Задача 18. Пусть $\sigma(z) = \frac{1}{1+e^{-z}}$ - сигмоида

Проверить: $\sigma' = \sigma(1-\sigma)$

$$\sigma'(z) = \frac{-(-e^{-z})}{(1+e^{-z})^2} = \frac{e^{-z}}{(1+e^{-z})^2}$$

$$\sigma(z)\cdot(1-\sigma(z)) = \frac{1}{1+e^{-z}}\cdot\left(1-\frac{1}{1+e^{-z}}\right) = \frac{1}{1+e^{-z}}\cdot\left(\frac{\cancel{1}+e^{-z}-\cancel{1}}{1+e^{-z}}\right) =$$

$$= \frac{1}{1+e^{-z}}\cdot\frac{e^{-z}}{1+e^{-z}} = \frac{e^{-z}}{(1+e^{-z})^2}$$

Отсюда $\sigma'(z) = \sigma(z)\cdot(1-\sigma(z))$

Задача 19 Задача классификации K классов $\{1,2..K\}$

Последний слой нейросети вычисляет softmax функцию.

$$g_k(s_1,s_2\dots s_k) = \frac{e^{s_k}}{\sum\limits_{l=1}^{k} e^{s_l}}$$ - softmax-функция

В качестве штрафа logloss-функция:

$$R^{(i)} = -\sum_{k=1}^{K} I(y^{(i)}=k)\cdot \ln g_k(s_1,s_2\dots s_k)$$

1) Док-ть: $\frac{\partial g_k}{\partial s_l} = g_k\cdot(I(k=l)-g_l)$

$$\frac{\partial g_k}{\partial s_l} = \frac{\partial}{\partial s_l}\left(\frac{e^{s_k}}{\sum\limits_{j=1}^{k} e^{s_j}}\right) = e^{s_k}\cdot\frac{-e^{s_l}}{\left(\sum\limits_{j=1}^{k} e^{s_j}\right)^2} = \frac{e^{s_k}}{\sum e^{s_j}}\cdot\frac{-e^{s_l}}{\sum e^{s_j}} =$$

$$= -g_k \cdot g_l = g_k\left(I(k=l) - g_l\right) \leftarrow \text{индикатор } = 0 \ (k \neq l)$$

$$k=l: \ \frac{\partial g_k}{\partial s_k} = \frac{\partial}{\partial s_k}\left(\frac{e^{s_k}}{\sum\limits_{j=1}^{K} e^{s_j}}\right) = \frac{e^{s_k}}{\sum\limits_{j=1}^{K} e^{s_j}} + \frac{e^{s_k} \cdot (-e^{s_k})}{\sum\limits_{j=1}^{K} e^{s_j}} =$$

$$= g_k - g_k^2 = g_k(1-g_k) = g_k\left(I(k=l) - g_l\right) \leftarrow \text{индикатор } = 1$$

② Док-ть: $\dfrac{\partial R^{(i)}}{\partial g_k} = -\dfrac{I(y^{(i)}=k)}{g_k}$

$$\frac{\partial R^{(i)}}{\partial g_k} = \frac{\partial}{\partial g_k}\left(-\sum_{k=1}^{K} I(y^{(i)}=k) \cdot \ln g_k(s_1, s_2 \dots s_K)\right) =$$

$$= -\sum_{j=1}^{K} I(y^{(i)}=j) \cdot \frac{\partial}{\partial g_k}\left(\ln g_j(s_1, s_2 \dots s_K)\right) =$$

$$= \underbrace{I(y^{(i)}=1) \cdot \frac{\partial}{\partial g_k} \cdot \ln g_1}_{=0} - \underbrace{I(y^{(i)}=2) \cdot \frac{\partial}{\partial g_k} \ln g_2}_{=0} - \dots - I(y^{(i)}=k) \cdot \frac{\partial}{\partial g_k} \ln g_k - \dots$$

$$- \underbrace{I(y^{(i)}=K) \cdot \frac{\partial}{\partial g_k} \ln g_K}_{=0} = -I(y^{(i)}=k) \cdot \frac{\partial}{\partial g_k} \ln g_k =$$

$$= -I(y^{(i)}=k) \cdot \frac{1}{g_k}$$

③ Док-ть: $\dfrac{\partial R^{(i)}}{\partial s_l} = g_l - I(l = y^{(i)})$

$$\frac{\partial R^{(i)}}{\partial s_l} = \sum_{j=1}^{K} \frac{\partial R^{(i)}}{\partial g_j} \cdot \frac{\partial g_j}{\partial s_l} = \sum_{j=1}^{K}\left(-\frac{I(y^{(i)}=j)}{g_j}\right) \cdot g_j\left(I(j=l) - g_l\right) =$$

$$= \sum_{j=1}^{K} I(y^{(i)}=j)\cdot(g_l - I(j=l)) = g_l \sum_{j=1}^{K} I(y^{(i)}=j) -$$

$$- \sum_{j=1}^{K} I(y^{(i)}=j)\cdot I(j=l) = g_l - I(y^{(i)}=l)$$