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Variational Methods in Biomedical Imaging
Part I: Modelling

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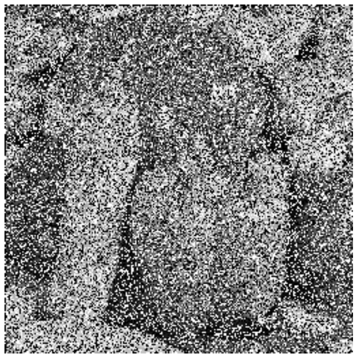
Restoration

Gaussian noise | impulse noise

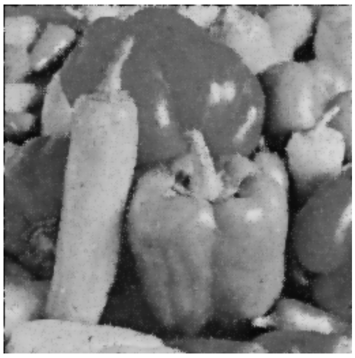
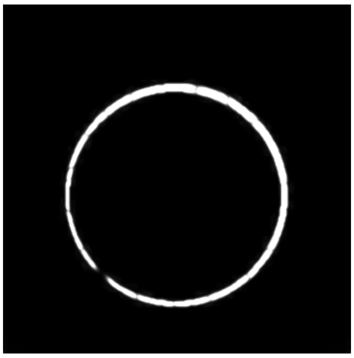
type I



type II

missing / incomplete data

Restoration

	Gaussian noise	impulse noise
type I		
type II		

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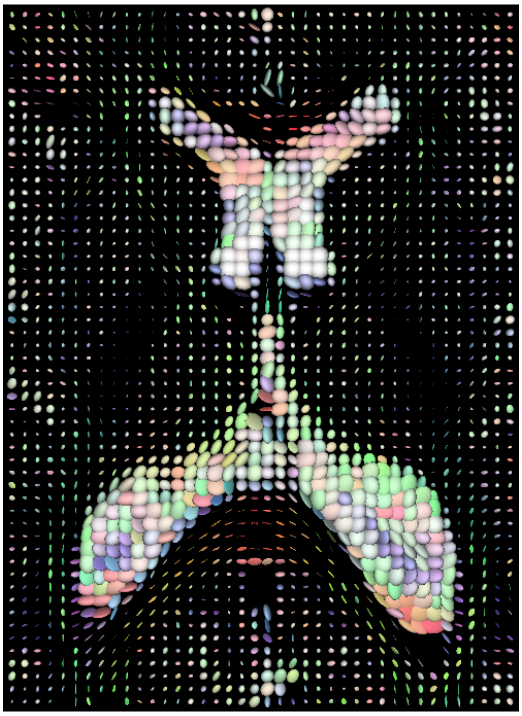
Motivation

Restoration

DT-MRI



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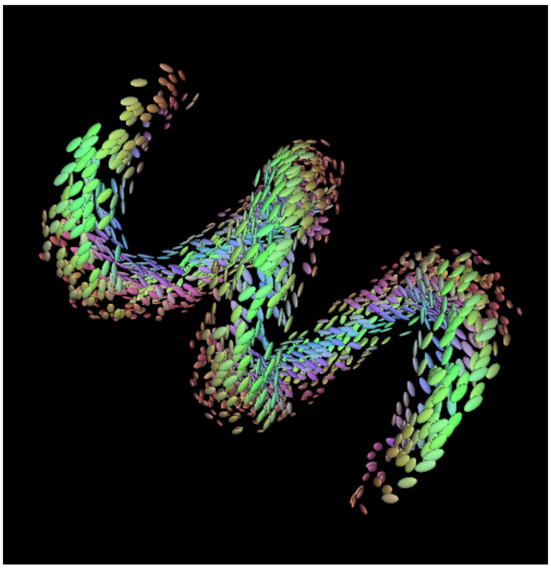

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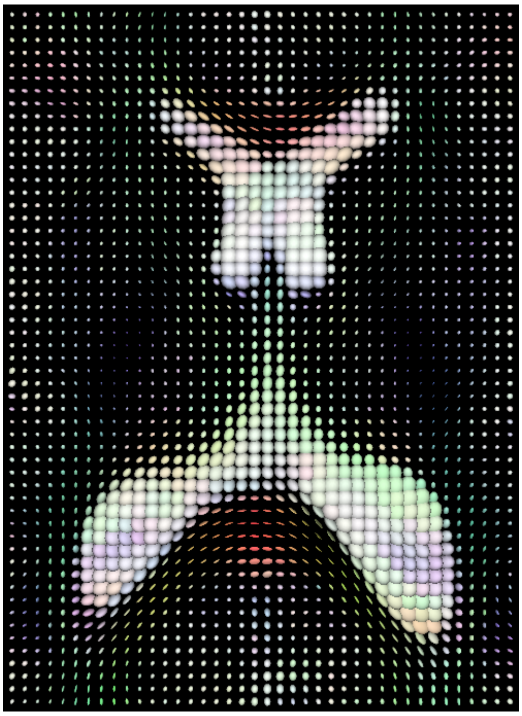
Restoration

