MA109 Tutorial Session

Week 4

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What's New this Wednesday

1. Tutorial Sheet 4
    - Q2.(a) 0 Reimann Integral and 0 function
    - Q2.(b) 0 Reimann Integral but non 0 function
    - A Useful Theorem
    - Q3. (ii) : Approximate Reimann Sums
    - Q3. (iv) : Approximate Reimann Sums
    - Another Useful Theorem
    - Q4. (b). (i) : Apply the Leibnitz Rule
    - Q4. (b). (ii) : Apply the Leibnitz Rule Again
    - Q6 : FTC I but Calculate (:

Q2. (a)

Question

Let $f : [a, b] \to \mathbb{R}$ *be Reimann integrable.*

1. *Given* $f(x) \geqslant 0$, *show that* $\int_a^b f(x)dx \geqslant 0$
2. *Given that* f *is continuous and non negative, and that* $\int_a^b f(x)dx = 0$, *show that* $f(x) = 0 \ \forall x \in [a, b]$

Q2. (a)

Proof for (1).

Consider an arbitrary partition $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$
Note that $m_i := \inf\limits_{x \in [x_{i-1}, x_i]} f(x) \geqslant 0$ since $f(x) \geqslant 0\ \forall x \in [x_{i-1}, x_i]$
So that, $L(P, f) = \sum\limits_{i=1}^{n} m_i(x_i - x_{i-1}) \geqslant 0$
Now, since $L(f) := \sup\limits_{P} L(P, f)$, by definition, $L(f) \geqslant L(P, f) \geqslant 0$
Also, f is Reimann integrable, hence Darboux Integrable
$\int_a^b f(x)dx = L(f) \geqslant 0$ ■

Q2. (a)

Proof for (2).

We shall prove this by contradiction. Assume that $\exists c \in [a, b]$ such that $f(c) > 0$
Claim : $\exists d \in (a, b)$ such that $f(d) > 0$.
If c were a or b, the $\epsilon - \delta$ definition would give such a d for $\epsilon = f(c)$ (show!)
Now, for d, let $\epsilon = f(d)/2$, then $\exists \delta > 0$ such that $(d - \delta, d + \delta) \subset (a, b)$ and
$x \in (d - \delta, d + \delta) \implies |f(x) - f(d)| < \epsilon \implies f(x) - f(d) > -f(d)/2 \implies f(x) > f(d)/2$
Consider the partition $P := \{a, d - \delta/2, d + \delta/2, b\}$
Here, $m_1 \geqslant 0$, $m_2 \geqslant f(d)/2 > 0$ and $m_3 \geqslant 0$
Hence, $L(P, f) = m_1(d - \delta/2 - a) + m_2\delta + m_3(b - d - \delta/2) \geqslant m_2\delta > 0$
Now, since $L(f) := \sup_P L(P, f)$, by definition, $L(f) \geqslant L(P, f) > 0$
Also, f is Reimann integrable, hence $\int_a^b f(x)dx = L(f) > 0$ which is a contradiction. ■

Q2. (b)

Question

Give an example of a Reimann integrable function, $f : [a, b] \to \mathbb{R}$ *such that* $f(x) \geqslant 0\ \forall x \in [a, b]$, $\int_a^b f(x)dx = 0$ *but* $f(x) \neq 0$ *for some* $x \in [a, b]$

Q2. (b)

Obviously, you cannot take f to be any continuous function (why?)

Consider the function $f : [0, 1] \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ 1 & x = 1 \end{cases}$$

f indeed is Reimann integrable and $\int_0^1 f(x)dx = 0$
This can be shown as follows

Q2. (b)

Proof.

We show that f is Darboux integrable.
Let $1 > \delta > 0$ and $P_\delta := \{0, 1-\delta, 1\}$
$L(P_\delta, f) = 0$ and $U(P_\delta, f) = \delta$
By definition, $L(f) \geqslant L(P_\delta, f) \geqslant 0$
And $U(f) := \inf_P U(P, f) \leqslant \inf_{0<\delta<1} U(P_\delta, f) = \inf_{0<\delta<1} \delta = 0$
We know that $L(f) \leqslant U(f)$ and we obtained that $L(f) \geqslant 0 \geqslant U(f)$
Hence, $L(f) = U(f) = 0$ and thus $\int_0^1 f(x)dx = 0$ ∎

Approximate Reimann Sums

Theorem

Let $f : [a, b] \to \mathbb{R}$ be Reimann integrable. Let P_n be a sequence of partitions such that $\lim_{n\to\infty} ||P_n|| = 0$ (written more subtly as $||P_n|| \to 0$). Let $P_n = \{a = x_0 < x_1 < \dots x_m = b\}$ (ofcourse m depends on n), and $t_i \in [x_{i-1}, x_i]$. Then

$$\lim_{n\to\infty} \sum_{i=0}^{m} f(t_i)(x_i - x_{i-1}) = \int_a^b f(x)dx$$

Indeed, the P_n and t_n combined are what are called **tagged partitions**. *And the theorem is another way of saying that if $||P_n|| \to 0$, $R(f, P_n, t_n) \to \int_a^b f$*

Proof of the Theorem

Proof.

We prove that if f is Reimann integrable and $||P_n|| \to 0$, then $R(f, P_n, t_n) \to \int_a^b f(x)dx$

To show our claim, that $R(f, P_n, t_n) \to \int_a^b f(x)dx$, proceed by definition.

