Fractional Delay Filter Design

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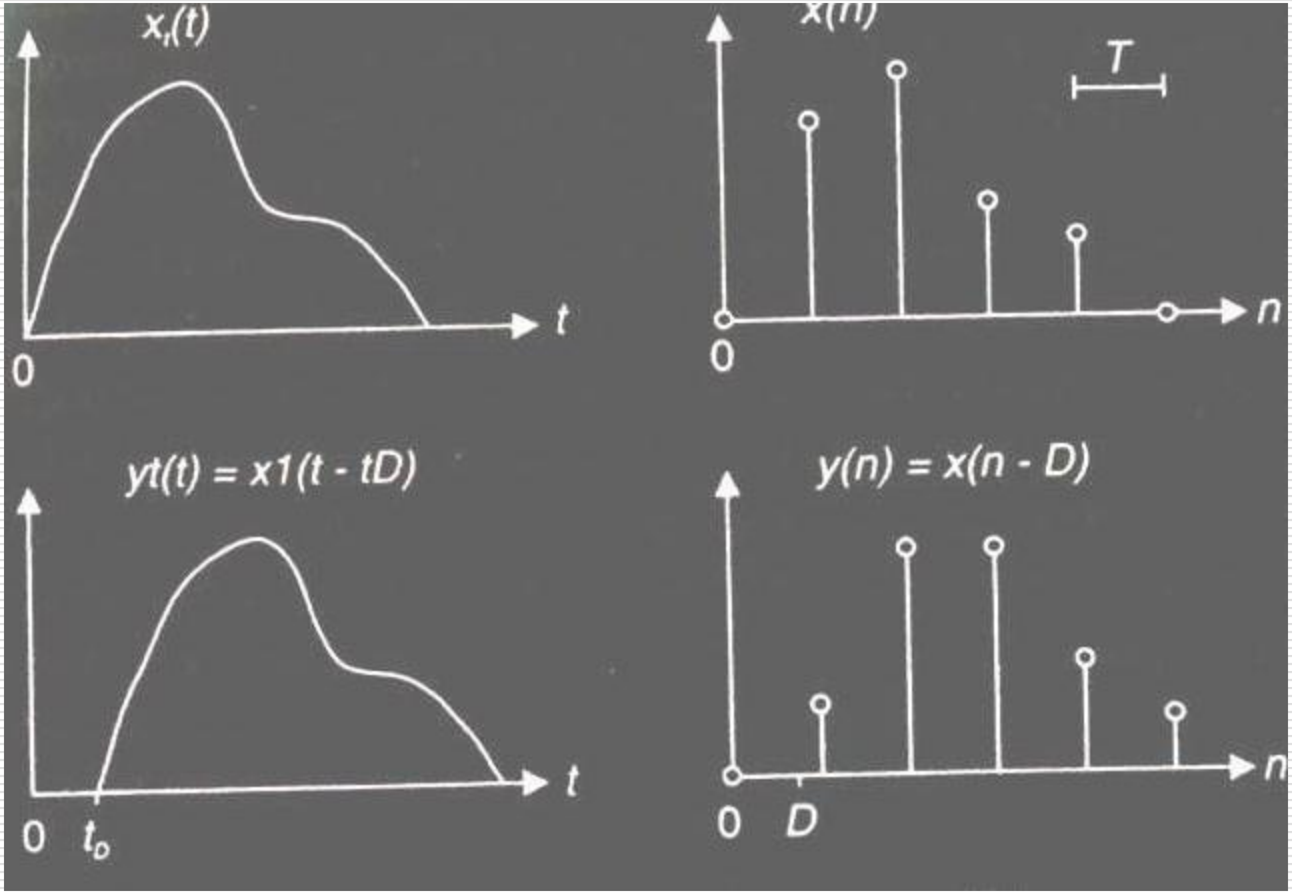
Preliminaries

- Delaying a continuous time signal is simple.
- $Y_c(t)=L_c\{x_c(t)\}=x_c(t-t_D)$
- Converting into discrete time by sampling at time instants t = nT.
- Y(n)=L{x(n)}=x(n-D) → meaningful only for integer values of D.
- D is a positive real number that can be split into the integer and fractional parts
- D=Int(D)+d

