Euler-Lagrange Equation 2D

$J(u) = \int (u(x,y) - f_0(x,y))^2 + a\,\Phi(u(x,y) - f_0(x,y)) + b\,\Psi(|\nabla u(x,y)|^2)\,dx$

```
Clear[x1, x2, a, b];
Lag[x1_, x2_, x3_] = (x1 - f[x, y]) ^2 + a Φ[x1 - f[x, y]] + b Ψ[x2^2 + x3^2];
L[x_, y_] =
 Lag[x1, x2, x3] /. x1 → u[x, y] /. x2 → ∂x u[x, y] /. x3 -> ∂y u[x, y] // FullSimplify
Lu[x_, y_] = ∂x1 Lag[x1, x2, x3] /. x1 → u[x, y] /. x2 → ∂x u[x, y] /. x3 -> ∂y u[x, y] //
  FullSimplify
Lux[x_, y_] = ∂x2 Lag[x1, x2, x3] /. x1 → u[x, y] /. x2 → ∂x u[x, y] /. x3 -> ∂y u[x, y] //
  FullSimplify
Luy[x_, y_] = ∂x3 Lag[x1, x2, x3] /. x1 → u[x, y] /. x2 → ∂x u[x, y] /. x3 -> ∂y u[x, y] //
  FullSimplify
```

$(f[x, y] - u[x, y])^2 + a\,\Phi[-f[x, y] + u[x, y]] + b\,\Psi\left[u^{(0,1)}[x, y]^2 + u^{(1,0)}[x, y]^2\right]$

$-2\,f[x, y] + 2\,u[x, y] + a\,\Phi'[-f[x, y] + u[x, y]]$

$2\,b\,\Psi'\left[u^{(0,1)}[x, y]^2 + u^{(1,0)}[x, y]^2\right] u^{(1,0)}[x, y]$

$2\,b\,\Psi'\left[u^{(0,1)}[x, y]^2 + u^{(1,0)}[x, y]^2\right] u^{(0,1)}[x, y]$

```
EL[x_] = Lu[x, y] - D[Lux[x, y], x] - D[Luy[x, y], y] // FullSimplify
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$$-2\,f[x, y] + 2\,u[x, y] + a\,\Phi'[-f[x, y] + u[x, y]] - 2\,b\,\Psi'\left[u^{(0,1)}[x, y]^2 + u^{(1,0)}[x, y]^2\right] \left(u^{(0,2)}[x, y] + u^{(2,0)}[x, y]\right) - 4\,b\,\Psi''\left[u^{(0,1)}[x, y]^2 + u^{(1,0)}[x, y]^2\right] \left(u^{(0,1)}[x, y]^2\,u^{(0,2)}[x, y] + 2\,u^{(0,1)}[x, y]\,u^{(1,0)}[x, y]\,u^{(1,1)}[x, y] + u^{(1,0)}[x, y]^2\,u^{(2,0)}[x, y]\right)$$

```
ϕ[x_, y_] = Exp[-(x^2 + y^2) / (2 σ^2)]
D[ϕ[x, y], x] // FullSimplify
D[ϕ[x, y], y] // FullSimplify
D[ϕ[x, y], {x, 2}] // FullSimplify
D[ϕ[x, y], {y, 2}] // FullSimplify
D[D[ϕ[x, y], x], y] // FullSimplify
D[D[ϕ[x, y], y], x] // FullSimplify
```

$$e^{\frac{-x^2-y^2}{2\sigma^2}}$$

$$-\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}\,x}{\sigma^2}$$

$$-\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}\,y}{\sigma^2}$$

$$\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}\,(x-\sigma)\,(x+\sigma)}{\sigma^4}$$

$$\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}\,(y-\sigma)\,(y+\sigma)}{\sigma^4}$$

$$\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}\,x\,y}{\sigma^4}$$

$$\frac{e^{-\frac{x^2+y^2}{2\sigma^2}}\,x\,y}{\sigma^4}$$