Noncooperative Games

COMP4418 Knowledge Representation and Reasoning

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Outline

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Prisoner's Dilemma

Both prisoners benefit if they cooperate. If one prisoner defects and the other does not, then the defecting prisoner gets scot-free!

	cooperate	defect
cooperate	2,2	0,3
defect	3,0	1,1

Setup

An n-player game (N, A, u) consists of

- Set of players $N = \{1, \ldots, n\}$
- $A = A_1 \times \cdots A_n$ where A_i is the action set of player i
    - $a \in A$ is an action profile.
    - $u = (u_1, \ldots, u_n)$ specifies a utility function $u_i : A \to \mathbb{R}$ for each player.

Bimatrix (2-player) Games

$$
\begin{array}{c|c|c|}
 & a_2^1 & a_2^2 \\
\hline
a_1^1 & u_1(a_1^1, a_2^1), u_2(a_1^1, a_2^1) & u_1(a_1^1, a_2^2), u_2(a_1^1, a_2^2) \\
\hline
a_1^2 & u_1(a_1^2, a_2^1), u_2(a_1^2, a_2^1) & u_1(a_1^2, a_2^2), u_2(a_1^2, a_2^2) \\
\hline
\end{array}
$$

- Actions of player $1 = A_1 = \{a_1^1, a_1^2\}$.
- Actions of player $2 = A_2 = \{a_2^1, a_2^2\}$.

Prisoner's Dilemma

Both prisoners benefit if they cooperate. If one prisoner defects and the other does not, then the defecting prisoner gets scot-free!

	cooperate	defect
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Penalty Shootout

Player 1 (Goal-keeper) wants to match; Player 2 (penalty taker) does not want to match.

	Left	Right
Left	+1,-1	-1,+1
Right	-1,+1	1,-1

Zero Sum Games

In zero-sum games, there are two players and for all action profiles $a \in A$, $u_1(a) + u_2(a) = 0$.

Example

	Left	Right
Left	+1,-1	-1,+1
Right	-1,+1	1,-1

	Heads	Tails
Heads	1	-1
Tails	-1	1

Rock-Paper-Scissors

Both players draw if they have the same action. Otherwise, playing Scissor wins against Paper, playing Paper wins against Rock, and playing Rock wins against Scissors.

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Battle of the Sexes

Player 1 (wife) prefers Ballet over Football. Player 2 (husband) prefers Football over Ballet. Both prefer being together than going alone.

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

Pareto Optimality

One outcome o' Pareto dominates another outcome o if o' all players prefer o' at least as much as o and at least one player strictly prefers o' to o.

Each game admits at least one Pareto optimal outcome.

Outline

Best Response

Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

Definition (Best Response)

$$a'_i \in BR_i(a_{-i})$$

iff

$$\forall a_i \in A_i, u_i(a'_i, a_{-i}) \geq u_i(a_i, a_{-i})$$

The best response of a player gives the player maximum possible utility.

Outline

Nash Equilibrium

Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$.

Definition (Best Response)

$a = (a_1, \ldots, a_n)$ is a (pure) **Nash equilibrium** iff

$$\forall i, a_i \in BR_i(a_{-i}).$$

A Nash equilibrium is an action profile in which each player plays a best response.

Weakly Dominated Actions

Let A_{-i} denote the set of action profiles of all players except player i.

$$A_{-i} = \prod_{j \in N \setminus \{i\}} A_j.$$

Definition

We say that action a_i is *weakly dominated* by action a_i' for player i if

$$\forall a_{-i} \in A_{-i} : u_i(a_i', a_{-i}) \geq u_i(a_i, a_{-i})$$

and

$$\exists a_{-i} \in A_{-i} : u_i(a_i', a_{-i}) > u_i(a_i, a_{-i}).$$

Note that a player will never a play a weakly dominated action in a best response. Hence a player will never a play a weakly dominated action in a Nash equilibrium.

Battle of the Sexes: Pure Nash Equilibria

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

What are the pure Nash equilibria of the game?

Weakly Dominated Actions

	C	D
A	2,1	3,0
B	0,0	3,2

Battle of the Sexes: Pure Nash Equilibria

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

Pure Nash equilibria:

- (Ballet, Ballet)
- (Football, Football)

Prisoner's Dilemma

	cooperate	defect
cooperate	2,2	0,3
defect	3,0	1,1

What are the pure Nash equilibria of the game?

Prisoner's Dilemma

	cooperate	defect
cooperate	2,2	0,3
defect	3,0	1,1

- The only Nash equilibrium is (defect, defect).
- The outcome of (defect,defect) is Pareto dominated by the outcome of (cooperate, cooperate).

