Lecture - 1 - Cake Cutting

Set of n agents $N = \{1, \ldots, n\}$

Resource : $\underbrace{\text{cake } [0,1]}$ (for this lecture)

↳ Divisible & each agent wants more and more

Each agent has a valuation f^n V_i

Assumptions on V_i :-

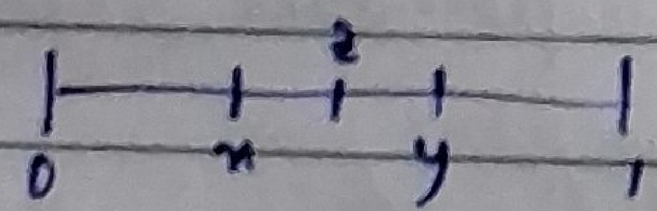
① **Normalisation :-**

$$\forall i \in N \quad V_i([0,1]) = 1$$

② **Divisibility :-**

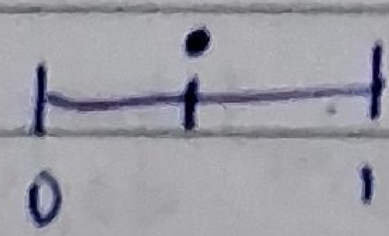
For any $[x,y] \in [0,1]$ and any $\lambda \in [0,1]$,

$$\exists\, z \in [x,y] \quad \text{s.t } V_i([x,z]) = \lambda V_i([x,y])$$



"every fraction is realisable by some cut"

⇒ No point mass



we cant break a point ☺

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③ Non-negativity

$\forall i \in N \quad \forall I \subseteq [0,1]$

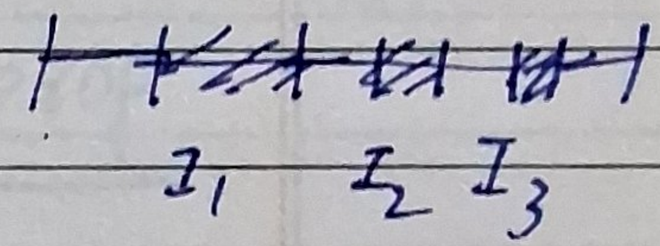
$v_i(I) \geq 0$ "we dont dislike any part of cake"

④ Additivity :-

$\forall i \in N$

$$v_i(I_1 \cup I_2 \cup \dots \cup I_k) = \sum_{j=1}^{k} v_i(I_j)$$

$I_1, I_2 \dots I_k$ are disjoint sets



Piece of cake :-

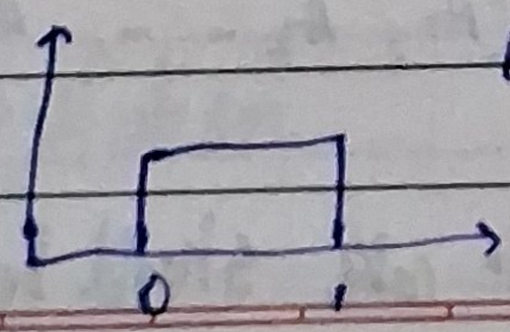
→ finite union of disjoint intervals

③ & ④ imply monotonicity

$$S \subseteq T \subseteq [0,1] \Rightarrow v_i(S) \leq v_i(T)$$

Types of Valuation functions :-

i) Uniform



density function

so $\int(\cdot)$ = valuation

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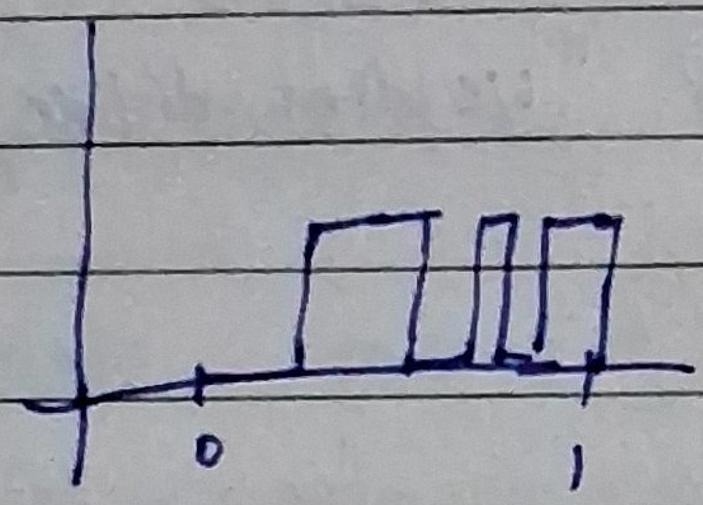
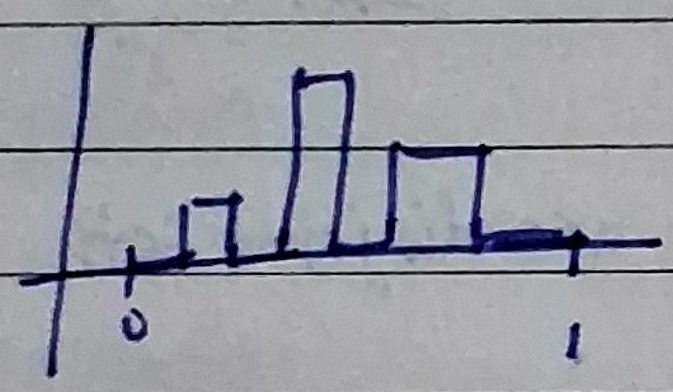
② piecewise uniform

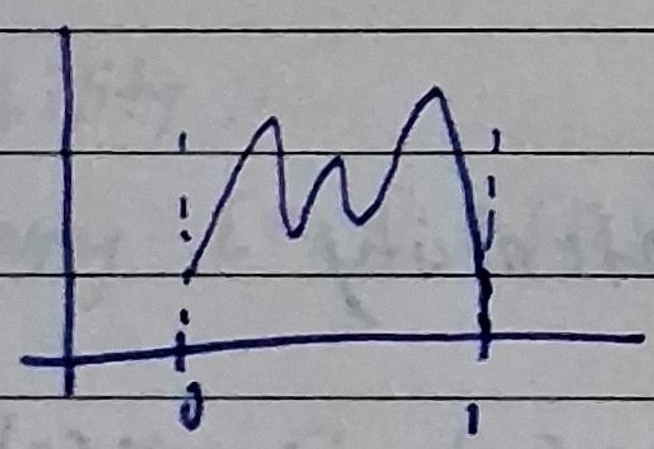


condition :- heights should be same

③ Piecewise const :-



④ Polynomial :-



Note:- The area under the curve for any Valuation f^n should be '1'

Goal

Piece of cake

Find an allocation $A = (A_1, A_2 - -, A_n)$ of $[0,1]$ that is "Fair"

(Whole cake should be ~~dist~~ Allocated)

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Fairness can have many definitions one of the popular metric is envy - freeness (EF)

Fairness notions :-

An allocation $A = \{A_1, A_2 \cdots A_n\}$

① **Proportionality :**

$\forall i \in N$

$v_i(A_i) \geq \frac{1}{n}$

② **Envy freeness**

$\forall i, j \in N$

$v_i(A_i) \geq v_i(A_j)$

prop $\not\Rightarrow$ EF

EF $\Rightarrow$ prop

spl case

for $n = 2$

prop $\Rightarrow$ EF

$$v_i(A_i) + \sum_{\substack{j \neq i \\ \forall j}} v_i(A_j) = 1$$

$$v_i(A_i) + (n-1)\, v_i(A_i) \geq 1$$

$$\boxed{v_i(A_i) \geq \frac{1}{n}}$$

Questions :-

① Existence

② Computation.

