Cooperative games

COMP4418 Knowledge Representation and Reasoning

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2019

Outline

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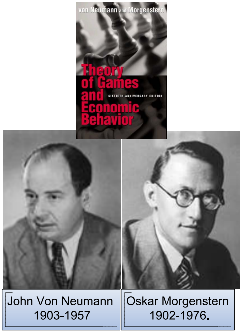
Coalitional games

'... we wish to concentrate on the alternatives for acting in cooperation with, or in opposition to, others, among which a player can choose. I.e. **we want to analyze the possibility of coalitions the question between which players, and against which player, coalitions will form***....' - von Neumann and O. Morgenstern*



Coalition games

Definition (Coalitional game)

- A **coalitional game** is a pair (N, v)
- $N = \{1, \ldots, n\}$ is the set of players
- $v : 2^N \rightarrow \mathbb{R}$ is a *valuation function* that associates with each coalition $S \subseteq N$ a value $v(S)$ where $v(\emptyset) = 0$.
- $v(S)$ can be considered as the value generated when players in coalition S cooperate.

Usual assumptions: valuations are *non-negative* and v is *monotonic* i.e., $S \subseteq T \subseteq N$ implies that $v(S) \leq v(T)$,

Coalitional game

Example

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	4	2	1	7	10	11	15

Simple coalitional game

Definition (Simple coalitional game)

- A **simple coalition game** is a monotone coalitional game (N, v) with $v : 2^N \to \{0, 1\}$ such that $v(N) = 1$.
- A coalition $S \subseteq N$ is **winning** if $v(S) = 1$ and **losing** if $v(S) = 0$.

- Also called simple voting game.

Outline

Solution concepts

$v(N)$ is the amount which the players can earn if they work together.
The aim is to divide $v(N)$ among the players in a stable or fair manner.

Definition (Payoffs)

A *payoff vector* $(x_1, \ldots, x_n) \in \mathbb{R}^N$ specifies for each player $i \in N$ the payoff x_i which is player i's share of $v(N)$.

Definition (Efficient payoffs)

A *payoff vector* $(x_1, \ldots, x_n) \in \mathbb{R}^N$ is **efficient** if $\sum_{i\in N} x_i = v(N)$, where x_i denotes player i's share of $v(N)$.
Notation: $x(S) = \sum_{i\in S} x_i$

Definition (Individual rational payoffs)

A payoff vector $x = (x_1, \ldots, x_n)$ satisfies **individual rationality** if $x_i \geq v(\{i\})$ for all $i \in N$.

Solution concepts

Definition (Solution concept)

A **solution concept** associates with each coalitional game (N, v) a set of *payoff vectors* $(x_1, \ldots, x_n) \in \mathbb{R}^N$ which are stable or fair in some sense.

Solution concepts

Definition (Solution concept)

A **solution concept** associates with each coalitional game (N, v) a set of *payoff vectors* $(x_1, \ldots, x_n) \in \mathbb{R}^N$ which are stable or fair in some sense.

Solution concepts: core, least core, nucleolus, and Shapley value

Solution concepts: core

Definition (Core)

A payoff vector $x = (x_1, \ldots, x_n)$ is in the **core** of a coalitional game (N, v) if for all $S \subset N$, $x(S) \geq v(S)$,

Solution concepts: core

Definition (Core)

A payoff vector $x = (x_1, \dots, x_n)$ is in the **core** of a coalitional game (N, v) if for all $S \subset N$, $x(S) \geq v(S)$, i.e., $e(x, S) \geq 0$.

Solution concepts: core

Definition (Core)

A payoff vector $x = (x_1, \ldots, x_n)$ is in the **core** of a coalitional game (N, v) if for all $S \subset N$, $x(S) \geq v(S)$, i.e., $e(x, S) \geq 0$.

Given a coalitional game (N, v) and payoff vector $x = (x_1, ..., x_n)$, the **excess of a coalition** S **under** x is defined by

$$e(x, S) = x(S) - v(S).$$

Recall that a payoff satisfies individual rationality if $x_i \geq v(\{i\})$ for all $i \in N$.

Question

Does the core satisfy individual rationality?

