MA109 Tutorial Session

Week 6

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with most (all) of the effort from

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Sophomore, Dept of CSE

Summary

Tutorial Sheet 6: 2,4,5,8,9

Sheet 6, Q2

Question

Find the directions in which the directional derivative of $f(x, y) = x^2 + \sin xy$ *at the point* $(1, 0)$ *has the value 1.*

Sheet 6, Q2

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Since the above function is a combination of polynomial and sinusoidal functions, $f(x, y)$ is continuous as well as differentiable in $\mathbb{R}^2$.

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Since the above function is a combination of polynomial and sinusoidal functions, $f(x, y)$ is continuous as well as differentiable in $\mathbb{R}^2$.

The directional derivative can be calculated using the total derivative. The total derivative at point $(1, 0)$ is given by

$$\nabla f(1,0) = \left(\frac{\partial f}{\partial x}(1,0), \frac{\partial f}{\partial y}(1,0)\right) = (2,1)$$

Sheet 6, Q2

The directional derivative in the unit direction $u = (u_1, u_2)$ is given by $\mathrm{D_u} f(1,0) = \nabla f(1,0) \cdot u = 2u_1 + u_2$.

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$$u_1^2 + u_2^2 = 1$$

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$$2u_1 + u_2 = 1$$
$$u_1^2 + u_2^2 = 1$$

This gives us $(u_1, u_2) = (0, 1)$ or $(0.8, -0.6)$. These are the required directions.

Sheet 6, Q4

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Find $\mathrm{D_u}F(2,2,1)$, *where* $F(x,y,z) = 3x - 5y + 2z$, *and u is the unit vector in the direction of the outward normal to the sphere* $g(x,y,z) = x^2 + y^2 + z^2 - 9 = 0$ *at (2, 2, 1).*

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As covered in class, the normal for a function $f(x,y,z) = 0$ is given by its gradient. So,

$$u = \frac{\nabla g(2,2,1)}{||\nabla g(2,2,1)||} = \left(\frac{2}{\sqrt{2^2+2^2+1^2}}, \frac{2}{\sqrt{2^2+2^2+1^2}}, \frac{1}{\sqrt{2^2+2^2+1^2}}\right) = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$$

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By the definition of direction derivative, we get

$$(\mathrm{D_u}F)(2,2,1) = \lim_{t\to 0} \frac{F(ut + (2,2,1)) - F(2,2,1)}{t} = \lim_{t\to 0} \frac{3(2t/3) - 5(2t/3) + 2(t/3)}{t} = -\frac{2}{3}$$

Sheet 6, Q5

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Similarly, differentiating with respect to y while keeping x constant gives us

$$\cos(x+y)+\cos(y+z)\left(1+\frac{\partial z}{\partial y}\right)=0. \qquad (**)$$

Sheet 6, Q5

Differentiating $(*)$ with respect to y gives us

$$-\sin(x+y) - \sin(y+z)\left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} + \cos(y+z)\frac{\partial^2 z}{\partial x \partial y} = 0.$$

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Thus, using $(*)$ and $(**)$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\cdot\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x}\right]$$

$$= \frac{1}{\cos(y+z)}\left[\sin(x+y) + \sin(y+z)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\left(-\frac{\cos(x+y)}{\cos(y+z)}\right)\right]$$

$$= \frac{\sin(x+y)}{\cos(y+z)} + \tan(y+z)\frac{\cos^2(x+y)}{\cos^2(y+z)}$$

Sheet 6, Q8

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(i) $f(x,y) = (x^2 - y^2)e^{-(x^2+y^2)/2}$

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$(x_0, y_0) \in \{(0, 0),\ (0, \sqrt{2}),\ (0, -\sqrt{2}),\ (-\sqrt{2}, 0),\ (\sqrt{2}, 0)\}$.
Now, we determine the exact nature using the second derivative test.

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Hence, in our case,

$$D = -e^{-x^2-y^2}\left(x^6 - x^4y^2 - 3x^4 - x^2y^4 + 22x^2y^2 - 8x^2 + y^6 - 3y^4 - 8y^2 + 4\right).$$

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Moreover, $f_{xx}(x, y) = e^{-\left(x^2+y^2\right)/2}(x^4 - x^2y^2 - 5x^2 + y^2 + 2)$

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Similarly, we get that the points $(\pm\sqrt{2}, 0)$ are points of local maxima as they have discriminant positive and f_{xx} negative.