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Existence has a clean proof.

→ Let $v_1, v_2 \dots v_n$ be valuen fns induced by 'Continious' value density fn then, it is possible to cut the cake at $n^2 - n$ places and reassemble them into an Allocation $A_1, \dots A_n$ such that $\forall i, j \in N$

$$v_i(A_j) = \frac{1}{n}$$

This partn is called perfect partn

every agents values every bundle as '$\frac{1}{n}$'

So this is Prop. and EF

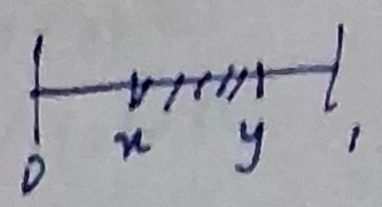
Unfortunately the above thm doesnt have a contructive ~~partition~~ proof.

Computing fair divisions :-

Robertson - Webb Query model :- (like an oracle)

i) Cut query $\rightarrow \underbrace{Cut_i(x, \alpha)}$

returns a point y such that $v_i(x, y) = \alpha$ according to i



$v_i([x, y]) = \alpha$.

'y' can be irrational

ii) Eval query

$\rightarrow eval_i[x, y]$ returns value of $[x, y]$ wrt i

so complexity is no. of RW queries

Cake-cutting algorithms :-

① I CUT YOU CHOOSE (for n = 2)

$Cut_1(0, \frac{1}{2}) \rightarrow$ Agent -1 cut the cake two equal peices (according to himself)

y

$\rightarrow$ Agent - 2 picks any one of a pieces (ofc the piece that it values more)

$eval(0, y)$

then it can chose whichever higher $eval(0, y)$ or $1 - eval(y, 1)$

$\therefore$ This allocation is EF and prop.

prop for $n > 2$

② Dubins spanier:-

→ there is knife going from 0 to 1, if for any $i \in N$ valuation reaches $1/n$, the knife stops and that particular part of cake is given to that person

so for every other person remaining

$$\text{value of cake} \geq 1 - \frac{1}{n}$$

$$\geq \frac{n-1}{n}$$

even if all others have same valuations then

$$\text{remaing vd}^n \geq \frac{n-1}{n} \times \frac{1}{n-1}$$

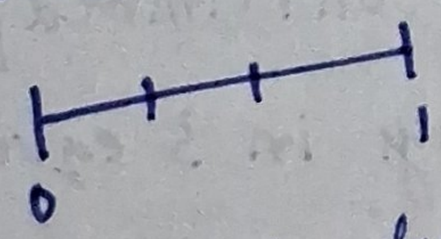
∴ prop^n is satisfied for every $i \in N$

EF for $n=3$:-

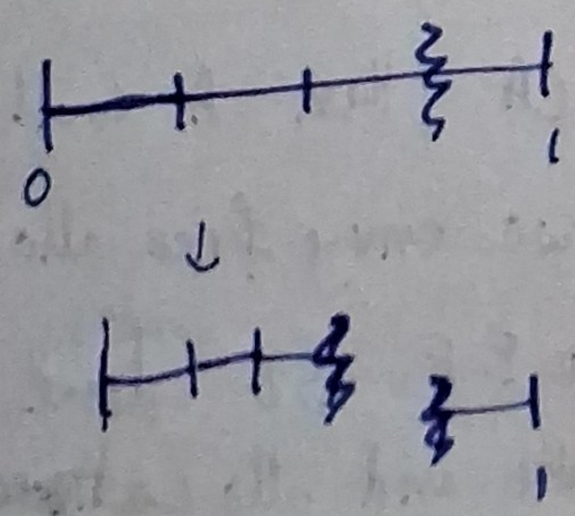
③ Selfridge - Conway :-

Agents = {A, B, C}

① Agent A divides the cake into 3 equal parts piece to itself



② Agent B divides his favorite piece such that its value is equal to its 2nd favorite



⑤ Agent c picks first then B then A

<u>condition</u> :- if c doesnt chose the trimmed part then B has to choose it.

~~This can be~~

c picks first so he cant envy others

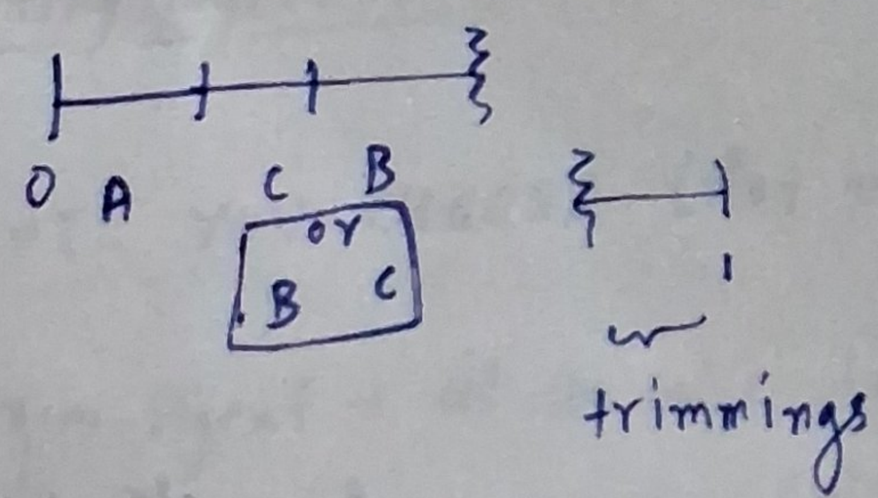
B created a two way tie so even he cant envy

A will be always remained with his 1/3 of the cake and other chose $\frac{1}{3}$rd or less acc to him so no envy

"A never gets the trimmed piece"

∴ EF allocation.