Solution concepts: core

Definition (Core)

A payoff vector $x = (x_1, \ldots, x_n)$ is in the **core** of a coalitional game (N, v) if for all $S \subset N$, $x(S) \geq v(S)$, i.e., $e(x, S) = x(S) - v(S) \geq 0$.

Formally proposed by Gillies (1959).

Donald Gillies

Solution concepts: core

Definition (Core)

A payoff vector $x = (x_1, \dots, x_n)$ is in the **core** of a coalitional game (N, v) if for all $S \subset N$, $x(S) \geq v(S)$.

Example

There are three people and it takes at least two people to complete the task.

$N = \{1, 2, 3\}$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	1	1	1

Question

Compute a core payoff of the game.

Solution concepts: least core

- For $\epsilon > 0$, a payoff vector x is in the ϵ-**core** if for all $S \subset N$, $e(x, S) \geq -\epsilon$.
- The **least core** is the intersection of all non-empty ϵ-cores.
- The **least core** is the refinement of the ϵ-core and is the solution of the following LP:

$$\begin{array}{ll} \min & \epsilon \\ \text{s.t.} & x(S) \geq v(S) - \epsilon \ \text{ for all } S \subset N, \\ & x_i \geq 0 \text{ for all } i \in N, \\ & \sum_{i=1,\ldots,n} x_i = v(N) \ . \end{array} \tag{1}$$

Solution concepts: least core

- For $\epsilon > 0$, an efficient payoff vector x is in the ϵ-**core** if for all $S \subset N$, $e(x, S) \geq -\epsilon$.
- The **least core** is the intersection of all non-empty ϵ-cores.
- The **least core** is the refinement of the ϵ-core and is the solution of the following LP:

$$\begin{array}{ll} \min & \epsilon \\ \text{s.t.} & x(S) \geq v(S) - \epsilon \ \text{ for all } S \subset N, \\ & x_i \geq 0 \text{ for all } i \in N, \\ & \sum_{i=1,\ldots,n} x_i = v(N)\ . \end{array} \tag{2}$$

Introduced in [Shapley and Shubik, 1966]

Solution concepts: nucleolus

Definition (Excess vector)

The **excess vector** $\theta(x)$ of a payoff vector x, is the vector $(e(x, S_1), ..., e(x, S_{2^n}))$ where $e(x, S_1) \leq e(x, S_2) \leq e(x, S_{2^n})$.

Example

Player 1 has a right hand glove, player 2 has a left hand glove and player 3 also has a left hand glove. A group of players has gets value 1 for a proper pair of gloves and 0 otherwise.

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

Compute the excess vector for payoff vector $(1/2, 1/4, 1/4)$

Solution concepts: nucleolus

Definition (Excess vector)

The **excess vector** $\theta(x)$ of a payoff vector x, is the vector $(e(x,S_1),...,e(x,S_{2^n}))$ where $e(x,S_1) \leq e(x,S_2) \leq e(x,S_{2^n})$.

Example

Player 1 *has a right hand glove, player* 2 *has a left hand glove and player* 3 *also has a left hand glove. A group of players has gets value* 1 *for a proper pair of gloves and* 0 *otherwise.*

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

Compute the excess vector for payoff vector $(1/2, 1/4, 1/4)$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0
$x(S)$	3/4	3/4	1	0	1/4	1/4	1/2	1/2
$e(x,S)$	(-1/4,	-1/4,	0,	0,	1/4,	1/4,	1/2,	1/2)

Solution concepts: nucleolus

Definition (Excess vector)

The **excess vector** of a payoff vector x, is the vector $(e(x,S_1),...,e(x,S_{2^n}))$ where $e(x,S_1) \leq e(x,S_2) \leq e(x,S_{2^n})$.

Definition (Nucleolus)

The **nucleolus** is the efficient payoff vector that has the largest excess vector lexicographically

Solution concepts: nucleolus

Definition (Excess vector)

The **excess vector** of a payoff vector x, is the vector $(e(x, S_1), ..., e(x, S_{2^n}))$ where $e(x, S_1) \leq e(x, S_2) \leq e(x, S_{2^n})$.

