$
$$= \frac{N-1}{N}\sigma_x^2 \Rightarrow \text{UNBIASED estimater of } \sigma_x^2$$

3 True

because we multiply $(1-p)^{k-1}$ by k

$$\mathbb{E}(x_i) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p\left[\sum_{k=1}^{\infty}(1-p)^{k-1} + \sum_{k=2}^{\infty}(1-p)^{k-1} + \sum_{k=3}^{\infty}(1-p)^{k-1} + \sum_{k=4}^{\infty}(1-p)^{k-1} + \dots\right]$$
$$= p\left[\frac{1}{p} + \frac{1-p}{p} + \frac{(1-p)^2}{p} + \frac{(1-p)^3}{p} + \dots\right]$$
$$= 1 + (1-p) + (1-p)^2 + \dots = \sum_{k=0}^{\infty}(1-p)^k$$

property of geometric series

$$= \frac{1}{1-(1-p)} = \frac{1}{p}$$