⑥ The trimmings are still there and should be distributed



→ The person who gets the untrimmed peice (B or C) here cuts the trimmings in 3 equal parts

→ Then ~~B — C — A~~

B A C

if B CA then A could have envied C, two envy free allocations union is also EF so done. we have divided the cake and allocated with EF for n = 3

→ There is a bound on no. of queries to EF division

$(n^2, n^{n^{n^{n^{n}}}})$ so it only depends on n

the valn fns doesn't matter for this bound

→ Important results : Aziz & Mackenzie proved that ∃ EF allocation with any no. of agents with v fn being additive, non atomic, can be discontinuous at finite pts with bound of $n^{n^{n^{n^{n}}}}$

Lecture - 2 :-

Fair division with indivisible goods :-

→ set of agents $N = \{1, 2, \dots, n\}$

→ Set of m indivisible goods $M = \{g_1, g_2 \dots g_m\}$

→ Valuation fns $v_i : \underbrace{2^M}_{\text{no. of subsets of } M} \to \mathbb{R}$

i) Additivity : for any $S \subseteq M$ and any agent $i \in N$

$$v_i(S) = \sum_{g_j \in S} v_i(g_j)$$

ii) Non-negativity :-

for any $g \in M$ and any agent $i \in N$

$$v_i(g) \geq 0$$

→ Allocation : $A = (\underbrace{A_1\ A_2 \dots A_n}_{\text{subsets of } M})$

A_i is disjoint

Fairness notions :-

An Allocation $A = (A_1, A_2 \dots A_n)$

① PROP orthonality :-

if $\forall i \in N$ $v_i(A_i) \geq \frac{1}{n} \left(\sum_{j \in (1, \dots m)} v_i(g_j)\right)$

② EF is same

③ Envy freeness upto 1 good (EF-1)

$\forall i, j \in N \ \exists g \in A_j$

$$v_i(A_i) \geq v_i(A_j / g)$$

Prop ↯ EF-1 are incompatable.

④ Maxmin share (MMS) :-

$A = (A_1, \dots A_n)$ is MMS fair if $v_i(A_i) \geq \mu_i \ \forall i \in N$

$\min_{K \in N} v_i(A_K)$, worst case of ith agent

so i tries to maximise it.

approx - MMS :-

$v_i(A_i) \geq \alpha \mu_i$

fraction

$$\mu_i = \max_{A \in \mathcal{A}} \min_{K \in N} v_i(A_K)$$

over all allocations

every i can build up different allocations

∴ There may not exist Allocation such that $v_i(A_i) \geq \mu_i$ for every $i \in N$

example

	a	b	c	d	e
1	6	6	4	4	4
2	6	3	3	3	0
3	6	6	12	0	0

$\mu_1 = 6 \quad ((6,6),(6,4))$

$\mu_2 = 3 \quad ((6,0),(3,3),3)$

$\mu_3 = 12 \quad ((6,6),12,(6,6))$

$A = \{ \underbrace{\{a,e\}}_{10}, \underbrace{\{b,d\}}_{6}, \underbrace{c}_{12} \}$ ∴ $\forall i \ v_i > \mu_i$

⇒ A is MMS-fair

for MMS threshold

in n allocations if μ_i is the worst then each of those other bundles at least should be $\frac{1}{n}$ (Total)

$$\sum_{j \in (1,\dots n)} v_i(A_j) = \underbrace{v_i(A_1) + v_i(A_2) \dots v_i(A_n)}_{\mu_i} \text{ then}$$

$$\geq n\mu_i$$

$$\mu_i \leq \frac{1}{n} \sum_{j=1}^{n} v_i(A_j)$$

$\therefore$ prop is upper bound of MMS

$\Rightarrow$ prop should imply MMS

$\rightarrow$ MMS is relaxation of prop

$\Rightarrow$ prop $\Rightarrow$ MMS

$$EF \Rightarrow \underbrace{EF\text{-}1 \Rightarrow \frac{1}{n}(MMS)}_{\text{how?}}$$

$A_{1,1}$ $A_{2,1}$

$A_{1,2}$ $A_{2,2}$

<u>Existence</u>:-

* EF, prop, MMS may not exist

$\rightarrow$ EF-1 always exists!

<u>Computation</u> :-

<u>Thm 1</u>. Even with two agents and identical valutations deciding whether there exists an EF allocation is NP-complete

we can redue partition problem to this

Thm :-

For $n=2$, an MMS allocation always exists

$A_1 \quad A_2$

$\underbrace{v_1(A_1)}_{\mu_1} \leq v_1(A_2)$

$v_2(A_2) \geq v_2(A_1)$

so 2 picks A_2

$\underbrace{v_2(A_2) \geq \frac{1}{2} \sum v_2(A_i)}_{\text{PROP } (\Rightarrow \text{ MMS})}$ (μ_2 can't be more than $\frac{1}{2} \sum v_2(A_i)$

Thm :-

Even with two agents and identical vals, deciding if $\mu_i \geq \theta$ is NP-complete.

→ Here Difficulty is finding out the best worst partition (finding out μ_i)

Algorithms :- [For EF-1]

① Greedy round robin :-

(i) Fix an ordering over agents

(ii) an agents pick one (their fav good) and then goes to the back of the line

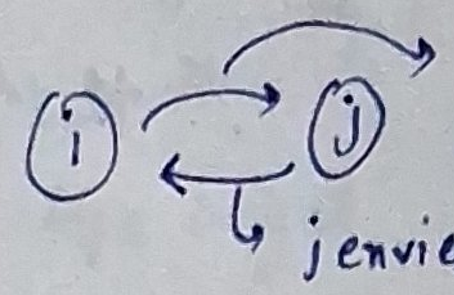
(iii) (ii) while goods are not empty

→ This is EF-1 we use additivity here to prove

② Envy Graph Algorithm :-

Given an allocation A the envy graph $G_A = (N, E)$

bidirectional ↑ vertices



if i envies j

→ can be cyclic

→ cycles can be resolved by swapping allocations

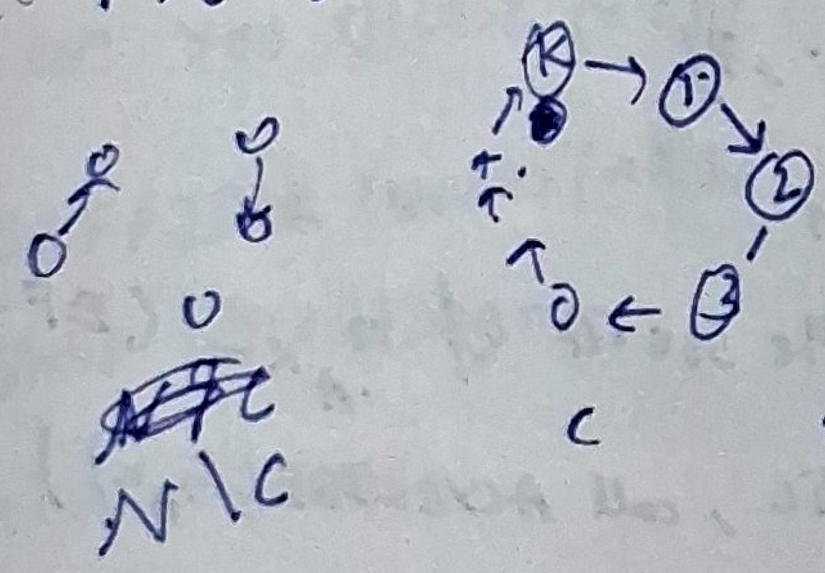
Lemma :-

Let A be an allocation such that G_A is cyclic, then one can transform it into an allocation B such that G_B is acyclic in poly^n time

$B = ACYCLIZE(A)$

Proof :-

Let $1 \to 2 \to 3 \text{ ---} \to K \to 1$ be some cycle in G_A



$B_i = A_{i+1} \bmod K \quad \forall\, i$ in cycle

$B_j = A_j \quad \forall\, j$ not in cycle

↑ new allocation

1 gets A_2 2 get A_3 ---- K gets A_1

(1) $i \in N \setminus C$
$j \in N \setminus C$

no change in $i \dots j$ — no. of edges unchanged

if there was an edge

"we say bundles, not agents"