DT-MRI



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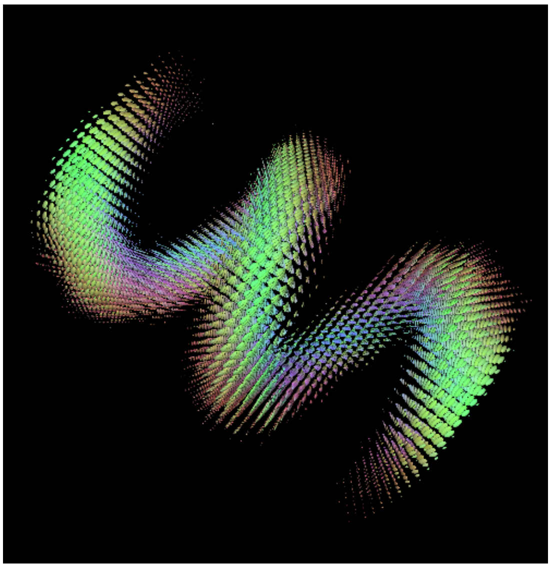

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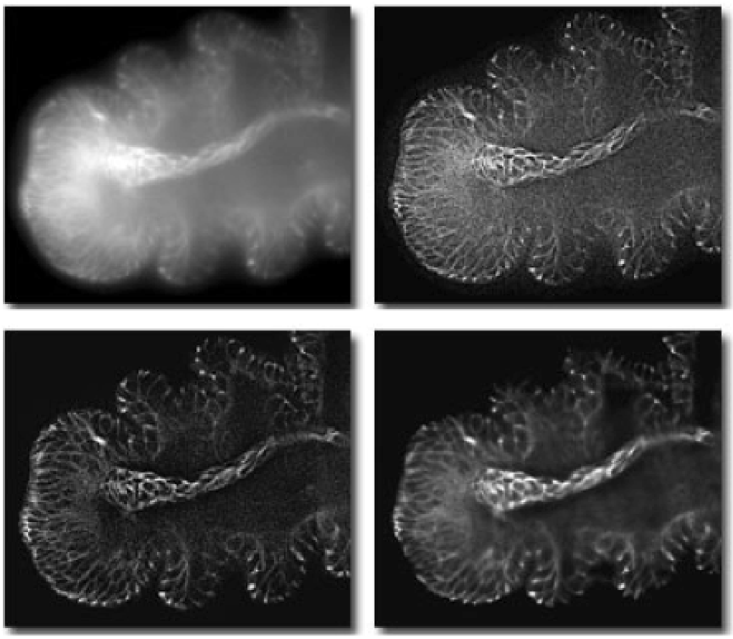
Deblurring