Definition (Nucleolus)

The **nucleolus** is the efficient payoff vector that has the largest excess vector lexicographically

$\theta(x) >_{lex} \theta(y)$ if the first coordinate in which the entry a in $\theta(x)$ is different than entry b in $\theta(y)$, it must be that $a > b$.

- The nucleolus is in the least core.
- It is in the core if the core is non-empty.
- The nucleolus is unique [Schmeidler, 1969]

Compute the nucleolus

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

Compute the nucleolus

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

- Nucleolus: $\gamma_1 = 1$; $\gamma_2 = 0$; $\gamma_3 = 0$;

Core of simple games

Definition (Vetoer)

A player i is a **vetoer** if $v(S) = 0$ for any $S \subseteq N \setminus \{i\}$.

Theorem

A simple game (N, v) has a non-empty core iff it has a vetoer. Moreover, an outcome $(x_1, \dots, x_n)$ is in the core iff $x_i = 0$ for all non-veto players.

Proof.

- Assume there exist at least one vetoer i. Set $x_i = 1$. Then consider any coalition S. If $v(S) = 0$, S cannot have an incentive to deviate. If $v(S) = 1$ then $i \in S$. Thus $x(S) = v(S)$.
- Assume there is no vetoer. Consider any payoff x. There exists a player i such that $x_i > 0$. Since i is not a vetoer, then $v(N \setminus \{i\}) = 1$. Thus $x(N \setminus \{i\}) < v(N \setminus \{i\})$.

□

Bonderva-Shapley Theorem

Definition (Balanced weights)

$$\lambda : 2^N \to \mathbb{R}^+$$

λ is **balanced** if $\forall i \in N$, $\sum_{S:i\in S} \lambda(S) = 1$.

Definition (Balanced game)

A game (N, v) is **balanced** if for all balanced weights, $v(N) \geq \sum_{S\subseteq N} \lambda(S)v(S)$.

Theorem (Bondareva [1963]; Shapley [1967])

A coalitional game has a non-empty core if and only if it is balanced.

Core of convex games I

Definition (Convex Game)

(N, v) is **convex** if

$$v(S \cup T) \geq v(S) + v(T) - v(S \cap T)$$

for all $S, T \subset N$.

Equivalently, (N, v) is **convex** if

$$v(A \cup \{i\}) - v(A) \geq v(B \cup \{i\}) - v(B)$$

for all $A, B \subseteq N \setminus \{i\}$ such that $B \subseteq A$.

Theorem (Shapley, 1971)

A convex game has a non-empty core.

Core of convex games II

Proof.

- $x_1 = v(\{1\})$, $x_2 = v(\{1,2\}) - v(\{1\})$,... $x_n = v(N) - v(N \setminus \{n\})$

We first show that $v(N) = \sum_{i \in N} x_i$

$$\begin{aligned}
x_1 &= & v(\{1\}) - v(\emptyset) \\
x_2 &= & v(\{1,2\}) - v(\{1\}) \\
x_i &= & v(\{1,2,\ldots,i\}) - v(\{1,2,\ldots,i-1\}) \\
x_n &= & v(\{1,\ldots,n\}) - v(\{1,2,\ldots,n-1\}) \\
\sum_{i \in N} x_i &= & v(\{1,\ldots,n\}) = v(N)
\end{aligned}$$

Hence $v(N) = \sum_{i \in N} x_i$ $\square$

Core of convex games I

Theorem (Shapley, 1971)

A convex game has a non-empty core.

Core of convex games II

Proof.

- $x_1 = v(\{1\})$, $x_2 = v(\{1,2\}) - v(\{1\})$,... $x_n = v(N) - v(N \setminus \{n\})$

Consider any coalition $C = \{j_1, \dots j_k\}$ such that $j_1 < \cdots < j_k$. We now show that $\sum_{i \in C} x_i \geq v(C)$.