Penalty Shootout

	Left	Right
Left	1	-1
Right	-1	1

What are the pure Nash equilibria of the game?

Penalty Shootout

	Left	Right
Left	1	-1
Right	-1	1

What are the pure Nash equilibria of the game?

A pure Nash equilibrium may not exist.

Complexity of a Computing a Pure Nash Equilibrium

Let us assume there are n players and each player has m actions.

- for each of the m^n possible action profiles, check whether some some player out of the n player has a different action among the m actions that gives more utility.
- Total number of steps: $O(m^n mn) = O(m^{n+1}n)$

Outline

Playing pure actions may not be a good idea

Example (Penalty Shootout)

	Left	Right
Left	1	-1
Right	-1	1

Mixed Strategies

Recall that the possible set of pure actions of each player $i \in N$ is A_i.

- A **pure strategy** is one in which exactly one action is played with probability one.
- A **mixed strategy**: more than one action is played with non-zero probability.

The set of strategies for player i is $S_i = \Delta(A_i)$ where $\Delta(A_i)$ is the set of probability distributions over A_i.

The set of all strategy profiles is $S = S_1 \times \cdots \times S_n$.

Mixed Strategies

We want to analyze the payoff of players under a mixed strategy profile:

$$u_i = \sum_{a \in A} u_i(a) Pr(a \mid s)$$

$$Pr(a \mid s) = \prod_{j \in N} s_j(a_j)$$

Mixed Strategies

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Example (Penalty Shootout)

	Left	Right
Left	1	-1
Right	-1	1

Consider the following strategy profile Player 1 plays Left with probability 0.1 and Right with probability 0.9. Player 2 players Left with probability 0.1 and Right with probability 0.9.

Question: What is the utility of player 1 under the strategy profile?

Mixed Strategies

We want to analyze the payoff of players under a mixed strategy profile:

$$u_i = \sum_{a \in A} u_i(a) Pr(a \mid s)$$

$$Pr(a \mid s) = \prod_{j \in N} s_j(a_j)$$

Example (Penalty Shootout)

	Left	Right
Left	1	-1
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Consider the following strategy profile Player 1 plays Left with probability 0.1 and Right with probability 0.9. Player 2 players Left with probability 0.1 and Right with probability 0.9.

Then $u_1 = (0.1 \times 0.1)1 + (0.1 \times 0.9)(-1) + (0.9 \times 0.1)(-1) + (0.9 \times 0.9)(1) = 0.01 - 0.09 - 0.09 + 0.81 = 0.64$.

Mixed Strategies

Definition (Best Response)

Best response: $s_i' \in BR_i(s_{-i})$ iff $\forall s_i \in S_i$, $u_i(s_i', s_{-i}) \geq u_i(s_i, s_{-i})$.

The best response of a player gives the player maximum possible utility.

Definition (Nash equilibrium)

$s = (s_1, \ldots, s_n)$ is a Nash equilibrium iff $\forall i \in N$, $s_i \in BR_i(s_{-i})$.

A Nash equilibrium is an action profile in which each player plays a best response.

Nash's Theorem

Theorem (Nash's Theorem)

A mixed Nash equilibrium always exists.

Battle of the Sexes

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

Battle of the Sexes

	Ballet	Football
Ballet	2,1	0,0
Football	0,0	1,2

- Let us assume that both players play their full support.
- Player 2 plays B with p and F with probability $1-p$.
- Player 1 must be indifferent between the actions it plays.

$$\begin{aligned} 2(p)+0(1-p) &= 0p+1(1-p) \\ p &= 1/3. \end{aligned}$$

- Player 1 plays B with q and F with probability $1-q$
- Player 2 must be indifferent between the actions it plays.

$$\begin{aligned} 1(q)+0(1-q) &= 0q+2(1-q) \\ q &= 2/3. \end{aligned}$$

Thus the mixed strategies $(2/3, 1/3), (1/3, 2/3)$ are in Nash equilibrium.

Exercise

Compute all the Nash equilibria of the following two player game and explain how you computed them.