Raw data

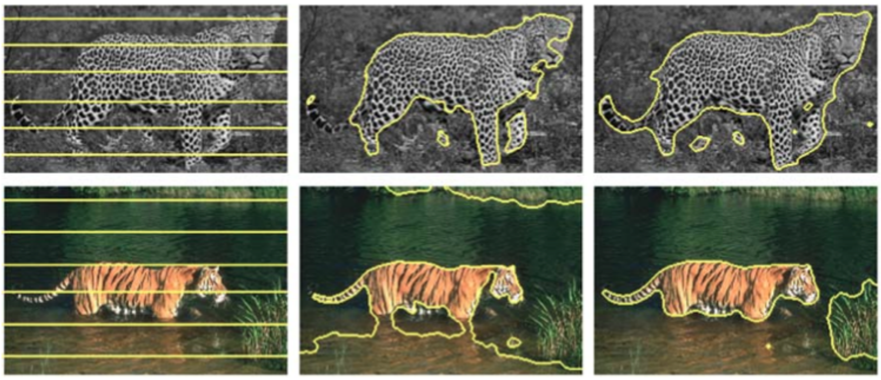
Nearest neighbours

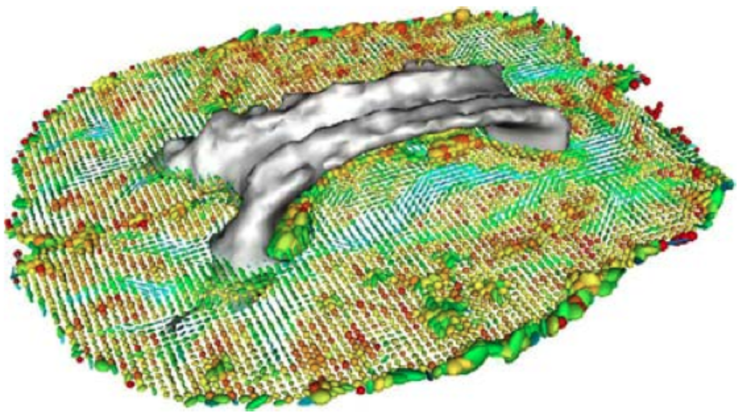


Inverse Wiener filter

Blind deconvolution

Segmentation





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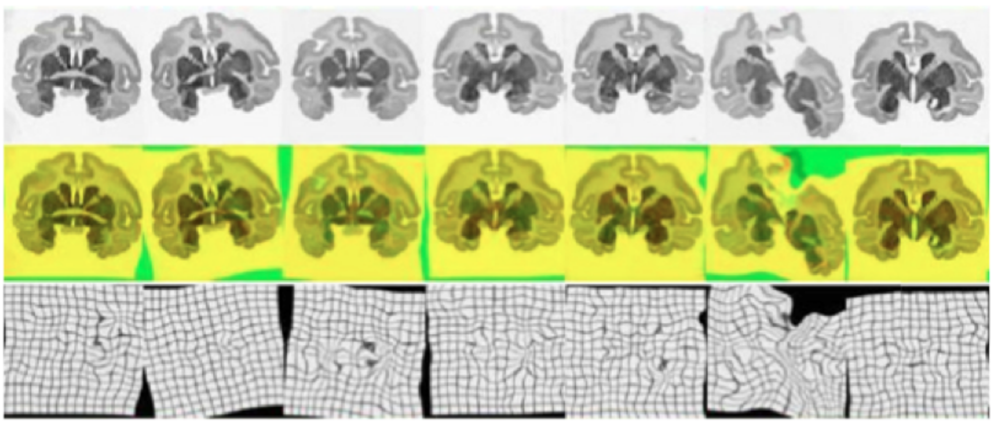
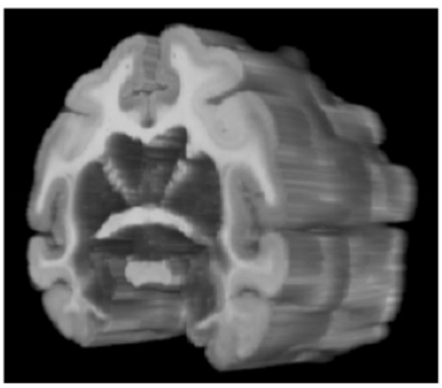
Registration





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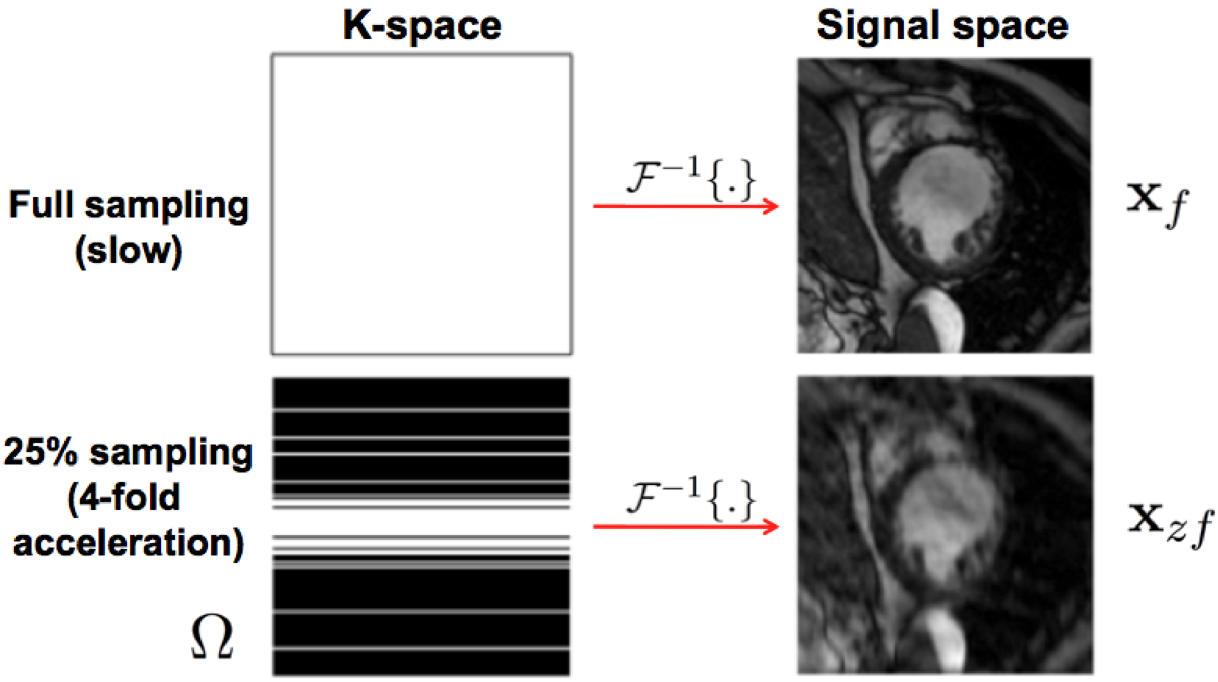
Reconstruction