Preliminaries



Preliminaries

- The problem can be solved by resampling process.
- First reconstructing the continuous bandlimited signal then resampling it after shifting [Gardner_90a, Gardner_90b].
- Related to interpolation in multirate filter design techniques[Chrochiere_75, Chrochiere_83, Vaidyanathan_90, Vaidyanathan_93] or sampling rate conversion in general[Ansari_83, Ansari_93, Bellanger_76, Ramstad_82, Ramstad_84,].
- Need not to perform construction and resampling explicitly.
- Can be reduced to appropriate linear filtering at chosen sampling rate.

Preliminaries

$$H_{id}(Z) = \frac{Y(z)}{X(z)} = z^{-D}\frac{X(z)}{X(z)} = z^{-D}$$

$$Z\left\{x(n-D)\right\} = z^{-D}X(z)$$

- z^{-D} can't be realized exactly for noninteger D.
- It must approximated in some way.

Preliminaries

- One approach is to construct a series of expansion for z^{-D} [Ramachandran_90].
- The more fruitful and general approach is to formulate the design objective in frequency domain.

Preliminaries

$$H_{id}(e^{jw}) = e^{-jwD}$$

$$\left|H_{id}(e^{jw})\right| = 1, \forall w$$

$$\arg\{H_{id}(e^{jw})\} = \Theta_{id}(w) = -Dw$$

