15/05 ~ San Isidro!

Cantor's Theory of Cardinality (Size)

Q: When do two sets, A and B, have the same size?

↳ A (kind of, given by Cantor): when the elements of the two sets can be paired off. (Theory of Cardinality).

Functions:

Dfn: If A, B are sets, a function $f: A \to B$ is a mapping that assigns to each $x \in A$ a unique element $f(x) \in B$.

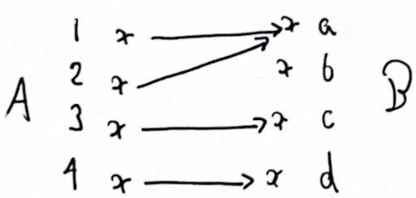
Dfn: Let $f: A \to B$.

1) If $C \subset A$, we define $f(C) = \{y \in B : \exists x \in C \text{ s.t. } y = f(x)\}$
$= \{f(x) : x \in C\}$

inverse image, not inverse function.

2) If $D \subset B$, we dfn $f^{-1}(D) = \{x \in A : f(x) \in D\}$. Inverse image always exist.

Ex) f:



$f(\{1,2\}) = \{a\}$ $\quad$ $f(\{1,3\}) = \{a,c\}$

$f^{-1}(\{a\}) = \{1,2\}$ $\quad$ $f^{-1}(\{a,c,d\}) = \{1,2,3,4\}$

Dfn: Let $f: A \to B$.

1) f is injective, one-to-one, if $f(x_1) = f(x_2) \to x_1 = x_2$ (or $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$)
2) f is surjective or onto if $f(A) = B$. Everything in B gets mapped by something in A.
$\forall y \in B, \exists x \in A : f(x) = y$
3) f is bijective if f is 1-1 and surjective

Dfn: If $f: A \to B$, $g: B \to C$

1) $g \circ f: A \to C$ is defined by $(g \circ f)(x) = g(f(x))$ → composition
2) if f is bijective, then $f^{-1}: B \to A$ by: if $y \in B$, then $f^{-1}(y) \in A$ is the unique element in A s.t. $f(f^{-1}(y)) = y$

* Bijections is what we'll mean when saying sets can be paired off!

__Cardinality:__

Dfn: Two sets A and B have the same cardinality if $\exists$ bijective function $f: A \to B$

• Notation: 1) If A, B have same cardinality, we write $|A| = |B|$

2) If $|A| = |\{1, \dots, n\}|$, we write $|A| = n$. The natural numbers up to n. → A is finite

3) If $\exists$ injective fnct $f: A \to B$, we write $|A| \leq |B|$

4) If $|A| \leq |B|$, but $|A| \neq |B|$, we write $|A| < |B|$.

Thm (Cantor-Schroder-Bernstein): If $|A| \leq |B|$ and $|B| \leq |A|$, $|A| = |B|$

Dfn: If $|A| = |\mathbb{N}|$, then A is countably infinite.

If A is finite or countably infinite, we say A is countable.

Otherwise, we say A is uncountable.

Thm: $|\{2n : n \in \mathbb{N}\}| = |\mathbb{N}|$ ← this means is countable

$|\{2n-1 : n \in \mathbb{N}\}| = |\mathbb{N}|$ interesante

Feynman: "There are twice as many numbers as numbers"

Pf: We want to find a bijective function from one set to the other:

One quick theorem before: **Thm**: If $|A| = |B|$, then $|B| = |A|$

Pf: Sps $|A| = |B|$. Then $\exists$ bijective fnct $f: A \to B$. Then $f^{-1}: B \to A$ is a bijection.

$\therefore |B| = |A|$. □

Thm: If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$.

Pf: Sps $|A|=|B|$ and $|B|=|C|$. Then $\exists$ bijections f: from $A \to B$ and $g: B \to C$. Let $h: A \to C$ be the fnct. $h(x) = (g \circ f)(x)$.

We want to prove h is a bijection.

①: h is 1-1. If $h(x_1) = h(x_2)$, then $x_1 = x_2$.

If $h(x_1) = h(x_2)$, then $g(f(x_1)) = g(f(x_2))$

Since g is 1-1, $f(x_1) = f(x_2)$.

