MA109 Tutorial Session

Week 5

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with most of the effort from

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What's New this Wednesday

1. Tutorial Sheet 5
   - Q2. Contour Lines and Level Curves
   - Q4. Continuity of Function Combinations
   - Q6. Partial Derivatives at 0
   - Q8. Continuity $\not\Rightarrow$ existence of partial derivatives
   - Q10. Existence of every directional derivative $\not\Rightarrow$ differentiability

Recap

Definition (Level Curve)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. Then the set $\{(x, y) \in \mathbb{R}^2 | f(x, y) = c\} \subseteq \mathbb{R}^2$ is called the level curve of f corresponding to c.

Definition (Contour Line)

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and $c \in \mathbb{R}$. Then the set $\{(x, y, c) \in \mathbb{R}^3 | f(x, y) = c\} \subseteq \mathbb{R}^3$ is called the contour line of f corresponding to c.

The difference is subtle. Simply stated, $\mathcal{C} = \mathcal{L} \times \{c\}$
Thus, I will only cover level curves in the following question.

Caution

Note

The "curve" and "line" in level curve and contour line is in not to be taken literally! A level curve only needs to be a subset of $\mathbb{R}^2$. Infact, you will see these mentioned as "level sets" and "contour plots" in different books.

Consider the function $f(x, y) = 0$. What is its level curve for $c = 0$?

Q2. (ii)

Question

Describe the level curves and the contour lines for $f(x, y) = x^2 + y^2$ corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$.

For $c < 0$, the level curves are empty sets as $f(x, y) \geq 0 \ \forall (x, y) \in \mathbb{R}^2$

For $c = 0$, the level curve is just the singleton set $\{(0,\ 0)\} \subset \mathbb{R}^2$

For $c > 0$, the level curve is the circle $\{(x,\ y) \in \mathbb{R}^2 : x^2 + y^2 = c\}$

For a complete solution, you should describe the level curve $\mathcal{L}$ explicitly!
For each value of c, you should also mention the contour line $\mathcal{C} = \mathcal{L} \times \{c\}$

Q2. (iii)

Question

Describe the level curves and the contour lines for $f(x, y) = xy$ corresponding to the values $c = -3, -2, -1, 0, 1, 2, 3, 4$.

For $c < 0$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the second and fourth quadrant.

For $c > 0$, level curves are rectangular hyperbolas $xy = c$ in the xy-plane with branches in the first and third quadrant.

For $c = 0$, the corresponding level curves are precisely the union of the x-axis and the y-axis

Recap

Definition (Euclidean norm)

Let $m \in \mathbb{N}$. We define the euclidean distance between $x, y \in \mathbb{R}^m$ by

$$||x - y|| = \sqrt{\sum_{i=1}^{m} (x_i - y_i)^2}$$

where $x = (x_1, x_2, \ldots, x_m)$ and $y_m = (y_1, y_2, \ldots, y_m)$

Recap

Definition (Convergence in $\mathbb{R}^m$)

Let (x_n) be a sequence in $\mathbb{R}^m$. If $\exists x \in \mathbb{R}^m$ such that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ for which

$$n \geq N \implies ||x_n - x|| < \epsilon$$

then we say (x_n) converges to x and write $x_n \to x$.

Something Extra

Definition (Metric Spaces)

A set X along with a function $d : X \times X \to \mathbb{R}$ is called a metric space if the distance function d satisfies the following :

1. $\forall x, y \in X, d(x, y) \geq 0$ *and* $d(x, y) = 0 \Leftrightarrow x = y$
2. $\forall x, y \in X, d(x, y) = d(y, x)$
3. $\forall x, y, z \in X, d(x, y) + d(y, z) \geq d(z, x)$

We will refer to $d(x, y) = d(y, x)$ *as* $||x - y|| = ||y - x||$

Something Extra

Definition (Convergence in General Metric Spaces)

Let (x_n) be a sequence in a metric space X. If $\exists x \in X$ such that $\forall \epsilon > 0, \exists N \in \mathbb{N}$ for which

$$n \geq N \implies ||x_n - x|| < \epsilon$$

Then the sequence (x_n) is said to converge to x and we write $x_n \to x$

Recap

The definition of continuity in functions from $\mathbb{R}^2 \to \mathbb{R}$ or in general from any metric space $X \to \mathbb{R}$ is parellel to that of functions from $\mathbb{R} \to \mathbb{R}$.
Thus, one intuitively expects that sequential continuity would also hold and be equivalent to the definition of continuity. This is indeed the case as we show.