$$\tau_g(w) = -\frac{\partial\Theta(w)}{\partial w}$$

$$\tau_p(w) = -\frac{\Theta(w)}{w}$$

$$\tau_{p,id}(w) = D$$

$$\tau_{g,id}(w) = D$$

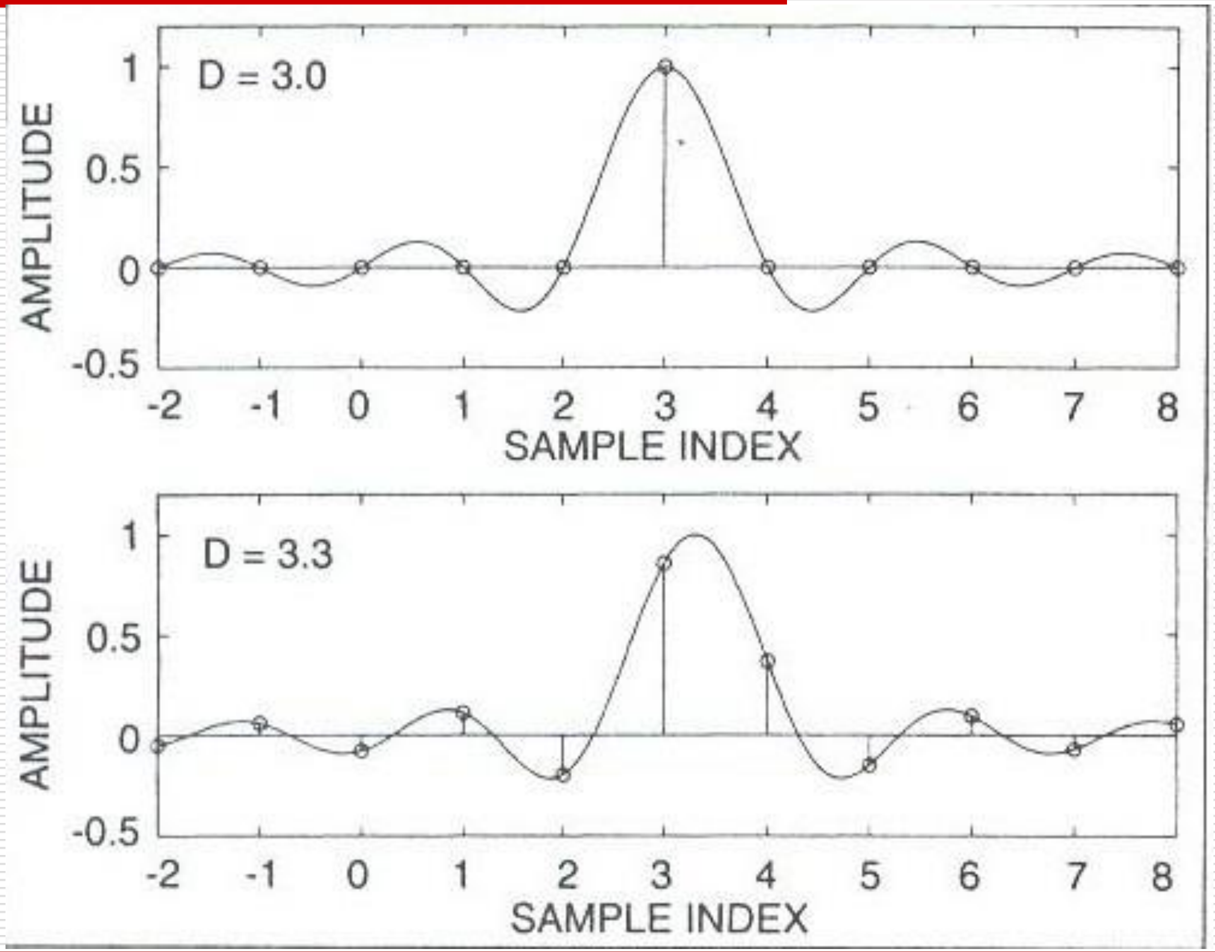
Preliminaries

- Assume that the discrete-time signal represents a bandlimited baseband signal.
- The implementation of a constant delay can be considered as an approximation of the ideal discrete-time linear phase allpass filter with unity magnitude and constant group delay.

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{jw}) e^{jwn} dw$$

$$h_{id}(n) = \frac{\sin\left[\pi(n-D)\right]}{\pi(n-D)} = \operatorname{sinc}(n-D)$$

Preliminaries



Preliminaries

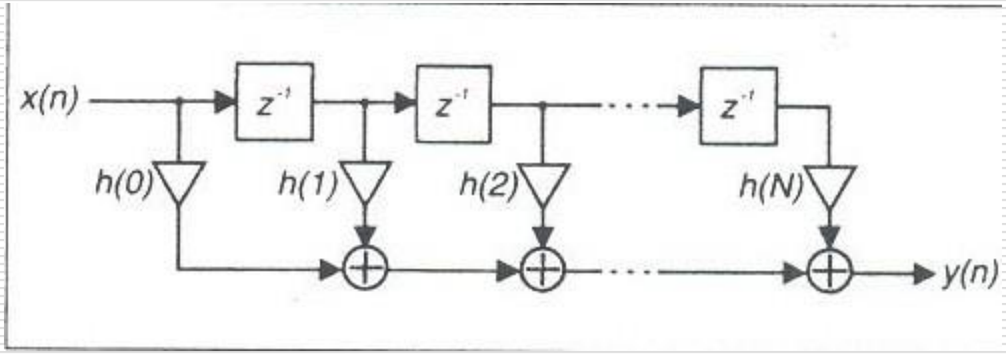
- When D is an integer, the impulse response reduces to a single impulse at n = D.
- For nonintegers values of D, the impulse response is an infinitely long, shifted and sampled version of the sinc function.
- It's also noncausal → impossible to implement in real-time applications.

Fractional Delay Approx. Using FIR FILTER

- An Nth-order FIR Filter (L=N+1).
- $E(e^{jw})$ must be minimized.

