Math 330: Introduction to Higher Math, Section 1, Spring 2009 — Final Exam, May 11

Name: .... *Solutions* ....

I	II	III	IV	V	VI	VII	TOTAL
11	10	14	23	14	14	14	100

I. Let X, Y, and Z be statements.
Are the following statements equivalent to "If X is true, then Y is true or Z is true"?
Please circle your answers.

1] If X is true and Y is false, then Z is true. .... $\boxed{\text{YES}}$ NO

2] If X is true and Z is false, then Y is true. .... $\boxed{\text{YES}}$ NO

3] If Y is false and Z is false, then X is false. .... $\boxed{\text{YES}}$ NO

4] If Y is false or Z is false, then X is false. .... YES $\boxed{\text{NO}}$

5] If Y is true or Z is true, then X is true. .... YES $\boxed{\text{NO}}$

6] If X is true, then Y is true and Z is true. .... YES $\boxed{\text{NO}}$

7] If X is false, then Y is false and Z is false. .... YES $\boxed{\text{NO}}$

8] If X is false, then Y is false or Z is false. .... YES $\boxed{\text{NO}}$

9] X is false or Y is true or Z is true. .... $\boxed{\text{YES}}$ NO

10] X is true and Y is true and Z is true. .... YES $\boxed{\text{NO}}$

11] X is true and Y is true, or X is true and Z is true. .... YES $\boxed{\text{NO}}$

II. For each of the following five questions, four possible answers are provided, but only one of them is correct: write the corresponding letter in the box!

II/1. Let $f\colon X \to Y$ be a function. Let x and x' be elements of X such that $f(x) = f(x')$.
What do we need to know about f to conclude that $x = x'$? .... $\boxed{B}$

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/2. Let $f\colon X \to Y$ be a function. Let x and x' be elements of X such that $x = x'$.
What do we need to know about f to conclude that $f(x) = f(x')$? .... $\boxed{A}$

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/3. Let $f\colon X \to Y$ be a function. Let y be an element of Y.
What do we need to know about f to conclude that $y = f(x)$ for some $x \in X$? .... $\boxed{C}$

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/4. Let $f\colon X \to Y$ be a function. Let y be an element of Y.
What do we need to know about f to conclude that $y = f(x)$ for exactly one $x \in X$? ..... $\boxed{D}$

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

II/5. Let $f\colon X \to Y$ be a function. Let y be an element of Y.
What do we need to know about f to conclude that $y = f(x)$ for at most one $x \in X$? .... $\boxed{B}$

A] Nothing: this is true for all functions f.
B] We need f to be injective.
C] We need f to be surjective.
D] We need f to be bijective.

III. Let a and b be integers, i.e., $a, b \in \mathbb{Z}$.

III/1. What exactly does it mean to say that "a is divisible by b", or equivalently that "b divides a"?

$\exists\, k \in \mathbb{Z}$ *s.t.* $a = kb$.

III/2. Is it true or false that for every natural number $n \in \mathbb{N}$, 6^n is not divisible by 5? ..... $\boxed{\textit{TRUE}}$
Prove your claim.

Since $6 = 5 + 1$, *by the division theorem* 6 *is not divisible by* 5.
Assume that for some $n \in \mathbb{N}$, 6^n *is not divisible by* 5. *Since* 5 *is a prime number and* 5 *does not divide* 6 *nor* 6^n, *then by Euclid's lemma* 5 *does not divide* $6 \cdot 6^n = 6^{n+1}$.
Therefore by induction $\forall\, n \in \mathbb{N}$, 6^n *is not divisible by* 5.

Alternative solution: prove by induction that $\forall n \in \mathbb{N}\ \exists k \in \mathbb{N}$ *s.t.* $6^n = 5k+1$, *which is obviously true for n=1, and which implies the claim by the division theorem.*
Assume that for some $n \in \mathbb{N}$, $\exists\, k \in \mathbb{N}$ *s.t.* $6^n = 5k + 1$.
Then $6^{n+1} = 6^n \cdot 6 = (5k + 1)6 = 5(6k + 1) + 1$.

IV. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers.

IV/1. What exactly does it mean to say that $(x_n)_{n\in\mathbb{N}}$ is convergent?

$$\exists\, \ell \in \mathbb{R} \textit{ s.t. } \forall\, \varepsilon \in \mathbb{R}_{>0} \ \exists\, N \in \mathbb{N} \textit{ s.t. } \forall\, n \in \mathbb{N}, \textit{ if } n \geq N \textit{ then } |x_n - \ell| < \varepsilon.$$

IV/2. Prove the following statement: If $(x_n)_{n\in\mathbb{N}}$ is convergent, then for every $\varepsilon \in \mathbb{R}_{>0}$ there exists an $N \in \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, if $m \geq N$ and $n \geq N$ then $|x_m - x_n| < \varepsilon$.

