Spring 2016
Torres

INF-510
Homework 1
August 9th, 2016

<u>**Deadline**</u>**: August 23rd of 2016**

Questions:

{i} [**Due: August 16th**] What is a: ...?

(a) Singular Matrix
(b) Vandermonde Matrix
(c) Symmetric Matrix
(d) Hermitian Matrix
(e) Skew-Hermitian Matrix
(f) Unitary Matrix
(g) Jacobian Matrix
(h) Projection Matrix
(i) Companion Matrix
(j) Jacobi Matrix
(k) Defective Matrix
(l) Toeplitz Matrix
(m) Circulant Matrix
(n) Hankel Matrix
(o) Hilbert Matrix
(p) Markov Matrix
(q) Differentiation Matrix (Matrices)
(r) Spectral Differentiation Matrix (Matrices)
(s) Chebyshev Differentiation Matrix (Matrices)

Describe all the properties you can find. For instance, what do you know about their eigenvalues? Are they invertible? If I have to solve a linear system of equation or an eigenvalue problem, does it help to know it belong to certain type of matrices?

{ii} [**Due: August 16th**] Let H_n be the $n \times n$ Hilbert matrix whose ij-th entry is defined as $1/(i+j-1)$, also let $\mathbf{1}_n$ be the vector of ones of dimension n. *Discuss the following questions*

(a) Find, as accurate as possible, the approximate solution $\tilde{\mathbf{x}}$ of the linear system $A\mathbf{x} = \mathbf{b}$, where $A = H_n$ and $\mathbf{b} = H_n \mathbf{1}_n$ for $n = 3 \dots 20$. Notice that we know a priori that the exact solution is just $\mathbf{x} = \mathbf{1}_n$, but (un)fortunately the computer can only give you $\tilde{\mathbf{x}}$.
(b) What is the relation between $\mathbf{x}$ and $\tilde{\mathbf{x}}$?
(c) What can we do now?

{iii} [**Due: August 16th**] *Solving a very (un)known problem*

(a) Implement a function that find the two roots of the quadratic equation $a\,x^2 + b\,x + c = 0$ given a, b y c, i.e. implement $x_{\pm} = \dfrac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$.

(b) What are the roots of $2x^2 + 10^9\,x + 1 = 0$? How many digits of significance can you get for the two roots? Is there any problem?

(c) Design a code that finds the corrects roots for $x^2 + B\,x + C = 0$, given $B \gg C$ to at least 2 digits of significance.

(d) Solve the previous equation using this new code and find the new roots. *I hope it works!*

(e) From the well-known solution $x_{\pm}$ of quadratic equation design an algorithm that approximates the two roots of $x^2 + B\,x + C = 0$, given $B \gg C$. *Hint: A Taylor expansion may work.*

{iv} **[Due: August 16th]** *A fix-point-iteration review. See* Numerical Analysis, 2nd edition by Timothy Sauer

(a) Definition 1: The real number r is a fix-point of the function $g(x)$ if $g(r) = r$.

(b) Algorithm 1: Fixed-Point-Iteration: Let x_0 be the initial guess. Compute $x_{i+1} = g(x_i)$, for $i = 0, 1, 2, 3, \ldots$. Notice that this fixed-point-iteration may or may not converge to r.

(c) Definition 2: Let $e = |r - x_i|$ be the error at iteration i. If $0 < \lim_{i\to\infty} \frac{e_{i+1}}{e_i} = S < 1$, the fixed-point-iteration $x_{i+1} = g(x_i)$ is said to obey linear convergence with rate S.

(d) Theorem 1: Assume that $g(x)$ is continuously differentiable and that $S = |g'(r)| < 1$. Then the fixed-point-iteration $x_{i+1} = g(x_i)$ converges at least linearly with rate S to the fixed point r for the initial guesses sufficiently close to r.

(e) Question 1: Prove that a continuously differentiable function $g(x)$ satisfying $|g'(x)| < 1$ on a closed interval cannot have two fixed points on that interval.

