COMP9020 Week 2
Sets, Relations, and Functions

- Textbook (R & W) - Ch. 1, Sec. 1.3-1.5, 1.7; Ch. 3., Sec. 3.1

Summary of topics

- Applications in Computer Science
- Introduction to Sets
- Formal Languages
- Introduction to Relations
- Introduction to Functions

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- Applications in Computer Science
- Introduction to Sets
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Applications of Sets, Formal Languages, Relations, and Functions

- Sets are the building blocks of nearly all mathematical structures
- Databases are collections of relations
- Any ordering is a relation
- Common data structures (e.g. graphs) are relations
- Functions/procedures/programs compute relations between their input and output
- Formal languages are essential for compilers and programming language design

Summary of topics

- Applications in Computer Science
- Introduction to Sets
- Formal Languages
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- Introduction to Functions

Sets

- A set is defined by the collection of its elements. Order and multiplicity of elements is not considered.
- We distinguish between an element and the set comprising this single element. Thus always $a \neq \{a\}$.
- Set $\emptyset = \{\}$ is empty (no elements);
- Set $\{\{\}\}$ is nonempty — it has one element.
- There is only one empty set; only one set consisting of a single a; only one set of all natural numbers.

Defining sets

Sets are typically described by:
(a) Explicit enumeration of their elements

$$
\begin{aligned}
S_1 &= \{a, b, c\} = \{a, a, b, b, b, c\} \\
&= \{b, c, a\} = \dots \quad \text{three elements} \\
S_2 &= \{a, \{a\}\} \quad \text{two elements} \\
S_3 &= \{a, b, \{a, b\}\} \quad \text{three elements} \\
S_4 &= \{\} \quad \text{zero elements} \\
S_5 &= \{\{\{\}\}\} \quad \text{one element} \\
S_6 &= \{\,\{\}, \{\{\}\}\,\} \quad \text{two elements}
\end{aligned}
$$

Defining sets

(b) Defining a subset of an existing "universal" set $\mathcal{U}$. Including:

- Specifying the properties their elements must satisfy. A typical description involves a **logical** property $P(x)$. For example, with $\mathcal{U} = \mathbb{N}$ and $P(x) =$ "x is even":

$$\{x : x \in \mathbb{N} \text{ and } x \text{ is even}\} = \{0, 2, 4, \ldots\}$$

- Using interval notation. For example, with $\mathcal{U} = \mathbb{Z}$:

$$[1, 5] = \{1, 2, 3, 4, 5\}$$

- Derived sets of integers

$$2\mathbb{Z} = \{\ 2x : x \in \mathbb{Z}\ \} \qquad \text{the even numbers}$$

$$3\mathbb{Z} + 1 = \{\ 3x + 1 : x \in \mathbb{Z}\ \}$$

Sidenote: Subsets

NB

- *$S \subseteq T$ — S is a* **subset** *of T; includes the case of $T \subseteq T$*
- *$S \subset T$ — a* **proper** *subset: $S \subseteq T$ and $S \neq T$*
- *$\emptyset \subseteq S$ for all sets S*
- $\mathbb{N}_{>0} \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- *An element of a set; and a subset of that set are two different concepts*

$$a \in \{a, b\}, \quad a \not\subseteq \{a, b\}; \qquad \{a\} \subseteq \{a, b\}, \quad \{a\} \notin \{a, b\}$$

Defining sets

(c) Constructions from other, already defined, sets

- Union ($\cup$), intersection ($\cap$), complement ($\cdot^c$), set difference ($\setminus$), symmetric difference ($\oplus$)
- Power set $\mathrm{Pow}(X) = \{ A : A \subseteq X \}$
- Cartesian product ($\times$)

Set Operations

Definition

$A \cup B$ – **union** (a or b):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (a and b):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set $\mathcal{U}$):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Set Operations

Other set operations

Definition

$A \setminus B$ – **set difference**, relative complement (a but not b):

$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (a and not b or b and not a; also known as a or b exclusively; a xor b):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Set Operations

There is a correspondence between set operations and logical operators (to be discussed in Week 6).