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Outline

- Preliminaries
- Calculus of Variations
- Euler-Lagrange Equations
- Examples
- Summary

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

Preliminaries

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

Energy-Based Methods

- Best performing approaches to numerous problems in image processing, computer vision, and biomedical imaging.
- Allow transparent modelling of problem-specific constraints and assumptions.
- Straightforward minimisation/maximisation procedure.

Variational Approach

$$
\begin{aligned}
f \quad &: \quad \Omega \subset \mathbb{R}^d \longrightarrow \mathbb{R} \qquad \text{initial image} \\
\hat{u} \quad &= \quad \arg\min \quad E(u) := \text{Data}(u, f) + \alpha \, \text{Smoothness}(u) \\
& \quad\;\; \text{regularisation parameter } \alpha > 0
\end{aligned}
$$

Partial Differential Equation (PDE) Approach

$$
\begin{aligned}
\partial_t u \quad &= \quad \mathbb{D}(u, \nabla u) \qquad \text{in } \Omega \times [0, T[\\
u(\cdot, 0) \quad &= \quad f(\cdot) \qquad \text{in } \Omega
\end{aligned}
$$

Preliminaries

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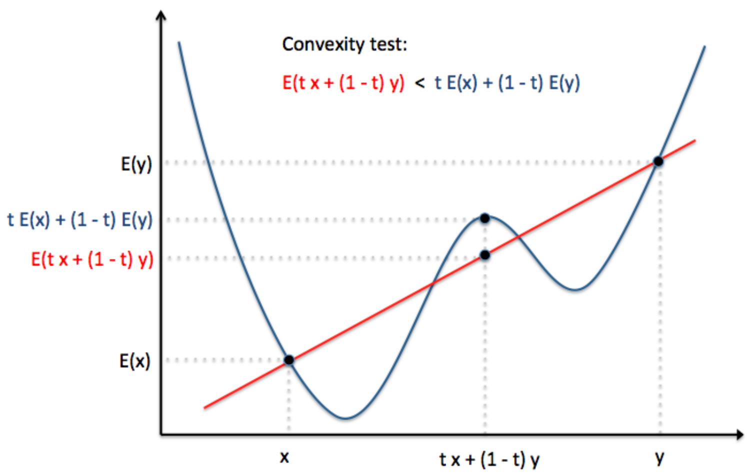
Convex Problem

- A global, optimal solution (minimum/maximum) can be found
- Independence from initialisation

Non-Convex Problem

- More realistic models can be devised
- No global solution, dependence on initialisation



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Continuous Methods

- Images are considered as continuous functions
- Differential operators easy to handle
- Optimality conditions found via calculus of variations

Discrete Methods

- Images considered as graphs with labelled nodes
- Differential operators handled in their discretised matriz form
- Optimality conditions found via standard calculus

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Variational Restoration

Bertero et al. 1988, Nordström 1990, Schnörr 1994, Charbonnier et al. 1994

$$E(u) := \int_\Omega \Big(\underbrace{(u-f)^2}_{\text{Data}} + \alpha \underbrace{\Psi\left(|\nabla u|^2\right)}_{\text{Smoothness}} \Big)\, dx$$

- The data term rewards similarity to the original image.
- The smoothness term penalises deviations from (piecewise) smoothness.
- The penalising function Ψ is differentiable and increasing: $\Psi'(s^2) > 0$.

