15/05

Chapter 3 - Exercises 1, 3, 7b, c, 9, 14e, f, 15, 20, 22

1. Let $F(x) = x^2$. Compute the first five points on the orbit of 1/2.

$x_0 = \frac{1}{2}$ $\quad x_1 = \frac{1}{4}$ $\quad x_2 = \frac{1}{16}$ $\quad x_3 = \frac{1}{256}$ $\quad x_4 = \frac{1}{65536}$ $\quad x_5 = \frac{1}{65536^2}$

3. Let $F(x) = x^2 - 2$. Compute $F^2(x)$ and $F^3(x)$.

$F^2(x) = F(F(x)) = (x^2-2)^2 - 2 \quad [\, x^4 - 4x^2 + 4 - 2 = x^4 - 4x^2 + 2 \,]$

$F^3(x) = ((x^2-2)^2-2)^2 - 2 \quad [\, (x^2-2)^4 - 4(x^2-2) + 4 \,]$

7. Find all real fixed points for the following functions

b) $F(x) = x^2 - 2$

recall: a fixed point is a point x_0 such that $F(x_0) = x_0$.

$\therefore x^2 - 2 = x$ $\qquad x = \frac{1 \pm \sqrt{1+8}}{2} = 2 ; -1$

$x^2 - x - 2 = 0$

c) $F(x) = x^2 + 1$

$x^2 - x + 1 = 0$ $\qquad x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2}$ $\quad \therefore$ no real fixed points

9. Let $F(x) = 1 - x^2$. Show that 0 lies on a 2-cycle for this function.

$x_0 = 0$ $\quad x_1 = F(0) = 1$ $\quad x_2 = F(1) = 0$ $\quad x_3 = F(0) = 1$ $\quad \ldots$

↳ Sequence $0, 1, 0, 1, \ldots$ $\therefore$ 2-cycle

14. For these seeds, discuss behavior of resulting orbit under D.

recall: $D(x) = 2x \bmod 1$, $\quad 0 \leq x < 1$.

e) $x_0 = 1/7$ $\quad x_1 = 2/7$ $\quad x_2 = 4/7$ $\quad x_3 = 1/7$ $\quad x_4 = 2/7$ $\quad x_5 = 4/7$

$\therefore$ for $x_0 = 1/7$, the orbit ~~has~~ has a 3-cycle.

f) $x_0 = \frac{1}{14}$ $x_1 = \frac{1}{7}$ $x_2 = \frac{2}{7}$ $x_3 = \frac{4}{7}$ $x_4 = \frac{1}{7}$ $x_5 = \frac{2}{7}$ ...

$\therefore x_0$ is eventually periodic, with the orbit having a cycle of period 3.

note that 1/14 is $2 \cdot 1/7$. If a seed x is periodic, ~~does~~ is any seed nx, for $n \in \mathbb{Z}$, also eventually periodic?

15. Give an explicit formula for $D^2(x)$ and $D^3(x)$. Can you write down a general formula for $D^n(x)$?

$D^2(x) = D(D(x)) = D(2x \bmod 1) = 2(2x \bmod 1) \bmod 1$

$D^2(0.4) = D(0.8) = 0.6$

$D^2(1.1) = D(D(1.1)) = D(0.2) = 0.4$

$D^2(1.2) = D(D(1.2)) = D(0.4) = 0.8$ $\therefore D^2(x) = 4x \bmod 1$

$D^3(x) = 2(2(2x \bmod 1) \bmod 1) \bmod 1$

$D^3(0.4) = D^2(0.8) = D(0.6) = 0.2$

$D^3(1.1) = D^2(0.2) = D(0.4) = 0.8$

$\therefore D^3(x) = 8x \bmod 1$

$\rightarrow D^n(x) = 2^n x \bmod 1$

20. Consider $T(x) = \begin{cases} 2x & \text{for } 0 \le x \le 1/2 \\ 2-2x & \text{for } 1/2 \le x \le 1 \end{cases}$ it's called tent map because of shape of graph on interval

Find all fixed points for T and T^2.

T: $2x = x$ $\quad 2-2x = x$

$x - 2x = 0$ $\quad x = 2/3$

$\rightarrow x = 0$

$\therefore 4x = x \rightarrow x = 0$

$4x - 2 = x \rightarrow x = 2/3$

it looks like it!

? will fixed points always be the same?

$T^2(x) = \begin{cases} 4x \\ 2-2(2-2x) = 4x-2 \end{cases}$

22. What does the graph of T^n look like?

$T^3 \begin{cases} 8x \\ 4(2-2x)-2 = \\ = -8x+6 \end{cases}$

$\therefore$ for n even, two parallel lines

for n odd, a tent map.