NB

$$A \cup B = B \quad \textit{iff} \quad A \cap B = A \quad \textit{iff} \quad A \subseteq B$$

Exercises

Exercises

$\boxed{1.4.4\ (d)}$ All subsets of $\{a, b\}$:

$\boxed{1.4.7\ (a)}$ $A \oplus A =$

$\boxed{1.4.7\ (b)}$ $A \oplus \emptyset =$

Exercises

Exercises

$\boxed{1.4.4\ (d)}$	All subsets of $\{a, b\} : \emptyset, \{a\}, \{b\}, \{a, b\}$
$\boxed{1.4.7\ (a)}$	$A \oplus A =$
$\boxed{1.4.7\ (b)}$	$A \oplus \emptyset =$

Exercises

Exercises

1.4.4 (d)	All subsets of $\{a, b\} : \emptyset, \{a\}, \{b\}, \{a, b\}$
1.4.7 (a)	$A \oplus A = \emptyset$
1.4.7 (b)	$A \oplus \emptyset =$

Exercises

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$\boxed{\text{1.4.4 (d)}}$ All subsets of $\{a, b\} : \emptyset, \{a\}, \{b\}, \{a, b\}$

$\boxed{\text{1.4.7 (a)}}$ $A \oplus A = \emptyset$

$\boxed{\text{1.4.7 (b)}}$ $A \oplus \emptyset = A$

Cardinality

Number of elements in a set X (various notations):

$$|X| = \#(X) = \mathrm{card}(X)$$

Fact

Always $|Pow(X)| = 2^{|X|}$

Exercises

Exercises

- $|\emptyset| =$
- $\mathrm{Pow}(\emptyset) =$
- $|\mathrm{Pow}(\emptyset)| =$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) =$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| =$
- $|\{a\}| =$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

Exercises

Exercises

- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) =$
- $|\mathrm{Pow}(\emptyset)| =$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) =$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| =$
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Exercises

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- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) = \{\emptyset\}$
- $|\mathrm{Pow}(\emptyset)| = 1$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) =$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| =$
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- $|[m, n]| =$

Exercises

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- $|\emptyset| = 0$
- $\mathrm{Pow}(\emptyset) = \{\emptyset\}$
- $|\mathrm{Pow}(\emptyset)| = 1$
- $\mathrm{Pow}(\mathrm{Pow}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$
- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| = 2$
- $|\{a\}| =$
- $\mathrm{Pow}(\{a\}) =$
- $|\mathrm{Pow}(\{a\})| =$
- $|[m, n]| =$

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- $|[m, n]| =$

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- $|\mathrm{Pow}(\mathrm{Pow}(\emptyset))| = 2$
- $|\{a\}| = 1$
- $\mathrm{Pow}(\{a\}) = \{\emptyset, \{a\}\}$
- $|\mathrm{Pow}(\{a\})| = 2$
- $|[m, n]| = n - m + 1$

Exercises

$\boxed{1.3.2}$ Find the cardinalities of sets

1. $|\{ \frac{1}{n} : n \in [1,4] \}| =$
2. $|\{ n^2 - n : n \in [0,4] \}| =$
3. $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| =$
4. $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| =$

Exercises

$\boxed{1.3.2}$ Find the cardinalities of sets

1. $|\{ \frac{1}{n} : n \in [1,4] \}| = 4$
2. $|\{ n^2 - n : n \in [0,4] \}| =$
3. $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| =$
4. $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| =$

Exercises

$\boxed{1.3.2}$ Find the cardinalities of sets

1. $|\{\ \frac{1}{n} : n \in [1, 4]\ \}| = 4$
2. $|\{\ n^2 - n : n \in [0, 4]\ \}| = 4$
3. $|\{\ \frac{1}{n^2} : n \in \mathbb{N}_{>0}$ and $2|n$ and $n < 11\ \}| =$
4. $|\{\ 2 + (-1)^n : n \in \mathbb{N}\ \}| =$

