Problem1

(a) Assume any element $(a, c) \in (R1;R2);R3$,

Then, there must be $(a, x) \in R1$, $(x, y) \in R2$, $(y, c) \in R3$.

$\therefore (x, c) \in R2;R3$

$(a, c) \in R1;(R2;R3)$

Therefore, $(R1;R2);R3 = R1;(R2;R3)$

(b) Assume any element $(a, b) \in R1$,

$R1 \subseteq S \times S$

Thus, $\{a,b\} \in S$,

$I = \{(x,x) : x \in S\}$

Therefore $\{(a, a), (b, b)\} \in I$

$\therefore (a, b) \in I;R1$ and $(a, b) \in R1;I$

So, $I;R1 = R1;I = R1$

(c) Assume $R1 = \{(a, b)\}$, $R2 = \{(b, c)\}$

$R1;R2 = (a, c)$

$(R1;R2)\leftarrow = (c, a)$

$R1\leftarrow = \{(b, a)\}$, $R2\leftarrow = \{(c, b)\}$

$R1\leftarrow ; R2\leftarrow = \emptyset$

Thus, $(R1;R2)\leftarrow \neq R1\leftarrow ;R2\leftarrow$.

(d) Assume $(R1 \cup R2);R3 = (a, b)$

Then, $(a, x) \in (R1 \cup R2)$ and $(x, b) \in R3$

$\because (a, x) \in (R1 \cup R2)$

$\therefore$There are three conditions

① $(a, x) \in R1$ and $(a, x) \notin R2$

Then, $(R1;R3) = (a, b)$, $(R2;R3) = \emptyset$

$(R1;R3) \cup (R2;R3) = (a, b) = (R1 \cup R2);R3$

② $(a, x) \in R2$ and $(a, x) \notin R1$

Then, $(R2;R3) = (a, b)$, $(R1;R3) = \emptyset$

$(R1;R3) \cup (R2;R3) = (a, b) = (R1 \cup R2);R3$

③ $(a, x) \in R1$ and $(a, x) \in R2$

Then, $(R1;R3) = (a, b)$ and $(R2;R3) = (a, b)$

$(R1;R3) \cup (R2;R3) = (a, b) = (R1 \cup R2);R3$

Therefore, $(R1 \cup R2);R3 = (R1;R3) \cup (R2;R3)$

(e) Assume $R1 = \{(b, a), (b, b)\}$, $R2 = \{(a, b)\}$, $R3 = \{(b, b)\}$

Thus, left hand: $R2 \cap R3 = \emptyset$,

$R1;(R2 \cap R3) = \emptyset$

Right hand: $(R1;R2) =(b, b)$, $(R1;R3) = (b, b)$

Then $(R1;R2) \cap (R1;R3) = (b, b) \neq \emptyset$

So, $R1;(R2 \cap R3) \neq (R1;R2) \cap (R1;R3)$

Problem 2

(a) Assume $R^j = R^{i+K}$,

$\because j >= i$

$\therefore k >= 0$

We assume $R^j = R^i$,

That is $R^{i+k} = R^i$,

Base case: $k = 0$, $R^{i+k} = R^{i+0} = R^i$

Inductive case: Assume $R^{i+k} = R^i$ hold,

$R^{i+k+1} = R^{i+k} \cup (R;R^{i+k})$

$\because R^{i+k} = R^i$ and $R^{i+1} := R^i \cup (R;R^i)$

$\therefore R^{i+k+1} = R^i \cup (R;R^i) = R^{i+1} = R^i$

Thus, if there is an i such that $R^i = R^{i+1}$, then $R^j = R^i$ for all $j \geq i$.

(b) 1. If $k >= i$, just as (a), $R^k = R^i$,

Then, $R^k \subseteq R^i$.

2. If $0 <= k < i$,

Let $i = k + x$

Thus, $R^k \subseteq R^i$ can be written as $R^k \subseteq R^{k+x}$

Base case: $x = 0$, $R^k = R^{k+0} \subseteq R^k$

$x=1$, $R^{k+1} = R^k \cup (R;R^k)$

thus, $R^k \subseteq R^{k+1}$

Inductive case: we assume $R^k \subseteq R^{k+x}$ hold,

$R^{k+x+1} = R^{k+x} \cup (R;R^{k+x})$

$R^{k+x} \subseteq R^{k+x+1}$

$\because R^k \subseteq R^{k+x}$

$\therefore R^k \subseteq R^{k+x+1}$

Thus, if there is an i such that $R^i = R^i+1$, then $R^i \subseteq R^i$ for all $k \geq 0$.