Note that by definition of Reimann integrability, we have $\forall \epsilon > 0\ \exists \delta > 0$ such that for any tagged partition (P, t), $||P|| < \delta \implies |R(f, P, t) - \int_a^b f(x)dx| < \epsilon$

Now, given $\epsilon > 0$, we obtain such a $\delta > 0$

Since $||P_n|| \to 0$, $\exists N \in \mathbb{N}$ such that $n \geqslant N \implies ||P_n|| < \delta$

Thus, by definition, $n \geqslant N \implies ||P_n|| < \delta \implies |R(f, P_n, t_n) - \int_a^b f(x)dx| < \epsilon$

Which is exactly what we wanted to show! ∎

Q3. (ii)

Question

Find the $\lim_{n\to\infty} S_n$ *where*

$$S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2}$$

by formulating it as the limit of an appropriate Reimann sum

Q3. (ii)

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{1+(i/n)^2} = \sum_{i=1}^{n} \frac{1}{1+(i/n)^2}\left(\frac{i}{n} - \frac{i-1}{n}\right)$$

Which is precisely $R(f, P_n, t_n)$ where

$$P_n = \{0 = x_0 < x_1 < \cdots < x_n = 1\} : x_i = \frac{i}{n}; t_i = \frac{i}{n}$$

$$f : [0,1] \to \mathbb{R}, f(x) = \frac{1}{1+x^2}$$

Note that $||P_n|| = \frac{1}{n} \to 0$ and f is Reimann integrable
Thus, by the theorem,

$$\lim_{n\to\infty} S_n = \int_0^1 \frac{1}{1+x^2} dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

Where the last part implicitly uses the Fundamental Theorem of Calculus II

Q3. (iv)

Question

Find the $\lim_{n\to\infty} S_n$ *where*

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$$

by formulating it as the limit of an appropriate Reimann sum

Q3. (iv)

$$S_n = \sum_{i=1}^{n} \cos\left[\pi\left(\frac{i}{n}\right)\right]\left(\frac{i}{n} - \frac{i-1}{n}\right)$$

Which is precisely $R(f, P_n, t_n)$ where

$$P_n = \{0 = x_0 < x_1 < \cdots < x_n = 1\} : x_i = \frac{i}{n}; t_i = \frac{i}{n}$$

$$f : [0,1] \to \mathbb{R}, f(x) = \cos(\pi x)$$

Note that $||P_n|| = \frac{1}{n} \to 0$ and f is Reimann integrable
Thus, by the theorem,

$$\lim_{n\to\infty} S_n = \int_0^1 \cos(\pi x)dx = \frac{\sin(\pi)}{\pi} - \frac{\sin(0)}{\pi} = 0$$

Where the last part implicitly uses the Fundamental Theorem of Calculus II

Leibnitz Integral Rule

Differentiate an integral whose limits are differentiable functions of the concerned variable. Note that here, the function involved is INDEPENDENT of the concerned variable.

Theorem (Leibnitz Integral Rule)

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $a, b : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define :

$$F(x) = \int_{a(x)}^{b(x)} f(t)dt$$

Then F is differentiable and $F'(x) = f(b(x))b'(x) - f(a(x))a'(x)$

Leibnitz Integral Rule

Proof.

Define

$$F_1(x) = \int_0^x f(t)dt$$

Since f is continuous, F_1 is differentiable by Fundamental Theorem of Calculus I

Also,

$$F(x) = \int_{a(x)}^{b(x)} f(t)dt = \int_0^{b(x)} f(t)dt - \int_0^{a(x)} f(t)dt = F_1(b(x)) - F_1(a(x))$$

Also, b and a are differentiable functions, hence F is differentiable by the chain rule

$$F'(x) = F_1'(b(x))b'(x) - F_1'(a(x))a'(x) = f(b(x))b'(x) - f(a(x))a'(x)$$

∎

Q4. (b). (i)

Question

Define

$$F(x) = \int_1^{2x} \cos(t^2)dt$$

Show that F *is differentiable and obtain* $\dfrac{dF}{dx}$

Q4. (b). (i)

Note that $\cos(x^2)$ is continous, and 1 and $2x$ are both differentiable.

The hypothesis of Leibnitz Integral Rule are satisfied, hence

$$\frac{dF}{dx} = F'(x) = 2\cos(4x^2) - 0cos(1) = 2\cos(4x^2)$$

Q4. (b). (ii)

Question

Define

$$F(x) = \int_0^{x^2} \cos(t)dt$$

Show that F is differentiable and obtain $\dfrac{dF}{dx}$

Q4. (b). (ii)

Note that $\cos(x)$ is continuous, and 0 and x^2 are both differentiable.

The hypothesis of Leibnitz Integral Rule are satisfied, hence

$$\frac{dF}{dx} = F'(x) = 2x\cos(x^2) - 0cos(0) = 2x\cos(x^2)$$

Q6

Question

Let $f : \mathbb{R} \to \mathbb{R}$ *be continuous and* $\lambda \in \mathbb{R} \setminus \{0\}$.
Define

$$g(x) = \frac{1}{\lambda}\int_0^x f(t)\sin[\lambda(x-t)]dt$$

Show that $g''(x) + \lambda^2 g(x) = f(x)$ *and* $g(0) = 0 = g'(0)$

Q6

Sketch.

Rearrange terms so that you do not have a function of x inside the integral.
(That is all there is to this question, rest is FTC I and plain calculation)

$$g(x) = \frac{1}{\lambda}\left[\sin(\lambda x)\int_0^x f(t)\cos(\lambda t)dt - \cos(\lambda x)\int_0^x f(t)\sin(\lambda t)dt\right]$$

Now note that f is continuous, so are $\cos(\lambda t)$ and $\sin(\lambda t)$, so that FTC I can be applied
Also, $\cos(\lambda x)$ and $\sin(\lambda x)$ are differentiable, so that chain rule can be applied
The rest is simple calculation. I leave that to you (: