DEPARTMENT OF MATHEMATICS
SCHOOL OF ADVANCED SCIENCES
Solution of homogeneous system of first order and second order differential equations by Matrix method.

Course Code: MAT2002 **Experiment: 4-A**
Course Name: Application of Differential and Difference Equations **Duration: 90 Minutes**

System of First Order Linear Differential Equations

A system of n linear first order differential equations in n unknowns (an $n \times n$ system of linear equations) has the general form:

$$\begin{cases} x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + g_1(t) \\ x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + g_2(t) \\ \vdots = \vdots + \vdots + \vdots + \vdots + \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + g_n(t) \end{cases} \tag{1}$$

where the coefficients a_{ij}'s are arbitrary constants, and g_i's are arbitrary functions of t. If every term g_i is constant zero, then the system is said to be homogeneous.

The system (1) is most often given in a shorthand format as a matrix-vector equation, in the form:

$$X' = AX + G$$

where $X' = [x_i']_{n\times 1}$, $A = [a_{ij}]_{n\times n}$, $X = [x_i]_{n\times 1}$, and $G = [g_i(t)]_{n\times 1}$.

If the coefficient matrix A has two distinct real eigenvalues λ_1 and λ_2 and their respective eigenvectors are X_1 and X_2, then the 2×2 system

$$X' = AX$$

has a general solution

$$X = C_1 X_1 e^{\lambda_1 t} + C_2 X_2 e^{\lambda_2 t}$$

System of Second Order Linear Differential Equations

Consider the system of second order linear differential equations of the form

$$\begin{cases} x_1'' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ x_2'' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots = \vdots + \vdots + \vdots + \vdots \\ x_n'' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases} \tag{2}$$

where the coefficients a_{ij}'s are arbitrary constants.

Then, the solution of (2), $X'' = AX$, is

$$X = PY$$

where Y is the solution of $Y'' = DY$, P is the modal matrix of A and D is it's diagonal matrix.

1. Solve:

$$x_1' = x_1 + 2x_2$$
$$x_2' = 0.5x_1 + x_2$$
$$x_1(0) = 16, x_2(0) = -2$$

MATLAB CODE

```
clc
clear

syms t C1 C2

A=input('Enter A: ');

[P,D]=eig(A);
L1=D(1);L2=D(4);

y1=C1*exp(L1*t);y2=C2*exp(L2*t);
Y=[y1;y2];
X=P*Y;

Cond=input('Enter the initial conditions [t0, x10,x20]: ');
t0=Cond(1);x10=Cond(2);x20=Cond(3);

eq1=subs(X(1)-x10,t0);eq2=subs(X(2)-x20,t0);
[C1, C2] = solve(eq1,eq2);

X=subs(X);
```

INPUT

```
Enter A: [1 2;0.5 1]
Enter the initial conditions [t0, x10,x20]: [0 16 -2]
```

OUTPUT

```
X =
10*exp(t/4503599627370496) + 6*exp(2*t)
3*exp(2*t) - 5*exp(t/4503599627370496)
```

2. The governing equations of a certain vibrating system are

$$x_1'' = 2x_1 + x_2$$
$$x_2'' = 9x_1 + 2x_2$$

Solve the system of equations by matrix method.

MATLAB CODE

```
clc
clear

A=input('Enter A: ');

[P D]=eig(A);

Sol1 = dsolve(['D2y = ',num2str(D(1)),'*y']);
Sol2 = dsolve(['D2y = ',num2str(D(4)),'*y']);

X = P*[Sol1;Sol2];

disp('x1=');disp(X(1))
disp('x2=');disp(X(2))
```

INPUT

```
Enter A: [-5 2;2 -2]
```

OUTPUT

```
x1=
(10^(1/2)*(C1*exp(5^(1/2)*t) + C2*exp(-5^(1/2)*t)))/10 - (10^(1/2)*
(C3*cos(t) + C4*sin(t)))/10

x2=
(3*10^(1/2)*(C3*cos(t) + C4*sin(t)))/10 + (3*10^(1/2)*
(C1*exp(5^(1/2)*t) + C2*exp(-5^(1/2)*t)))/10
```

Exercise

3. Solve the following:
   (a) $x_1' = 3x_1 - 2x_2; x_2' = 2x_1 - 2x_2; x_1(0) = 1, x_2(0) = -1.$
   (b) $x_1' = -x_2 + x_3; x_2' = 4x_1 - x_2 - 4x_3; x_3' = -3x_1 - x_2 + 4x_3.$

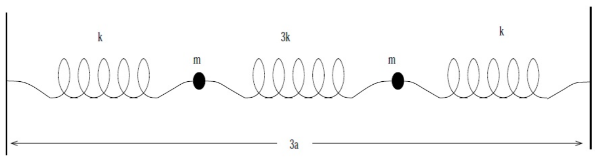
4. Solve the following:
   (a) $x_1'' = -5x_1 + 2x_2; x_2'' = 2x_1 - 2x_2.$
   (b) $x_1'' + 2x_1 - x_2 = 0; x_2'' - x_1 + 2x_2 = 0.$

5. Two particles of equal mass $m = 1$ move in one dimension at the junction of three springs. The springs each have unstretched length $a = 1$ and have spring stiffness constants, k, $3k$ and k (with $k = 1$) respectively see Figure. Applying Newton's second law and Hooke's, this mass-spring system gives rise to the differential equation system

$$x_1'' = -4x_1 + 3x_2$$

$$x_2'' = 3x_1 - 4x_2$$

   Find the displacements $x_1(t)$ and $x_2(t)$.



6. Reduce the third order equation $y''' + 2y'' - y' - 2y = 0$ to the system of first order linear equations and solve by matrix method.

7. Consider tanks T_1 and T_2 which contain initially 100 gallons of water each. In T_1 water is pure whereas 150 pounds of salt is dissolved in T_2. By circulating the liquid at the rate of 2 gallons per minute and stirring, the amount of salt $y_1(t)$ in T_1 and $y_2(t)$ in T_2 change with time t, find the amount of salt in the two tanks after a time t.