13/05

1. Sets

Definition (Dfn): a set is a collection of objects called elements/members.

Dfn: the empty set is the set with no elements, denoted by $\emptyset$.

Some notation we'll use: • $\Rightarrow$ "implies"

Dfn:

1) A set A is a subset of B, $A \subset B$, if $a \in A \Rightarrow a \in B$.
2) Two sets are equal, $A = B$, if $A \subset B$ and $B \subset A$.
3) A is a proper subset of B, $A \subsetneq B$, if $A \subset B$ and $A \neq B$.

Ex)

1) $\mathbb{N} = \{1, 2, 3, 4, \dots\}$
2) $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$
3) $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$ Note: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \ (\subset \mathbb{C})$
4) $\mathbb{R} = \mathbb{Q}$ along w/ irrationals [this will be our first goal]
5) Odd numbers $= \{2m - 1 : m \in \mathbb{N}\}$

Dfn

1) The union of A, B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$
2) The intersection of A, B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$
3) The set difference of A with respect to B, is the set
   $A \setminus B = \{x \in A : x \notin B\}$
4) The complement of A is the set $A^c = \{x : x \notin A\}$
5) Two sets are disjoint if ~~their~~ $A \cap B = \emptyset$.

Thm (De Morgan): if A, B, C are sets, then

1) $(B \cup C)^c = B^c \cap C^c$

2) $(B \cap C)^c = B^c \cup C^c$

3) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

4) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Pf (1): Let B, C be sets. We want to show that $(B \cup C)^c \subset B^c \cap C^c$ and $B^c \cap C^c \subset (B \cup C)^c$.

- 1st thing WTS: $(B \cup C)^c \subset B^c \cap C^c$.
  Let $x \in (B \cup C)^c$. Then $x \notin B \cup C \Rightarrow x \notin B$ and $x \notin C$. $\Rightarrow$
  $\Rightarrow x \in B^c$ and $x \in C^c \Rightarrow x \in B^c \cap C^c$. Thus, $(B \cup C)^c \subset B^c \cap C^c$.
- 2nd thing WTS: $B^c \cap C^c \subset (B \cup C)^c$
  Let $x \in$ ~~$(B \cup C)^c$~~ $B^c \cap C^c$. Then $x \in B^c$ and $x \in C^c \Rightarrow x \notin B$ and $x \notin C \Rightarrow$
  $\Rightarrow x \notin B \cup C \Rightarrow x \in (B \cup C)^c$. Thus, $B^c \cap C^c \subset (B \cup C)^c$. $\square$

2. Induction

$\mathbb{N} = \{1, 2, 3, \dots\}$ has an ordering

Axiom (Well ordering property of $\mathbb{N}$): if $S \subset \mathbb{N}$ and $S \neq \emptyset$, then S has a least element. i.e. $\exists x \in S$ s.t. $x \leq y$, $\forall y \in S$.

Thm (Induction of Pascal Holes). Let $P(n)$ be a statement depending on $n \in \mathbb{N}$.
Assume
1) (Base case) $P(1)$ is true
2) (Inductive step) If $P(m)$ is true, then $P(m+1)$ is true

Then, $P(n)$ is true $\forall n \in \mathbb{N}$.

Pf: Let $S = \{n \in \mathbb{N} : P(n) \text{ is not true}\}$ WTS: $S = \emptyset$. We will show this by contradiction.
Sps $S \neq \emptyset$. By the Well Ordering Prop., S has a least element $x \in S$.
Since $P(1)$ is true, $1 \notin S \Rightarrow x \neq 1$. In particular, $x > 1$.

ince x is the least element of S, and $x-1<x$, then $x-1 \notin S$. Thus, by dfn of S, $P(x-1)$ is true. However, by (2) $\Rightarrow P(x)$ is true $\Rightarrow x \notin S$.

Then, from $S \neq \emptyset$, we arrived at a false statement. $\rightarrow\leftarrow$

Thus, $S = \emptyset$. $\square$

Using induction. we want to prove $\forall n \in \mathbb{N}, P(n)$ is true. $\boxed{!}$ I know this.

<u>Thm</u>: $\forall c \neq 1, \forall n \in \mathbb{N}, 1+c+c^2+\cdots+c^n = \dfrac{1-c^{n+1}}{1-c}$. This is $P(n)$.

<u>Pf</u>: We do this by induction.

$P(1)$: $1+c^1 = \dfrac{1-c^{1+1}}{1-c}$. It holds.

Assume $P(n)$ holds for $n=k$. $1+c+c^2+\cdots+c^k = \dfrac{1-c^{k+1}}{1-c}$. We want to show it holds for $n=k+1$.

$$1+c+c^2+\cdots+c^k+c^{k+1} = \frac{1-c^{k+1}}{1-c} + c^{k+1}$$
$$= \frac{1-c^{k+1}+c^{k+1}-c^{k+2}}{1-c} = \frac{1-c^{(k+1)+1}}{1-c}$$

Thus, $P(n)$ holds for $n=k+1$. Then, by induction, $P(n)$ holds for all $n \in \mathbb{N}$.

<u>Thm</u>: if $c \geq -1$, then $\forall n \in \mathbb{N}$ $(1+c)^n \geq 1+nc$

<u>Pf</u>: we do this by induction

$P(1)$: $(1+c)^1 \geq 1+c$. It holds. Assume $P(n)$ holds for $n=k$. Now, for $n=k+1$:

$$(1+c)^{k+1} = (1+c)(1+c)^k \geq (1+c)(1+kc)$$
$$= 1+(k+1)c + kc^2$$
$$\geq 1+(k+1)c.$$

Then, $P(n)$ holds for $n=k+1$. By induction, $P(n)$ holds $\forall n \in \mathbb{N}$.