Exercises

$\boxed{1.3.2}$ Find the cardinalities of sets

1. $|\{ \frac{1}{n} : n \in [1, 4] \}| = 4$
2. $|\{ n^2 - n : n \in [0, 4] \}| = 4$
3. $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| = 5$
4. $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| =$

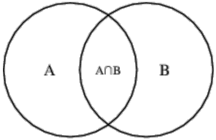
Exercises

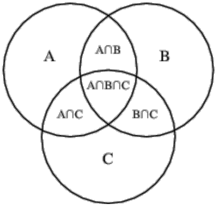
$\boxed{1.3.2}$ Find the cardinalities of sets

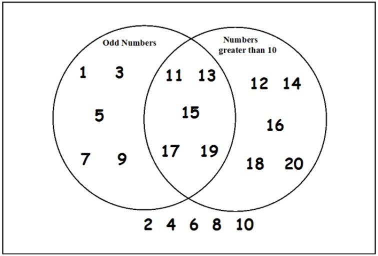
1. $|\{ \frac{1}{n} : n \in [1, 4] \}| = 4$
2. $|\{ n^2 - n : n \in [0, 4] \}| = 4$
3. $|\{ \frac{1}{n^2} : n \in \mathbb{N}_{>0} \text{ and } 2|n \text{ and } n < 11 \}| = 5$
4. $|\{ 2 + (-1)^n : n \in \mathbb{N} \}| = 2$

Venn Diagrams

A simple graphical approach to reason about the algebraic properties of set operations.







Exercises

$\boxed{1.4.8}$ Relate the cardinalities $|A \cup B|$, $|A \cap B|$, $|A \setminus B|$, $|A \oplus B|$, $|A|$, $|B|$

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$\boxed{1.4.8}$ Relate the cardinalities $|A \cup B|$, $|A \cap B|$, $|A \setminus B|$, $|A \oplus B|$, $|A|$, $|B|$

- $|A \cup B| = |A| + |B| - |A \cap B|$
  hence $|A \cup B| + |A \cap B| = |A| + |B|$
- $|A \setminus B| = |A| - |A \cap B|$
- $|A \oplus B| = |A| + |B| - 2|A \cap B|$

Cartesian Product

$S \times T \stackrel{\text{def}}{=} \{\ (s,t) : s \in S,\ \ t \in T\ \}$ where (s,t) is an **ordered** pair

$\bigtimes_{i=1}^{n} S_i \stackrel{\text{def}}{=} \{\ (s_1, \ldots, s_n) : s_k \in S_k,\ \text{for } 1 \leq k \leq n\ \}$

$S^2 = S \times S, \quad S^3 = S \times S \times S, \ldots, \quad S^n = \times_1^n S, \ldots$

$\emptyset \times S = \emptyset$, for every S
$|S \times T| = |S| \cdot |T|, \quad |\times_{i=1}^{n} S_i| = \prod_{i=1}^{n} |S_i|$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned}
A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\
&= \{(0, a), (1, a), (0, b), (1, b)\} \\
B \times A &= \\
A^2 &= \\
A^3 &=
\end{aligned}$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 =$$

$$A^3 =$$

Examples

Examples

Let $A = \{0,1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0,a),(0,b),(1,a),(1,b)\} \\ &= \{(0,a),(1,a),(0,b),(1,b)\} \end{aligned}$$

$$B \times A = \{(a,0),(b,0),(a,1),(b,1)\} \neq A \times B$$

$$A^2 = \{(0,0),(0,1),(1,0),(1,1)\}$$

$$A^3 =$$

Examples

Examples

Let $A = \{0, 1\}$ and $B = \{a, b\}$

$$\begin{aligned} A \times B &= \{(0, a), (0, b), (1, a), (1, b)\} \\ &= \{(0, a), (1, a), (0, b), (1, b)\} \end{aligned}$$

$$B \times A = \{(a, 0), (b, 0), (a, 1), (b, 1)\} \neq A \times B$$

$$A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$$

$$\begin{aligned} A^3 = \{&(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), \\ &(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \end{aligned}$$