Whittaker 1923, Tikhonov 1963:

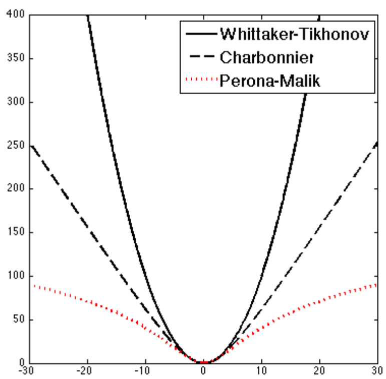
$$\Psi(s^2) = s^2$$

Charbonnier et al. 1994:

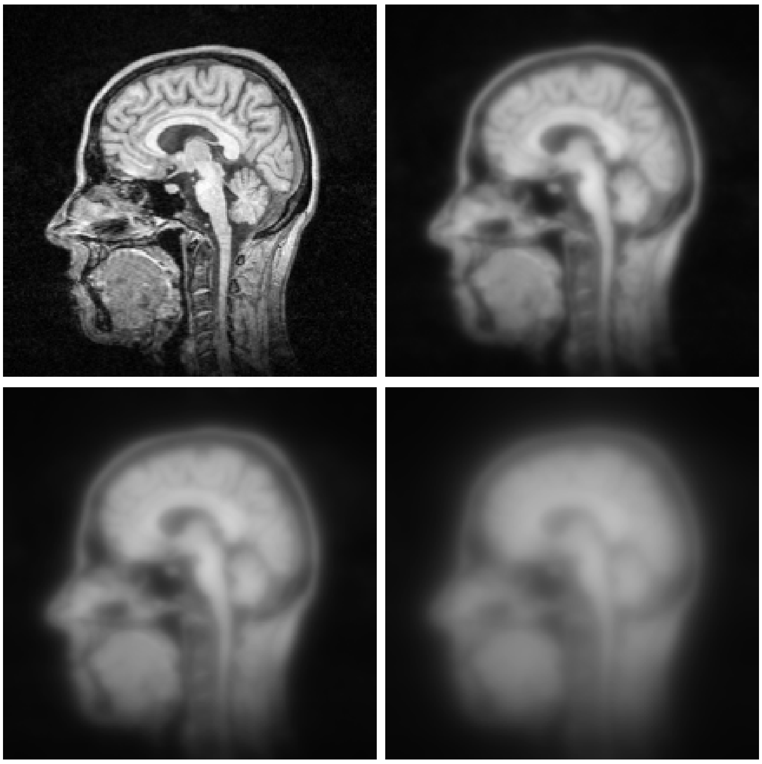
$$\Psi(s^2) = 2\lambda^2\sqrt{1 + s^2/\lambda^2} - 2\lambda^2$$

Perona and Malik 1990:

$$\Psi(s^2) = 2\lambda^2 \log\left(1 + s^2/\lambda^2\right)$$



Preliminaries



Variational restoration with the Whittaker-Tikhonov penaliser. **Top left to bottom right:** Regularisation parameters $\alpha = 0, 5, 20, 100$.

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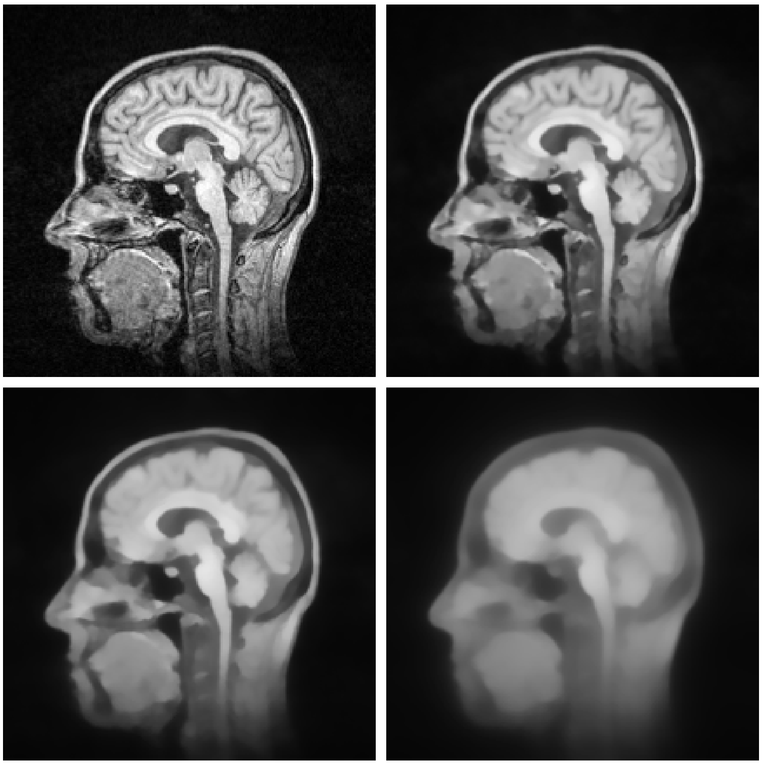
Preliminaries



Variational restoration with the Charbonnier penaliser ($\lambda = 2$). **Top left to bottom right:** Regularisation parameters $\alpha = 0, 10, 30, 100$.