$$H(z) = \sum_{n=0}^{N} h(n) z^{-n}$$

$$E(e^{jw}) = H(e^{jw}) - H_{id}(e^{jw})$$



Least Squared Error Solutions for FIR Filters

- Direct Least Squared Integral Error [Parks_87, Papoulis_77, Kay_83]

$$E_1 = \frac{1}{\pi}\int_0^{\pi} \left|E(e^{jw})\right|^2 dw = \frac{1}{\pi}\int_0^{\pi} \left|E(e^{jw}) - H_{id}(e^{jw})\right|^2 dw$$

$$E_1 = \sum_{n=-\infty}^{\infty} \left|h(n) - h_{id}(n)\right|^2$$

$$h(n) = \begin{cases} \sin c(n-D), M \le n \le M+N \\ 0, otherwise \end{cases}$$

$$E_1 = \sum_{n=-\infty}^{M-1} \left|h_{id}(n)\right|^2 + \sum_{n=M+N+1}^{\infty} \left|h_{id}(n)\right|^2$$

$$M_{opt} = \begin{cases} Round(D) - \frac{N}{2}, even \quad n \\ Int(D) - \frac{N-1}{2}, odd \quad n \end{cases}$$

Filter order increase → error decreases.

The smallest error → D is placed at the "centre of gravity"

Windowing Methods [Cain_94a, Cain_94b, Cain_95]

- The performance of the truncated sinc function is not acceptable.
- To reduce the Gibbs phenomenon, use window functions.
- A bell-shaped windows can be used.
- More emphasis on middle values and reduces the peak magnitude error.
- Wider transition band.

$$h(n)=\begin{cases} W(n-D)\operatorname{sinc}(n-D), 0 \le n \le N \\ 0, otherwise \end{cases}$$

Smooth Transition Band Functions
[Parks_87, Burrus_92]

- The Gibbs phenomenon is caused by discontinuity.
- It's essential that the desired response be smooth.
- Prescribe a passband $[0,\omega_p]$ with ideal response e^{-jDw}, a stopband $[\omega_s,\pi]$ with the ideal response 0 and a Pth-order spline in transition band.

Smooth Transition Band Functions
[Parks_87, Burrus_92]

$$h(n)=\left\{\frac{\sin\left[\frac{n-D}{2P}(\omega_s-\omega_p)\right]}{\left[\frac{n-D}{2P}(\omega_s-\omega_p)\right]}\right\}^p \frac{\sin\left[\frac{n-D}{2}(\omega_s+\omega_p)\right]}{\left[\frac{\pi(n-D)}{2}\right]}$$

$$n=0,1,2,\ldots,N$$

General Least Squares FIR Approx.

[Nguyen_76, Pei_92, Lang_92, Burrus_95]

$$E_4 = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\left|E(e^{j\omega})\right|^2 d\omega$$

$$= \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\left|H(e^{j\omega}) - H_{id}(e^{j\omega})\right|^2 d\omega$$

$$h = [h(0)\, h(1) \ldots h(N)]^T$$

$$e = [1 e^{-j\omega} \ldots e^{-j\omega N}]^T$$

$$C = \mathrm{Re}\{ee^H\} = \begin{bmatrix} 1 & \cos\omega & \ldots & \cos N\omega \\ \cos\omega & 1 & \ldots & \cos(N-1)\omega \\ \vdots & & \ddots & \vdots \\ \cos N\omega & \cos(N-1)\omega & \ldots & 1 \end{bmatrix}$$

Error is defined in the lowpass frequency band $[0, \alpha\pi]$

Superscript 'H' : Hermitian operation→

transposition with conjunction

General Least Squares FIR Approx.

$$E_4 = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)[h^T e - H_{id}(e^{j\omega})][h^T e - H_{id}(e^{j\omega})]^* d\omega$$

$$= \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)[h^T C h - 2h^T \operatorname{Re}\{H_{id}(e^{j\omega})e^*\} + \left|H_{id}(e^{j\omega})\right|^2]d\omega$$

$$E_4 = h^T P h - 2h^T p_1 + p_0$$

$$P = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega) C d\omega$$

$$p_1 = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)[\operatorname{Re}\{H_{id}(e^{j\omega})\}c - \operatorname{Im}\{H_{id}(e^{j\omega})\}s]d\omega$$

$$p_0 = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\left|H_{id}(e^{j\omega})\right|^2 d\omega$$

General Least Squares FIR Approx.

