Tutorial Session

Week 2

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What's New this Wednesday

1. Tutorial Sheet 1
   - Q13. (ii) : Continuity
   - Q15 : Differentiability
   - Q18 : Functional Equations
   - Optional : Q7
   - Optional : Q10
2. Tutorial Sheet 2
   - Q3 : Rolle's Theorem
   - Q5 : Langrange's Mean Value Theorem
3. Something Extra

Tutorial Sheet 1

AHH SHIT, HERE WE GO AGAIN!

Q13. (ii)

Question

Discuss the continuity of $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} x \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Q13. (ii)

Claim : f is continuous on $\mathbb{R}$

Proof.

Consider any $y \neq 0$, $1/x$ is a rational function, hence continuous at y. sin is continuous on $\mathbb{R}$, so $\sin(1/x)$ is continuous at y since this is a composition. Similarly, x is continuous on $\mathbb{R}$, so x is continuous at y. Product of 2 continous functions is continuous, showing f is continuous at all $y \neq 0$.

All we need to show is that f is continuous at 0.
Note that $|f(x) - f(0)| = |x \sin(1/x)| \leq |x|$
Given any $\epsilon > 0$, chose $\delta = \epsilon$.
$|x - 0| < \delta \implies |f(x) - f(0)| \leq |x| < \delta = \epsilon$.
This shows f is continuous at 0 and the proof is completed. $\square$

Q15

Question

Show that the function $f : \mathbb{R} \to \mathbb{R}$ *given by*

$$f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable on $\mathbb{R}$. *Comment on the continuity of* f'.

Q15

First we show f is differentiable on $\mathbb{R}$.

Proof.

Consider any $y \neq 0$, $1/x$ is a rational function, hence differentiable at y. sin is continuous on $\mathbb{R}$, so $\sin(1/x)$ is differentiable at y since this is a composition. Similarly, x^2 is differentiable on $\mathbb{R}$, so x is differentiable at y. Product of 2 differentiable functions is differentiable, showing f is differentiable at all $y \neq 0$.

All we need to show is that f is differentiable at 0.
We claim $f'(0) = 0$. The proof follows as :
Note that $\frac{f(x)-f(0)}{x-0} = \frac{x^2\sin(1/x)}{x} = x\sin(1/x)$
By the same argument as in last question, $\lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} x\sin(1/x) = 0$. $\square$

Q15

Claim : f' is continuous on $\mathbb{R} \setminus \{0\}$ and discontinuous at 0

Proof.

For $x \neq 0$, we simply use the chain rule to obtain f'.

$$f'(x) = \begin{cases} -\cos(1/x) + 2x\sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

f' is continuous on $\mathbb{R} \setminus \{0\}$ (why?)
For the purpose of contradiction, assume f' is continuous at 0.
We proved $x\sin(1/x)$ is continuous on $\mathbb{R}$, hence at 0.
We obtain $\lim_{x \to 0} f'(x) = \lim_{x \to 0} \cos(1/x) = 0$.
This is a contradiction (why?) and thus our claim is proven. □

Explanation

Question

Show that the limit $\lim_{x \to 0} \cos(1/x)$ *does not exist.*

Proof.

We use the sequence definition of limits. Consider the sequence $\{a_n\}$ formed as $a_n = \frac{1}{n\pi}$. It is easy to verify $a_n \to 0$ and $\forall n \in \mathbb{N}$, $a_n \neq 0$.
Consider the sequence $\cos(1/a_n) = cos(n\pi) = (-1)^n$
This shows that the limit does not exist, as if it did, the sequences should have converged to the limit. Since the sequence diverges, this would form a contradiction. $\square$

Q18

Question

Let $f : \mathbb{R} \to \mathbb{R}$ *satisfy* $f(x+y) = f(x)f(y)\,\forall x, y \in \mathbb{R}$. *If* f *is differentiable at 0, show that* f *is differentiable on* $\mathbb{R}$ *and* $f'(c) = f(c)f'(0)$.

