MA109 Tutorial Session

Week 4

Dhruv Arora

Sophomore, Dept of CSE

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What's New this Wednesday

1. Tutorial Sheet 4
   - Q2.(a) 0 Reimann Integral and 0 function
   - Q2.(b) 0 Reimann Integral but non 0 function
   - A Useful Theorem
   - Q3. (ii) : Approximate Reimann Sums
   - Q3. (iv) : Approximate Reimann Sums
   - Another Useful Theorem
   - Q4. (b). (i) : Apply the Leibnitz Rule
   - Q4. (b). (ii) : Apply the Leibnitz Rule Again
   - Q6 : FTC I but Calculate (:

Q2. (a)

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Question

Let $f : [a, b] \to \mathbb{R}$ *be Reimann integrable.*

Q2. (a)

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1. *Given* $f(x) \geqslant 0$*, show that* $\int_a^b f(x)dx \geqslant 0$

Q2. (a)

Question

Let $f : [a, b] \to \mathbb{R}$ *be Reimann integrable.*

1. *Given* $f(x) \geqslant 0$, *show that* $\int_a^b f(x)dx \geqslant 0$
2. *Given that* f *is continuous and non negative, and that* $\int_a^b f(x)dx = 0$, *show that* $f(x) = 0 \ \forall x \in [a, b]$

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Note that $m_i := \inf_{x \in [x_{i-1}, x_i]} f(x) \geqslant 0$ since $f(x) \geqslant 0 \; \forall x \in [x_{i-1}, x_i]$

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So that, $L(P, f) = \sum\limits_{i=1}^{n} m_i(x_i - x_{i-1}) \geqslant 0$

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Also, f is Reimann integrable, hence Darboux Integrable

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Also, f is Reimann integrable, hence Darboux Integrable
$\int_a^b f(x)dx = L(f) \geqslant 0$ ■

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Claim : $\exists d \in (a, b)$ such that $f(d) > 0$.
If c were a or b, the $\epsilon - \delta$ definition would give such a d for $\epsilon = f(c)$ (show!)

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If c were a or b, the $\epsilon - \delta$ definition would give such a d for $\epsilon = f(c)$ (show!)
Now, for d, let $\epsilon = f(d)/2$, then $\exists \delta > 0$ such that $(d - \delta, d + \delta) \subset (a, b)$ and
$x \in (d - \delta, d + \delta) \implies |f(x) - f(d)| < \epsilon \implies f(x) - f(d) > -f(d)/2 \implies f(x) > f(d)/2$

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Consider the partition $P := \{a, d - \delta/2, d + \delta/2, b\}$

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Consider the partition $P := \{a, d-\delta/2, d+\delta/2, b\}$
Here, $m_1 \geqslant 0$, $m_2 \geqslant f(d)/2 > 0$ and $m_3 \geqslant 0$

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Consider the partition $P := \{a, d - \delta/2, d + \delta/2, b\}$
Here, $m_1 \geqslant 0$, $m_2 \geqslant f(d)/2 > 0$ and $m_3 \geqslant 0$
Hence, $L(P, f) = m_1(d - \delta - a) + m_2(2\delta) + m_3(b - d - \delta) \geqslant 2m_2\delta > 0$

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Hence, $L(P, f) = m_1(d - \delta - a) + m_2(2\delta) + m_3(b - d - \delta) \geqslant 2m_2\delta > 0$
Now, since $L(f) := \sup_P L(P, f)$, by definition, $L(f) \geqslant L(P, f) > 0$
Also, f is Reimann integrable, hence $\int_a^b f(x)dx = L(f) > 0$ which is a contradiction. ■

Q2. (b)

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Question

Give an example of a Reimann integrable function, $f : [a, b] \to \mathbb{R}$ *such that* $f(x) \geqslant 0 \ \forall x \in [a, b]$, $\int_a^b f(x)dx = 0$ *but* $f(x) \neq 0$ *for some* $x \in [a, b]$

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Consider the function $f : [0, 1] \to \mathbb{R}$ such that

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This can be shown as follows

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$L(P_\delta, f) = 0$ and $U(P_\delta, f) = \delta$

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$L(P_\delta, f) = 0$ and $U(P_\delta, f) = \delta$
By definition, $L(f) \geqslant L(P_\delta, f) \geqslant 0$

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$L(P_\delta, f) = 0$ and $U(P_\delta, f) = \delta$
By definition, $L(f) \geqslant L(P_\delta, f) \geqslant 0$
And $U(f) := \inf_P U(P, f) \leqslant \inf_{0<\delta<1} U(P_\delta, f) = \inf_{0<\delta<1} \delta = 0$

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We know that $L(f) \leqslant U(f)$ and we obtained that $L(f) \geqslant 0 \geqslant U(f)$

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And $U(f) := \inf_P U(P, f) \leqslant \inf_{0<\delta<1} U(P_\delta, f) = \inf_{0<\delta<1} \delta = 0$
We know that $L(f) \leqslant U(f)$ and we obtained that $L(f) \geqslant 0 \geqslant U(f)$
Hence, $L(f) = U(f) = 0$ and thus $\int_0^1 f(x)dx = 0$ ∎

Approximate Reimann Sums

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Theorem

Let $f : [a, b] \to \mathbb{R}$ be Reimann integrable. Let P_n be a sequence of partitions such that $\lim_{n\to\infty} ||P_n|| = 0$ (written more subtly as $||P_n|| \to 0$). Let $P_n = \{a = x_0 < x_1 < \dots x_m = b\}$ (ofcourse m depends on n), and $t_i \in [x_{i-1}, x_i]$. Then

$$\lim_{n\to\infty} \sum_{i=0}^{m} f(t_i)(x_i - x_{i-1}) = \int_a^b f(x)dx$$

Indeed, the P_n and t_n combined are what are called **tagged partitions**. *And the theorem is another way of saying that if $||P_n|| \to 0$, $R(f, P_n, t_n) \to \int_a^b f$*

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We prove that if f is Reimann integrable and $||P_n|| \to 0$, then $R(f, P_n, t_n) \to \int_a^b f(x)dx$

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To show our claim, that $R(f, P_n, t_n) \to \int_a^b f(x)dx$, proceed by definition.

