Problem 1

(a). Assume {p,q,r} in P_{ROG},

Firstly, we can easily prove that p≡p,

Thus, F is reflexive

Secondly, assume p≡q, then p ↔ q

p	q	p → q	q → p	p ↔ q	q ↔ p
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	T	T

As the above truth table, from p ↔ q, we can get q ↔ p,

Thus, F is symmetric

Finally, assume p≡q and q≡r, then p ↔ q and q ↔ r

p	q	r	p→q	q→p	q→r	r→q	p→r	r→p	p↔q	q↔r	p↔r
T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	F	T	T	F	F
T	F	T	F	T	T	F	T	T	F	F	T
T	F	F	F	T	T	T	F	T	F	T	F
F	T	T	T	F	T	T	T	F	F	T	F
F	T	F	T	F	F	T	T	T	F	F	T
F	F	T	T	T	T	F	T	F	T	F	F
F	F	F	T	T	T	T	T	T	T	T	T

As the above truth table, from p ↔ q and q ↔ r, we can get p ↔ r,

Thus, F is transitive.

As a result we can get that the logical equivalence relation, ≡, is an equivalence relation on F .

(b).

p	q	r	p→q	p→r	p∧q	p∧r
T	F	F	F	F	F	F

As the above truth table, {p→q, p→r, p∧q, p∧r} in [⊥].

(c). (i). v(¬φ) = !v(φ)

= !v(φ')

= v(¬φ')

Thus, ¬φ ≡¬φ'

(ii). v(φ∧ψ) = v(φ) && v(ψ)

= v(φ') && v(ψ')

= v(φ'∧ψ')

Thus, φ∧ψ≡φ'∧ψ'

(iii). v(φ∨ψ) = v(φ) || v(ψ)

= v(φ') || v(ψ')

= v(φ'∨ψ')

Thus, φ∨ψ≡φ'∨ψ'

(d). $T = \{[\varphi] : \varphi \in F\}$

$[\varphi] \wedge [\psi] : [\varphi \wedge \psi]$

$[\varphi] \vee [\psi]: [\varphi] \vee [\psi]$

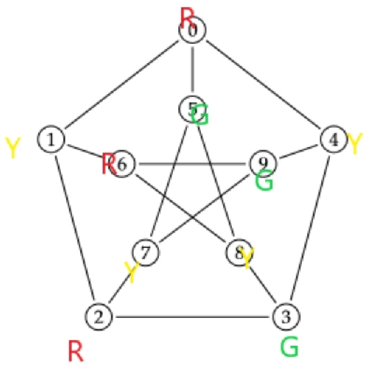
$[\varphi]' : [\neg\varphi]$

$0 : \varnothing$

$1 : F$

Problem 2

(a).



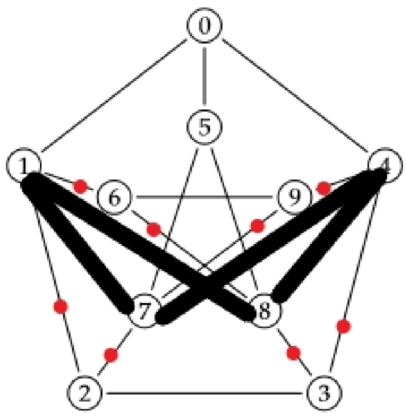
As above picture show,

$\chi(G) = 3$,

$\because \chi(K5) = 5$ and $\chi(G) \geqslant \kappa(G)$.

$\therefore$ The Petersen graph does not contain a subdivision of K5

(b).