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- Applications in Computer Science
- Introduction to Sets
- Formal Languages
- Introduction to Relations
- Introduction to Functions

Formal Languages: Symbols

Σ — **alphabet**, a finite, nonempty set

Examples (of various alphabets and their intended uses)

$\Sigma = \{a, b, \ldots, z\}$ for single words (in lower case)
$\Sigma = \{\sqcup, -, a, b, \ldots, z\}$ for composite terms
$\Sigma = \{0, 1\}$ for binary integers
$\Sigma = \{0, 1, \ldots, 9\}$ for decimal integers

The above cases all have a natural ordering; this is not required in general, thus the set of all Chinese characters forms a (formal) alphabet.

Formal Languages: Words

Definition

word — any finite string of symbols from Σ
empty word — λ

Example

$w = aba$, $w = 01101\ldots1$, etc.

length(w) — # of symbols in w
length(aaa) $= 3$, length(λ) $= 0$
The only operation on words (discussed here) is **concatenation**, written as juxtaposition $vw, wvw, abw, wbv, \ldots$

NB

$\lambda w = w = w\lambda$
length(vw) $=$ length(v) $+$ length(w)

Examples

Examples

Let $w = abb$, $v = ab$, $u = ba$

- $vw = ababb$
- $ww = abbabb = vubb$
- $w\lambda v = abbab$
- $\text{length}(vw) = \text{length}(ababb) = 5$

Formal Languages: Languages

Notation: Σ^k — set of all words of length k
We often identify $\Sigma^0 = \{\lambda\}$, $\Sigma^1 = \Sigma$
Σ^* — set of all words (of all lengths)
Σ^+ — set of all nonempty words (of any positive length)

$$\Sigma^* = \Sigma^0 \cup \Sigma^1 \cup \Sigma^2 \cup \ldots; \quad \Sigma^{\leq n} = \bigcup_{i=0}^{n} \Sigma^i$$

$$\Sigma^+ = \Sigma^1 \cup \Sigma^2 \cup \ldots = \Sigma^* \setminus \{\lambda\}$$

Definition

A **language** is a subset of Σ^*.

Typically, only the subsets that can be formed (or described) according to certain rules are of interest. Such a collection of 'descriptive/formative' rules is called a **grammar**.

Example (Decimal numbers)

The "language" of all numbers written in decimal to at most two decimal places can be described as follows:

- $\Sigma = \{-, ., 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Consider all words $w \in \Sigma^*$ which satisfy the following:
  - w contains at most one instance of $-$, and if it contains an instance then it is the first symbol.
  - w contains at most one instance of $.$, and if it contains an instance then it is preceeded by a symbol in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, and followed by either one or two symbols in that set.
  - w contains at least one symbol from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

NB

According to these rules 123*,* 123.0 *and* 123.00 *are all (distinct) words in this language.*

Example (HTML documents)

Take $\Sigma = \{$"<html>", "</html>", "<head>", "</head>", "<body>", $\ldots\}$.

The (language of) **valid HTML documents** is loosely described as follows:

- Starts with "<html>"
- Next symbol is "<head>"
- Followed by zero or more symbols from the set of HeadItems (defined elsewhere)
- Followed by "</head>"
- Followed by "<body>"
- Followed by zero or more symbols from the set of BodyItems (defined elsewhere)
- Followed by "</body>"
- Followed by "</html>"

Exercises

$\boxed{1.3.10}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ —

(f) $\{\ w \in \Sigma^* : \text{length}(w) \leq 4\ \}$ where $\Sigma = \{a, b, c\}$
$|\Sigma^{\leq 4}| =$

Exercises

$\boxed{1.3.10}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ — $|\Sigma^*| = \infty$