(f) Question 2: Given that $f(x)$ has a root near x_0. Derive three different fix-point-iterations that may converge to $f(r) = 0$ and state the restrictions of $f(x)$ needed, if any. Assume that $f(x)$ has as many derivatives as you may need.

(g) Question 3: Derive a unsuccessful and a successful fix-point-iteration for finding the root of $x^3 + x = 1$ near $x_0 = 1$.

(h) Algorithm 2: Newton's method: Let x_0 be the initial guess. Compute $x_{i+1} = \widehat{g}(x_i)$, for $i = 0, 1, 2, 3, \ldots$, where $\widehat{g}(x) = x - (f'(x))^{-1}\, f(x)$ and $(f'(x))^{-1}$ is the inverse of $f(x)$, i.e. $(f'(x))^{-1} = 1/f'(x)$.

(i) Definition 3: Let $e = |r - x_i|$ be the error at iteration i. If $\lim_{i\to\infty} \frac{e_{i+1}}{e_i^2} = M < \infty$, the method is said to be quadratically convergent.

(j) Question 4: Prove that Newton's method is quadratically convergent as long as $f'(r) \neq 0$.

(k) Question 5: Explain when Newton's method show linear convergence and also explain how it can be fixed.

{v} *A simple ODE*

(a) Design a numerical method to approximate the following Boundary Value Problem: $\varepsilon y''(x) + (1+\varepsilon) y'(x) + y(x) = 0$ for $0 < x < 1$, and $y(0) = 0$, $y(1) = 1$.

(b) Plot the numerical approximation against it analytical solution for $\varepsilon = 1, 10^{-3}, 10^{-8}$ and 10^{-14}.

(c) Is your numerical approximation valid for all x?

(d) Is your approximation accurate?

(e) What does it happen as $\varepsilon \to 0$?

{vi} Let $X \in \mathbb{R}^{m\times n}$ with $m \gg n$. Its reduced singular value decomposition is $U\,\Sigma\,V^*$ ($X = U\,\Sigma\,V^*$). Compute the reduced singular value decomposition of the following matrix $(I - \mathbf{v}_1\,\mathbf{v}_1^*)U\,\Sigma\,V^*$, where $\mathbf{v}_1$ is the first column of V and I is the identity matrix. In summary, you need to find $\widetilde{U}\,\widetilde{\Sigma}\,\widetilde{V}^*$ such that its product give you $(I - \mathbf{v}_1\,\mathbf{v}_1^*)U\,\Sigma\,V^*$. *Hint: This new SVD depends on the original SVD!*

{vii} Let $X \in \mathbb{R}^{m\times n}$ with $m \gg n$, prove that maximum of the following maximization problem:

$$\begin{aligned} \underset{\mathbf{w}\in\mathbb{R}^n}{\text{maximize}} \quad & \|X\,\mathbf{w}\|_2^2 \\ \text{subject to} \quad & \|\mathbf{w}\|_2 = 1. \end{aligned}$$

is obtained for $\mathbf{w} = \mathbf{v}_1$, where $\mathbf{v}_1$ is the first column of V matrix from the singular value decomposition of X, i.e. $X = U\,\Sigma\,V^*$.

{viii} Repeat the same procedure from the previous question but for the following maximization problem:

$$\begin{aligned} \underset{\mathbf{w}\in\mathbb{R}^n}{\text{maximize}} \quad & \|(I - \mathbf{v}_1\,\mathbf{v}_1^*)X\,\mathbf{w}\|_2^2 \\ \text{subject to} \quad & \|\mathbf{w}\|_2 = 1. \end{aligned}$$

where $\mathbf{v}_1$ is the first column of V and I is the identity matrix.

{ix} **[Due: August 16th]** Let

$$H(x) = \begin{cases} 1 & \text{if } 0.5 \le x \le 0.5 \\ 0 & \text{otherwise} \end{cases}.$$

Compute $F(x) = \displaystyle\int_{-\infty}^{\infty} \widetilde{H}_a(y)\,\widetilde{H}_b(x-y)\,dy$, where $\widetilde{H}_a(x) = \dfrac{1}{a}\,H\left(\dfrac{x}{a}\right)$.

