MA109 Tutorial Session

Week 3

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What's New this Wednesday

1. Tutorial Sheet 2
   - Q8. (ii) : Fitting a Function
   - Q8. (iii) : Fitting a Function
   - Q10. (i) : Sketching a Function
   - Q11 : Another Curve Fitting

2. Tutorial Sheet 3
   - Q1. (ii) : Taylor Series for $\tan^{-1}$
   - Q2 : Taylor Series with an offset
   - Q4 : Convergence of e^x
   - Q5 : $\int \frac{e^x}{x}$

Q8. (ii)

Question

Find a function $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

1. $f''(x) > 0 \forall x \in \mathbb{R}$
2. $f'(0) = 1$ *and* $f'(1) = 2$

or otherwise show such a function cannot exist.

Q8. (ii)

A lot of functions satisfy the constraints.
You could fit a 2 degree polynomial with ease.
I will fit an exponential function (because, strictly convex).

Consider ae^{bx}, with the given constraints,

Verify that $\frac{2^x}{\ln 2}$ satisfies.

To be fair, you haven't defined ln or e yet, so try to fit a quadratic!

$x + \frac{x^2}{2}$ works!

Q8. (iii)

Question

Find a function $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

1. $f''(x) \geq 0 \forall x \in \mathbb{R}$
2. $f'(0) = 1$ *pause*
3. $f(x) \leq 100 \forall x > 0$

or otherwise show such a function cannot exist.

Q8. (iii)

Claim : Such a function does not exist.

Proof. (by Contradiction).

Assume such a function exists, let it be f.
$f(0) \leq 100$ (why?), so that $x_0 = 100 - f(0) \geq 0$
$f''(x) \geq 0$ on $\mathbb{R} \implies f'(x)$ is monotonically increasing on $\mathbb{R}$
$\forall x > 0\ f'(x) \geq f'(0) = 1 > 0 \implies f$ is strictly increasing on $(0, \infty)$
Pick a $y > x_0$, now $\frac{f(y)-f(0)}{y-0} = f'(x)$ for some $x \in (0, y)$
Note that $f(0) < f(y) \leq 100$ so $0 < f(y) - f(0) \leq 100 - f(0)$ also, $y > 100 - f(0)$
Thus, $f'(x) < 1$ which contradicts the fact that $\forall x > 0\ f'(x) \geq 1$ $\square$

Q10. (i)

Question

Sketch the function defined on $\mathbb{R}$ given by

$$y = f(x) = 2x^3 + 2x^2 - 2x - 1$$

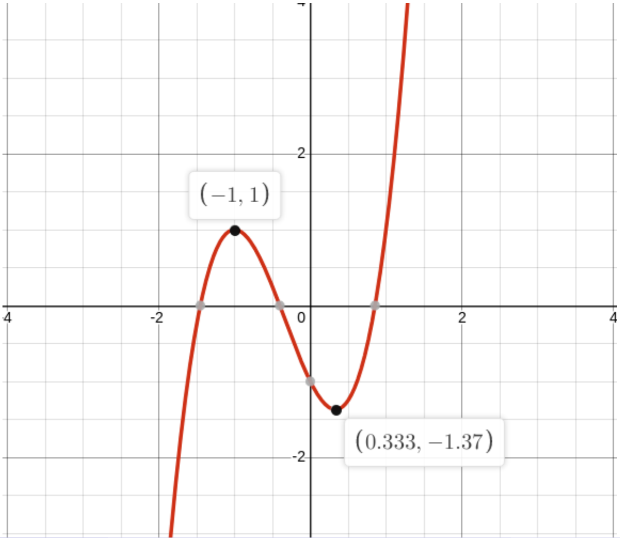
after identifying

1. *Intervals of increase/ decrease*
2. *Intervalse of concavity and convexity*
3. *Points of local maxima/minima*
4. *Points of inflection*
5. *Asymptotes*

Q10. (i)

The function is polynomial, hence smooth!
The function does not have global extrema. (why?)
Get a reference point : $f(0) = -1$ (in general, try to get a few more)
$f'(x) = 6x^2 + 4x - 2 = 2(3x - 1)(x + 1)$ is ≥ 0 on $(-\infty, -1] \cup [1/3, \infty)$ and ≤ 0 on $[-1, 1/3]$
These are intervals of increase and decrease
$\{-1, 1/3\}$ are points of (possible) local extrema
$f''(x) = 12x + 4$ which is < 0 (concave) on $(-\infty, -1/3)$ and > 0 (convex) on $(1/3, \infty)$
So, indeed -1 is a point of local maxima, $1/3$ a point of local minima
Also, $-1/3 = [-1 + 1/3]/2$ (show this for any cubic) a point of inflection.

Q10. (i)



Q11

Question

Sketch a continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying the following*

1. $f(-2) = 8, f(0) = 4, f(2) = 0$
2. $f'(x) > 0$ *for* $|x| > 2$ *and* $f'(x) < 0$ *for* $|x| < 2$
3. $f''(x) < 0$ *for* $x < 0$ *and* $f''(x) > 0$ *for* $x > 0$

Q11

Note : I didn't mention $f'(2) = f'(-2) = 0$ (why?)

We saw something similar just now. A cubic **might** have such properties!!