$$c = [1 \ \cos\omega \ \ldots . \ \cos N\omega]^T$$

$$s = [0 \ \sin\omega \ \ldots . \ \sin N\omega]^T$$

$$2Ph = 2p_1 = 0$$

$$h = P^{-1} p_1$$

$$P_{k,l} = \frac{1}{\pi} \int_0^{\alpha\pi} \cos[(k-l)\omega] d\omega = \alpha \sin c[\alpha(k-l)]$$

$$k, l = 1, 2, \ldots, L$$

$$P_{l,k} = \frac{1}{\pi} \int_0^{\alpha\pi} \cos[(k-D)\omega] d\omega = \alpha \sin c[\alpha(k-D)]$$

$$k = 1, 2, \ldots, L$$

The unique minimum-error solution is found by setting the derivative with respect to h to 0.

The computational cost is $(N+1)^3$ → increased.

Stochastic Least-Mean-Squared (LMS) Error [Oetken_75]

$$E_6 = E\{|y(n) - y_{id}(n)|^2\} = E\{|x(n) * [h(n) - h_{id}(n)]|^2\}$$

$$= \frac{1}{\pi}\int_0^{\pi} S_{xx}(\omega)\left|H(e^{j\omega}) - H_{id}(e^{j\omega})\right|^2$$

$$W(\omega) = S_{xx}(\omega)$$

$$P_{k,l} = \frac{1}{\pi}\int_0^{\pi} S_{xx}(\omega)\cos[(k-l)\omega]$$

$$= E\{x(n)x(n+k+1)\} = r_{xx}(k-l)$$

Maximally Flat FIR FD Filter Design: Lagrange Interpolation [Oetken_79, Laine_88, Liu_90, Liu_92]

$$\left.\frac{d^n E(e^{j\omega})}{d\omega^n}\right|_{\omega=\omega_0} = 0 \ for \ n = 0, 1, 2, \ldots, N$$

$$\sum_{k=0}^{N} k^n h(k) = D^n$$

$$Vh = V$$

$$V = \begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ 0 & 1 & 2 & & N \\ 0 & 1 & 2^2 & & N^2 \\ \vdots & & & \ddots & \vdots \\ 0 & 1 & 2^N & \ldots & N^N \end{bmatrix}$$

$$v = [1 \ D \ \ D^2 \ \ \ldots. \ \ D^N]^T$$

The error function can be maximally flat at a certain frequency.

The solution is equal to the classical Lagrange interpolation formula.

Maximally Flat FIR FD Filter Design: Lagrange Interpolation

For integer values of the desired delay, the approximation error is set to zero.

$$h(z) = z^{-D} \quad for \;\; D = 0, 1, 2, \ldots, N$$

$$h(n) = \prod_{\substack{k=0 \\ k \neq n}}^{N} \frac{D-k}{n-k} \quad for \;\; n = 0, 1, 2, \ldots, N$$

$$h(0) = 1 - D, \, h(1) = D$$

The case N=1 corresponds to linear interpolation.

Fractional Delay Approx. Using Allpass Filters

$$A(z) = \frac{z^{-N} D(z^{-1})}{D(z)}$$

$$\Theta_A(\omega) = \arg\{A(e^{j\omega})\} = -N\omega + 2\Theta_D(\omega)$$

$$\Theta_D(\omega) = \arg\{\frac{1}{D(e^{j\omega})}\} = \arctan\left\{\frac{\sum_{k=0}^{N} a_k \sin(k\omega)}{\sum_{k=0}^{N} a_k \cos(k\omega)}\right\}$$

$$= \arctan\left\{\frac{a^T s}{a^T c}\right\}$$

$$a = [a_0 \;\; a_1 \;\; a_2 \; \ldots. \;\; a_N]^T$$

Fractional Delay Approx. Using Allpass Filters

$$\tau_{g,A}(\omega) = -\frac{d\Theta_A(\omega)}{d(\omega)} = N - 2\tau_{g,D}(\omega)$$

$$\tau_{g,D}(\omega) = -\frac{d\Theta_D(\omega)}{d(\omega)} = \frac{a^T G \Lambda a}{a^T G a}$$

$$G = cc^T + ss^T$$

$$\Lambda = diag[0\ 1 \ldots\ N]$$

$$\tau_{p,A}(\omega) = -\frac{\Theta_A(\omega)}{(\omega)} = N - 2\tau_{p,D}(\omega)$$

Fractional Delay Approx. Using Allpass Filters

- The phase, phase delay and group delay are all related to the filter coefficients in a very nonlinear manner.
- One can not expect as simple design formulas for the allpass filter coefficients as for FIR filters.
- Only iterative optimization techniques.