(2) $i \in N \setminus C$
$j \in C$

the bundles don't change so these edges remain same

just the owner of bundles change

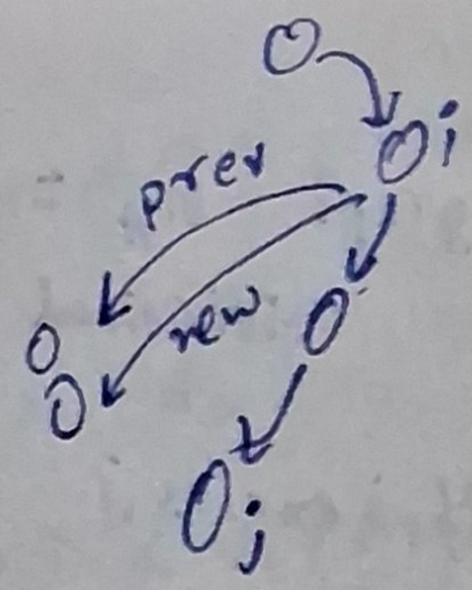
∴ edges get shifted

no. of edges unchanged

(3) $i \in C$
$j \in N \setminus C$

no new edges (may decrease but not increase)

④ $i \in C$, $j \in C$



∴ no new edges

but edges in cycle are all gone

∴ # edges strictly decreases

<u>Lemma</u> :-

ACYCLISE preserves EF1

we dont change bundles of if A was EF1 then B is EF1

<u>Algo</u> :- for a DAG, ∃ source (we can give a good tr source and EF-1 is preserved)

(1) Fix an ordering $g_1 > g_2 \cdots > g_n$

(2) initialize $A^{(0)} = \{\phi, \phi, \ldots, \phi\}$

(3) For $K = 1, 2, \ldots m$

→ assign g_K to the source of $G_{A^{K-1}}$ (EF-1 is maintained)

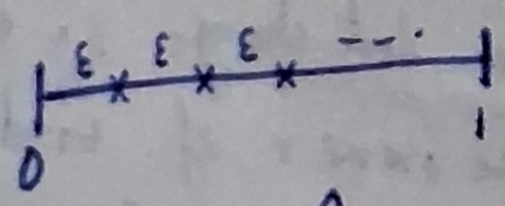
(If becomes cyclic, call ACYCLISE(G_{A^K}))

(4) Return G_{A^m}

<u>Approx-envy freeness</u> :-

→ An alloc^n $A = (A_1 \ldots A_n)$ of $[0,1]$ is ε-EF if $v_i(A_i) \geq v_i(A_j) - \varepsilon$ $\forall j \in N, j$

cut, $(0, \varepsilon)$



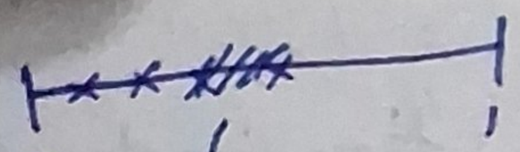
each player does $\left\lceil \frac{1}{\varepsilon} \right\rceil$ marks

2 also does the same thing

⋮

n " " " " "

So total marks $= \frac{n}{\varepsilon}$



$v(x)$ atmost ε by any agent

treating this as an indivisible good (b/w consecutive marks)

$$v_i(A_i) \geq v_i(A_i \setminus g)$$

$$= v_i(A_i) - \underbrace{v_i(g)}_{\text{atmost } \varepsilon}$$

$$= v_i(A_i) - \varepsilon$$

call the EF-1 Algo to get EF-1 division

Lecture-3 :- APPROX ENVY FREE cake cutting & Rent division

Back to cake-cutting

<u>till now</u> :-

1. I cut you CHOOSE ($n=2$)
2. SELFRIDGE LONWAY ($n=3$)
3. BRAMS TAYLOR (any n)
4. Aziz MacKenzie

In this lecture cake cutting with <u>connectedness</u>

<u>QNS</u>:-

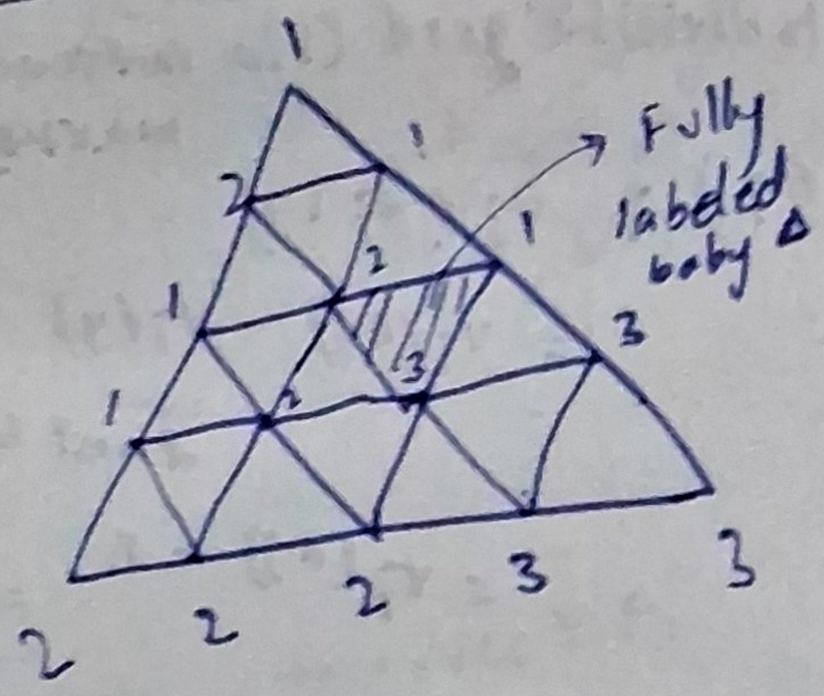
Q) Does $\exists$ a connected EF division?
→ "<u>Yes</u>" (Stromquist '80) but no finite algo

[Read su'99]

Q) Algorithms (No finite algo)

Q) connected "approx" EF division in finite steps?
→ Yes (su'99)

Sperner's Lemma :-



① Triangulation in elementary Δs

② sperner's labeling
- Main vertices have distinct labels
- Boundary vertices inherit the labels of the adj main vertices

For Internal vertices
we can assign any label (1, 2, 3)

Lemma :-
→ ∃ atleast one fully labeled elementary Δ [in fact odd]