Theorem

Let X be a metric space, $x \in X$ and $f : X \to \mathbb{R}$ be a function. Then, f is continuous at x iff $\forall$ sequences (x_n) such that $x_n \to x$, $f(x_n) \to f(x)$.

Recap

Proof. (Forward).

Let $f : X \to \mathbb{R}$ be a continuous function and (x_n) be a sequence in X that converges to x.
Given $\epsilon > 0$, $\exists \delta > 0$ such that $\forall y$ for which $||y - x|| < \delta$, $|f(y) - f(x)| < \epsilon$
Obtain this $\delta > 0$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $||x_n - x|| < \delta$ (why?)
Thus, $\forall n \geq N$, $|f(x_n) - f(x)| < \epsilon$ and so $f(x_n) \to f(x)$. ∎

Recap

Proof. (Backward).

We will proceed via contrapositive. Let f be a function that is not continuous at x.
Thus, $\exists \epsilon > 0$ such that $\forall \delta > 0$, $\exists y$ for which $||y - x|| < \delta$ but $|f(y) - f(x)| \geq \epsilon$.
Construct a sequence (x_n) by chosing x_n to be such a y for $\delta = \frac{1}{n}$.
It is easy to see $x_n \to x$ (why?). Also, $f(x_n) \not\to f(x)$ (why?)
Thus, we are done proving the contrapositive ■

Q4

Question

Suppose $f, g : \mathbb{R} \to \mathbb{R}$ *are continuous functions. Show that each of the following functions of* $(x, y) \in \mathbb{R}^2$ *are continuous:*

1. $f(x) \pm g(y)$
2. $f(x)g(y)$
3. $max\{f(x), g(y)\}$
4. $min\{f(x), g(y)\}$

Q4. (i)

Proof.

Let (x_n, y_n) be any sequence such that $(x_n, y_n) \to (x, y)$

Thus, $x_n \to x$ and $y_n \to y$

Furthermore, by continuity of f and g, $f(x_n) \to f(x)$ and $g(y_n) \to g(y)$

And by the theorem for sequences, $f(x_n) \pm g(y_n) \to f(x) \pm g(y)$

Since (x_n, y_n) was arbitrary, we are done. ∎

Q4. (ii)

Proof.

Let (x_n, y_n) be any sequence such that $(x_n, y_n) \to (x, y)$

Thus, $x_n \to x$ and $y_n \to y$

Furthermore, by continuity of f and g, $f(x_n) \to f(x)$ and $g(y_n) \to g(y)$

And by the theorem for sequences, $f(x_n)g(y_n) \to f(x)g(y)$

Since (x_n, y_n) was arbitrary, we are done. ∎

Q4. (iii)

Proof.

Let (x_n, y_n) be any sequence such that $(x_n, y_n) \to (x, y)$

Thus, $x_n \to x$ and $y_n \to y$

Furthermore, by continuity of f and g, $f(x_n) \to f(x)$ and $g(y_n) \to g(y)$

And by the theorem for sequences, $f(x_n) \pm g(y_n) \to f(x) \pm g(y)$

Thus, we also have $|f(x_n) - g(y_n)| \to |f(x) - g(y)|$ (recall convergence theorem for $|x_n|$)

Now recall $\max\{a, b\} = \frac{(a+b)+|a-b|}{2}$. Hence $\max\{f(x_n), g(y_n)\} \to \max\{f(x), g(y)\}$

Since (x_n, y_n) was arbitrary, we are done. ∎

Q4. (iv)

Yeah, this is getting boring, let us skip!

Q6. (ii)

Question

Examine the function given by

$$f(x,y) = \begin{cases} 0 & \textit{where } (x,y) = (0,0) \\ \frac{\sin^2(x+y)}{|x|+|y|} & \textit{otherwise} \end{cases}$$

for the existence of partial derivatives at $(0,0)$.