$$
\begin{aligned}
\sum_{i=1}^{k} x_{j_i} =& \sum_{i=1}^{k} (v(\{1, \dots, j_i\}) - v(\{1, \dots, j_i - 1\})) \\
\geq& \sum_{i=1}^{k} (v(\{j_1, \dots, j_i\}) - v(\{j_1, \dots, j_{i-1}\})) \\
=& (v(j_1) - v(\emptyset)) + \\
& (v(\{j_1, j_2\}) - v(j_1)) + \\
& \cdots + \\
& (v(\{j_1, \dots, j_k\}) - v(\{j_1, \dots, j_{k-1}\})) \\
=& v(\{j_1, \dots, j_k\}) = v(C)
\end{aligned}
$$

$\square$

Solution concepts: Shapley value

Definition (Shapley value)

$$\phi_i(N,v) = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|S|!)(|N|-|S|-1)!(v(S \cup \{i\}) - v(S))$$

- $v(S \cup \{i\}) - v(S)$: marginal contribution of player i to coalition S
- Shapley value of a player is his expected marginal contribution in a uniformly random permutation
- Introduced by Shapley [1953]

$$\underbrace{\underbrace{\overbrace{S}^{|S|!}}_{v(S)} \qquad\qquad i}_{v(S\cup\{i\})} \qquad\qquad \overbrace{N \setminus (S \cup \{i\})}^{(|N|-|S|-1)!}$$

Solution concepts: Shapley value

- $S_\pi(i) = \{j \mid \pi(j) < \pi(i)\}$ $S_\pi(i)$ is the set of players that come before i in permutation π.
- $\Delta_\pi^G(i) = v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))$ is the marginal contribution of player i in permutation π.

Definition (Shapley value)

$$\phi_i(G) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \Delta_\pi^G(i)$$

Introduced by Shapley [1953]

Shapley value of a simple game

$$\underbrace{\overbrace{S}^{|S|!}}_{v(S)=0} \quad\quad i \quad\quad \overbrace{N \setminus (S \cup \{i\})}^{(|N|-|S|-1)!}$$

$$\underbrace{\hphantom{S \quad\quad i}}_{v(S\cup\{i\})=1}$$

- **Shapley value**

$$\phi_i = \frac{\#\text{ permutations in which } i \text{ has a marginal contribution of 1}}{|N|!}.$$

Compute the Shapley value

$$\phi_i = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|S|!)(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))$$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

Compute the Shapley value

$$\phi_i = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|S|!)(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))$$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

- 1 2 3
- 1 3 2
- 2 1 3
- 2 3 1
- 3 1 2
- 3 2 1

Shapley value: $\phi_1 = 4/6$; $\phi_2 = 1/6$; $\phi_3 = 1/6$

Compute the Shapley value

$$\phi_i = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|S|!)(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))$$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	500	500	750	1000

Shapley value of player 2:

- 213: $v(\{2\}) - v(\emptyset) = 0$
- 231: $v(\{2\}) - v(\emptyset) = 0$
- 123: $v(\{1,2\}) - v(\{1\}) = 500$
- 321: $v(\{3,2\}) - v(\{3\}) = 500$
- 312: $v(\{1,2,3\}) - v(\{1,3\}) = 250$
- 132: $v(\{1,2,3\}) - v(\{1,3\}) = 250$

$$\phi_2 = (500 + 500 + 250 + 250)/6 = 250.$$

Compute the Shapley value

$$\phi_i = \frac{1}{|N|!} \sum_{S \subseteq N \setminus \{i\}} (|S|!)(|N| - |S| - 1)!(v(S \cup \{i\}) - v(S))$$

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{2,3\}$	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	500	500	750	1000

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- 213: $v(\{2\}) - v(\emptyset) = 0$
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- 132: $v(\{1,2,3\}) - v(\{1,3\}) = 250$

$$\phi_2 = (500 + 500 + 250 + 250)/6 = 250.$$

$$\phi_1 = \phi_3 = 375$$

Shapley value: efficiency

Shapley satisfies efficiency.

