MA109 Tutorial Session

Week 6

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with most (all) of the effort from

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Summary

Tutorial Sheet 6: 2,4,5,8,9

Sheet 6, Q2

Question

Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ *at the point* $(1, 0)$ *has the value 1.*

Since the above function is a combination of polynomial and sinusoidal functions, $f(x, y)$ is continuous as well as differentiable in $\mathbb{R}^2$.

The directional derivative can be calculated using the total derivative. The total derivative at point $(1, 0)$ is given by

$$\nabla f(1,0) = \left(\frac{\partial f}{\partial x}(1,0), \frac{\partial f}{\partial y}(1,0)\right) = (2,1)$$

Sheet 6, Q2

The directional derivative in the unit direction $u = (u_1, u_2)$ is given by
$\mathrm{D_u} f(1,0) = \nabla f(1,0) \cdot u = 2u_1 + u_2$.
We have to solve the following equations:

$$2u_1 + u_2 = 1$$
$$u_1^2 + u_2^2 = 1$$

This gives us $(u_1, u_2) = (0,1)$ or $(0.8, -0.6)$. These are the required directions.

Sheet 6, Q4

Question

Find $\mathrm{D_u}F(2,2,1)$, *where* $F(x,y,z) = 3x - 5y + 2z$, *and u is the unit vector in the direction of the outward normal to the sphere* $g(x,y,z) = x^2 + y^2 + z^2 - 9 = 0$ *at (2, 2, 1).*

As covered in class, the normal for a function $f(x,y,z) = 0$ is given by its gradient. So,

$$u = \frac{\nabla g(2,2,1)}{||\nabla g(2,2,1)||} = \left(\frac{2}{\sqrt{2^2+2^2+1^2}}, \frac{2}{\sqrt{2^2+2^2+1^2}}, \frac{1}{\sqrt{2^2+2^2+1^2}}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

By the definition of direction derivative, we get

$$(\mathrm{D_u}F)(2,2,1) = \lim_{t\to 0} \frac{F(ut + (2,2,1)) - F(2,2,1)}{t} = \lim_{t\to 0} \frac{3(2t/3) - 5(2t/3) + 2(t/3)}{t} = -\frac{2}{3}$$

Sheet 6, Q5

Question

Given $\sin(x+y)+\sin(y+z)=1$, *find* $\frac{\partial^2 z}{\partial x \partial y}$ *provided* $\cos(y+z)\neq 0$.

We are given that $\sin(x+y)+\sin(y+z)=1$ and $\cos(y+z)\neq 0$.

Differentiating with respect to x while keeping y constant gives us

$$\cos(x+y)+\cos(y+z)\frac{\partial z}{\partial x}=0. \qquad (*)$$

Similarly, differentiating with respect to y while keeping x constant gives us

$$\cos(x+y)+\cos(y+z)\left(1+\frac{\partial z}{\partial y}\right)=0. \qquad (**)$$

Sheet 6, Q5

Differentiating $(*)$ with respect to y gives us

$-\sin(x+y) - \sin(y+z)\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0.$

Thus, using $(*)$ and $(**)$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\cdot\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right]$$

$$= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\right]$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}$$

Sheet 6, Q8

Question

Analyse the following functions for local maxima, local minima and saddle points:
(i) $f(x, y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$

Note that the above function is defined on $Domain = \mathbb{R}^2$.
Thus, every point is an interior point of *Domain*. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.
For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.
Note that $f_x(x, y) = xe^{1/2\left(-x^2-y^2\right)}\left(-x^2 + y^2 + 2\right)$.
Also, $f_y(x, y) = ye^{1/2\left(-x^2-y^2\right)}\left(-x^2 + y^2 - 2\right)$.
solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that
$(x_0, y_0) \in \{(0, 0),\ (0, \sqrt{2}),\ (0, -\sqrt{2}),\ (-\sqrt{2}, 0),\ (\sqrt{2}, 0)\}$.
Now, we determine the exact nature using the second derivative test.