(f) $\{\ w \in \Sigma^* : \text{length}(w) \leq 4\ \}$ where $\Sigma = \{a, b, c\}$
$|\Sigma^{\leq 4}| =$

Exercises

$\boxed{1.3.10}$ Number of elements in the sets (cont'd)

(e) Σ^* where $\Sigma = \{a, b, c\}$ — $|\Sigma^*| = \infty$

(f) $\{ \ w \in \Sigma^* : \text{length}(w) \leq 4 \ \}$ where $\Sigma = \{a, b, c\}$
$|\Sigma^{\leq 4}| = 3^0 + 3^1 + \ldots + 3^4 = \frac{3^5-1}{3-1} = \frac{243-1}{2} = 121$

Set Operations for Languages

Languages are sets, so the standard set operations ($\cap$, $\cup$, $\setminus$, $\oplus$, etc) can be used to build new languages.
Two set operations that apply to languages uniquely:

- Concatenation (written as juxtaposition):
  $XY = \{xy \ : \ x \in X \text{ and } y \in Y\}$
- Kleene star: X^* is the set of words that are made up by concatenating 0 or more words in X

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$

Set Operations for Languages

Example

Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$

Set Operations for Languages

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- $\{\lambda\}^* =$

Set Operations for Languages

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- $\{\lambda\}^* = \{\lambda\}$

Set Operations for Languages

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- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* =$

Set Operations for Languages

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Let $A = \{aa, bb\}$ and $B = \{\lambda, c\}$ be languages over $\Sigma = \{a, b, c\}$.

- $A \cup B = \{\lambda, c, aa, bb\}$
- $AB = \{aa, bb, aac, bbc\}$
- $AA = \{aaaa, aabb, bbaa, bbbb\}$
- $A^* = \{\lambda, aa, bb, aaaa, aabb, bbaa, bbbb, aaaaaa, \ldots\}$
- $B^* = \{\lambda, c, cc, ccc, cccc, \ldots\}$
- $\{\lambda\}^* = \{\lambda\}$
- $\emptyset^* = \{\lambda\}$

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Relations and Functions

Relations are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are *related*. These objects may

- influence one another (each other for binary relations; self(?) for unary)
- share some common properties
- correspond to each other precisely when some constraints are satisfied

Functions capture the idea of transforming *inputs* into *outputs*.

In general, functions and relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.

Relations

Definition

An **n-ary relation** is a subset of the cartesian product of n sets.

$$R \subseteq S_1 \times S_2 \times \ldots \times S_n$$

To show tuples related by R we write:

$$(x_1, x_2, \ldots, x_n) \in R \quad \text{or} \quad R(x_1, x_2, \ldots, x_n)$$

If $n = 2$ we have a **binary** relation $R \subseteq S \times T$ and to show pairs related by R we write:

$$(x, y) \in R \quad \text{or} \quad R(x, y) \quad \text{or} \quad xRy$$

Examples

Examples

- Equality: $=$
- Inequality: $\leq$, $\geq$, $<$, $>$, $\neq$
- Divides relation: $|$
- Element of: $\in$
- Subset, superset: $\subseteq$, $\subset$, $\supseteq$, $\supset$
- Congruence modulo n: $m = p \pmod{n}$

Database Examples

Example (Course enrolments)

S = set of CSE students
(S can be a subset of the set of all students)
C = set of CSE courses
(likewise)
E = enrolments = $\{\ (s, c) : s \text{ takes } c\ \}$

$$E \subseteq S \times C$$

In practice, almost always there are various 'onto' (nonemptiness) and 1–1 (uniqueness) constraints on database relations.

Example (Class schedule)

C = CSE courses

T = starting time (hour & day)

R = lecture rooms

S = schedule =

$$\{\, (c, t, r) : c \text{ is at } t \text{ in } r \,\} \subseteq C \times T \times R$$

Example (sport stats)

$$R \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$

Defining Relations

Just as with sets R can be defined by

- explicit enumeration of interrelated k-tuples (ordered pairs in case of binary relations);
- properties that identify relevant tuples within the entire $S_1 \times S_2 \times \ldots \times S_k$;
- construction from other relations.