- $S_\pi(i) = \{j \mid \pi(j) < \pi(i)\}$
- $\Delta_\pi^G(i) = v(S_\pi(i) \cup \{i\}) - v(S_\pi(i))$
- $a_i = \pi^{-1}(i)$ for $i \in N$ a_i is the player who appears in position i in π. Then,

$$\begin{aligned}\sum_{i=1}^{n} \Delta_\pi^G(i) =& v(\{a_1\}) - v(\emptyset)) + v(\{a_1, a_2\}) - v(\{a_1\}) + \cdots + \\ & v(\{a_1, \ldots, a_n\}) - v(\{a_1, \ldots, a_{n-1}\}) = v(N)\end{aligned}$$

$$\sum_{i=1}^{n} \phi_i(G) = \frac{1}{n!} \sum_{i=1}^{n} \sum_{\pi \in \Pi_N} \Delta_\pi^G(i) = \frac{1}{n!} \sum_{\pi \in \Pi_N} \sum_{i=1}^{n} \Delta_\pi^G(i) = \frac{1}{n!} \sum_{\pi \in \Pi_N} v(N) = v(N)$$

Shapley value: characterization

- The **symmetry axiom** says that players which make the same contribution should get the same payoff.
  $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\} \Rightarrow \phi_i = \phi_j$
- The **dummy player axiom** says that players which make no contribution should get no payoff: if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, $\Rightarrow \phi_i = 0$.
- $(N, v_1 + v_2)$ is the game such that $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for all $S \subseteq N$. **Additivity axiom** says that
  $\forall i \in N$, $\phi_i(N, v_1 + v_2) = \phi_i(N, v_1) + \phi_i(N, v_2)$

Theorem (Shapley, 1953)

The Shapley value uniquely satisfies efficiency, symmetry, dummy player, and additivity.

Shapley value: characterization

- The **symmetry axiom** says that players which make the same contribution should get the same payoff.
  $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\} \Rightarrow \phi_i = \phi_j$
- The **dummy player axiom** says that players which make no contribution should get no payoff: if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N \setminus \{i\}$, $\Rightarrow \phi_i = 0$.
- $(N, v_1 + v_2)$ is the game such that $(v_1 + v_2)(S) = v_1(S) + v_2(S)$ for all $S \subseteq N$. **Additivity axiom** says that
  $\forall i \in N$, $\phi_i(N, v_1 + v_2) = \phi_i(N, v_1) + \phi_i(N, v_2)$

Theorem (Shapley, 1953)

The Shapley value uniquely satisfies efficiency, symmetry, dummy player, and additivity.

Shapley value: another characterization

- The **symmetry axiom** says that players which make the same contribution should get the same payoff.
  $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \setminus \{i, j\} \Rightarrow \phi_i = \phi_j$
- A solution ϕ satisfies **marginality** if for every pair of games (N, v) and (N, w) and every player i, if

$$v(S \cup \{i\}) - v(S) = w(S \cup \{i\}) - w(S), \forall S \subseteq N \setminus \{i\},$$

  then

$$\phi_i(N, v) = \phi_i(N, w).$$

Theorem (Young, 1985)

The Shapley value uniquely satisfies efficiency, symmetry, and marginality.

Fairness versus stability

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Glove Game

- Shapley value: $\phi_1 = 4/6$; $\phi_2 = 1/6$; $\phi_3 = 1/6$
- Nucleolus: $\gamma_1 = 1$;$\gamma_2 = 0$; $\gamma_3 = 0$;

Banzhaf index for Simple Games

Definition (Banzhaf index)

- A player i is **critical** in a coalition C if the player's exclusion results in C changing from winning to losing.
- **Banzhaf value** η_i of a player i is the number of coalitions for which i is critical.
- **Banzhaf index**

$$\beta_i = \frac{\eta_i}{\sum_{i \in N} \eta_i}$$

John Banzhaf

Compute the Banzhaf indices

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$$\beta_i = \frac{\eta_i}{\sum_{i \in N} \eta_i}$$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Game

Compute the Banzhaf indices

Definition (Banzhaf index)

- A player i is **critical** in a coalition C if the player's exclusion results in C changing from winning to losing.
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- **Banzhaf index**

$$\beta_i = \frac{\eta_i}{\sum_{i \in N} \eta_i}$$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Game

- Banzhaf indices: ?

Compute the Banzhaf indices

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- A player i is **critical** in a coalition C if the player's exclusion results in C changing from winning to losing.
- **Banzhaf value** η_i of a player i is the number of coalitions for which i is critical.
- **Banzhaf index**

$$\beta_i = \frac{\eta_i}{\sum_{i \in N} \eta_i}$$

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

Table: Game

- Banzhaf indices: ?
  $\beta_1 = 3/5$; $\beta_2 = 1/5$; $\beta_3 = 1/5$.