Sheet 6, Q8

Recall that $D := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.
Hence, in our case,

$$D = -e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4\right).$$

Moreover, $f_{xx}(x, y) = e^{-\left(x^2+y^2\right)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$
For $(x_0, y_0) = (0, 0)$, it is clear that it is a saddle point for f as discriminant is $-4 < 0$.
Note that if $x = 0$, the discriminant reduces to $-e^{-y^2}(y^6 - 3y^4 - 8y^2 + 4)$.
Substituting $y = \pm\sqrt{2}$ gives us that the discriminant is positive with f_{xx} positive and hence, the points are points of local minima.
Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima as they have discriminant positive and f_{xx} negative.

Sheet 6, Q8

Question

Analyse the following functions for local maxima, local minima and saddle points:

(ii) $f(x, y) = x^3 - 3xy^2$

Note that the above function is defined on $Domain = \mathbb{R}^2$.

Thus, every point is an interior point of *Domain*. Moreover, it can be seen that the partial derivatives of all orders exist and are continuous everywhere.

For (x_0, y_0) to be a point of extrema or a saddle point, it must be the case that $(\nabla f)(x_0, y_0) = (0, 0)$.

Note that $f_x(x, y) = 3x^2 - 3y^2$.

Also, $f_y(x, y) = -6xy$.

Thus, solving $(\nabla f)(x_0, y_0) = (0, 0)$ gives us that $(x_0, y_0) = (0, 0)$.

Sheet 6, Q8

Recall that $D := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.

Hence, in our case,

$$D(x_0, y_0) = -36(x_0^2 + y_0^2).$$

Thus, for $(x_0, y_0) = (0, 0)$, we get the discriminant is 0.

Hence, we get that the discriminant test is inconclusive!

This means that we must turn to some other analytic methods of determining the nature.

Note that $f(\delta, 0) = \delta^3$ for all $\delta \in \mathbb{R}$.

Take any $r > 0$. Take the points $(r/2, 0)$ and $(-r/2, 0)$. For these, $||(x, y)|| < r$ but $f(-r/2, 0) < f(0, 0) < f(r/2, 0)$

This gives us that $(0, 0)$ is saddle point.

Sheet 6, Q9

Question

Find the absolute maximum and absolute minimum of $f(x,y) = (x^2 - 4x)\cos y$ *for* $1 \leq x \leq 3$, $-\pi/4 \leq y \leq \pi/4$

Note that the domain is a closed and bounded set. As f is continuous on the domain, f does achieve a maximum and a minimum. Further since $f(x,y) \in C^\infty$, the points where it can achieve the maximum and minimum are the critical points, and the boundary.

Note that $f_x(x,y) = (2x-4)\cos y$ and $f_y(x,y) = -\left(x^2 - 4x\right)\sin y$ for interior points (x,y).

Thus, the only critical point is $p_1 = (2,0)$.

Sheet 6, Q9

Now we restrict ourselves to the boundaries to find the local extrema.

"Right boundary:" This is the line segment $x = 3, -\pi/4 \leq y \leq \pi/4$.

The function now reduces to $-3\cos y$ on this segment.

Using our theory from one-variable calculus, we get that we need to check the points $(3, 0),\ (3, \pi/4),\ (3, -\pi/4)$. (How?)

Similarly, considering the "left boundary" gives the points $(1, 0),\ (1, \pi/4),\ (1, -\pi/4)$.

Now, we look at the "top boundary.". The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.

Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4),\ (2, \pi/4),\ (3, \pi/4)$.

Similarly, checking the "bottom boundary" gives us the points $(1, -\pi/4),\ (2, -\pi/4),\ (3, -\pi/4)$.

Sheet 6, Q9

(x_0, y_0)	$(2,0)$	$(3,0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\dfrac{-3}{\sqrt{2}}$	$\dfrac{-4}{\sqrt{2}}$	$\dfrac{-3}{\sqrt{2}}$

(x_0, y_0)	$(1,0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\dfrac{-3}{\sqrt{2}}$	$\dfrac{-4}{\sqrt{2}}$	$\dfrac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\text{min}} = -4$ at $(2, 0)$ and $f_{\text{max}} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$.