Certamen N°2 INF-510 - Take Home
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Directions: **You have to show all the work to earn full credit**. Partial credit is available for progress toward the solution. Final answers without explanation will received 0 points. Good luck!

1. **[25 points]** In class we discussed the use of the Chebyshev differentiation matrix $\widetilde{D}_N^2$ when building the Laplacian operator for a square domain $[-1,1]^2$ with 0 boundary conditions. Our first attempt for the discretization gave us the matrix equation $\widetilde{D}_N^2\,U + U\,(\widetilde{D}_N^2)^T = f(X,Y)$ for $U, \widetilde{D}_N^2, f(X,Y) \in \mathbb{R}^{N-1,N-1}$. After some algebraic manipulation and the use of the function vec$(\cdot)$ we were able to translate the linear system of equation into a classic for $A\,\vec{x} = \vec{b}$, where $A = L_N = I \otimes \widetilde{D}_N^2 + \widetilde{D}_N^2 \otimes I$, $\vec{x} = \mathrm{vec}(U)$ and $\vec{b} = \mathrm{vec}(f(X,Y))$. Although, it is an elegant solution, we don't really need to build A and $\vec{b}$ to solve the linear system of equation.

   (a) Build a matrix-free algorithm based on GMRes that obtains U without requiring to build A, i.e. you must only use $\widetilde{D}_N^2\,U + U\,(\widetilde{D}_N^2)^T = f(X,Y)$.

   (b) Build a matrix-free algorithm based on GMRes that obtains an approximation of the first 10 eigenvalues and eigenfunctions of A, i.e. you must find an estimation of the eigenvalues of $\widetilde{D}_N^2\,U + U\,(\widetilde{D}_N^2)^T$. Compare with the approximations obtains by using L_N obtained in class. *Do you recall the Ritz values?*.

2. **[25 points]** [**From Spectral Methods in MatLab, by Lloyd N. Trefethen**] Modify Program 20 to make use of *Chebyshev* matrices instead of the FFT. Make sure to do this **elegantly**, using matrix-matrix multiplications rather than explicit loops. How much faster is it? Show the same outputs and compare time for different N. For instance, $N = 24, 48, 96$ and 192.

3. **[25 points]** [**From Spectral Methods in MatLab, by Lloyd N. Trefethen**] The KdV equation (10.3) is closely related to the *Burgers equation*, $u_t + (u^2)_x = \varepsilon\,u_{xx}$, where $\varepsilon > 0$ is a constant [Whi74]. Modify Program 27 to solve this equation for $\varepsilon = 0.25$ by a Fourier spectral method on $[-\pi,\pi]$ with an integrating factor. Take $u(x,0)$ equal to $\sin^2(x)$ in $[-\pi,0]$ and to 0 in $]0,\pi]$, and produce plots at times $0, 0.5, 1, \ldots, 3$, with a sufficiently small time step, for $N = 64, 128$ and 256. For $N = 256$, how small a value of ε can you take without obtaining unphysical oscillations?

4. **[25 points]** Consider the differential-integral operator:

$$K\,u = -u''(x) + 4\,\pi^2 \int_0^1 u(s)\,ds \qquad u(0) = u(1) = 0.$$

   (a) Build a numerical approximation of the operator K using Chebyshev points. *Hint: It may be helpful if you look at the Clenshaw-Curtis quadrature from chapter 12 in the textbook, specifically to the code* `clancurt.m`.

   (b) Find an accurate approximation of the first 5 eigenvalues and eigenfunctions of K. Plot them.

   (c) Solve numerically the following equation $K\,u = \exp(x)\,\sin(x)$ with $u(0) = u(1) = 0$, if it can't be done, explain why.