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Standard Calculus:

- Considers real-valued *functions* $f(x)$, i.e. mappings from a *number* x into $\mathbb{R}$.
- If f has a minimum in ξ, then it necessarily holds $f'(\xi) = 0$.
- If f is strictly convex and $f'(\xi) = 0$, then ξ is the unique minimum of f.

Calculus of Variations:

- Considers real-valued *functionals* $E(u)$, i.e. mappings from a *function* $u(x)$ into $\mathbb{R}$.
- If E is minimised by a function v, then v has to satisfy necessarily a so-called Euler-Lagrange equation. This is a partial differential equation in v.
- If the E is strictly convex and satisfies the Euler-Lagrange equation, then v is the unique minimum of E.

Calculus of Variations in 1-D

Let $x \in [a, b]$ and assume that $v(x)$ is a sufficiently differentiable minimiser of E. We embed $v(x)$ into the family:

$$u(x, \varepsilon) := v(x) + \varepsilon h(x)$$

with some perturbation function $h(x)$.
Since $v(x)$ minimises $E(u)$, we know that the scalar-valued function

$$g(\varepsilon) := E(u(x, \varepsilon)) = E(v + \varepsilon h)$$

has a minimum in $\varepsilon = 0$. Therefore, we have

$$0 = g'(0) = \frac{d}{d\varepsilon} E(v + \varepsilon h)\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_a^b F(x, \underbrace{v + \varepsilon h}_{u(\cdot,\varepsilon)}, \underbrace{v' + \varepsilon h'}_{u'(\cdot,\varepsilon)})\, dx\Big|_{\varepsilon=0}$$

This gives the Euler-Lagrange equation

$$\boxed{F_u - \frac{d}{dx} F_{u'} \;=\; 0}$$

Note that this equation does not depend on the perturbation h.

This gives

$$F_{u'}(x, v, v')h(x)\Big|_{x=a}^{x=b} \;=\; 0\,,$$

which holds for arbitrary perturbations h (also with $h(a) \neq 0$ and $h(b) \neq 0$).

Thus, one obtains the natural boundary conditions

$$\boxed{F_{u'} \;=\; 0}$$

at the image boundaries $x = a$ and $x = b$.

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3-D energy functional with 1^{st}-order regularisation

Let $\Omega \subseteq \mathbb{R}^3, x = (x_1, x_1, x_3)^\top$, $\nabla u = (u_{x_1}, u_{x_2}, u_{x_3})^\top$.

The minimiser, $u : \Omega \to \mathbb{R}$, of the energy functional

$$E(u) := \int_\Omega F(x, u, \nabla u)\; dx$$

satisfies necessarily the Euler-Lagrange equation

$$F_u - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} F_{u_{x_i}} = 0 \qquad \Leftrightarrow \qquad F_u - \operatorname{div} \begin{pmatrix} F_{u_{x_1}} \\ F_{u_{x_2}} \\ F_{u_{x_3}} \end{pmatrix} = 0$$

with $F_u := \frac{\partial F}{\partial u}$, $u_{x_i} := \frac{\partial u}{\partial x_i}$, and the natural boundary conditions

$$\eta^\top \begin{pmatrix} F_{u_{x_1}} \\ F_{u_{x_2}} \\ F_{u_{x_3}} \end{pmatrix} = 0$$

at the image boundary $\partial\Omega$ with normal vector η.

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1-D energy functional with p^{th}-order regularisation

Let $\Omega := [a, b]$, $a, b \in \mathbb{R}$, and $p \in \mathbb{N}$. The minimiser of

$$E(u) := \int_a^b F(x, u, u^{(1)}, u^{(2)}, \ldots, u^{(p)})\, dx$$

satisfies necessarily the Euler-Lagrange equation

$$F_u - \frac{\partial}{\partial x} F_{u^{(1)}} + \frac{\partial}{\partial x^2} F_{u^{(2)}} - \cdots + (-1)^p \frac{\partial}{\partial x^p} F_{u^{(p)}} = 0 \qquad \left(\sum_{k=0}^{p} (-1)^k \frac{\partial}{\partial x^k} F_{u^{(k)}} = 0 \right)$$

with the natural boundary conditions

$$\sum_{k=j}^{p} \left(-\frac{\partial}{\partial x} \right)^{k-j} F_{u^{(k)}} = 0$$

for all $j = \{1, \ldots, p\}$ at the boundary $x \in \{a, b\}$.