(c) Base case: $n = 0$,

$R^0;R^m = I;R^m = R^m$, hold. Problem1 (b)

Inductive case: Assume $R^n;R^m = R^{n+m}$,

$R^{n+1};R^m = R^n \cup (R;R^n);R^m$

$= R^n;R^m \cup (R;R^n);R^m$ problem1 (d)

$= R^n;R^m \cup R;(R^n;R^m)$ problem1 (a)

$= R^{n+m} \cup (R;R^{n+m})$ $R^n;R^m = R^{n+m}$

$= R^{n+m+1}$

Therefore, hold.

(d) Firstly, prove $R^0 \in R^i$

Base case: $i = 1$, $R^1 = R^0 \cup (R;R^0)$

$R^0 \in R^i$ hold,

Inductive case: Assume $R^0 \in R^i$,

$R^{i+1} = R^i \cup (R; R^i)$

$\because R^0 \in R^i$,

$\therefore R^0 \in R^{i+1}$, hold

Therefore, $R^0 \in R^i$

$\because (a, b) \in R^{k+1}$

If $k = 0$,

Then, $(a, b) \in R^1 = R^0 \cup (R;R^0)$

$= I \cup (R;I)$

$= I \cup R$

$\because I = \{(x, x) : x \in S\}$

$\therefore (a, b) \in R$ and $(b, b) \in I = R^0$

$\because R^0 \in R^{i-1}$ $(i >= 1)$

$\therefore (a, b) \in R;R^{i-1}$

Therefore, $(a, b) \in R^i = R^{i-1} \cup (R; R^{i-1})$

Thus, $R^K = R^{k+1}$.

(e)

(f). Let $Z = (R \cup R\leftarrow)$,

Then, $(R \cup R\leftarrow)^k = Z^k$

①. When $k = 0$, $Z^0 = I = \{(x,x) : x \in S\}$

$\because Z^0 \in Z^k$

$\therefore Z^k$ is reflective

②.

Problem3

(a) Binary tree:

①. Empty tree

②. A node with left_tree and right_tree.

(b) count(T):

①. $T = \emptyset$, count(t) = 0,

②. count(left_tree) + count(right_tree) +1

(c) leaves(T):

① no successors, return 1

② leaves(left_tree) + leaves(right_tree)

(d) internal(t):

①. No successor: 0

②. 1 successor: internal(successor)

③. 2 successors: internal(left_tree) + internal(right_tree)

(e) Base case: when node = 1, leaves(1) = 1, internal(1) = 0,

Then, leaves(T) = 1 + internal(T), hold.

Inductive case: Assume leaves(T) = 1 + internal(T),

①. If leaves(T+1) = leaves(T),
Then, internal(T+1) = internal(T)
Thus, internal(T+1) = 1 + internal(T+1), hold

②. If leaves(T+1) = leaves(T) + 1,
Then, internal(T+1) = internal(T) + 1
Thus, internal(T+1) = 1 + internal(T+1), hold.

Problem4

(a) Define hi Alpha as HA, lo Alpha as LA
Define hi Bravo as HB, lo Bravo as LB
Define hi Charlie as HC, lo Charlie as LC
Define hi Delta as HD, lo Delta as LD

(i).(HA∪LA)∩(HB∪LB)∩(HC∪LC)∩(HD∪LD)
(ii).((¬HA∩LA)∪(HA∩¬LA))∩((¬HB∩LB)∪(HB∩¬LB))∩((¬HC∩LC)∪(HC∩¬LC))∩((¬HD∩LD)∪(HD∩¬LD))
(iii).((HA∩LB)∪(HB∩LA))∩((HB∩LC)∪(HC∩LB))∩((HC∩LD)∪(HD∩LC))

(b) (i).HA∩LB∩HC∩LD is true

(ii).Alpha and Charlie use hi, Bravo and Delta use lo.