Q18

Proof.

Put $x = y = 0$ in the functional equation.
$f(0) = f^2(0) \implies f(0) \in \{0, 1\}$
Case 1 $f(0) = 0$ shows $f(x) = 0 \, \forall x \in \mathbb{R}$ (why?).
This satisfies the hypothesis (verify!)
The other case is $f(0) = 1$. We are given $\lim\limits_{h \to 0} \frac{f(h) - f(0)}{h - 0} = \lim\limits_{h \to 0} \frac{f(h) - 1}{h}$ exists and $= f'(0)$.
Note that $\lim\limits_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim\limits_{h \to 0} \frac{f(c)f(h) - f(c)}{h} = f(c) \lim\limits_{h \to 0} \frac{f(h) - 1}{h}$.
This limit exists and therefore f is differentiable on $\mathbb{R}$.
Also, $f'(c) = f(c)f'(0)$. $\square$

Try to Show

Given the hypothesis of the previous question and $f(0) = 1$, show $f(x) = a^x$ for some $a \in \mathbb{R}^+$.

THIS IS PURELY FOR FUN

Optional : Q7

Question

Let $f : (a, b) \to \mathbb{R}$. *Let* $c \in (a, b)$.
Show that the following are equivalent :

1. f *is differentiable at* c
2. $\exists \delta > 0, \alpha \in \mathbb{R}$ *and* $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ *such that*

$$f(c+h) = f(c) + h\alpha + h\epsilon_1(h)\,\forall h \in (-\delta, \delta)$$

   and $\lim\limits_{h \to 0} \epsilon_1(h) = 0$.
3. $\exists \alpha \in \mathbb{R}$ *such that* $\lim\limits_{h \to 0} \left| \frac{f(c+h) - f(c) - h\alpha}{h} \right| = 0$

Optional : Q7

Theorem (Carathéodory Lemma)

$f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$ iff $\exists \delta > 0, \alpha \in \mathbb{R}$ and $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $f(c + h) = f(c) + h\alpha + h\epsilon_1(h)$ and $\lim_{h\to 0} \epsilon_1(h) = 0$. Then, $\alpha = f'(c)$.

Proof ($\Rightarrow$).

Let $\alpha = f'(c)$. For any δ small enough such that $(c - \delta, c + \delta) \subset (a, b)$, define

$$\epsilon_1(h) = \begin{cases} \frac{f(c+h)-f(c)}{h} - \alpha & h \neq 0 \\ 0 & h = 0 \end{cases}$$

It is trivial to see $\lim_{h\to 0} \epsilon_1(h) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \alpha = \alpha - \alpha = 0$ $\square$

Optional : Q7

Proof ($\Leftarrow$).

Let there be an ϵ_1, δ and α as in the hypothesis.
Then for $0 < h < \delta$, $\epsilon_1(h) = \frac{f(c+h)-f(c)}{h} - \alpha$.
Now since $\lim_{h\to 0} \epsilon_1(h)$ exists, $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ exists and $= \alpha$.
Which is same as $f'(c) = \alpha$. $\square$

Optional : Q7

We show f is differentiable at c iff $\exists \alpha \in \mathbb{R}$ such that $\lim_{h \to 0} \left| \frac{f(c+h) - f(c) - h\alpha}{h} \right| = 0$

Proof (⇒).

Let f be differentiable at c. $\alpha := f'(c)$.

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} - \alpha = 0$$

$$\therefore \lim_{h \to 0} \frac{f(c+h) - f(c) - h\alpha}{h} = 0$$

$$\therefore \lim_{h \to 0} \left| \frac{f(c+h) - f(c) - h\alpha}{h} \right| = 0 \text{ (why?)}$$

□

Optional : Q7

Proof (⇐).