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m-channel, n-D energy functional with p^{th}-order regularisation

Let $u \in \mathbb{R}^m$, $x \in \Omega \subseteq \mathbb{R}^n$, $\beta \in \{x_1, \ldots, x_n\}^p$, $\mathcal{D}^\beta u := \partial_{\beta_p} \ldots \partial_{\beta_1} u$.
The minimiser, $u^i : \Omega \to \mathbb{R}$, $i \in \{1, \ldots, m\}$, of the energy functional

$$E(u) := \int_\Omega F(x, u, \mathcal{D}u, \ldots, \mathcal{D}^p u)\, dx$$

satisfies necessarily the Euler-Lagrange equation

$$\sum_{|\beta| \le p} (-1)^{|\beta|}\, \mathcal{D}^{\tilde{\beta}} F_{\mathcal{D}^\beta u^i} = 0 \quad \text{for all} \;\; i \in \{1, \ldots, m\}$$

with the natural boundary conditions

$$\sum_{\substack{k \le |\beta| \le p \\ (\beta_1, \ldots, \beta_{k-1}) = \gamma}} (-1)^{|\beta| - k} \left(\partial_{\beta_{k+1}} \ldots \partial_{\beta_{|\beta|}} F_{\mathcal{D}^\beta u^i} \right) \eta_{\beta_k} \;=\; 0$$

for all $i \in \{1, \ldots, m\}$, $k \in \{1, \ldots, p\}$, and $\gamma \in \{x_1, \ldots, x_n\}^{k-1}$
at the image boundary $\partial\Omega$ with normal vector η.

Example 1

2-D Variational Restoration: 1^{st}-order regularisation

$$E(u) := \int_\Omega \underbrace{\left((u-f)^2 \; + \; \alpha \, \Psi\left(|\nabla u|^2\right)\right)}_{F(x,u,\nabla u)} \, dx$$

In general, we do admit *non-convex* penalisers Ψ.

The minimiser $u : \Omega \to \mathbb{R}$ satisfies necessarily

$$\begin{aligned}
0 &= F_u - \partial_{x_1} F_{u_{x_1}} - \partial_{x_2} F_{u_{x_2}} \\
0 &= 2(u-f) - \partial_{x_1}\left(2\alpha \, \Psi'(|\nabla u|^2) u_{x_1}\right) - \partial_{x_2}\left(2\alpha \, \Psi'(|\nabla u|^2) u_{x_2}\right) \\
0 &= (u-f) - \alpha \, \mathrm{div}\left(\Psi'(|\nabla u|^2)\nabla u\right)
\end{aligned}$$

with the Neumann boundary conditions

$$0 = \eta^\top \begin{pmatrix} F_{u_{x_1}} \\ F_{u_{x_2}} \end{pmatrix} = \eta^\top \left(\underbrace{2\alpha \, \Psi'(|\nabla u|^2)}_{>0} \nabla u \right) = \eta^\top \nabla u = \partial_\eta u$$

Example 1

2-D Variational Restoration: 1^{st}-order regularisation

The Euler-Lagrange equation corresponding to the elliptic PDE

$$0 \quad = \quad \underbrace{(u-f)}_{\text{reaction term}} \quad - \quad \underbrace{\alpha \, \mathrm{div}\left(\Psi'(|\nabla u|^2)\nabla u\right)}_{\text{diffusion term}} \quad =: \; \nabla_u E$$

can be seen as the steady state ($t \to \infty$) of the PDE $\dfrac{\partial u}{\partial t} = -\mu \, \nabla_u E$, with $\mu > 0$

$$\frac{\partial u}{\partial t} \quad = \quad \mu \left(\alpha \, \mathrm{div}\left(\Psi'(|\nabla u|^2)\nabla u\right) - (u-f) \right)$$

which is of parabolic type.

In the absence of the reaction term, one obtains a pure *nonlinear diffusion* PDE (Perona-Malik type)

$$\begin{aligned}
\partial_t u &= \mathrm{div}\left(\Psi'(|\nabla u|^2)\nabla u\right) && \text{in } \Omega \times [0, T[\\
u(\cdot, 0) &= f(\cdot) && \text{in } \Omega
\end{aligned}$$

Example 2

2-D Variational Restoration: combined 1^{st}- and 2^{nd}-order regularisation
Didas 2004

$$E(u) := \int_\Omega \underbrace{\left((u-f)^2 \;+\; \alpha\,\Psi_1\left(|\nabla u|^2\right) \;+\; \beta\,\Psi_2\left((\triangle u)^2\right)\right)}_{F(x,u,\nabla u,\triangle u)}\,dx$$