Proof :-

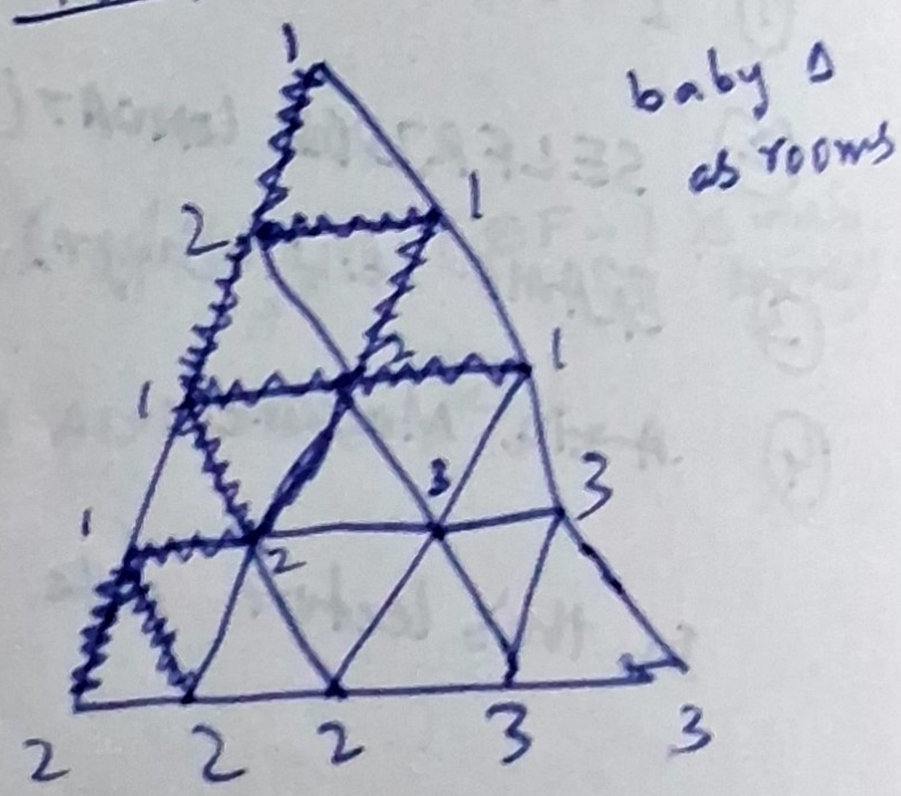
Observations :-

1, # doors on boundary is odd (atleast 1)

1 → 2
or
1 → 2 → 1 → 2

2, A room can have 0, 1, 2 doors

House :-



→ treat edges 1-2 (ann) as doors

→ each baby Δ can have '0' or '1' or '2' doors
$^{3}\Delta_{3}$ $^{1}\Delta_{3}$ $^{1}\Delta_{1}$
↑ we want these fully labeled

→ ∃ atleast one walk such that we reach fully labeled Δ^{k}

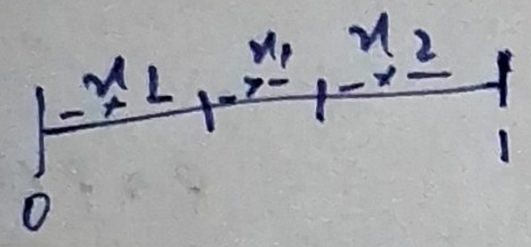
→ walk such that 1 is to left
2 is to right

Cake - cutting using spencers lemma (forest simmons)

cake : $[0, 1]$ $\quad N = \{1, 2 .. n\}$ set of agents

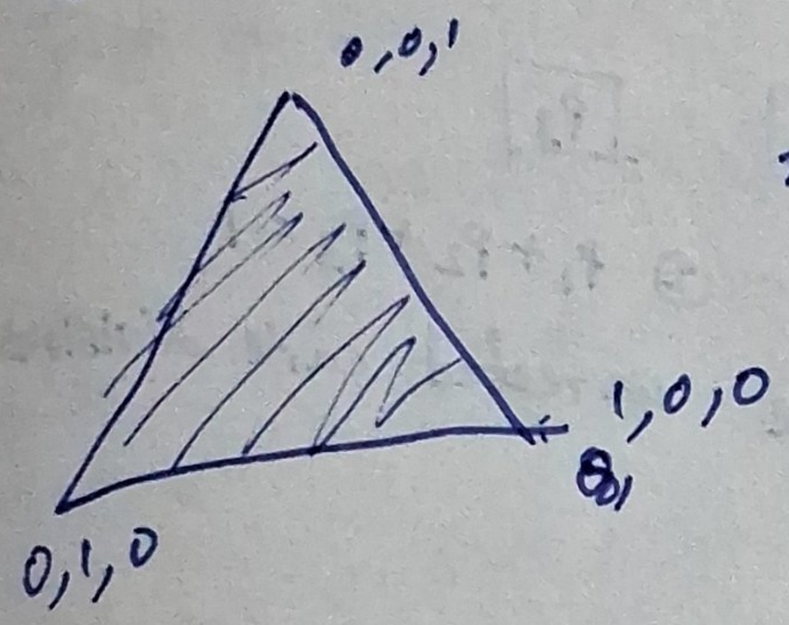
envy freeness : $v_i(A_i) \geq v_i(A_j)$

$n=3$ (for the lecture) (it generalizes to any n)



