Tutorial Session Week 2

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December 2, 2020

What's New this Wednesday

- 🚺 Tutorial Sheet 1
 - Q13. (ii) : Continuity
 - Q15 : Differentiability
 - Q18 : Functional Equations
 - Optional : Q7
 - Optional : Q10
- Tutorial Sheet 2
- Q3 : Rolle's Theorem
 - Q5 : Langrange's Mean Value Theorem
- a do : EanBrange a Mean value i necient
- Something Extra

Tutorial Sheet 1

AHH SHIT, HERE WE GO AGAIN!

Q13. (ii)

Question

Discuss the continuity of $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Claim: f is continuous on \mathbb{R}

Proof.

Consider any $y \neq 0$, 1/x is a rational function, hence continuous at y. sin is continuous on \mathbb{R} , so $\sin(1/x)$ is continuous at y since this is a composition. Similarly, x is continuous on \mathbb{R} , so x is continuous at y. Product of 2 continuous functions is continuous, showing f is continuous at all $y \neq 0$.

All we need to show is that f is continuous at 0.

Note that $|f(x) - f(0)| = |x \sin(1/x)| \le |x|$

Given any $\epsilon >$ 0, chose $\delta = \epsilon$.

$$|x-0| < \delta \implies |f(x)-f(0)| \le |x| < \delta = \epsilon.$$

This shows f is continuous at 0 and the proof is completed.



Question

Show that the function $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable on \mathbb{R} . Comment on the continuity of f'.

First we show f is differentiable on \mathbb{R} .

Proof.

Consider any $y \neq 0$, 1/x is a rational function, hence differentiable at y. sin is continuous on \mathbb{R} , so $\sin(1/x)$ is differentiable at y since this is a composition. Similarly, x^2 is differentiable on \mathbb{R} , so x is differentiable at y. Product of 2 differentiable functions is differentiable, showing f is differentiable at all $y \neq 0$.

All we need to show is that f is differentiable at 0.

We claim f'(0) = 0. The proof follows as :

Note that
$$\frac{f(x) - f(0)}{x - 0} = \frac{x^2 \sin(1/x)}{x} = x \sin(1/x)$$

By the same argument as in last question, $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x \sin(1/x) = 0$.

Claim: f' is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at 0

Proof.

For $x \neq 0$, we simply use the chain rule to obtain f'.

$$f'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f' is continuous on $\mathbb{R} \setminus \{0\}$ (why?)

For the purpose of contradiction, assume f' is continuous at 0.

We proved $x \sin(1/x)$ is continuous on \mathbb{R} , hence at 0.

We obtain $\lim_{x\to 0} f'(x) = \lim_{x\to 0} \cos(1/x) = 0$.

This is a contradiction (why?) and thus our claim is proven.



Explanation

Question

Show that the limit $\lim \cos(1/x)$ does not exist.

Proof.

We use the sequence definition of limits. Consider the sequence $\{a_n\}$ formed as $a_n = \frac{1}{n\pi}$. It is easy to verify $a_n \to 0$ and $\forall n \in \mathbb{N}$, $a_n \neq 0$.

Consider the sequence $\cos(1/a_n) = \cos(n\pi) = (-1)^n$

This shows that the limit does not exist, as if it did, the sequences should have converged to the limit. Since the sequence diverges, this would form a contradiction.

Question

Let $f: \mathbb{R} \to \mathbb{R}$ satisfy $f(x+y) = f(x)f(y) \forall x, y \in \mathbb{R}$. If f is differentiable at 0, show that f is differentiable on \mathbb{R} and f'(c) = f(c)f'(0).

Proof.

Put x = y = 0 in the functional equation.

$$f(0) = f^2(0) \implies f(0) \in \{0, 1\}$$

Case 1 f(0) = 0 shows $f(x) = 0 \forall x \in \mathbb{R}$ (why?).

This satisfies the hypothesis (verify!)

The other case is f(0) = 1. We are given $\lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim_{h \to 0} \frac{f(h) - 1}{h}$ exists and f'(0) = f'(0).

Note that
$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0} \frac{f(c)f(h)-f(c)}{h} = f(c)\lim_{h\to 0} \frac{f(h)-1}{h}$$
.

This limit exists and therefore f is differentiable on \mathbb{R}

Also,
$$f'(c) = f(c)f'(0)$$
.



Try to Show

Given the hypothesis of the previous question and f(0) = 1, show $f(x) = a^x$ for some $a \in \mathbb{R}^+$.

THIS IS PURELY FOR FUN

Question

Let $f:(a,b)\to\mathbb{R}$. Let $c\in(a,b)$.

Show that the following are equivalent:

- f is differentiable at c

$$f(c+h) = f(c) + h\alpha + h\epsilon_1(h) \forall h \in (-\delta, \delta)$$

- and $\lim_{h\to 0} \epsilon_1(h) = 0$.
- $\exists \alpha \in \mathbb{R} \text{ such that } \lim_{h \to 0} \left| \frac{f(c+h) f(c) h\alpha}{h} \right| = 0$

Theorem (Carathéodory Lemma)

$$f:(a,b) \to \mathbb{R}$$
 is differentiable at $c \in (a,b)$ iff $\exists \delta > 0, \alpha \in \mathbb{R}$ and $\epsilon_1:(-\delta,\delta) \to \mathbb{R}$ such that $f(c+h) = f(c) + h\alpha + h\epsilon_1(h)$ and $\lim_{h\to 0} \epsilon_1(h) = 0$. Then, $\alpha = f'(c)$.