How to fit a cubic? Observe that now $f'(x) = c(x-2)(x+2)$, $c > 0$

Verify that this satisfies conditions (2) and (3)

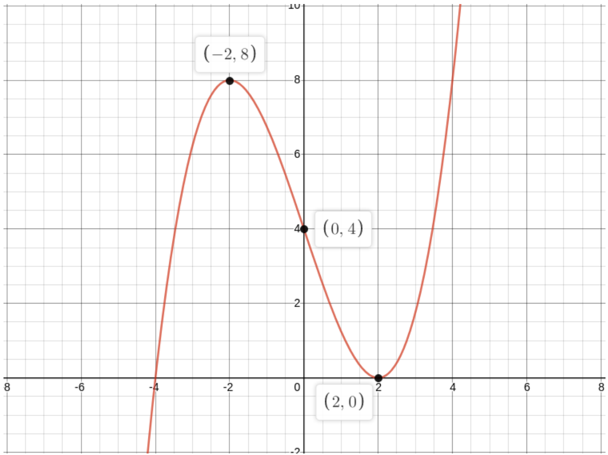
Now the cubic will be of form $\frac{cx^3}{3} - 4cx + d$

$d = 4$ and $c = 3/4$ satisfy all constraints.

Note

This is not always guarenteed to work. Say, I change $f(0) = 3$ instead. Can you still fit a cubic? Can you fit another function? Can you fit another polynomial?

Q11



Q1. (ii)

Question

Give the n^{th} taylor polynomial and remainder of $\tan^{-1}(x)$ *about* 0 *when* $|x| < 1$

Q1. (ii)

Claim : The complete taylor expansion is $\tan^{-1}(x) = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$

Sketch. (Hacky).

We find $f^{(n)}(x)$ and thus $f^{(n)}(0)$

Observe $f'(x) = f^{(1)}(x) = \frac{1}{1+x^2} = \frac{1}{x-i} - \frac{1}{x+i}$

Show by induction that $f^{(n)}(x) = \frac{(n-1)!(-1)^{n-1}}{2i}\left(\frac{1}{(x-i)^n} - \frac{1}{(x+i)^n}\right)$

$$f^{(n)}(0) = \begin{cases} 0 & n \text{ is even} \\ (n-1)!(-1)^{(n-1)/2} & n \text{ is odd} \end{cases}$$

□

Q1. (ii)

$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}$

To show convergence, you need to show $R_n(x) \to 0$ (try!)

A better method will be using integration like in Q5

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2} dt$$

Can you write a power series for $\frac{1}{1+x^2}$ with ease?

The Rigorous Method

Note that $\sum\limits_{n=0}^{\infty}(-t^2)^n = \frac{1}{1+t^2}$ for $|t| < 1$

This forms the power series expansion of $\frac{1}{1+t^2}$, so it can be integrated term by term

$$\tan^{-1} x = \int_0^x \frac{1}{1+t^2}dt = \int_0^x \sum_{n=0}^{\infty}(-t^2)^n = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

This is the Taylor series expansion of $\tan^{-1}(x)$[1]

What is the remainder $R_n(x)$? There is no closed form solution.

Note that $R_{2m-1}(x) = R_{2m}(x) = \sum\limits_{n=m}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$

You can write it as an integral!

[1] I remember there was a doubt regarding this in class. The point is, if a function has a power series expansion about 0, it can be differentiated term by term as well. Can you use this to show that $a_n = f^{(n)}(0)/n!$?

Q2

Question

Write Taylor series of $f(x) = x^3 - 3x^2 + 3x - 1$ *about 1.*

Taylor series about 1 will be

$$P_\infty(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$f(x) = (x-1)^3$ hence $f^{(n)}(1) = 0 \forall n \neq 3$
Verify that indeed $f(x) = P_\infty(x)$

Q4

Question

Show that the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$ *converges.*

Proof.

Pick an $N > 2|x|$, now for $n > N$, $\left|\frac{x^{n+1}}{(n+1)!}\right| \leq \frac{1}{2}\frac{|x|^n}{n!}$

This can be seen as $|x/(n+1)| \leq |x/N| < 1/2$, also, note $\frac{x^{n+1}}{(n+1)!} \leq \left|\frac{x^{n+1}}{(n+1)!}\right|$

Consider any $n > m > N$, then $\sum_{k=n}^{m} \frac{x^k}{k!} \leq \sum_{k=n}^{m} \frac{|x|^k}{k!} \leq \sum_{k=N+1}^{\infty} \frac{|x|^k}{k!}$

Using the inequality proved, show that $\sum_{k=N+1}^{\infty} \frac{|x|^k}{k!} \leq \frac{|x|^N}{N!}$ (form a GP)

Since $\lim_{n\to\infty} |x|^n/n! = 0$, we can find for any given $\epsilon > 0$, a $M \in \mathbb{N}$ such that $|x|^n/n! < \epsilon \forall n \geq M$

Chose $N_0 = max\{N, M\}$. Show that this satisfies the cauchy definition. □

Q5

Question

Write down a series for $\int \frac{e^x}{x} dx$

We write $\frac{e^x}{x} = \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}$

i.e. $\frac{e^x}{x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$, so that

$\int \frac{e^x}{x} = \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C$