Let $\exists \alpha \in \mathbb{R}$ as in the hypothesis.

$$\lim_{h\to 0} \left| \frac{f(c+h)-f(c)-h\alpha}{h} \right| = 0$$

$$\therefore \lim_{h\to 0} \frac{f(c+h)-f(c)-h\alpha}{h} = 0 \text{ (why?)}$$

$$\therefore \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \alpha = 0$$

$$\therefore \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \alpha$$

□

Explanation

Felt a bit hacky, didn't it? Formalize using this:

Theorem

Let $f : (a, b) \to \mathbb{R}$ *and* $c \in (a, b)$ *then* $\lim_{x \to c} f(x) = 0$ *iff* $\lim_{x \to c} |f(x)| = 0$

Proof.

Left as an exercise because it is indeed very simple. □

Optional : Q10

Question

Show that any continuous function $f : [0,1] \to [0,1]$ *has a fixed point.*

Geez, but what is a fixed point ??

Definition (Fixed Point)

Let $f : X \to Y$ *be a function. Then,* $x \in X \cap Y$ *is a fixed point of* f *if* $f(x) = x$.

Now this is a simple question, isn't it?

Optional : Q10

Proof.

Define $g : [0,1] \to \mathbb{R}$ by $g(x) = f(x) - x$
g is continuous on $[0,1]$ (why?)
$g(0) = f(0) \geq 0$ and $g(1) = f(1) - 1 \leq 0$
If either $g(0) = 0$ or $g(1) = 0$, we are done. Else $g(0) > 0 > g(1)$
Now by the IVP (remember, g was continuous), $\exists c \in (0,1)$ such that $g(c) = 0$.
i.e. $\exists c \in (0,1)$ such that $f(c) = c$. This completes the proof. $\square$

Tutorial Sheet 2

FINALLY, GODDAMNIT!

Q3 : Rolle's Theorem

Question

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a)f(b) < 0$ and $f'(x) \neq 0 \, \forall x \in (a, b)$, then $\exists! x_0 \in (a, b)$ such that $f(x_0) = 0$.

Note that we want to show :

1. Existence
2. Uniqueness

Q3 : Rolle's Theorem

Proof (Existence).

This is fairly straightforward.
Remember, f is continuous on $[a, b]$.
There are 2 possible cases, namely $f(a) < 0 < f(b)$ and $f(a) > 0 > f(b)$. In both, the intermediate value property guarentees $\exists c \in (a, b)$ such that $f(c) = 0$. $\square$

Q3 : Rolle's Theorem

Proof (Uniqueness).

We shall prove via contradiction
Assume $\exists x_1, x_2 \in (a, b)$ such that $f(x_1) = f(x_2) = 0$
WLOG $x_1 < x_2$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2).
Moreover $f(x_1) = f(x_2)$. So the hypothesis of Rolle's theorem is satisfied.
Thus, $\exists x' \in (x_1, x_2) \subset (a, b)$ such that $f'(x') = 0$.
This is a contradiction and we are done. $\square$

Q5 : Langrange's Mean Value Theorem

Question

Show that $\forall a, b \in \mathbb{R}$, $|\sin(a) - \sin(b)| \leq |a - b|$

Proof.

For $a = b$, the claim is trivial. WLOG, assume $a < b$.
sin is continuous and differentiable on $\mathbb{R}$. (it is a "nice" function)
Thus, it is continuous on $[a, b]$ and differentiable on (a, b).
The hypothesis of LMVT are satisfied.
$\therefore \sin(a) - \sin(b) = \cos(c)(a - b)$ for some $c \in (a, b)$
$\therefore |\sin(a) - \sin(b)| = |\cos(c)||a - b| \leq |a - b|$ and the proof is complete. $\square$

Generalized Mean Value Theorem

Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then, $\exists c \in (a, b)$ such that $[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$.

Try to show this using Rolle's Theorem

Note that LMVT is a special case where $g(x) = x$.