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To show our claim, that $R(f, P_n, t_n) \to \int_a^b f(x)dx$, proceed by definition.
Note that by definition of Reimann integrability, we have $\forall \epsilon > 0 \; \exists \delta > 0$ such that for any tagged partition (P, t), $||P|| < \delta \implies |R(f, P, t) - \int_a^b f(x)dx| < \epsilon$

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Now, given $\epsilon > 0$, we obtain such a $\delta > 0$

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Now, given $\epsilon > 0$, we obtain such a $\delta > 0$

Since $||P_n|| \to 0$, $\exists N \in \mathbb{N}$ such that $n \geqslant N \implies 0 < ||P_n|| < \delta$

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Since $||P_n|| \to 0$, $\exists N \in \mathbb{N}$ such that $n \geqslant N \implies 0 < ||P_n|| < \delta$

Thus, by definition, $n \geqslant N \implies ||P_n|| < \delta \implies |R(f, P_n, t_n) - \int_a^b f(x)dx| < \epsilon$

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We prove that if f is Reimann integrable and $||P_n|| \to 0$, then $R(f, P_n, t_n) \to \int_a^b f(x)dx$
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Note that by definition of Reimann integrability, we have $\forall \epsilon > 0\ \exists \delta > 0$ such that for any tagged partition (P, t), $||P|| < \delta \implies |R(f, P, t) - \int_a^b f(x)dx| < \epsilon$
Now, given $\epsilon > 0$, we obtain such a $\delta > 0$
Since $||P_n|| \to 0$, $\exists N \in \mathbb{N}$ such that $n \geqslant N \implies 0 < ||P_n|| < \delta$
Thus, by definition, $n \geqslant N \implies ||P_n|| < \delta \implies |R(f, P_n, t_n) - \int_a^b f(x)dx| < \epsilon$
Which is exactly what we wanted to show! ■

Q3. (ii)

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Question

Find the $\lim_{n\to\infty} S_n$ *where*

$$S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2}$$

by formulating it as the limit of an appropriate Reimann sum

Q3. (ii)

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$$S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2} = \sum_{i=1}^{n}\frac{1}{1+(i/n)^2}\left(\frac{i}{n}-\frac{i-1}{n}\right)$$

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Which is precisely $R(f, P_n, t_n)$ where

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Which is precisely $R(f, P_n, t_n)$ where

$$P_n = \{0 = x_0 < x_1 < \cdots < x_n = 1\} : x_i = \frac{i}{n}; t_i = \frac{i}{n}$$

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$$f : [0,1] \to \mathbb{R}, f(x) = \frac{1}{1+x^2}$$

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$$f : [0,1] \to \mathbb{R}, f(x) = \frac{1}{1+x^2}$$

Note that $||P_n|| = \frac{1}{n} \to 0$ and f is Reimann integrable

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$$S_n = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)^2} = \sum_{i=1}^{n}\frac{1}{1+(i/n)^2}\left(\frac{i}{n} - \frac{i-1}{n}\right)$$

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$$f : [0,1] \to \mathbb{R}, f(x) = \frac{1}{1+x^2}$$

Note that $||P_n|| = \frac{1}{n} \to 0$ and f is Reimann integrable
Thus, by the theorem,

$$\lim_{n\to\infty} S_n = \int_0^1 \frac{1}{1+x^2}dx = \arctan(1) - \arctan(0) = \frac{\pi}{4}$$

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Where the last part implicitly uses the Fundamental Theorem of Calculus II

Q3. (iv)

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Question

Find the $\lim_{n\to\infty} S_n$ *where*

$$S_n = \frac{1}{n}\sum_{i=1}^{n} \cos\left(\frac{i\pi}{n}\right)$$

by formulating it as the limit of an appropriate Reimann sum

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$$S_n = \sum_{i=1}^{n} \cos\left[\pi\left(\frac{i}{n}\right)\right]\left(\frac{i}{n} - \frac{i-1}{n}\right)$$

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Q3. (iv)

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Where the last part implicitly uses the Fundamental Theorem of Calculus II

Leibnitz Integral Rule

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Theorem (Leibnitz Integral Rule)

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $a, b : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Define :

$$F(x) = \int_{a(x)}^{b(x)} f(t)dt$$

Then F is differentiable and $F'(x) = f(b(x))b'(x) - f(a(x))a'(x)$

Leibnitz Integral Rule

Proof.

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∎

Q4. (b). (i)

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Question

Define

$$F(x) = \int_1^{2x} \cos(t^2)dt$$

Show that F *is differentiable and obtain* $\frac{dF}{dx}$

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The hypothesis of Leibnitz Integral Rule are satisfied, hence

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Q6

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Show that $g''(x) + \lambda^2 g(x) = f(x)$ *and* $g(0) = 0 = g'(0)$

Q6

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Also, $\cos(\lambda x)$ and $\sin(\lambda x)$ are differentiable, so that chain rule can be applied
The rest is simple calculation. I leave that to you (: