19/05

Finishing up cardinality:

Dfn: If A is a set, $\mathcal{P}(A) = \{B : B \subset A\}$

Ex): $\mathcal{P}(\emptyset) = \{\emptyset\}$

Thm (Cantor): If A is a set, then $|A| < |\mathcal{P}(A)|$.

Rmk: $\therefore |\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| \ldots$

Pf: Let A be a set. Define $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$.

Then if $f(x) = f(y) \Rightarrow \{x\} = \{y\} \Rightarrow x = y$. $\therefore f$ is 1-1.

$\therefore |A| \leq |\mathcal{P}(A)|$

Now, we show that $|A| \neq |\mathcal{P}(A)|$. We do this by contradiction.

Assume $|A| = |\mathcal{P}(A)|$. Then $\exists$ bijection $g : A \to \mathcal{P}(A)$.

Define a set B, $B \subset A$, $B = \{x \in A : x \notin g(x)\}$.

Since $B \subset A$, then $B \in \mathcal{P}(A)$. Since g is surjective, $\exists\, b \in A$ s.t.

$g(b) = B$. *

Case 1: $b \in g(b)$

If $b \in g(b) = B \Rightarrow b \in B \Rightarrow b \notin g(b)$. Contradiction

Case 2: $b \notin g(b)$

If $b \notin g(b) \Rightarrow b \in B \Rightarrow b \in g(b)$ * by this. Contradiction.

Thus, we have shown that $b \in g(b) \Leftrightarrow b \notin g(b)$. Contradiction.

Thus, $|A| \neq |\mathcal{P}(A)|$.

The Real Numbers

Goal: describe $\mathbb{R}$.

Thm (without proof): there exists a unique ordered field containing $\mathbb{Q}$ with the least upper bound property, which we denote by $\mathbb{R}$.

Ordered sets / fields

Dfn: an ordered set is a set S with a relation $<$ s.t. ("an order")

1) $\forall x, y \in S$, either $x = y$, $x < y$, or $y < x$
2) If $x < y$, and $y < z$, then $x < z$.

Ex) $\mathbb{Z}$ $[m < n$ if $n - m \in \mathbb{N}]$

$\mathbb{Q}$ $[q < r$ if $\exists m, n \in \mathbb{N} : r - q = \frac{m}{n}]$

Non Ex) $S = P(\mathbb{N})$. We define a relation $A \prec B$ if $A \subset B$.
Then, $\prec$ satisfies (2) $\because$ if $A \subset B$ and $B \subset C$, then $A \subset C$ i.e. $A \prec C$.
But $\prec$ does not satisfy (1). $\because$ $\{0\} \neq \{1\}$ but neither $\{0\} \prec \{1\}$ or $\{1\} \prec \{0\}$ hold.

Ex) Dictionary ordering of $\mathbb{Q} \times \mathbb{Q}$

We say $(a, b) < (q, r)$ if either $a < q$ or $a = q$ and $b < r$.

Then $<$ is an order on $\mathbb{Q} \times \mathbb{Q}$.

Dfn: Let S be an ordered set. Let $E \subset S$.

1) If $\exists b \in S$ s.t. $\forall x \in E$, $x \leq b$, then E is bounded above and b is an upper bound for E.

2) If $b \in S$ s.t. $\forall x \in E$, $b \le x$, then E is bounded below and b is a lower bound for E.

3) We call $b_0 \in S$ the least upper bound for E if

a) b_0 is an upper bound for E

b) if b is any upper bound for E, then $b_0 \le b$.

We also call b_0 the supremum of E, $b_0 = \sup E$.

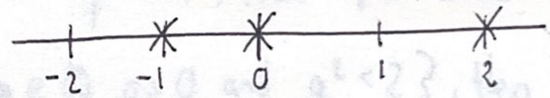
4) We call $b_0 \in S$ the greatest lower bound for E if,

a) b_0 is a lower bound for E.

b) if b is any lower bound for E, then $b \le b_0$.

b_0 is the infimum of E, $b_0 = \inf E$

Ex) $S = \mathbb{Z}$, $E = \{-1, 0, 2\}$



- UB's: $2, 3, 4, 5 \ldots$ $\sup E = 2$
- LB's: $-1, -2, -3, \cdots$ $\inf E = -1$

Ex) $S = \mathbb{Q}$, $E = \{q \in \mathbb{Q} : 0 \le q \le 1\}$

- $\sup E = 1$ • $\inf E = 0$

Ex) $S = \mathbb{Q}$, $E = \{q \in \mathbb{Q} : 0 < q < 1\}$

- $\sup E = 1$, but $1 \notin E$. • $\inf E = 0$, but $0 \notin E$.

__Dfn__: An ordered set S has the least upper bound property if every $E \subset S$ which is nonempty and bounded above has a supremum in S.

Ex) $S = \{0\}$

$S = \{0,1\} \longrightarrow$ $E = \{0\} \Rightarrow \sup E = 0 \in S$
$E = \{1\} \Rightarrow \sup E = 1 \in S$
$E = \{0,1\} \Rightarrow \sup E = 1 \in S$

__recall__: we don't consider $\{1,0\}$ ∵ it is an ordered set.

Ex) $S = \{-1,-2,-3,-4,\dots\}$

If $E \subset S$, E nonempty, then $-E = \{-x : x \in E\} \subset \mathbb{N}$.

By the well-order prop. of $\mathbb{N}$, $\exists m \in -E$ s.t. $m \leq -x \ \forall x \in E$.

$\Rightarrow -m \in E$ and $\forall x \in E$, $x \leq -m \Rightarrow -m = \sup E$

__Claim:__ $\mathbb{Q}$ does not have the least upper bound property.

Basically, if $E = \{q \in \mathbb{Q} : q > 0,\text{ and } q^2 < 2\}$, then $\sup E$ DNE in $\mathbb{Q}$.

__Thm:__ If $x \in \mathbb{Q}$ and $x = \sup\{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$, then $x \geq 1$ and $x^2 = 2$.

__Pf:__ Let $E = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$ and suppose we have $x \in \mathbb{Q}$ s.t. $x = \sup E$.

Since $1 \in E$, ~~and $1^2 < 2$~~ and $x = \sup E \Rightarrow 1 \leq x$.

We now prove $x^2 \geq 2$ by contradiction. Assume $x^2 < 2$.

Define $h = \min\left\{\frac{1}{2}, \frac{2-x^2}{2(2x+1)}\right\} < 1$. Then $h > 0$. We now prove $x+h \in E$.

We compute
$$
\begin{aligned}
(x+h)^2 &= x^2 + 2xh + h^2 \\
&< x^2 + 2xh + h \quad ∵ h < 1 \\
&= x^2 + (2x+1)h \\
&\leq x^2 + (2x+1)\left(\frac{2-x^2}{2(2x+1)}\right) \leq x^2 + 2 - x^2 = 2
\end{aligned}
$$

$\Rightarrow (x+h)^2 < 2 \Rightarrow x+h \in E$. But $x+h \in E$ and $x+h > x \Rightarrow x \neq \sup E$ $\rightarrow\leftarrow$

Contradiction. $\therefore x^2 \geq 2$.

We now show $a^2 = 2$. Since $a^2 \geq 2$, this means $a^2 = 2$ or $a^2 > 2$.

We show $a^2 > 2$ cannot hold, by contradiction.

Assume $a^2 > 2$ (we'll show it next lecture).