Certamen N°1 INF-510
Claudio Torres - Ma.27.09.16

Nombre:_____ Rol:_____

Directions: *You have 90 minutes to finish this test.* ***You have to show all the work to earn full credit****. Partial credit is available for progress toward the solution. Final answers without explanaition will received 0 points. Good luck!*

1. **[20 points]** Let $A \in \mathbb{C}^{n\times n}$ be a positive definite square matrix with eigenvalues $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_n$ and you also know that the matrix is normal, i.e. $\kappa_2(V) = \|V\|_2\|V^{-1}\|_2 = 1$. A well know results for the convergence of GMRes says that the norm of the relative residual of a linear system of equations at the n-th iteration for a positive definite matrix is bounded by $\mathcal{C}\,\kappa_2(V)\,\rho^n$, where $\rho = \dfrac{\sqrt{\kappa_2(A)}-1}{\sqrt{\kappa_2(A)}+1}$.

   (a) Compute the condition number for the matrix A.

   (b) Consider you want to solve the following linear system of equations $(A^2 + 2\,A + I)\mathbf{x} = \mathbf{b}$. Compute the condition number for this linear system of equations.

   (c) Which of the two previously computed condition numbers is smaller? Justify clearly.

   (d) Write a pseudo-code based of GMRes that solve the linear system proposed in **(b)**. Your code must receive as parameter A, b and the number of iterations for GMRes. Consider the following constraints: you have used almost all the memory so you can't store another matrix in memory and you need to return the approximation after 50 iterations at most.

   (e) Do you need a preconditioner for the linear system of equations in **(b)**? If so, propose one. Otherwise, justify clearly.

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2. **[20 points]** Consider the initial value problem: $u'' - u = \varepsilon t u$, $t > 0$, $u(0) = 1$, and $u'(0) = -1$. Find the 2-term perturbation approximation for $0 < \varepsilon \ll 1$ and a 8-term Taylor series approximation (about $t = 0$). Hint: Explain each step clearly, and please don't show just final steps. You are allowed to use an ODE solver but first you must state what ODE you are solving.
[Bonus] Implemente un método numérico y resuelva el problema hasta un tiempo *razonable*. Entregue el jupyter notebook respectivo.

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3. **[20 points]** Una masa m cuelga de un resorte y tiene una velocidad inicial V desde la posición de equilibrio. El desplazamiento $x = x(t)$ está gobernado por:

$$m\,x'' = -a\,x\,|x'| - k\,x \tag{1}$$
$$x(0) = 0 \tag{2}$$
$$x'(0) = V \tag{3}$$

donde $-a\,x\,|x'|$ es un termino de amortiguamiento no lineal y $-k\,x$ es la fuerza de restauración lineal del resorte. Si la fuerza de restauración es menor que la fuerza de amortiguamiento, determine las escalas de tiempo y espaciales apropiadas para no-dimensionalizar el problema.
[Bonus] Implemente un método numérico y resuelva el problema hasta un tiempo *razonable*. Entregue el jupyter notebook respectivo.

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4. **[20 points]** Find the rescaling for the roots of

$$\varepsilon^2 x^3 + x^2 + 2x + \varepsilon = 0$$

and the first two terms in the approximation for each root. Consider $0 < \varepsilon \ll 1$.

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5. **[20 points]** Use a singular perturbation method to obtain an uniform approximation solution for the following problem:

$$\begin{aligned} \varepsilon\, y'' + x^{1/3}\, y' + y &= 0 \\ y(0) &= 0 \\ y(1) &= \exp(-3/2) \end{aligned}$$

consider $0 < \varepsilon \ll 1$ and $0 < x < 1$.