Assignment due date: Feb 10, 2017, 11:59pm

Hand-in to be submitted electronically in PDF format with code to the CDF server by the above due date

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I hereby affirm that all the solutions I provide, both in writing and in code, for this assignment are my own. I have properly cited and noted any reference material I used to arrive at my solution and have not share my work with anyone else. I am also aware that should my code be copied from somewhere else, whether found online, from a previous or current student and submitted as my own, it will be reported to the department.

Nan Li

Signature

(Note: -3 marks penalty for not completing properly the above section)

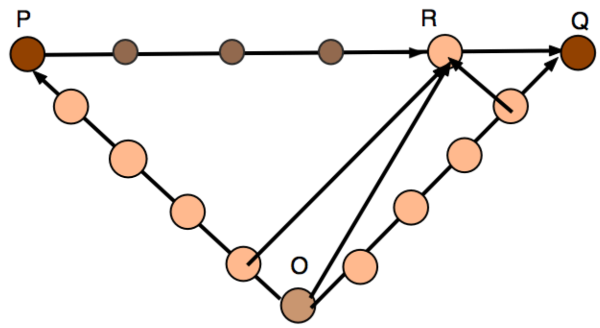
Part 1 total marks: 50
Part 2 total marks: 50
Total: 100

1)

Since R is the point on the line segment that is four times as far from P as it is from Q, this indicates that $\overrightarrow{PR} = \frac{4}{5}\overrightarrow{PQ}$.

Then:

$\vec{r} = \overrightarrow{OR} = \overrightarrow{OP} + \overrightarrow{PR} = \overrightarrow{OP} + \frac{4}{5}\overrightarrow{PQ} = \frac{1}{5}\overrightarrow{OP} + \frac{4}{5}\left(\overrightarrow{OP} + \overrightarrow{PQ}\right) = \frac{1}{5}\overrightarrow{OP} + \frac{4}{5}\overrightarrow{OQ} = \frac{1}{5}\vec{p} + \frac{4}{5}\vec{q}$



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Let P be (a,b), Q be (c,d).

The function of the line segment of PQ is:

$$y = \frac{d-b}{c-a}(x-a) + b$$

$$\Rightarrow \ (c-a)y = (d-b)(x-a) + b(c-a)$$

$$(c-a)(y-b) = (d-b)(x-a)$$

$$\frac{y-b}{d-b} = \frac{x-a}{c-a} = t$$

$$\Rightarrow \ \begin{cases} y-b = t(d-b) \\ x-a = t(c-a) \end{cases} \ \Rightarrow \ \begin{cases} y = t(d-b)+b \\ x = t(c-a)+\ a \end{cases}$$

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The function of the line segment of PQ is:

$$y = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) + y_1 \qquad ①$$

For $R(x,y)$:

If $x \notin [x_1, x_2]$ or $y \notin [y_1, y_2]$, then R is not on the line segment

If $x \in [x_1, x_2]$ or $y \in [y_1, y_2]$, and $R(x,y)$ satisfy equation ①, then R is on the line segment.

2)

Let $P(x_1, y_1), Q(x_2, y_2)$

$$\vec{n} * \overrightarrow{PQ} = 0$$

$\Rightarrow$ the normals are
$(y_2 - y_1, -x_2 + x_1)$ and $(y_1 - y_2, x_2 - x_1)$

Since it must be on the left wrt the direction of the line segment,
Therefore, the desired normal is

$$\frac{y_1 - y_2}{\|\overrightarrow{PQ}\|}, \frac{x_2 - x_1}{\|\overrightarrow{PQ}\|} \quad if\ x_2 - x_1 \geq 0$$

else the normal is

$$\left(\frac{y_2 - y_1}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}, \frac{-x_2 + x_1}{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}\right)$$

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First, to find if the projection of $\overrightarrow{PR}$ onto $\overrightarrow{PQ}$ is on the line segment:
Let O' on the line of P,Q.