Relation R as Correspondence From S to T

Given $R \subseteq S \times T$, $A \subseteq S$, and $B \subseteq T$.

- Relational image of A, $R(A)$:

$$R(A) \stackrel{\text{def}}{=} \{t \in T \ : \ (s,t) \in R \text{ for some } s \in A\}$$

- Converse relation $R^{\leftarrow} \subseteq T \times S$:

$$R^{\leftarrow} \stackrel{\text{def}}{=} \{(t,s) \in T \times S \ : \ (s,t) \in R\}$$

- Relational pre-image of B, $R^{\leftarrow}(B)$:

$$R^{\leftarrow}(B) \stackrel{\text{def}}{=} \{s \in S \ : \ (s,t) \in R \text{ for some } t \in B\}$$

Observe that $(R^{\leftarrow})^{\leftarrow} = R$.

Exercises

Exercises

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3,4\}$, $X = [1,4]$

- $\mid$ on X:
- $\in$ on $X \times \{A, B, C\}$:
- $\subseteq^{\leftarrow}$ on $\{A, B, C, X\}$:
- $<$ (2) (on X):

Exercises

Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$

- $|$ on X:
  $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4))\}$
- $\in$ on $X \times \{A, B, C\}$:
- $\subseteq^{\leftarrow}$ on $\{A, B, C, X\}$:
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Exercises

Exercises

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3,4\}$, $X = [1,4]$

- $|$ on X:
  $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4))\}$
- $\in$ on $X \times \{A, B, C\}$:
  $\{(1,A),(2,A),(2,B),(3,B),(3,C),(4,C)\}$
- $\subseteq^{\leftarrow}$ on $\{A, B, C, X\}$:
- $<$ (2) (on X):

Exercises

Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$

- $|$ on X:
  $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4))\}$
- $\in$ on $X \times \{A, B, C\}$:
  $\{(1, A), (2, A), (2, B), (3, B), (3, C), (4, C)\}$
- $\subseteq^{\leftarrow}$ on $\{A, B, C, X\}$:
  $\{(A, A), (X, A), (B, B), (X, B), (C, C), (X, C), (X, X)\}$
- $<$ (2) (on X):

Exercises

Exercises

Let $A = \{1,2\}$, $B = \{2,3\}$, $C = \{3,4\}$, $X = [1,4]$

- $\mid$ on X:
  $\{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4))\}$
- $\in$ on $X \times \{A, B, C\}$:
  $\{(1,A),(2,A),(2,B),(3,B),(3,C),(4,C)\}$
- $\subseteq^{\leftarrow}$ on $\{A, B, C, X\}$:
  $\{(A,A),(X,A),(B,B),(X,B),(C,C),(X,C),(X,X)\}$
- $<$ (2) (on X): $\{3,4\}$

Summary of topics

- Applications in Computer Science
- Introduction to Sets
- Formal Languages
- Introduction to Relations
- Introduction to Functions

Functions

Definition

A **function**, $f : S \to T$, is a binary relation $f \subseteq S \times T$ such that for all $s \in S$ there is *exactly one* $t \in T$ such that $(s, t) \in f$.

We write $f(s)$ for the unique element related to s.

A **partial function** $f : S \nrightarrow T$ is a binary relation $f \subseteq S \times T$ such that for all $s \in S$ there is *at most one* $t \in T$ such that $(s, t) \in f$. That is, it is a function $f : S' \longrightarrow T$ for $S' \subseteq S$

Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$. To emphasise where a specific element is sent, we can write $f : x \mapsto y$, which means the same as $f(x) = y$

		Symbol	
S	**domain** of f	$\mathrm{Dom}(f)$	(inputs)
T	**co-domain** of f	$\mathrm{Codom}(f)$	(*possible* outputs)
$f(S)$	**image** of f	$\mathrm{Im}(f)$	(*actual* outputs)

$= \{ f(x) : x \in \mathrm{Dom}(f) \}$

Important!

