14/05

3.1-3.4 Orbits & Iteration

Dfn: For a function F, $F^n(x)$ is the n-fold composition of F with itself, or, $F^n(x)$ is the n^{th} iterate of F evaluated at x.

Ex) If $F(x) = x^2+1$, then $F^2(x) = F(F(x)) = (x^2+1)^2+1$

Dfn: Given $x_0 \in \mathbb{R}$, we define the orbit of x_0 under F to be the sequence $x_0, x_1 = F(x_0), x_2 = F^2(x_0), \dots, x_n = F^n(x_0)$.
The point x_0 is called the seed of the orbit.

Ex) If $F(x) = \sqrt{x}$ and $x_0 = 256$, then:

$x_0 = 256$

$x_1 = \sqrt{256} = 16$

$x_2 = \sqrt{16} = 4$

$x_3 = \sqrt{4} = 2$

$x_4 = \sqrt{2} = 1.41\dots$

Dfn[1]: A fixed point is a point x_0 that satisfies $F(x_0) = x_0$.

↳ Orbit of a fixed point: $x_0, x_0, x_0, \dots$

Dfn[2]) The point x_0 is periodic if $F^n(x_0) = x_0$ for some $n>0$. The least such n is called the prime period of the orbit.

∴ If x_0 is periodic with prime period n, then orbit of x_0 is just a repeating sequence: $x_0, F(x_0), \dots, F^{n-1}(x_0), x_0, F(x_0), \dots, F^{n-1}(x_0), \dots$

∴ If x_0 has prime period k, then x_0 is fixed by $F^{nk}(x_0)$ for $n \in \mathbb{N}, n\geq 1$.

Dfn[3+4]) A point x_0 is called eventually fixed or eventually periodic if x_0 is not fixed/periodic, but some point on the orbit of x_0 is fixed or periodic.

Ex.1: For $F(x) = x^3$, if $x=1 \Rightarrow F(1) = 1,\ F^2(1) = 1,\ F^3(1) = 1$

Ex.2: For $F(x) = x^2-1$, if $x = 0 \Rightarrow F(0) = -1,\ F^2(0) = F(-1) = 0,\ F^3(x) = F(0) = -1$ ↗ 2-cycle

Ex.3: For $F(x) = x^2$, if $x = -1 \Rightarrow F(-1) = 1,\ F^2(-1) = 1,\ F^3(-1) = 1$

Ex.4: For $F(x) = x^2-1$, if $x=1 \Rightarrow F(1) = 0,\ F^2(1) = -1,\ F^3(1) = 0,\ F^4(1) = -1$ ↗ cycle of period 2.

Some ~~fo~~ simple functions can have orbits of great complexity!

Ex) Consider $F(x) = x - 2$.

n	$x=0$	$x=0.1$	$x=0.01$	$x=0.001$
0	0	0.1	0.01	0.001
1	-2	-1.99	-1.999	-1.999
2	2	1.960	1.999	1.999
3	2	1.842	1.998 1.993	1.999
4	2	1.393	1.971	1.999
5	2	-0.597	1.898	1.999
6	2	-1.996		1.999
7	2	1.986	1.604	1.996
8	2	1.943	0.573	1.984
9	2	1.776	-1.671	1.938
10	2	1.154	0.793	1.755

3.5 ~ The Doubling Function

$$D(x) = \begin{cases} 2x & 0 \le x < 1/2 \\ 2x-1 & 1/2 \le x < 1 \end{cases}$$

$\therefore$ Domain is half open, half closed $[0,1)$

or, $D(x) = 2x \bmod 1$ $\therefore$ $D(x)$ is fractional part of $2x$.

Ex) $D(0.3) = 0.6$ $D(0.6) = 1.2 - 1 = 0.2$

3.6 ~ Experiment

See Github. We consider the functions

a. $F(x) = x^2 - 2$, for $-2 < x < 2$

b. $G(x) = x^2 + c$, for some $c < -2$

c. The doubling function D.