Outline

Coalitional game representations

- Mathematically interesting to examine valuation functions which have more structure
- Need for succinct representations
- Modeling requirements

Some representations: weighted voting games, graph games, and marginal contribution nets.

Weighted Voting Games

Definition (Weighted voting game)

- Players, $N = \{1, ..., n\}$ with corresponding voting weights $\{w_1, ..., w_n\}$
- **Quota**, $0 \le q \le \sum_{1 \le i \le n} w_i$
- $v(S) = 1$ if and only if $\sum_{i \in S} w_i \ge q$.
- Notation: $[q; w_1, ..., w_n]$

Example

S	$\{1,2\}$	$\{1,3\}$	$\{1,2,3\}$	$\emptyset$	$\{2\}$	$\{3\}$	$\{1\}$	$\{2,3\}$
$v(S)$	1	1	1	0	0	0	0	0

$$[3; 2, 1, 1]$$

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Question

Can every simple game be represented by a weighted voting game?

Weighted Voting Games

Proposition

Every simple game cannot be represented by a weighted voting game.

Proof.

- Consider the simple game (N, v) where $N = \{1, 2, 3, 4\}$ and the minimal winning coalitions are $\{1, 2\}$, $\{1, 4\}$, $\{2, 3\}$.
- Assume (N, v) can be represented by a weighted voting game.
- $w_1 + w_4 \geq q$, $w_2 + w_4 < q \implies w_1 > w_2$
- $w_1 + w_3 < q$; $w_2 + w_3 \geq q \implies w_2 > w_1$

$\square$

Shapley value and Banzhaf value

Consider a weighted voting game in which the quota is 12 and the countries have the following weights:

- France: 4
- Germany: 4
- Italy: 4
- Belgium: 2
- Netherlands: 2
- Luxembourg: 1

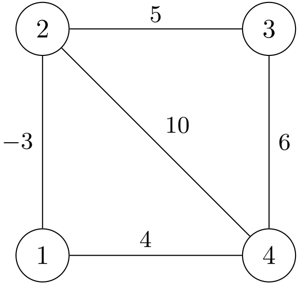
What is the Banzhaf and Shapley value of Luxembourg?

Graph game

Definition (Graph game)

Graph game: Let $G = (V, E, w)$ be a weighted undirected graph. The **graph game** for $S \subseteq N$, corresponding to G is the coalitional game (N, v) with

- $N = V$
- for each $S \subseteq N$, the value $v(S)$ is the sum of the weight of the edges in the subgraph induced by S.



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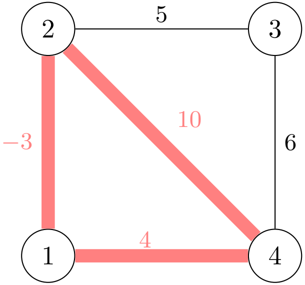

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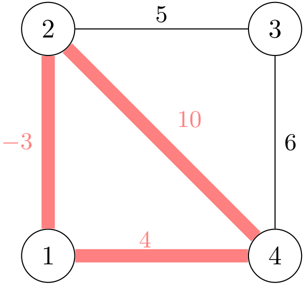

This representation is not complete (fully expressive).

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Xiaotie Deng

Christos Papadimitriou

Marginal Contribution Nets

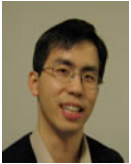
Definition (Marginal Contribution Nets)

- Valuation function represented as **rules**: pattern $\rightarrow$ value.
- Pattern is conjunction of players (negation of a player is allowed).
- Value of a coalition is the sum over the values of all the rules that apply to the coalition.

Example

$x_1 \wedge x_2 \rightarrow 4$, $x_1 \rightarrow 1$, $\neg x_3 \rightarrow 2$. Then we have $v(\{1,2\}) = 4+1+2 = 7$ as all three rules apply to coalition $\{1,2\}$.