Least Squares Design of Allpass Filters
[Lang_92, Lang_94, Laakso_93, Nguyen_94]

Approximate LS Phase Error Design

$$\Delta\Theta(\omega) = \Theta_{id}(\omega) - \Theta_A(\omega) = 2\arctan\{\frac{a^T s_\beta}{a^T c_\beta}\}$$

$$s_\beta = [\sin\{\beta(\omega)\} \quad \sin\{\beta(\omega)-\omega\} \quad \ldots \quad \sin\{\beta(\omega)-N\omega\}]^T$$

$$c_\beta = [\cos\{\beta(\omega)\} \quad \cos\{\beta(\omega)-\omega\} \quad \ldots \quad \cos\{\beta(\omega)-N\omega\}]^T$$

$$\beta(\omega) = \frac{1}{2}[\Theta_{id}(\omega) + N\omega]$$

$$\Theta_{id}(\omega) = -D\omega = -(N+d)\omega$$

$$\beta(\omega) = -\frac{\omega d}{2}$$

Approximate LS Phase Error Design

$$E = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\left|\Delta\Theta(\omega)\right|^2 d\omega$$

$$E = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\frac{\left|2a^T s_\beta\right|^2}{\left|a^T c_\beta\right|^2} d\omega = \frac{4}{\pi}\int_0^{\alpha\pi} W(\omega)\frac{a^T s_\beta s_\beta{}^T a}{a^T c_\beta c_\beta{}^T a} d\omega$$

Approximating arctan(x) = x.

$$= \frac{4}{\pi}\int_0^{\alpha\pi} W(\omega)\frac{a^T S_\beta(\omega) a}{a^T C(\omega) a} d\omega$$

$$E = a^T\left[\frac{4}{\pi}\int_0^{\alpha\pi} W(\omega)\frac{S_\beta(\omega)}{a_0{}^T C(\omega) a_0} d\omega\right] a = a^T P a$$

Approximate LS Phase Error Design

$$P = \frac{4}{\pi}\int_0^{\alpha\pi} W(\omega) S_\beta(\omega) d\omega$$

$$P_{k,l} = \frac{4}{\pi}\int_0^{\alpha\pi} \{\cos[(k-l)\omega] - \cos[(N-(k+l+d)\omega]\} d\omega$$

$$= 4\alpha\{\mathrm{sin}c[\alpha(k-l)] - sic[\alpha(N-(k+l+d))]\}$$

$$k, l = 1, 2, \ldots, L$$

$$P^{(q)} = \frac{4}{\pi}\int_0^{\alpha\pi} W(\omega) \frac{S_\beta(\omega)}{a^{(q-1)T} C_\beta(\omega) a^{(q-1)}} d\omega$$

LS Phase Delay Error Design of Allpass Filters

$$E = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\left|\Delta\tau_p(\omega)\right|^2 d\omega = \frac{1}{\pi}\int_0^{\alpha\pi} W(\omega)\left|\frac{\Delta\Theta(\omega)}{\omega}\right|^2 d\omega =$$

$$= \frac{1}{\pi}\int_0^{\alpha\pi} \frac{W(\omega)}{\omega^2}\left|\Delta\Theta(\omega)\right|^2 d\omega$$

$$P_{k,l} = \frac{4}{\pi}\int_0^{\alpha\pi} \frac{1}{\omega^2}\{\cos[(k-l)\omega] - \cos[N-(k+l+d)\omega]\}d\omega$$

$$k,l = 1,2,\ldots.L$$

$$\int \frac{\cos(ax)}{x^2} = -aSi(ax) - \frac{\cos(ax)}{x}$$

$$Si(x) = \int_0^x \frac{\sin t}{t}dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!} \quad t \geq 0$$