	D	E
A	2,3	8,5
B	6,6	4,2

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A	2,3	8,5
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(A,E) and (B,D) are in PNE

Player 2 plays D with probability p and E with probability $1-p$. When player 1 is indifferent between her actions

$$2p + 8(1-p) = 6p + 4(1-p)$$
$$p = 1/2$$

Player 1 plays A with probability q and B with probability 1-q. Player 2 is indifferent between her actions

$$3q + 6(1-q) = 5q + 2(1-q)$$
$$q = 2/3$$

Support Enumeration Algorithm

For 2-player games, a support profile can be checked for Nash equilibria as follows:

$$\sum_{a_{-i} \in A_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) = U^* \quad \forall i \in N, a_i \in B_i$$

$$\sum_{a_{-i} \in A_{-i}} s_{-i}(a_{-i}) u_i(a_i, a_{-i}) \leq U^* \quad \forall i \in N, a_i \notin B_i$$

$$s_i(a_i) \geq 0 \quad \forall i \in N, a_i \in B_i$$

$$s_i(a_i) = 0 \quad \forall i \in N, a_i \notin B_i$$

$$\sum_{a_i \in A_i} s_i(a_i) = 1 \quad \forall i \in N$$

When there are more than two players, the constraints are not linear.

Complexity of Computing Nash Equilibrium

PPAD (Polynomial Parity Arguments on Directed graphs) is a complexity class of computational problems for which a solution always exists because of a parity argument on directed graphs.
The class PPAD introduced by Christos Papadimitriou in 1994.
Representative PPAD problem: Given an exponential-size directed graph with no isolated nodes and with every node having in-degree and out-degree at most one described by a polynomial-time computable function $f(v)$ that outputs the predecessor and successor of v, and a node s with degree 1, find a $t \neq s$ that is either a source or a sink.

Theorem (Daskalakis et al., Chen & Deng; 2005)

The problem of finding a Nash equilibrium is PPAD-complete.

- It is believed that P is not equivalent to PPAD.
- PPAD-hardness is viewed as evidence that the problem does not admit an efficient algorithm.

Outline

Maxmin

Definition (Maxmin strategy)

The **maxmin strategy** for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

It is the strategy that maximizes i's worst-case payoff assuming that all other players want to minimize i's payoff.

Definition (Maxmin value)

The **maxmin value** for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Maxmin Strategy

$$\max U_i^* \text{ s.t}$$

$$\sum_{a_i^j \in A_i} s_i(a_i^j) u_i(a_i^j, a_{-i}) \geq U_i^* \qquad \forall i \in N, a_{-i} \in A_{-i}$$

$$s_i(a_i^j) \geq 0 \qquad \forall a_i^j \in A_i$$

$$\sum_{a_i^j \in A_i} s_i(a_i^j) = 1$$

Minmax Strategy

Definition (Minmax strategy)

The **minmax strategy** for player i in a 2 player game is $\arg\min_{s_i}\max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

It is the strategy that minimizes $-i$'s best-case payoff.
The **minmax value** for player $-i$ is $\min_{s_i}\max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Minmax I

Theorem (Minimax theorem (von Neumann, 1928))

In any finite two-player zero-sum game, in any Nash equilibrium, each player receives a payoff that is equal to both his maxmin value and his minmax value.

Proof:

- Let (s'_i, s'_{-i}) be any Nash equilibrium and v_i be the payoff of i in the equilibrium.
- Let $\bar{v}_i$ be the maxmin value of i and $\underline{v_i}$ be the minmax value of i.

We first prove that $v_i = \bar{v}_i$

- We already know that $\bar{v}_i \leq v_i$ because if $\bar{v}_i > v_i$, then player i can deviate from (s'_i, s'_{-i}) by playing her maxmin strategy. We prove that $\bar{v}_i \geq v_i$
- Assume for contradiction that $\bar{v}_i < v_i$.
- Since (s'_i, s'_{-i}) is a Nash eq, $v_{-i} = \max_{s_{-i}} u_{-i}(s'_i, s_{-i})$.
- Equivalently: $-i$ minimizes the negative of his payoff:

$$-v_i = \min_{s_{-i}} -u_i(s'_i, s_{-i})$$

Minmax II

- Since the game is zero-sum: $v_i = -v_{-i}$ and $u_i = -u_{-i}$. Thus

$$v_i = \min_{s_{-i}} u_i(s'_i, s_{-i})$$

- Recall that $\bar{v_i} = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$. Hence

$$\bar{v_i} = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) \geq \min_{s_{-i}} u_i(s'_i, s_{-i}) = v_i$$

Similarly, we can prove that $v_i = \underline{v_i}$

Outline

Reading

- K. Leyton-Brown and Y. Shoham, Essentials of Game Theory: A Concise Multidisciplinary Introduction. Synthesis Lectures on Artificial Intelligence and Machine Learning. Morgan & Claypool Publishers, 2008. `www.gtessentials.org`
- Y. Shoham and K. Leyton-Brown. Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations. 2009. `http://www.masfoundations.org`