Since f is 1-1, $x_1 = x_2$. $\therefore h$ is injective

②: h is surjective. $h(A) = C$, $\forall z \in C, \exists x \in A$ s.t. $h(x) = z$.

Let $z \in C$. Since g is surjective, $\exists y \in B$ s.t. $g(y) = z$. Since f is surjective, $\exists x \in A$ s.t. $f(x) = y$. Then $h(x) = g(f(x)) = g(y) = z$. $\therefore h$ is surjective

$\therefore h$ is bijective. $\square$

Now, back to Thm: $|\{2n : n \in \mathbb{N}\}| = |\mathbb{N}|$

Pf: We want to find a bijective function from one set to the other

Let $f: \mathbb{N} \to \{2n : n \in \mathbb{N}\}$. That is, $f(n) = 2n$, $n \in \mathbb{N}$.

recall that $|\{2n : n \in \mathbb{N}\}| = |\mathbb{N}|$ is the same as $|\mathbb{N}| = |\{2n : n \in \mathbb{N}\}|$.

①: Let $f(n_1) = f(n_2)$. Then $2n_1 = 2n_2 \to n_1 = n_2$

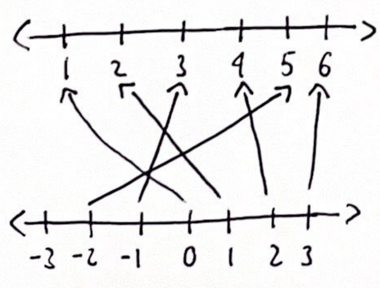
$\therefore f$ is injective

②: Let m be an even integer, $m \in \{2k : k \in \mathbb{N}\}$. Then $\exists n \in \mathbb{N}$ s.t. $m = 2n$.

Then $f(n) = 2n = m$. $\therefore f$ is onto.

$\therefore f$ is bijection $\therefore$ the two sets have the same cardinality!

Thm: $|\mathbb{Z}| = |\mathbb{N}|$ (we'll prove it in HW)



basically, pstv. ints. get mapped to even numbers and ngtv. ints. get mapped to odd numbers.

Thm: $|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|$ (prove it in HW too)

remark: $\forall q \in \mathbb{Q}$, we have $q = \dfrac{p_1^{a} p_2^{b} \cdots p_n^{r_n}}{q_1^{s_1} q_2^{s_2} \cdots q_m^{s_m}}$ *, $r_j, s_k \in \mathbb{N}$, $\forall j,k$ $q_j \neq p_k$

basically, the fundamental theorem of arithmetic.

* this gets mapped to $p_1^{2r_1} \cdots p_n^{2r_n} q_1^{2s_1 - 1} \cdots q_m^{2s_m - 1}$

Corollary: $|\mathbb{Q}| = |\mathbb{N}|$

Pf: (Sketch) We have $|\{q \in \mathbb{Q} : q > 0\}| = |\{r \in \mathbb{Q} : r < 0\}|$

since $f(q) = -q$ is a bijection from 1st set to 2nd set.

Thus $|\{r : \mathbb{Q} : r < 0\}| = |\mathbb{N}|$. Then $\exists f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ and $g : \{r \in \mathbb{Q} : r < 0\} \to \mathbb{N}$. Define $h : \mathbb{Q} \to \mathbb{Z}$ by $h(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } x > 0 \\ -g(x) & \text{if } x < 0 \end{cases}$

Then h is a bijection.

$\therefore |\mathbb{Q}| = |\mathbb{Z}| \to |\mathbb{Q}| = |\mathbb{N}|$

Q: Is there a set A s.t. $|\mathbb{N}| < |A|$?

Dfn: If A is a set, the power set of A is $\mathcal{P}(A) = \{B : B \subset A\}$

That is, the set of all subsets.

Thm: If $|A| = n$, then $|\mathcal{P}(A)| = 2^n$ (prove it in HW)

Thm (Cantor): If A is a set, then $|A| < |\mathcal{P}(A)|$

$\therefore \exists$ a set ~~greater +~~ with cardinality greater $|\mathbb{N}|$.

A: **Thm:** $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| \overset{*}{<} |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$

↳ infinity of infinitudes

* **Q:** Is there a set A s.t. $|\mathbb{N}| < |A| < |\mathcal{P}(\mathbb{N})|$?

Continuum hypothesis.

↳ Outside of the scope of the course.