Maximally Flat Group Delay Design of Allpass Filters [Thiran_71]

- In 1917, Thiran proposed an analytic solution for an all-pole lowpass filter with a maximally flat group delay response at the zero frequency.
- The group delay of an all-pass filter is twice that of the corresponding all-pole filter.
- The solution for the allpass filter coefficients approximating the delay D=N+d is:

$$a_k = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{D-N+n}{D-N+k+n} \quad for\ k = 0,1,2,\ldots,N$$

References

1. [Ansari_83] R. Ansari and B. Liu, “Efficient sampling rate conversion using recursive IIR digital filters”, IEEE Trans. Acoust. Speech Signal Processing, vol. 31, no.6, pp. 1366-1373, Dec 1983.
2. [Ansari_93] R. Ansari and B. Liu, “Multirate Signal Processing”, Handbook of digital signal processing, New York, John Wiley & sons, 1993, pp.981-1084.
3. [Bellanger_76] M. G. Bellanger, G.Bennerot and M. Coudreuse, “Digital Filtering by polyphase network”, IEEE Trans. Acoust. Speech Signal Processing, vol. 24, no. 2, pp. 199-126, April 1976.
4. [Burrus_92] C. S. Burrus, A. W. Sowieto and R. A. Gopinath, “Least square error FIR filter design with transition bands”, IEEE Trans. Signal Processing, vol. 40, no. 6, pp. 1327-1340, June 1992.
5. [Burrus_95] C. S. Burrus, “Multiband Least Square FIR Filter Design”, IEEE Trans. Signal Processing, vol. 43, no. 2, pp. 412-421, Feb. 1995.
6. [Cain_94a] G. D. Cain and A. Yardim, “The tunable fractional delay filter”, Fourth Cost 229 Workshop on Adaptive Methods and Emergent Techniques for Signal Processing and Communications, pp. 9-24, Ljubljana, Slovenia, April 5-7, 1994.
7. [Cain_94b] G. D. Cain, A. Yardim and P. Henry, “New FIR fractional delayors for effective signal interpolation”, in Proc. DSP-94 ESA Workshop on DSP for Space Applications, pp.42-49, London, Sept. 26-28, 1994.
8. [Cain_95] G. D. Cain, A. Yardim and P. Henry, “Offset windowing for FIR fractional-sample delay”, in Proc. IEEE Int. Conf. Acoust. Speech Signal Processing, vol. 2, pp. 1276-1279, Detroit, Michigan, May 9-12, 1995.

References

9. [Crochiere_75] R. E. Chrochier, L. R. Rabiner and R. R. Shively, "A novel implementation of digital phase shifters", Bell Syst. Tech, J., vol. 54, no. 8, pp. 1497-1502, Oct. 1975.
10. [Crochiere_83] R. E. Chrochier, L. R. Rabiner, "Multirate Digital Signal Processing", Englewood Cliffs, New Jersey: Prentice-Hall, 1983.
11. [Gardner_90a] F. M. Gardner, "Timing Adjustment via interpolation in digital demodulators", European Space Agency Report, California, June 1990.
12. [Gardner_90b] F. M. Gardner and L. Erup, "Interpolation for timing adjustment in digital modems", in Proc. Second Int. Workshop on Digital Signal Processing Techniques Applied to Space Communications, Turin, Italy, Sept. 24-25, 1990.
13. [Kay_83] S. Kay, "Some results in linear interpolation theory", IEEE Trans. Acoust. Speech Signal Processing, vol. 31, pp. 746-749, June 1983.
14. [Laakso_93] T. I. Laakso, T. Q. Nguyen and D. Koilpillai, "Designing allpass filters using the eigenfilter method", IEEE Int. Conf. Acoust. Speech Signal Processing, vol. 3, pp.77-80, April 1993.
15. [Laakso_95] T. I. Laakso, V. Valimaki and J. Henriksson, "Tunable downsampling using fractional delay filters with applications to digital TV transmission", IEEE Int. Conf. Acoust. Speech Signal Processing, vol. 2, pp.1304-1307, May 1995.
16. [Laine_88] U. K. Laine, "Digital modeling of a variable length acoustic tube", Nordic Acoustical Meeting, pp. 165-158, June 1988
17. [Lang_92] M. Lang and T. Laakso, "Design of allpass filters for phase approximation and equalization using LSEE error criterion", IEEE Int. Symp. Circuits Syst. Pp.2417-2420, May 1992.