$x_1, x_2, x_3 \geq 0$

$\sum x_i = 1$

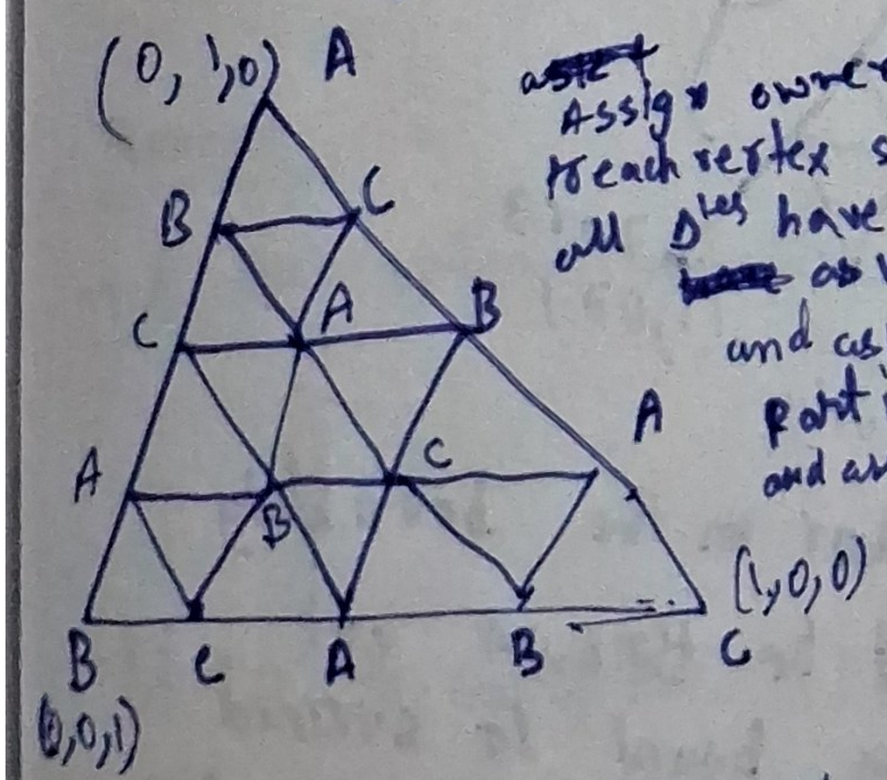


2d - plane

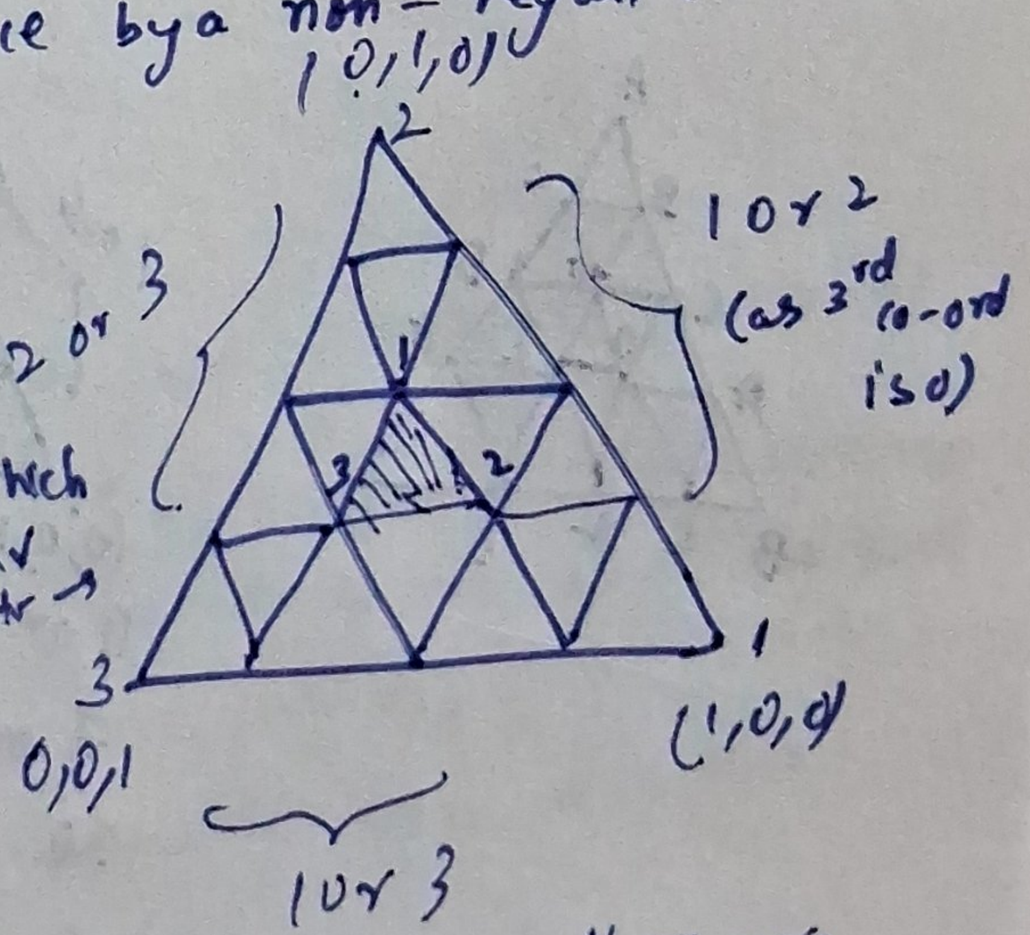
Assumptions :-

① Given any cut (x_1, x_2, x_3), each agent can ~~pic~~ point out his fav piece

② each player values every piece by a non-negative real no.



~~assi~~ Assign owners A, B, C to each vertex such that all Δ^{les} have A, B, C 2 or 3 ~~[illegible]~~ as vertices and ask them, which part is your fav and assign that tr →



connected (approx EF)

fully labeled Δ^{le} means, there are 3 nearby allocations where each agent likes different allocations (pieces)

if that nearness = 0 then EF

Closedness :-

$X = (x_1, x_2, x_3)$

Point in Δ (0,1,0) (1,0,0) (0,0,1)

Valuations are topologically closed for any sequence $\{x^{(n)}\}$ if an agent prefers piece x at each of $x^{(1)}, x^{(2)}, \ldots x^{(n)}$ then that agent will prefer x at the final limit

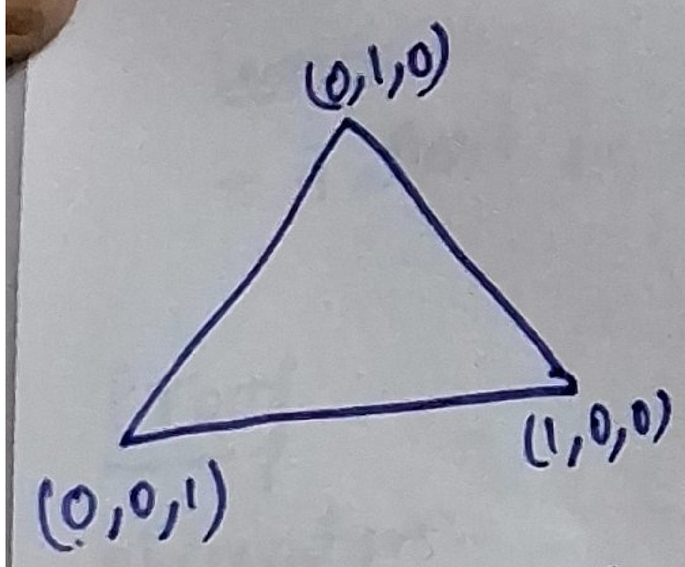
Rent-Division :-

Three friends

Rent = 1

$p \to$ price

Three rooms →



$\boxed{P_1}$ $\boxed{P_2}$ $\boxed{P_3}$

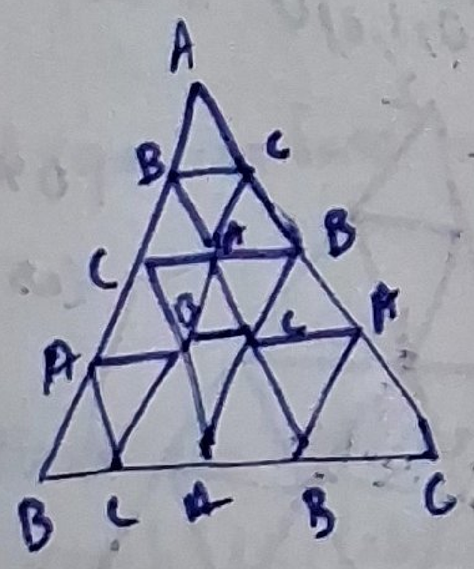
① $P_1, P_2, P_3 \geq 0$ ② $P_1 + P_2 + P_3 = 1$