Q6. (ii)

$$f_x(0,\ 0) = \lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} = \lim_{h \to 0} \left(\frac{\sin^2(h)}{h|h|} \right)$$

This limit does not exist (why?)

Hint : Consider the two sequences with n^{th} term given by $1/n$ and $-1/n$.

Observe that f is symmetric in x and y, hence $f_y(0,\ 0)$ also does not exist.

Q8

Question

Let $f(0,0) = 0$ *and*

$$f(x,y) = \begin{cases} x\sin(1/x) + y\sin(1/y) & \textit{if } x \neq 0,\ y \neq 0 \\ x\sin(1/x) & \textit{if } x \neq 0,\ y = 0 \\ y\sin(1/y) & \textit{if } x = 0,\ y \neq 0 \end{cases}$$

Show that none of the partial derivatives of f *exist at* $(0,0)$ *although* f *is continuous at* $(0,0)$.

Q8

Claim : f is continuous at $(0,0)$

Proof.

We only need to prove that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Given any $\epsilon > 0$, $\exists\delta > 0$ such that $0 < |x| < \delta \implies |x\sin(1/x)| < \epsilon/2$. (why?)

Also note that $|x| \leq ||(x,y)||$ and $|y| \leq ||(x,y)||$

Let $0 < ||(x,y)|| < \delta$, then

- if $x = 0$, $|f(x,y)| = |y\sin(1/y)| < \epsilon/2 < \epsilon$
- if $y = 0$, $|f(x,y)| = |x\sin(1/x)| < \epsilon/2 < \epsilon$
- otherwise, $|f(x,y)| = |x\sin(1/x) + y\sin(1/y)| \leq |x\sin(1/x)| + |y\sin(1/y)| < \epsilon$

Clearly, $|f(x,y)| < \epsilon$.

Thus, $0 < ||(x,y) - (0,0)|| < \delta \implies |f(x,y) - f(0,0)| < \epsilon$ and the proof is complete. ■

Q8

Now let us show that the partial derivatives don't exist.

$$f_x(0,\ 0) = \lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$$

Which we know, does not exist

Again, note that f is symmetric in x and y. Hence, the other partial derivative does not exist.

Q10

Question

Let $f(x, y) = 0$ *if* $y = 0$ *and*

$$f(x, y) = \frac{y}{|y|}\sqrt{x^2 + y^2} \text{ if } y \neq 0$$

Show that f *is continuous at* $(0, 0)$*,* $D_u f(0, 0)$ *exists for every vector* u*, yet* f *is not differentiable at* $(0, 0)$*.*

Q10

Proof. (Continuity).

Continuity of f follows easily from $\epsilon - \delta$ condition

$|f(x,y) - f(0,0)| = \left|\sqrt{x^2+y^2}\right|$ for $y \neq 0$ and $|f(x,y) - f(0,0)| = 0$ for $y = 0$

Thus, in general, we have that $|f(x,y) - f(0,0)| \leq \left|\sqrt{x^2+y^2}\right| = ||(x,y)||$

Let $\delta := \epsilon$ and we are done. ∎

Q10

Proof. (Directional Derivatives).

For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f(0 + tu_1, 0 + tu_2) - f(0,0)}{t} = \begin{cases} 0 & u_2 = 0 \\ \frac{u_2}{|u_2|} & u_2 \neq 0 \end{cases}$$

Hence, $(\mathrm{D}_{\mathrm{u}} f)(0,0)$ exists for all $\mathbf{u}$. Thus, all directional derivatives exist ■

Q10

For f to be differentiable, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h,0+k) - f(0,0) - f_x(0,0)h - f_y(0,0)k}{\sqrt{h^2+k^2}} = 0$$

For $(h,k) \neq (0,0)$, we have that

$$\frac{f(0+h,0+k) - f(0,0) - 0h - 1k}{\sqrt{h^2+k^2}} = \frac{k}{|k|} - \frac{k}{\sqrt{h^2+k^2}}$$

Consider the sequence $((h_n,k_n))$ given by $h_n = k_n = \frac{1}{n}$.
Obviously $(h_n,k_n) \to (0,0)$ but

$$\frac{k_n}{|k_n|} - \frac{k_n}{\sqrt{h_n^2+k_n^2}} = 1 - \frac{1}{\sqrt{2}}$$

Which clearly does not converge to 0.