Assume that (x_n) *converges to* $\ell \in \mathbb{R}$.
For any $\varepsilon > 0$ *let* $\varepsilon' = \frac{\varepsilon}{2}$. *Then* $\exists\, N \in \mathbb{N}$ *s.t.* $\forall\, n \geq N,\ |x_n - \ell| < \varepsilon'$.
Hence if $m \geq N$ *and* $n \geq N$, $|x_m - \ell| < \varepsilon'$ *and* $|x_n - \ell| < \varepsilon'$, *and therefore by the triangle inequality* $|x_m - x_n| \leq |x_m - \ell| + |\ell - x_n| = |x_m - \ell| + |x_n - \ell| < \varepsilon' + \varepsilon' = \varepsilon$.

IV/3. What is the contrapositive of the statement in IV/2?

If $\exists\, \varepsilon \in \mathbb{R}_{>0}$ *s.t.* $\forall\, N \in \mathbb{N}$ $\exists\, m, n \in \mathbb{N}$ *s.t.* $m \geq N$ *and* $n \geq N$ *and* $|x_m - x_n| \geq \varepsilon$ *then* (x_n) *is divergent.*

IV/4. Use IV/3 to show that the sequence $x_n = (-1)^n$ is divergent.

Choose $\varepsilon = 1$, *and for any* $N \in \mathbb{N}$ *take* $n = N$ *and* $m = N + 1$. *Then* $|x_m - x_n| = 2 > 1 = \varepsilon$. *Therefore* (x_n) *is divergent by IV/3.*

V. For each natural number $n \in \mathbb{N}$, define $x_n = \sum_{j=1}^{n} \frac{1}{j^2}$.

V/1. Prove that for all $n \in \mathbb{N}$, $x_n \leq 2 - \frac{1}{n}$.

$x_1 = 1 \leq 2 - 1$.
Assume that for some $n \in \mathbb{N}$, $x_n \leq 2 - \frac{1}{n}$.
Then $x_{n+1} = x_n + \frac{1}{(n+1)^2} \leq 2 - \frac{1}{n} + \frac{1}{(n+1)^2} = 2 - \frac{n^2+n+1}{n(n+1)^2} < 2 - \frac{n^2+n}{n(n+1)^2} = 2 - \frac{1}{n+1}$.
Therefore by induction $\forall\, n \in \mathbb{N}$, $x_n \leq 2 - \frac{1}{n}$.

V/2. Does the sequence $(x_n)_{n\in\mathbb{N}}$ defined above converge in $\mathbb{R}$? .... $\boxed{\textit{YES}}$
Prove your claim.

By V/1 the sequence (x_n) is bounded above by 2.
Since $x_{n+1} = x_n + \frac{1}{(n+1)^2} > x_n$, (x_n) is increasing.
Therefore (x_n) is convergent.

VI. VI/1. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ as follows: for all $x, y \in \mathbb{R}$, declare $x \sim y$ if and only if $x - y \in \mathbb{Z}$. Is $\sim$ an equivalence relation? .... $\boxed{\textit{YES}}$
Prove your claim.

For all $x, y, z \in \mathbb{R}$*:*

- $x \sim x$ *since* $x - x = 0 \in \mathbb{Z}$*;*
- *if* $x \sim y$ *then* $x - y \in \mathbb{Z}$*, hence* $-(x - y) = y - x \in \mathbb{Z}$*, and so* $y \sim x$*;*
- *if* $x \sim y$ *and* $y \sim z$ *then* $x-y \in \mathbb{Z}$ *and* $y-z \in \mathbb{Z}$*, hence* $(x-y)+(y-z) = x-z \in \mathbb{Z}$*, and so* $x \sim z$*.*

VI/2. More generally, let A be a subset of $\mathbb{R}$ and define a relation $\backsim$ on $\mathbb{R}$ by declaring $x \backsim y$ if and only if $x - y \in A$. What conditions must A satisfy in order for $\backsim$ to be an equivalence relation?

- $0 \in A$*;*
- $\forall\, a \in A,\ -a \in A$*;*
- $\forall\, a, b \in A,\ a + b \in A$*.*

VII. Suppose that you have a set A and a subset $B \subseteq A$ such that $B \neq A$.

VII/1. What exactly do these conditions mean?

$B \subseteq A$

$$\forall\, b \in B,\ b \in A.$$

$B \neq A$

$$\exists\, a \in A \text{ s.t. } a \notin B.$$

VII/2. Given A and B satisfying the above conditions, is it possible for A and B to have the same cardinality, i.e., $A \simeq B$? .... $\boxed{F}$

A] No, it is not possible for any A.
B] Yes, it is possible for any A.
C] Yes, but only if A is empty.
D] Yes, but only if A is not empty.
E] Yes, but only if A is finite.
F] Yes, but only if A is infinite.
G] Yes, but only if A is countable.
H] Yes, but only if A is uncountable.

VII/3. Prove that your answer to VII/2 is correct.

It is possible for infinite sets, e.g., $\mathbb{Z} \simeq \mathbb{N}$ and $\mathbb{R} \simeq \mathbb{R}_{>0}$. On the other hand, suppose that A is finite and $B \subseteq A$ and $B \neq A$. Then B is finite and $|B| < |A|$. Since for finite sets A and B, $A \simeq B$ if and only if $|A| = |B|$, it follows that $A \not\simeq B$.

Have a great summer!