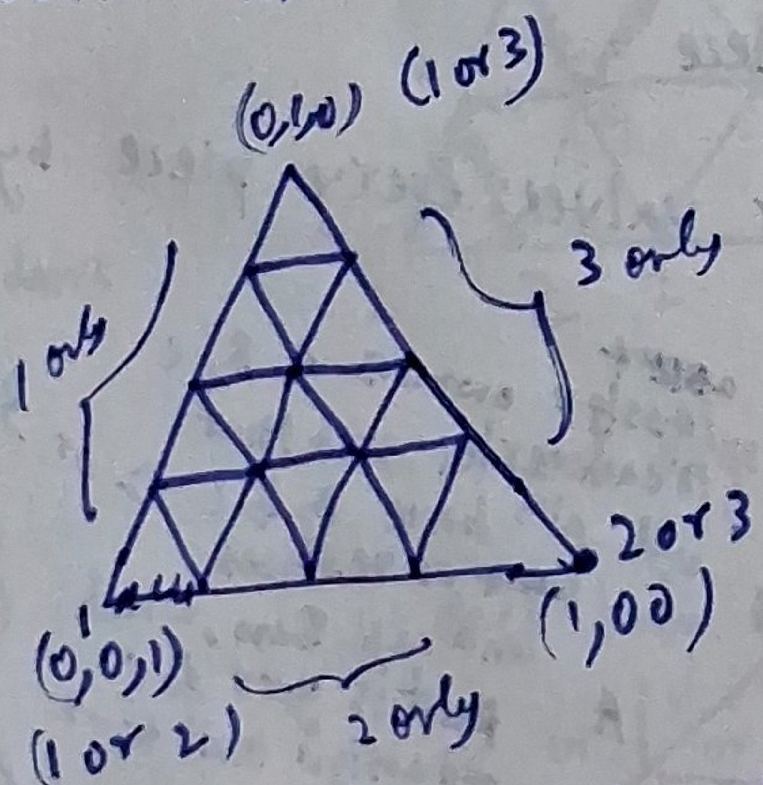
same like connected cake divisions

Assumptions:

① Ability to pick fav room given any (P_1, P_2, P_3)

② free room is better than not free rooms





∃ only one door on the boundary

so we cant be thrown out

So we are bound to succeed

Lecture - 4 :-

Nash Social Welfare :-

$N = \{1, 2 \ldots, n\}$

$M = \{g_1, \ldots g_m\}$

valuations

$$\begin{array}{cc} & \text{goods} \\ \text{Agents} & \begin{bmatrix} - & - & - & \cdot \\ - & - & - & \cdot \\ \vdots & & \ddots & \vdots \end{bmatrix} \end{array}$$

Given an allocation A

$$NSW(A) = \left[v_1(A_1) \cdot v_2(A_2) \cdots v_n(A_n)\right]^{1/n}$$

always A^* exists for an instance of valuations

$$A^* = \arg\max_{A} NSW(A)$$

$\downarrow$

Nash optimal Allocation

→ when NSW (A) is ↑, fairness ↑, think of it some no. that sum up to a number and we want highest no. when we multiple all of those, its better to divide equally.

10

$5 \cdot 5 = 25$

$9 \cdot 1 = 9$

Theorem :-

→ A Nash optimal allocation is $\overbrace{\text{EF-1}}^{\text{fair}}$ and $\overbrace{\text{PO}}^{\text{efficient}}$ (Pareto optimality)

$\downarrow$

∄ other allocation that improves happiness of one agent without decreasing any other

→ if PO is satisfied, it does not necessarily mean it is fair

eg → Give all items to 1 agent

Thm :-

Maximizing NSW is NP-Complete

↳ is $\underbrace{\text{APX-complete}}_{\text{even approx}}$ NSW is NP-Complete

Goal

special case of NSW maximization

$v_i(g_j) = 0$ or 1 $\forall$ i and j

Thm :

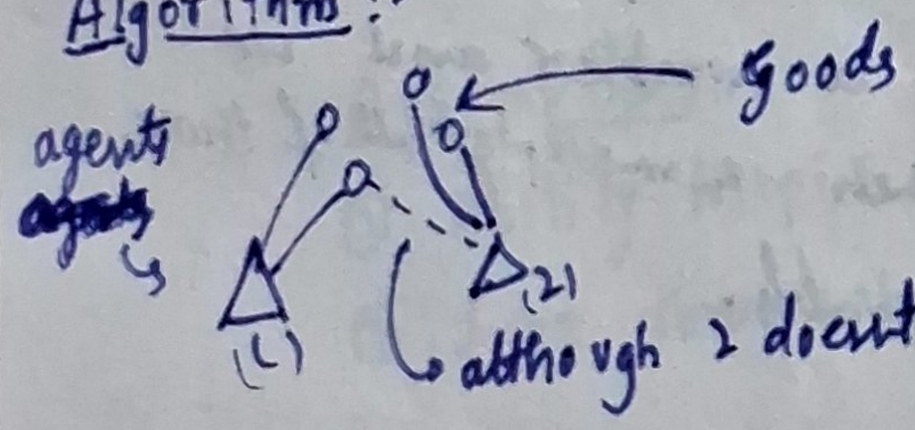
For binary valn, a nash optimal allocation can be computed in polynomial time...

Few observations.

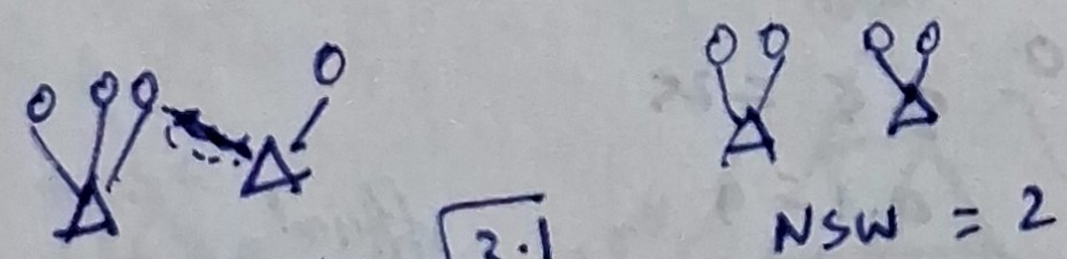
① Each good is assigned to an agent that valves it at 1

② There must be some allocation A with $NSW(A) > 0$

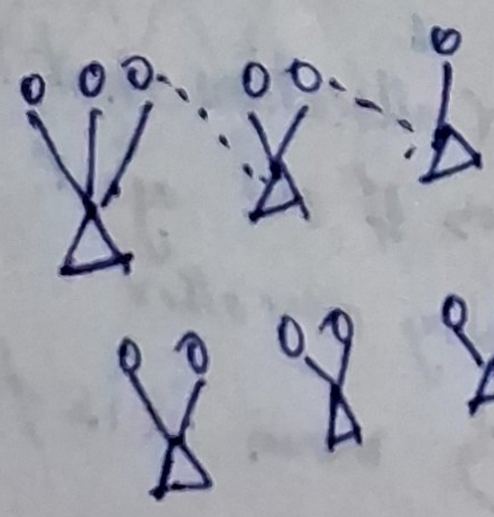
Algorithm :-



eg



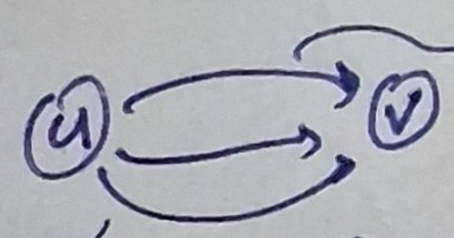
$NSW = \sqrt{3 \cdot 1}$ $\quad$ $NSW = 2$



$NSW = \sqrt{6}$ $\quad$ $NSW = 2$

NOTE → a single swap did not improve NSW here but a chain of swaps did.