The minimiser $u : \Omega \to \mathbb{R}$ satisfies necessarily

$$\begin{aligned} 0 &= F_u - \partial_{x_1} F_{u_{x_1}} - \partial_{x_2} F_{u_{x_2}} \\ &\qquad + \partial_{x_1 x_1} F_{u_{x_1 x_1}} + \partial_{x_1 x_2} F_{u_{x_2 x_1}} + \partial_{x_2 x_1} F_{u_{x_1 x_2}} + \partial_{x_2 x_2} F_{u_{x_2 x_2}} \\ 0 &= 2(u-f) - \partial_{x_1}\left(2\alpha\,\Psi_1'(|\nabla u|^2) u_{x_1}\right) - \partial_{x_2}\left(2\alpha\,\Psi_1'(|\nabla u|^2) u_{x_2}\right) \\ &\qquad + \partial_{x_1 x_1}\left(2\beta\,\Psi_2'((\triangle u)^2)\triangle u\right) + \partial_{x_2 x_2}\left(2\beta\,\Psi_2'((\triangle u)^2)\triangle u\right) \\ 0 &= (u-f) - \alpha\,\mathrm{div}\,\left(\Psi_1'(|\nabla u|^2)\nabla u\right) + \beta\,\triangle\left(\Psi_2'((\triangle u)^2)\triangle u\right) \end{aligned}$$

Example 2

2-D Variational Restoration: combined 1^{st}- and 2^{nd}-order regularisation
Didas 2004

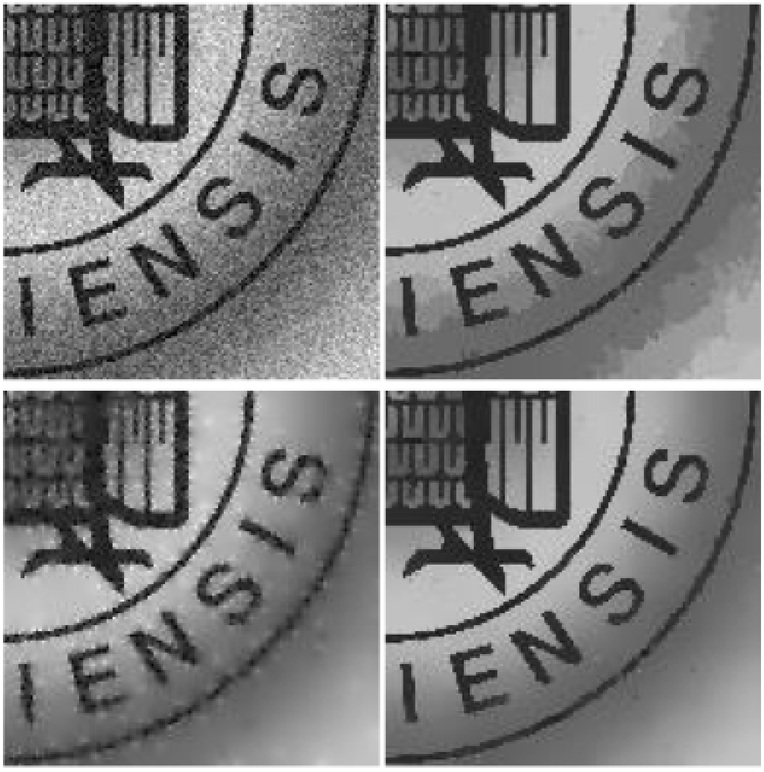
With the corresponding parabolic PDE

$$\frac{\partial u}{\partial t} = (f-u) + \alpha\,\mathrm{div}\,\left(\Psi_1'(|\nabla u|^2)\nabla u\right) - \beta\,\triangle\left(\Psi_2'((\triangle u)^2)\triangle u\right)$$

and the natural boundary conditions

$$\begin{aligned} 0 &= \eta^\top \begin{pmatrix} F_{u_{x_1}} - \partial_{x_1} F_{u_{x_1 x_1}} - \partial_{x_2} F_{u_{x_1 x_2}} \\ F_{u_{x_2}} - \partial_{x_1} F_{u_{x_2 x_1}} - \partial_{x_2} F_{u_{x_2 x_2}} \end{pmatrix} \\ 0 &= \eta^\top \begin{pmatrix} F_{u_{x_1 x_1}} \\ F_{u_{x_1 x_2}} \end{pmatrix} \\ 0 &= \eta^\top \begin{pmatrix} F_{u_{x_2 x_1}} \\ F_{u_{x_2 x_2}} \end{pmatrix} \end{aligned}$$

Example 2



Variational restoration with higher-order regularisation. **Top left:** Noisy input image, **Top right:** 1^{st}-order, **Bottom left:** 2^{nd}-order, **Bottom right:** combined 1^{st}- and 2^{nd}-order. Source: Didas 2004.

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Summary

Summary

- Variational methods consist of two energy terms, a data similarity and a smoothness constraint.
- The energy minimiser must satisfy the Euler-Lagrange equation.
- Partial differential equations of either elliptic or parabolic type need to be solved.
- This formalism allows an arbitrary combination of different similarity terms and smoothness terms.
- Discretisation issues remain open.

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

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