$$\overrightarrow{PO'} = \frac{\overrightarrow{PR} \cdot \overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} \frac{\overrightarrow{PQ}}{\|\overrightarrow{PQ}\|} = \left(\frac{(x_2 - x_1)^2 (x - x_1)}{\|\overrightarrow{PQ}\|^2}, \frac{(y_2 - y_1)^2 (y - y_1)}{\|\overrightarrow{PQ}\|^2}\right)$$

From 1) part 3, we can check if O' is on the line segment, if not, the distance between an arbitrary point $R(x, y)$ and the line segment will be:

$$d = \min\big(d(P,R), d(Q,R)\big) = \min\left(\sqrt{(x_1 - x)^2 + (y_1 - y)^2}, \sqrt{(x_2 - x)^2 + (y_2 - y)^2}\right)$$

If O' is on the line segment, then the distance will be:

$$d = \frac{|(y_2 - y_1)x - (x_2 - x_1)y - y_1 x_1 + y_1 x_2|}{\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}$$
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B.

1). Rotation $- R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Translation $- T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix}$

$$RT = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ tx & ty. & 1 \end{bmatrix}$$

$$TR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ tx & ty & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ tx\cos\theta - ty\sin\theta & tx\sin\theta + ty\cos\theta & 1 \end{bmatrix}$$

$\therefore RT \neq TR$. these 2 tranformations do not commute.

2). First Rotation $- R_1 = \begin{bmatrix} \cos\theta_1 & \sin\theta_1 & 0 \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Second Rotation $- R_2 = \begin{bmatrix} \cos\theta_2 & \sin\theta_2 & 0 \\ -\sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$R_1R_2 = \begin{bmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 & 0 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R_2R_1$$

$\therefore$ $R_1R_2 = R_2R_1$, These 2 tranformations commute.

3). Translation -- $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$

Reflection - $R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$TR = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ -t_x & t_y & 1 \end{bmatrix}$$

$$RT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ t_x & t_y & 1 \end{bmatrix}$$

$TR \neq RT$.

$\therefore$ These 2 transformations do not commute.

4). Shear wrt to x-axis - $S = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

uniform scaling - $U = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$SU = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S & aS & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$US = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S & Sa & 0 \\ 0 & S & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$US = SU$.

$\therefore$ These 2 transformations commute.

5). Rotation - R = $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Non-uniform scaling = N = $\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$RN = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x\cos\theta & -\sin\theta S_y & 0 \\ S_x\sin\theta & \cos\theta S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$NR = \begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_x\cos\theta & -S_x\sin\theta & 0 \\ S_y\sin\theta & S_y\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$RN \neq NR$

∴ They do not comute.

C

1).

The steps can be seen as:

① non-uniform scaling with $S_x = \frac{2}{3}$, $S_y = \frac{1}{2}$.
② reflection w.r.t y-axis
③ rotate 45° clockwise
④ translation with $T_x = 5$, $t_y = 1.5$.

Then the sequence is $U \cdot Re \cdot R \cdot T$.

2).

Let the affine transformation be:

$$f(\vec{a}) = A\vec{a} + \vec{b}$$

$$\begin{aligned} f(T(\alpha,\beta)) &= f(\vec{p_1} + \alpha\vec{d_1} + \beta\vec{d_2}) \\ &= A(\vec{p_1} + \alpha\vec{d_1} + \beta\vec{d_2}) + \vec{b} \\ &= (A\vec{p_1} + \vec{b}) + \alpha A\vec{d_1} + \beta A\vec{d_2} \\ &= (A\vec{p_1} + \vec{b}) + \alpha\vec{d_1'} + \beta\vec{d_2'} \qquad (\text{Let } \vec{d_1'} = A\vec{d_1},\ \vec{d_2'} = A\vec{d_2}) \end{aligned}$$

By the proof of "lines and parallelism are preserved under affine transformation" on page 12 of 418 notes.pdf, $\vec{d_1'}$ and $\vec{d_2'}$ are not parallel.
Since T is a triangle, then $f(T(\alpha,\beta))$ also forms a triangle.

3).

Yes. By dividing the homogeneous points with some number n, ~~that~~ which equals the last component of the coordinates, and drop the last component.
Adv: with homogeneous coordinates, affine transformations become matrices and composition of them becomes matrix multiplication, which helps computing.