{x} Use the Fourier transform to solve the following PDE:

$$\begin{aligned} u_t - k\,u_{xx} = 0,\ \ x \in \mathbb{R} \text{ and } t > 0 \\ u(x,0) = f(x),\ \ x \in \mathbb{R}\ . \end{aligned}$$

Assume f and $u \in L^2(\mathbb{R})$.

{xi} Let $\ddot{y}(t) - \mu\,(1 - y^2(t))\,\dot{y}(t) + y(t) = 0$, with $y(0) = 2$, $\dot{y}(0) = 0$ and $\mu = 1234$.

(a) Approximate the solution by means of a Taylor series expansion about $t = 0$.

(b) Implemente a numerical solver for it. *You mat not use scipy but you may use numpy.*

(c) Compare both solutions and comment on the comparison.

{xii} **[Due: August 16th]** Let γ by a positively oriented circular path with center at 0 and radious $a > 2$, compute the following:

- $\displaystyle\int_\gamma \frac{\exp(z)}{z}\,dz = 2\pi\,\mathrm{i}$
- $\displaystyle\int_\gamma \frac{\exp(z)}{z\,(z-1)}\,dz = 2\pi\,\mathrm{i}\,e$
- $\displaystyle\int_\gamma \frac{\exp(z)}{z^3}\,dz = \pi\,\mathrm{i}$

{xiii} **[Due: August 16th]** Show that the following Laurent expnasion is valid in $1 < |z| < 2$:

$$\frac{1}{(z-1)(z-2)} = \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n},$$

and draw and sketch of the region. Does it exist a expansion when $|z| > 2$? If so, please compute it.

{xiv} Considering the following inner product:

$$\langle p(x), q(x)\rangle = \int_{-1}^{1} \overline{p(x)}\,q(x)\,dx$$

(a) Let $A = [1|x|x^2|\dots|x^{n-1}]$ be the "matrix" whose "columns" are the monomials x^j, for $j = 0,\dots,n-1$. *We agreed in class that you have to do it for* $n = 5$ *manually and for* $n = 1000$ *computationally.* Each column is a function in $L^2[-1,1]$. Compute the QR decomposition of A

(b) Let $A = [1|\sin(2\,\pi\,x)|\sin(4\,\pi\,x)|\dots|\sin(2\,\pi\,(n-1)\,x)]$ be the "matrix" whose "columns" are the functions 1 and $\sin(2\,\pi\,j\,x)$, for $j = 1,\dots,n-1$. Each column is a function in $L^2[-1,1]$. Compute the QR decomposition of A

(c) Do part (a) numerically. Make sure you understand what you are doing since this is a important concept that links symbolic computing with numerical computing.

{xv} Let $f(x) = \sum_{i=1}^{n} \alpha_j \operatorname{sinc}(x - x_i)$, where $\operatorname{sinc}(x) = \dfrac{\sin(x)}{x}$. Compute the total number of operations need for evaluation $f(x)$ at x_j, for $j = 1\dots m$. Also implemente this algorithm and validate your estimation.

{xvi} **[Due: August 16th]** Let $\mathbf{x}_i = \langle x_i, y_i\rangle$, for $i = 1 : n$, a set of points that describe a simple polygon. Derive an algorithm that computes the area enclosed by it exactly.

Instructions:

(a) The homework may be done in Jupyter Notebooks. Any other language must be discussed with the instructor.

(b) The theoretical part of the homework must be written in LaTeX and the computational part in Jupyter Notebook.

(c) The structure must be the following

Only once	Title, name, email and rol.
For each question	A small description of the problem and assumptions.
For each question	Discussion of the solution (include numerical experiments here). *Please be brief but clear.*
For each question	Conclusions.
For each question	References.

(d) The final work is personal but I do encourage you to discuss partial results with your classmates.

(e) Any exception, must be discuss with the instructor in advance.

(f) If you don't follow these instructions, you will get a 0.