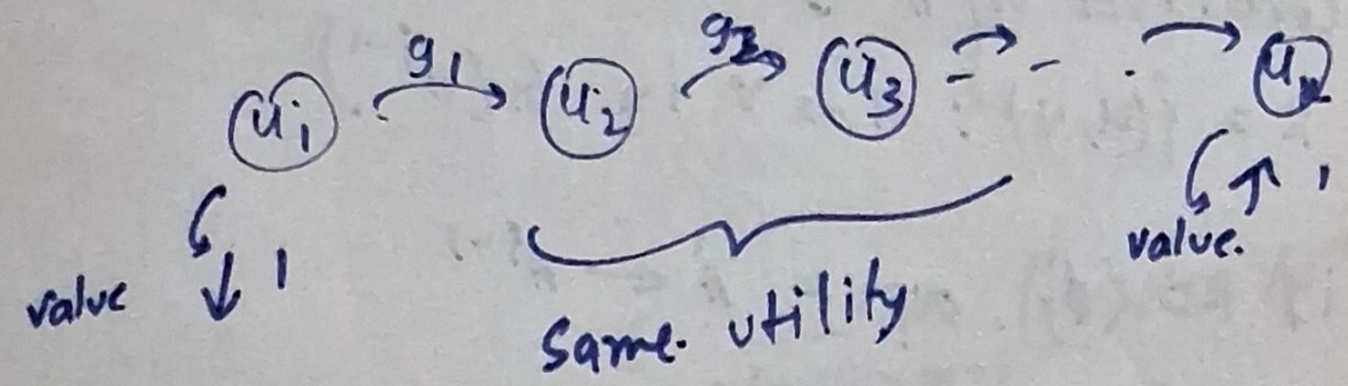
For any Allocation A, define $G(A)$, $v \to (1, 2, \dots n)$ agents



u has a good that v values if at 1 then $\exists$ a edge

potential swaps $\Rightarrow$ 3 Goods of u are valued at 1 by v

→ A path $p = (u_1, \dots u_k)$ denotes chain of swaps



instead of searching over paths
we can check for pairs of agents
(search space ↓)

<u>Lemma. 1.</u>

→ Let A be a sub optimal allocation, then $\exists$ a pair of agents $u, v \in N$ such that (a) v is reachable from u in $G(A)$

(b) Reallocating along <u>ANY</u> path p from u to v leads to a new allocation $A' = A(P)$ that satisfies

$$\ln NSW(A^*) - \ln NSW(A') \le \left(1 - \frac{1}{m}\right) \cdot \left[\ln NSW(A^*) - \ln NSW(A)\right]$$

~~no good~~

$m \to$ no. of goods

"the gap between $NSW(A^*)$ and $NSW(A)$ decreases by a fraction $\frac{1}{m}$"

→ doesnt mean that every pair will bring an improvement.

ALG - Binary :- (Iterative algorithm)

i/p An Allocation A

o/p Another Allocation A'

Init $A^0 \leftarrow A$

For $i = 1$ to $2m(n+1)\ln(mn)$

- construct $G(A^{i-1})$
- $R = \{(u,v) : u \to v \text{ in } G(A^{i-1})\}$
- if $R = \{\phi\}$, output $A' \leftarrow A^{i-1}$

otherwise

for each pair $(u,v) \in R$

Let $A^{i-1}(u,v)$ be allocation obtained by reallocation along $u \rightsquigarrow v$

If $\max_{(u,v)\in R} NSW(A^{i-1}(u,v)) > NSW(A^{i-1})$

$A^i \leftarrow \arg\max_{A^{i-1}(u,v),\,(u,v)\in R} NSW(A^{i-1}(u,v))$

else

$A' \leftarrow A^{i-1}$

Thm :- A' is Nash optimal

Proof :-

~~for an~~

at the end of iteration i

$$\underbrace{\ln NSW(A^*) - \ln NSW(A^i)}_{\text{final gap}} \le \left(1 - \frac{1}{m}\right)^i \underbrace{(\ln NSW\, A^* - \ln NSW\, A^0)}_{\text{initial gap}}$$

Since $i \le 2m(n+1)\ln(mn)$

Lecture - 5 1.

OPEN PROBLEMS :

① Query complexity :

$n=2$ CUT AND CHOOSE → 2 queries

$n=3$ SELFRIDGE - CONWAY → 14 queries

$n \geq 4$ Aziz - Mackenzie → $O(n^{n^{n^{n^{n}}}})$

lower bound → $\Omega(n^2)$

Q) make the Gap narrow

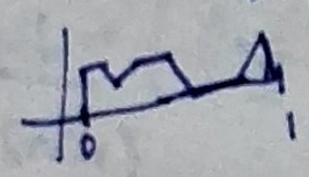
Q) improved bounds on special classes of valuations

↳ polynomial (deg d) → $O(n^2 d)$ (Branzei)

↳ piecewise linear

↳ $O(n^6 \, k \ln k)$

no. of break pts



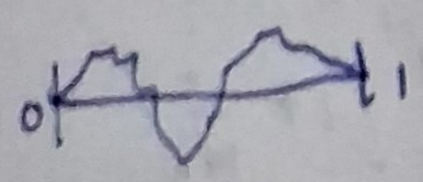
Q) Power of simple algorithms

$n=4$ Amanatidis et al improve the query complexity from ~600 (AM) to ~200

② Query complexity of approx EF with connectedness worst case $\left(\frac{1}{\varepsilon}\right)^{n-2}$ queries (Deng et al)

Q) For what classes of valuations does the "path following" algorithm require poly $(n, \frac{1}{\varepsilon})$ queries

③ Cake Cutting with both positive and negative value



Known :-

① $n=3$, a connected EF allocation exists (Segal Halevi)

② $n=4$ or prime " (Meunier and Zerbib)

Q) existence results??
computation ??

Indivisible Goods :-

① **Nash social welfare**

→ Nash optimal is EF-1 and PO

→ maximizing NSW is NP-hard

A psvedo-Polynomial time algorithm for EF-1 + PO exists

↳ Polynomial in (n, m, v_{ij}) (Barman et al)

For bounded valuations, the algo is poly time.

——— NSW is still NP-hard

Q) complexity of EF1 + PO ?

② MAXMIN SHARE (MMS)

→ Need not always exist (Procassia & Wang)

Q) what is the best approx-MMS that is guaranteed to exists ? Best answer till now $\frac{3}{4}$ MMS always exists

Q) what is the best approx MMS that can be computed in poly time ?

→ for any const $\varepsilon > 0$

$\left(\frac{3}{4} - \varepsilon\right)$ - MMS can be computed in poly time.

Q) Existence / computation of appox MMS + PO allocation

③ **Stronger fairness notions:**

EFx → Envy freeness up to the least positively valued good

An allocation $A = (A_1, A_2 \ldots A_n)$ satisfies EFx

if $\forall i, j \in N$, for any $g \in A_j$ such that $v_i(g) > 0$

$\forall g \in A_j$ (any 1)

$v_i(A_i) \geq A_i(A_j / \{g\})$

Q) (Do EFx allocations always exists) ??

EF → EFx → EF-1