The domain and co-domain are critical aspects of a function's definition.

$$f : \mathbb{N} \to \mathbb{Z} \quad \text{given by} \quad f(x) \mapsto x^2$$

and

$$g : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad g(x) \mapsto x^2$$

are different functions even though they have the same behaviour!

Composition of Functions

Composition of functions is described as

$$g \circ f : x \mapsto g(f(x)), \quad \text{requiring } \operatorname{Im}(f) \subseteq \operatorname{Dom}(g)$$

Composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f, \quad \text{can write } h \circ g \circ f$$

Composition of Functions

If a function maps a set into itself, i.e. when $\mathrm{Dom}(f) = \mathrm{Codom}(f)$ (and thus $\mathrm{Im}(f) \subseteq \mathrm{Dom}(f)$), the function can be composed with itself — **iterated**

$$f \circ f, f \circ f \circ f, \ldots, \quad \text{also written } f^2, f^3, \ldots$$

Identity function on S

$$\mathrm{Id}_S(x) = x, x \in S; \mathrm{Dom}(\mathrm{Id}_S) = \mathrm{Codom}(\mathrm{Id}_S) = \mathrm{Im}(\mathrm{Id}_S) = S$$

For $g : S \longrightarrow T \quad g \circ \mathrm{Id}_S = g,\ \mathrm{Id}_T \circ g = g$

Extension: Composition of Binary Relations

If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ then the composition of R_1 and R_2 is the relation:

$$R_1; R_2 := \{(a, c) : \quad \text{there is a } b \in T \text{ such that} \\ (a, b) \in R_1 \text{ and } (b, c) \in R_2\}.$$

Note that if $f : S \to T$ and $g : T \to S$ are functions then $f; g = g \circ f$.

Exercises

Exercises

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 + 3$ and $g(n) = 5n - 11$. What is:

- $f \circ g(n) =$
- $g \circ f(n) =$
- $g^2(n) =$

Exercises

Exercises

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 + 3$ and $g(n) = 5n - 11$. What is:

- $f \circ g(n) = (5n - 11)^2 + 3 = 25n^2 - 110n - 118$
- $g \circ f(n) =$
- $g^2(n) =$

Exercises

Exercises

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 + 3$ and $g(n) = 5n - 11$. What is:

- $f \circ g(n) = (5n - 11)^2 + 3 = 25n^2 - 110n - 118$
- $g \circ f(n) = 5(n^2 + 3) - 11 = 5n^2 + 4$
- $g^2(n) =$

Exercises

Exercises

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 + 3$ and $g(n) = 5n - 11$. What is:

- $f \circ g(n) = (5n - 11)^2 + 3 = 25n^2 - 110n - 118$
- $g \circ f(n) = 5(n^2 + 3) - 11 = 5n^2 + 4$
- $g^2(n) = 5(5n - 11) - 11 = 25n - 66$

Properties of Functions

Function is called **surjective** or **onto** if every element of the codomain is mapped to by at least one x in the domain, i.e.

$$\mathrm{Im}(f) = \mathrm{Codom}(f)$$

Examples (of functions that are surjective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- Floor, ceiling

Examples (of functions that are not surjective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x^2$
- $f : \{a, \ldots, e\}^* \longrightarrow \{a, \ldots, e\}^*$ with $f(w) \mapsto awe$

Injective Functions

Function is called **injective** or **1–1** (**one-to-one**) if different x implies different $f(x)$, i.e.

$$\text{If } f(x) = f(y) \text{ then } x = y$$

Examples (of functions that are injective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- set complement (for a fixed universe)

Examples (of functions that are not injective)

- absolute value, floor, ceiling
- length of a word

Function is **bijective** if it is both surjective and injective.

Converse of a function

Question

$f^{\leftarrow}$ is a relation; when is it a function?