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This gives us that $(0, 0)$ is saddle point.

Sheet 6, Q9

Question

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Note that $f_x(x, y) = (2x - 4)\cos y$ and $f_y(x, y) = -\left(x^2 - 4x\right)\sin y$ for interior points (x, y).

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Question

Find the absolute maximum and absolute minimum of $f(x,y) = (x^2 - 4x)\cos y$ *for* $1 \le x \le 3$, $-\pi/4 \le y \le \pi/4$

Note that the domain is a closed and bounded set. As f is continuous on the domain, f does achieve a maximum and a minimum. Further since $f(x,y) \in C^\infty$, the points where it can achieve the maximum and minimum are the critical points, and the boundary.

Note that $f_x(x,y) = (2x-4)\cos y$ and $f_y(x,y) = -\left(x^2 - 4x\right)\sin y$ for interior points (x,y).

Thus, the only critical point is $p_1 = (2,0)$.

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Now we restrict ourselves to the boundaries to find the local extrema.

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The function now reduces to $-3\cos y$ on this segment.

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The function now reduces to $-3\cos y$ on this segment.

Using our theory from one-variable calculus, we get that we need to check the points $(3, 0)$, $(3, \pi/4)$, $(3, -\pi/4)$.

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Using our theory from one-variable calculus, we get that we need to check the points $(3, 0)$, $(3, \pi/4)$, $(3, -\pi/4)$.

Similarly, considering the "left boundary" gives the points $(1, 0)$, $(1, \pi/4)$, $(1, -\pi/4)$.

Now, we look at the "top boundary.". The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.

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"Right boundary:" This is the line segment $x = 3, -\pi/4 \leq y \leq \pi/4$.

The function now reduces to $-3\cos y$ on this segment.

Using our theory from one-variable calculus, we get that we need to check the points $(3,0),\ (3,\pi/4),\ (3,-\pi/4)$.

Similarly, considering the "left boundary" gives the points $(1,0),\ (1,\pi/4),\ (1,-\pi/4)$.

Now, we look at the "top boundary.". The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.

Once again, using our theory from one-variable calculus, we get that we need to check the points $(1,\pi/4),\ (2,\pi/4),\ (3,\pi/4)$.

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Now we restrict ourselves to the boundaries to find the local extrema.

"Right boundary:" This is the line segment $x = 3, -\pi/4 \leq y \leq \pi/4$.

The function now reduces to $-3\cos y$ on this segment.

Using our theory from one-variable calculus, we get that we need to check the points $(3, 0)$, $(3, \pi/4)$, $(3, -\pi/4)$. (How?)

Similarly, considering the "left boundary" gives the points $(1, 0)$, $(1, \pi/4)$, $(1, -\pi/4)$.

Now, we look at the "top boundary.". The function there reduces to $\frac{x^2-4x}{\sqrt{2}}$.

Once again, using our theory from one-variable calculus, we get that we need to check the points $(1, \pi/4)$, $(2, \pi/4)$, $(3, \pi/4)$.

Similarly, checking the "bottom boundary" gives us the points $(1, -\pi/4)$, $(2, -\pi/4)$, $(3, -\pi/4)$.

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(x_0, y_0)	$(2,0)$	$(3,0)$	$(3,\pi/4)$	$(2,\pi/4)$	$(1,\pi/4)$
$f(x_0, y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$

(x_0, y_0)	$(1,0)$	$(1,-\pi/4)$	$(2,-\pi/4)$	$(3,-\pi/4)$	
$f(x_0, y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

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(x_0, y_0)	$(2, 0)$	$(3, 0)$	$(3, \pi/4)$	$(2, \pi/4)$	$(1, \pi/4)$
$f(x_0, y_0)$	-4	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$
(x_0, y_0)	$(1, 0)$	$(1, -\pi/4)$	$(2, -\pi/4)$	$(3, -\pi/4)$	
$f(x_0, y_0)$	-3	$\frac{-3}{\sqrt{2}}$	$\frac{-4}{\sqrt{2}}$	$\frac{-3}{\sqrt{2}}$	

Thus, we get that $f_{\text{min}} = -4$ at $(2, 0)$ and $f_{\text{max}} = -\frac{3}{\sqrt{2}}$ at $(1, \pm\pi/4)$ and $(3, \pm\pi/4)$.