Proof (\Rightarrow) .

Let $\alpha = f'(c)$. For any δ small enough such that $(c - \delta, c + \delta) \subset (a, b)$, define

$$\epsilon_1(h) = \begin{cases} \frac{f(c+h) - f(c)}{h} - \alpha & h \neq 0\\ 0 & h = 0 \end{cases}$$

It is trivial to see $\lim_{h\to 0} \epsilon_1(h) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \alpha = \alpha - \alpha = 0$



Proof (\Leftarrow) .

Let there be an ϵ_1, δ and α as in the hypothesis.

Then for $0 < h < \delta$, $\epsilon_1(h) = \frac{f(c+h) - f(c)}{h} - \alpha$.

Now since $\lim_{h\to 0} \epsilon_1(h)$ exists, $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists and $=\alpha$.

Which is same as $f'(c) = \alpha$.



We show
$$f$$
 is differentiable at c iff $\exists \alpha \in \mathbb{R}$ such that $\lim_{h \to 0} \left| \frac{f(c+h) - f(c) - h\alpha}{h} \right| = 0$

Proof (\Rightarrow) .

Let f be differentiable at c. $\alpha := f'(c)$.

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} - \alpha = 0$$

$$\therefore \lim_{h\to 0} \frac{f(c+h)-f(c)-h\alpha}{h} = 0$$

$$\therefore \lim_{h \to 0} \left| \frac{f(c+h) - f(c) - h\alpha}{h} \right| = 0 \text{ (why?)}$$



Proof (\Leftarrow) .

Let $\exists \alpha \in \mathbb{R}$ as in the hypothesis.

$$\lim_{h\to 0} \left| \frac{f(c+h)-f(c)-h\alpha}{h} \right| = 0$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c) - h\alpha}{h} = 0 \text{ (why?)}$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} - \alpha = 0$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$



Explanation

Felt a bit hacky, didn't it? Formalize using this:

Theorem

Let
$$f:(a,b)\to\mathbb{R}$$
 and $c\in(a,b)$ then $\lim_{x\to c}f(x)=0$ iff $\lim_{x\to c}|f(x)|=0$

Proof.

Left as an exercise because it is indeed very simple.



Question

Show that any continuous function $f:[0,1] \rightarrow [0,1]$ has a fixed point.

Geez, but what is a fixed point ??

Definition (Fixed Point)

Let $f: X \to Y$ be a function. Then, $x \in X \cap Y$ is a fixed point of f if f(x) = x.

Now this is a simple question, isn't it?

Proof.

Define $g:[0,1]\to\mathbb{R}$ by g(x)=f(x)-x g is continuous on [0,1] (why?) $g(0)=f(0)\geq 0$ and $g(1)=f(1)-1\leq 0$ If either g(0)=0 or g(1)=0, we are done. Else g(0)>0>g(1)

Now by the IVP (remember, g was continuous), $\exists c \in (0,1)$ such that g(c) = 0.

i.e. $\exists c \in (0,1)$ such that f(c) = c. This completes the proof.

Tutorial Sheet 2

FINALLY, GODDAMNIT!

Q3: Rolle's Theorem

Question

Let $f:[a,b] \to \mathbb{R}$ be continuous on [a,b] and differentiable on (a,b). If f(a)f(b) < 0 and $f'(x) \neq 0 \ \forall x \in (a,b)$, then $\exists ! x_0 \in (a,b)$ such that $f(x_0) = 0$.

Note that we want to show:

- Existence
- Uniqueness

Q3: Rolle's Theorem

Proof (Existence).

This is fairly straightforward.

Remember, f is continuous on [a, b].

There are 2 possible cases, namely f(a) < 0 < f(b) and f(a) > 0 > f(b). In both, the intermediate value property guarentees $\exists c \in (a, b)$ such that f(c) = 0.



Proof (Uniqueness).

We shall prove via contradiction

Assume $\exists x_1, x_2 \in (a, b)$ such that $f(x_1) = f(x_2) = 0$

WLOG $x_1 < x_2$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) .

Moreover $f(x_1) = f(x_2)$. So the hypothesis of Rolle's theorem is satisfied.

Thus, $\exists x' \in (x_1, x_2) \subset (a, b)$ such that f'(x') = 0.

This is a contradiction and we are done.



Question

Show that $\forall a, b \in \mathbb{R}$, $|\sin(a) - \sin(b)| \le |a - b|$

Proof.

For a = b, the claim is trivial. WLOG, assume a < b.

sin is continuous and differentiable on \mathbb{R} . (it is a "nice" function)

Thus, it is continuous on [a, b] and differentiable on (a, b).

The hypothesis of LMVT are satisfied.

$$\therefore \sin(a) - \sin(b) = \cos(c)(a - b)$$
 for some $c \in (a, b)$

$$|\sin(a) - \sin(b)| = |\cos(c)||a - b| \le |a - b|$$
 and the proof is complete.



Generalized Mean Value Theorem

Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on [a, b] and differentiable on (a, b). Then, $\exists c \in (a, b)$ such that [g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c).

Try to show this using Rolle's Theorem

Note that LMVT is a special case where g(x) = x.