References

18. [Lang_94] M. Lang and T. Laakso, "Simple and robust method for the design of allpass filters using least square phase error criterion", IEEE Trans. Circuits Syst.II, Jan. 1994.
19. [Liu_90] G. S. Liu and C. H. Wei, "Programmable fractional sample delay filter with Lagrange interpolation", Electronics Letters, vol. 26, no. 19, pp. 1608-1610, Sept. 1990.
20. [Liu_92] G. S. Liu and C. H. Wei, "A new variable fractional sample delay filter with nonlinear interpolation", IEEE Trans. Circuits Syst.II, Feb. 1992.
21. [Nguyen_93] T. Q. Nguyen, "The design of arbitrary FIR filters using the eigenfilter method", IEEE Trans. Signal Processing, Mar. 1992.
22. [Nguyen_94] T. Q. Nguyen, T. Laakso and R. D. Koilpillai, "Eigenfilter approach for the design of allpass filter approximating a given phase response", IEEE Trans. Signal Processing, Sept. 1994.
23. [Oetken_75] G. Oetken, T. W. Parks and H. W. Schussler, "New results in the design of digital interpolators", IEEE Trans. Acoust. Speech Signal Processing, vol. 23, pp. 301-309, June 1975.
24. [Oetken_79] G. Oetken, "A new approach for the design of digital interpolator filters", IEEE Trans. Acoust. Speech Signal Processing, vol. 27, pp. 637-643, Dec. 1979.
25. [Papoulis_77] A. Papoulis, "Signal Analysis", New York: McGraw-Hill, 1977.
26. [Parks_87] W. Parks and C. S. Burrus, "Digital Filter Design", New York:John Wiley & Sons, 1987.
27. [Pei_92] S. C. Pei and J. J. Shyu, "Eigen-approach for designing FIR filters and allpass phase equalizers with prescribed magnitude and phase response", IEEE Trans. Circuits Syst.II, Mars. 1992.

References

28. [Ramachandran_90] V. Ramachandran, C. S. Gargour and M. Ahmadi, "Generation of digital transfer functions of the FIR type approximating $z^{-p/q}$", IEEE Int. Symp. Circuits Syst., May 1990.
29. [Ramstad_82] T. A. Ramstad, "Sample rate conversion by arbitrary ratios", IEEE Int. Conf. Acoust. Speech Signal Processing, vol. 1, pp. 101-104, May 1982.
30. [Ramstad_84] T. A. Ramstad, "Digital methods for conversion between arbitrary sampling frequencies", IEEE Trans. Acoust. Speech Signal Processing, vol. 32, pp. 577-591, June 1984.
31. [Tarczynski_94] A. Tarczynski, W. Kozinski and G. D. Cain, "Sampling rate conversion using fractional sample delay", IEEE Int. Conf. Acoust. Speech Signal Processing, vol. 3, pp. 285-288, April 1994.
32. [Thiran_71] J. P. Thiran, "Recursive digital filters with maximally flat group delay", IEEE Trans. Circuit Theory, vol. CT-18, pp. 659-664, Nov. 1971.
33. [Vaidyanathan_90] P. P. Vaidyanathan, "Multirate digital filters, filter banks, polyphase networks and applications: a tutorial", Proc. IEEE. Vol. 78, no. 1, pp. 56-93, Jan. 1990.
34. [Vaidyanathan_93] P. P. Vaidyanathan, "Multirate Systems and Filter Banks", Englewood Cliffs, New Jersey: Prentice-Hall, 1993

Merci Beaucoup!