This representation is complete (fully expressive) and was introduced by Ieong and Shoham [2005]



Marginal Contribution Nets

Example

1. $x_1 \wedge x_2 \longrightarrow 5$
2. $x_2 \longrightarrow 2$
3. $x_3 \longrightarrow 4$
4. $x_2 \wedge \neg x_3 \longrightarrow -2$

- $v(\{1\}) = 0$ (no rules apply)
- $v(\{2\}) = 0$ (rules 2 and 4 apply)
- $v(\{3\}) = 4$ (rules 3 applies)
- $v(\{1,2\}) = 5$ (rules 1, 2, 4 apply)
- $v(\{1,3\}) = 4$ (rule 3 applies)
- $v(\{2,3\}) = 6$ (rules 2 and 3 apply)
- $v(\{1,2,3\}) =$

Marginal Contribution Nets

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- $v(\{1,2\}) = 5$ (rules 1, 2, 4 apply)
- $v(\{1,3\}) = 4$ (rule 3 applies)
- $v(\{2,3\}) = 6$ (rules 2 and 3 apply)
- $v(\{1,2,3\}) = 11$ (rules 1, 2, and 3 apply)

Proposition

MC-nets are universally expressive.

Proof.

For each coalition S we can have a separate rule where literal x_i is in the rule if $i \in S$ and literal $\neg x_i$ is in the rule if $i \notin S$. The value of the rule is the value of coalition S.

Not that the rule only applies to its corresponding coalition. $\square$

Outline

Computational issues

- How to represent the valuation function succinctly?

For a given game G and solution concept X

- Is X empty for G?
- Compute a payoff in X for G.
- Is a payoff in X for G?

Computing the payoffs

- Core: LP with an exponential number of constraints:

$$\begin{array}{ll} \min & x(N) \\ \text{s.t.} & x(S) \geq v(S) \ \text{ for all } S \subseteq N \\ & x_i \geq 0 \text{ for all } i \in N, \end{array}$$

- Shapley value involves an exponential number of permutations

WVGs

- Deciding if a player is a dummy: coNP-complete [Prasad and Kelly, 1990]. Implies that computing the Shapley value and Banzhaf indices is NP-hard.
- Checking core non-emptiness/checking if an outcome is in the core: polynomial-time (since weighted voting games are simple games).
- Computing a least core payoff is coNP-hard [Elkind et al., 2007]

Hard problems become polynomial-time solvable if weights are bounded (use of dynamic programming).

Graph games

Theorem (Deng and Papadimitriou [1994])

- *Computing Shapley: in polynomial time. A player gets half the payoff from its edges:* $\phi_i = \sum_{i \neq j} w(\{i,j\})/2$
- *However, determining emptiness of the core is NP-complete.*
- *Checking whether a specific outcome is in the core is coNP-complete.*

Marginal Contribution Nets

Theorem (Ieong and Shoham [2005])

- *Shapley value: in polynomial time.*
- *Checking whether an outcome is in the core is coNP-complete*
- *Checking whether the core is non-empty is coNP-hard.*

A complete representation, but not necessarily succinct.

Marginal Contribution Nets

Proposition

Shapley value of an MC-nets can be computed in linear time.

Proof.

- By additivity of the Shapley value, it is sufficient to compute the Shapley value of each game induces by a single rule separately and then adding the Shapley values.
- Consider a rule for which the value is x. Let us say there are p positive literals and s negative literals. For all players corresponding to positive literals, their marginal value is x if it appears after all players corresponding to positive literals and before all players corresponding to negative literals. The Shapley value of a positive player is $((p-1)!s!/(p+s)!) \times x$
- For all players corresponding to negative literals, the player will be responsible for cancelling the application of the rule if all positive literals come before the negative literals in the ordering, and the negative player is the first among the negative players.
- The Shapley value of a negative player is $(p!(s-1)!/(p+s)!) \times (-x)$

Outline

Summary

Coalitional games model how and when coalitions form; how to distribute payoffs.

Solution concept	Existence	Uniqueness
Core	-	-
Least Core	✓	-
Nucleolus	✓	✓
Shapley value	✓	✓

Table: Solution concepts for coalitional games

Some representations of coalitional games: WVGs, graph games, marginal contribution nets.

Further Reading

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