As above picture shows,

Connecting 1 and 7 with replacing 2

Connecting 4 and 8 with replacing 3

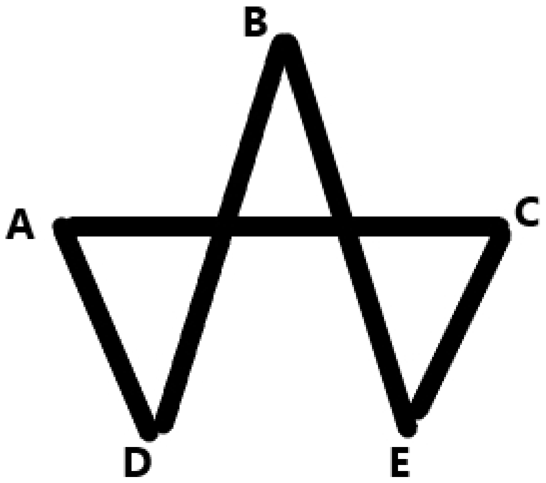
Connecting 1 and 8 with replacing 6

Connecting 4 and 7 with replacing 9

Then, we get $K_{3,3}$

Problem 3

(a) (i).



Define Defence against the Dark Arts as A

Define Potions as B

Define Herbology as C

Define Transfiguration as D

Define Charms as E

Define edge AC as Harry can take A and C both

Define edge AD as Harry can take A and D both

Define edge BD as Harry can take B and D both

Define edge BE as Harry can take B and E both

Define edge CE as Harry can take C and E both

(ii). Find a vertex with maximum edges, which means that vertex has maximum number of classes he can take

(b) There five solutions:

① Take both A and C

② Take both A and D

③ Take both B and D

④ Take both B and E

⑤ Take both C and E

Problem 4

(a) T(n) :

$T(0) = 1$

$T(1) = 1$

$T(n) = T(n-1) * (4*n - 2) / (n+1)$

(b) Assume the layer of full binary tree is k and the number of nodes is n

Count(n):

$n = 1 \qquad k = 0$

$n = 1 + 2^k \qquad k > 0$

As 1 is odd and 2^k is even,

Thus, $1 + 2^k$ is odd.

As a result, a full binary tree must have an odd number of nodes.

(c) $T(n) = \frac{1}{n+1}\binom{2n}{n}$

$B(n) = \frac{1}{T(n)}\binom{2n}{n} - 1$

(d)

Problem 5

(a) $p_1(n+1) = (p_2(n)+p_4(n))/3$

$p_2(n+1) = (p_1(n)+p_3(n))/2 + p_4(n)/2$

$p_3(n+1) = (p_2(n)+p_4(n))/3$

$p_4(n+1) = (p_1(n)+p_3(n))/2 + p_2(n)/2$

(b) Let $p_i(n+1) = p_i(n)$,

Then, we can get

$p_1(n) = (p_2(n)+p_4(n))/3$ ①

$p_2(n) = (p_1(n)+p_3(n))/2 + p_4(n)/2$ ②

$p_3(n) = (p_2(n)+p_4(n))/3$ ③

$p_4(n) = (p_1(n)+p_3(n))/2 + p_2(n)/2$ ④

from ① and ③, we get $p_1(n) = p_3(n)$ ⑤

from ② and ⑤, $p_2(n) = p_1(n) + p_4(n)/2$ ⑥

from ② and ⑤, $p_4(n) = p_1(n) + p_2(n)/2$ ⑦

from ⑥ - ⑦, $p_2(n) = p_4(n)$ ⑧

from ⑥ and ⑧, $p_2(n) = 2\, p_1(n)$ ⑨

$p_1(n)+ p_2(n)+ p_3(n)+ p_4(n)$

$= p_1(n) + 2\, p_1(n) + p_1(n) + 2\, p_1(n)$

$= 6\, p_1(n) = 1$

Then , we get $p_1(n) = p_3(n) = 1/6$, $p_2(n) = p_4(n) = 1/3$.

(c) $d(v_1\text{-}v_2) = p_2(n)*1 = 1/3$

$d(v_1\text{-}v_4) = p_4(n)*1 = 1/3$

$d(v_1 - v_3) = p_3(n)*2 + p_3(n)*2 = (1/6)*2 + (1/6)*2 = 2/3$